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Tridendriform structure on combinatorial Hopf algebras ☆

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A B S T R A C T

We extend the definition of tridendriform bialgebra by introducing a parameter q. The subspace of primitive elements of a q-tridendriform bialgebra is equipped with an associative product and a natural structure of brace algebra, related by a distributive law. This data is called q-Gerstenhaber–Voronov algebras. We prove the equivalence between the categories of conilpotent q-tridendriform bialgebras and of q-Gerstenhaber–Voronov algebras. The space spanned by surjective maps between finite sets, as well as the space spanned by parking functions, have a natural structure of q-tridendriform bialgebra, denoted $\mathbf{ST}(q)$ and $\mathbf{PQSym}(q)^*$, in such a way that $\mathbf{ST}(q)$ is a sub-tridendriform bialgebra of $\mathbf{PQSym}(q)^*$. Finally we show that the bialgebra of $\mathcal{M}$-permutations defined by T. Lam and P. Pylyavskyy comes from a q-tridendriform algebra which is a quotient of $\mathbf{ST}(q)$.

Introduction

Some associative algebras admit finer algebraic structures. Dendriform algebras were introduced by J.-L. Loday in [7] as associative algebras whose product splits into two binary operations satisfying some relations. In particular, any associative product induced somehow by the shuffle product is an example of dendriform structure. The algebraic operad describing dendriform algebras is regular, so it is determined by the free dendriform algebra on one element, which is the algebra of planar binary rooted trees described in [9]. The natural question which arises is the existence of a regular operad

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such that the free algebra spanned by one element has, as underlying vector space, the space spanned by all planar rooted trees. Here are two examples of such an operad:

(1) In [4], Frédéric Chapoton defined a $\mathcal{K}$-algebra as a differential graded dendriform algebra equipped with an extra associative product and a boundary map, satisfying certain conditions. When considering the free K-algebra on one element, the differential homomorphism on planar trees coincides with the co-boundary map of the associahedron.
(2) In a joint work with J.-L. Loday, see [10], the second author introduced the notion of tridendriform algebra, which is an associative algebra such that the product splits into three operations.

In fact the free $\mathcal{K}$-algebra is the associated graded algebra of the free tridendriform algebra.

In this paper, we define the notion of q-tridendriform algebra which is a parametrized tridendriform algebra. The advantage of this notion is that it permits us to deal simultaneously with tridendriform algebras (when $q = 1$) and the notion of $\mathcal{K}$-algebras (obtained when $q = 0$). Mimicking the definition of dendriform bialgebra given in [15], a q-tridendriform bialgebra is a bialgebra such that the associative product comes from a q-tridendriform structure, which satisfies certain compatibility relations with the coproduct.

Our main motivation to study this type of bialgebras are the following examples:

(1) Given a positive integer n, let $[n]$ denote the set $\{1, \dots, n\}$. We define a q-tridendriform bialgebra structure on the space spanned by all surjective maps from $[n]$ to $[r]$, for all positive integers $r \leqslant n$, which we denote by $\mathbf{ST}(q)$. As a vector space $\mathbf{ST}(q)$ is spanned by all the faces of the permutohedron.
(2) In [12] and [11], J.-C. Novelli and J.-Y. Thibon define the 1-tridendriform bialgebra $\mathbf{PQSym}^*$ of parking functions. This structure is generalized to any q. The natural map which associates to any parking function a surjective map is called the standardization, its dual induces a monomorphism of q-tridendriform bialgebras from $\mathbf{ST}(q)$ to $\mathbf{PQSym}^*(q)$, which differs from the one defined in [11] for $q = 1$. In a forthcoming paper, we apply this homomorphism to prove that $\mathbf{PQSym}^*(q)$ is free as a tridendriform algebra, as was conjectured in [11].
(3) The bialgebra $\mathcal{M}MR$ of big multi-permutations defined by T. Lam and P. Pylyavskyy in [6] comes from a 1-tridendriform bialgebra structure, which may be generalized to a q-tridendriform bialgebra $\mathcal{M}MR(q)$, for any q. We prove that the q-tridendriform bialgebra $\mathcal{M}MR(q)$ is a quotient of $\mathbf{ST}(q)$.

Any dendriform algebra H may be equipped with a brace algebra structure (see [14]), in such a way that whenever H is a dendriform bialgebra the subspace $Prim(H)$ of primitive elements of H is a sub-brace algebra. Moreover, the category of conilpotent dendriform bialgebras and the category of brace algebras are equivalent (see [3] and [15]). We extend these results to q-tridendriform bialgebras by introducing the notion of q-Gerstenhaber–Voronov algebras, denoted GV_q-algebras, which are brace algebras (B, M_{1n}) equipped with an associative product $\cdot$ which satisfies the distributive law:

$$M_{1n}(x \cdot y; z_1, \dots, z_n) = \sum_{0 \leqslant i \leqslant j \leqslant n} q^{j-i} M_{1i}(x; z_1, \dots, z_i) \cdot z_{i+1} \cdot \dots \cdot z_j \cdot M_{1(n-j)}(y; z_{j+1}, \dots, z_n).$$

As any q-tridendriform bialgebra has a natural structure of dendriform algebra, we show that we can associate to any q-tridendriform algebra a GV_q-algebra which has the same underlying vector space. Following the results described in [15], we prove that:

(1) the subspace of primitive elements of a q-tridendriform bialgebra H is a sub-GV_q-algebra of H,
(2) the free q-tridendriform algebra spanned by a vector space V is isomorphic, as a coalgebra, to the cotensor coalgebra of the free GV_q-algebra spanned by V,
(3) the category of conilpotent q-tridendriform bialgebras is equivalent to the category of GV_q-algebras.

Our result gives a good triple of operads for the theory of generalized bialgebras studied by Loday, cf. [8].

Let us point out that, applying Chapoton's results, the operad of GV_0-algebras may be equipped with a differential in such a way that we recover the operad $\mathcal{S}_2$ described in [16], also called *homotopy G-algebra* in [5].

The paper is organized as follows. The first section gives the definition of q-tridendriform bialgebra, illustrated by some examples. In the next section we prove the structure theorem for conilpotent q-tridendriform bialgebras and GV_q-algebras, which generalizes the Cartier–Milnor–Moore Theorem in our context. In the last section we describe the q-tridendriform structures of the bialgebras of parking functions and of big multi-permutations and prove that there exists a diagram of q-tridendriform bialgebras:

$$\mathbf{PQSym}^*(q) \hookleftarrow \mathbf{ST}(q) \twoheadrightarrow \mathcal{MMR}(q)$$

Notations

All vector spaces and algebras are over a field $\mathbb{K}$. Given a set X, we denote by $\mathbb{K}[X]$ the vector space spanned by X. For any vector space V, we denote by $V^{\otimes n}$ the tensor product of $V \otimes \cdots \otimes V$, n times, over $\mathbb{K}$. In order to simplify notation, we shall denote an element of $V^{\otimes n}$ indistinctly by $x_1 \otimes \cdots \otimes x_n$ or $(x_1, \ldots, x_n)$.

A *coalgebra* over $\mathbb{K}$ is a vector space C equipped with a linear homomorphism $\Delta : C \longrightarrow C \otimes C$ which is coassociative. A counit of a coalgebra (C, Δ) is a linear homomorphism $\epsilon : C \longrightarrow \mathbb{K}$ such that $\mu \circ (\epsilon \otimes Id_C) \circ \Delta = id_C = \mu \circ (Id_C \otimes \epsilon) \circ \Delta$, where μ denotes the action of $\mathbb{K}$ on C. The kernel of ϵ is denoted by $\overline{C}$.

For any coalgebra (C, Δ) the image of an element $x \in C$ under Δ is denoted using the Sweedler' s notation $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$.

Let (C, Δ, ϵ) be a counital coalgebra such that $C = \mathbb{K} \oplus \overline{C}$, an element $x \in C$ is *primitive* if $\Delta(x) = x \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes x$. The subspace of primitive elements of C is denoted $Prim(C)$. There exists a natural filtration on $\overline{C}$ given by:

- $F_1(C) = Prim(C)$,
- $F_n(C) := \{x \in \overline{C} \mid \overline{\Delta}(x) \in F_{n-1}C \otimes F_{n-1}C\}$,

where $\overline{\Delta}(x) = \Delta(x) - 1_{\mathbb{K}} \otimes x - x \otimes 1_{\mathbb{K}}$.

Definition. The counital coalgebra C is said to be *conilpotent* if

$$C = \mathbb{K} \oplus \bigcup_{n \geqslant 1} F_n C.$$

Given a vector space V, we denote by $T^c(V)$ the space $T(V) = \bigoplus_{n \geqslant 0} V^{\otimes n}$ equipped with the coalgebra structure given by deconcatenation:

$$\Delta^c(x_1 \otimes \cdots \otimes x_n) := \sum_{i=0}^{n} (x_1 \otimes \cdots \otimes x_i) \otimes (x_{i+1} \otimes \cdots \otimes x_n),$$

for $x_1, \ldots, x_n \in V$.

Let n be a natural number, the ordered set $\{1, \ldots, n\}$ is denoted by $[n]$. If $J = \{j_1, \ldots, j_k\} \subseteq [n]$ and $r \geqslant 1$, we denote by $J + r$ the set $\{j_1 + r, \ldots, j_k + r\}$. A *composition* of n is an ordered set $\underline{n} = (n_1, \ldots, n_r)$ of positive integers such that $\sum_{i=1}^{r} n_i = n$; while a *partition* of n is a sequence of non-negative integers $\lambda = (l_1, \ldots, l_r)$ such that $\sum_{i=1}^{r} l_i = n$.

The symmetric group of permutations of n elements is denoted by S_n. Given a composition $\underline{n} = (n_1, \dots, n_r)$ of n, an *$\underline{n}$-shuffle* is a permutation $\sigma \in S_n$ such that $\sigma(n_1 + \cdots + n_i + 1) < \cdots < \sigma(n_1 + \cdots + n_{i+1})$, for $0 \leqslant i \leqslant r-1$. We denote by $Sh(n_1, \dots, n_r)$ the set of all $\underline{n}$-shuffles.

Consider the set of maps between finite sets. We identify a function $f : [n] \longrightarrow [r]$, with its image $(f(1), \dots, f(n))$.

Given a map $f : [n] \longrightarrow [r]$ and a subset $J = \{i_1 < \cdots < i_k\} \subseteq [n]$, the restriction of f to J is the map $f|_J := (f(i_1), \dots, f(i_k))$. Similarly, for a subset K of $[r]$, the co-restriction of f to K is the map $f|^K := (f(j_1), \dots, f(j_l))$, where $\{j_1 < \cdots < j_l\} := \{i \in [n] / f(i) \in K\}$.

For any map $f : [n] \longrightarrow [r]$, let $max(f)$ be the maximal element in the image of f. If $g \in \mathcal{F}_m$ is another map, then fg is the element in $\mathcal{F}_{n+m}$ such that

$$fg(i) := \begin{cases} f(i), & \text{for } 1 \leqslant i \leqslant n, \\ g(i-n), & \text{for } n+1 \leqslant i \leqslant n+m. \end{cases}$$

We denote by $\cap(f, g)$ the cardinal of the intersection $Im(f) \cap Im(g)$.

1. Tridendriform bialgebras

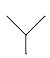
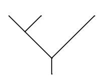
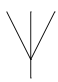
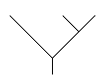
We introduce the definition of q-tridendriform algebra in such a way that specializing in $q = 1$ we get the definition of tridendriform algebra given in [10], while for $q = 0$ we get the definition of $\mathcal{K}$-algebra described in [4]. Our main goal is to study the tridendriform algebra structures of the space of parking functions defined in [11] and of the space of multipermutations introduced in [6], which we treat in the next sections. We give in the present section some other examples. The first one is described in [10] for $q = 1$ and in [4] for $q = 0$, while the second one is studied in [13] for $q = 1$ and in [4] for $q = 0$.

1.1. Definition. A *q-tridendriform algebra* is a vector space A together with three operations $\prec : A \otimes A \to A$, $\cdot : A \otimes A \to A$ and $\succ : A \otimes A \to A$, satisfying the following relations:

(1) $(a \prec b) \prec c = a \prec (b \prec c + b \succ c + q b \cdot c)$,
(2) $(a \succ b) \prec c = a \succ (b \prec c)$,
(3) $(a \prec b + a \succ b + q a \cdot b) \succ c = a \succ (b \succ c)$,
(4) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(5) $(a \succ b) \cdot c = a \succ (b \cdot c)$,
(6) $(a \prec b) \cdot c = a \cdot (b \succ c)$,
(7) $(a \cdot b) \prec c = a \cdot (b \prec c)$.

Note that the operation $* := \prec + q \cdot + \succ$ is associative. Moreover, given a q-tridendriform algebra $(A, \prec, \cdot, \succ)$, the space A equipped with the binary operations $\prec$ and $\succeq := q \cdot + \succ$ is a dendriform algebra, as defined by J.-L. Loday in [7].

1.2. Examples. a) The free tridendriform algebra. Let T_n denote the set of planar rooted trees with $n+1$ leaves. For instance,

$T_0 = \{|\}$, $T_1 = \{$  $\}$, $T_2 = \{$  ,  ,  $\}$.

The tree with $n+1$ leaves and a unique vertex (the root) is called the *n-corolla*, and denoted by c_n.

Given trees $t^1, \dots, t^r$, let $\bigvee(t^1, \dots, t^r)$ be the tree obtained by joining the roots of $t^1, \dots, t^r$, ordered from left to right, to a new root. It is easy to see that any tree $t \in T_n$ may be written in a unique way as $t = \bigvee(t^1, \dots, t^r)$, with $t^i \in T_{n_i}$ and $\sum_{i=1}^r n_i + r - 1 = n$. On the space $\mathbb{K}[T_\infty]$ spanned by the set $T_\infty := \bigcup_{n \geqslant 1} T_n$, we define operations $\prec$, $\cdot$ and $\succ$ recursively as follows:

$$t \succ | = t \cdot | = | \cdot t = | \prec t = 0, \quad \text{for all } t \in T_\infty,$$
$$| \succ t = t \prec | = t, \quad \text{for all } t \in T_\infty,$$
$$t \prec w = \bigvee\bigl(t^1, \ldots, t^{r-1}, t^r * w\bigr),$$
$$t \cdot w := \bigvee\bigl(t^1, \ldots, t^{r-1}, t^r * w^1, w^2, \ldots, w^l\bigr),$$
$$t \succ w := \bigvee\bigl(t * w^1, w^2, \ldots, w^l\bigr),$$

for $t = \bigvee(t^1, \ldots, t^r)$ and $w = \bigvee(w^1, \ldots, w^l)$, where $*$ is the associative product $* = \prec + q \cdot + \succ$ previously defined.

Note that, even if we need to consider the element $| \in T_0$ as the identity for the product $*$ in order to define the tridendriform structure on $\mathbb{K}[T_\infty]$, the elements $| \prec |$, $| \cdot |$ and $| \prec |$ are not defined.

Following [4] and [10], it is immediate to verify that the data $(\mathbb{K}[T_\infty], \prec, \cdot, \succ)$ is the free q-tridendriform algebra spanned by the unique element of T_1.

For any vector space V, the q-tridendriform structure of $\mathbb{K}[T_\infty]$ extends naturally to the space $Tridend_q(V) := \bigoplus_{n \geqslant 1} \mathbb{K}[T_n] \otimes V^{\otimes n}$ as follows:

$$(t \otimes v_1 \otimes \cdots \otimes v_n) \circ (w \otimes u_1 \otimes \cdots \otimes u_m) := (t \circ w) \otimes v_1 \otimes \cdots \otimes v_n \otimes u_1 \otimes \cdots \otimes u_m;$$

where $\circ$ is replaced either by $\succ$, or $\prec$, or $\cdot$, respectively. In this case, $Tridend_q(V)$ is the free q-tridendriform algebra spanned by V (see [4] and [10]).

b) The algebra of surjective maps. Let $\mathbf{ST}_n^r$ be the set of surjective maps from $[n]$ to $[r]$, for $1 \leqslant r \leqslant n$, and let $\mathbf{ST}_n := \bigcup_{r=1}^n \mathbf{ST}_n^r$. Given $f : [n] \longrightarrow [r]$ there exists a unique surjective map $std(f) \in \mathbf{ST}_n^r$ such that $f(i) < f(j)$ if, and only if, $std(f)(i) < std(f)(j)$, for $1 \leqslant i, j \leqslant n$. The map $std(f)$ is called the standardization of f.

For example if $f = (2, 3, 3, 5, 7)$, then $std(f) = (1, 2, 2, 3, 4)$.

Let $\times : \mathbf{ST}_n^r \times \mathbf{ST}_m^s \longrightarrow \mathbf{ST}_{n+m}^{r+s}$ be the map

$$(\alpha, \beta) \mapsto \alpha \times \beta := \bigl(\alpha(1), \ldots, \alpha(n), \beta(1) + r, \ldots, \beta(m) + r\bigr).$$

Let $\mathbf{ST}(q)$ be the vector space $\mathbf{ST} := \bigoplus_{n \geqslant 1} \mathbb{K}[\mathbf{ST}_n]$ equipped with the operations $\succ$, $\cdot$ and $\prec$ defined as follows:

$$f \succ g := \sum_{max(h) < max(k)} q^{\cap(h,k)} hk,$$
$$f \cdot g := \sum_{max(h) = max(k)} q^{\cap(h,k)-1} hk,$$
$$f \prec g := \sum_{max(h) > max(k)} q^{\cap(h,k)} hk,$$

where the sums are taken over all pairs of maps (h, k) verifying that hk is surjective, $std(h) = f$ and $std(k) = g$, for $f \in \mathbf{ST}_n$ and $g \in \mathbf{ST}_m$.

For example, if $\alpha = (1, 2, 1) \in \mathbf{ST}_3$ and $\beta = (2, 1) \in \mathbf{ST}_2$, then

$$\alpha \succ \beta = (1, 2, 1, 4, 3) + q(1, 2, 1, 3, 2) + q(1, 2, 1, 3, 1) + (1, 3, 1, 4, 2) + (2, 3, 2, 4, 1),$$
$$\alpha \cdot \beta = q(1, 2, 1, 2, 1) + (1, 3, 1, 3, 2) + (2, 3, 2, 3, 1),$$
$$\alpha \prec \beta = q(1, 3, 1, 2, 1) + (1, 4, 1, 3, 2) + q(2, 3, 2, 2, 1) + (2, 4, 2, 3, 1) + (3, 4, 3, 2, 1).$$

To check that $(\mathbf{ST}(q), \succ, \cdot, \prec)$ is a q-tridendriform algebra we refer to [4] and to [13].

c) Rota–Baxter algebras. Let $(A, \cdot)$ be an associative algebra over $\mathbb{K}$. A *Rota–Baxter operator of weight q* on A (see [1]) is a linear map $R : A \to A$ verifying that:

$$R(x) \cdot R(y) = R\big(R(x) \cdot y\big) + R\big(x \cdot R(y)\big) + qR(x \cdot y),$$

for $x, y \in A$. The data $(A, \cdot, R)$ is called an associative Rota–Baxter algebra of weight q.

Any Rota–Baxter algebra A of weight q has a natural structure of q-tridendriform algebra with the associative product $\cdot$ and the operations $\prec$ and $\succ$ given by:

$$x \prec y := x \cdot R(y),$$

$$x \succ y := R(x) \cdot y,$$

for $x, y \in A$.

Let $(A, \prec, \cdot, \succ)$ be a q-tridendriform algebra and let $A_+ := A \oplus \mathbb{K}$. We denote by $\epsilon : A_+ \longrightarrow \mathbb{K}$ the projection on the second term. For any $x \in A$, we fix $x \succ 1_{\mathbb{K}} = x \cdot 1_{\mathbb{K}} = 1_{\mathbb{K}} \cdot x = 1_{\mathbb{K}} \prec x = 0$ and $1_{\mathbb{K}} \succ x = x = x \prec 1_{\mathbb{K}}$.

1.3. Definition. A q-tridendriform bialgebra over $\mathbb{K}$ is a q-tridendriform algebra H equipped with a linear homomorphism $\Delta : H_+ \longrightarrow H_+ \otimes H_+$ verifying the following conditions:

(1) $\Delta(1_{\mathbb{K}}) = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$,
(2) $(\epsilon \otimes Id) \circ \Delta(x) = 1_{\mathbb{K}} \otimes x$ and $(Id \otimes \epsilon) \circ \Delta(x) = x \otimes 1_{\mathbb{K}}$, for all $x \in H$,
(3) $\Delta(x \succ y) := \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \succ y_{(2)})$,
(4) $\Delta(x \cdot y) := \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \cdot y_{(2)})$,
(5) $\Delta(x \prec y) := \sum (x_{(1)} * y_{(1)}) \otimes (x_{(2)} \prec y_{(2)})$,

where $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$ for all $x \in H$, and by convention:

- $(x * y) \otimes (1_{\mathbb{K}} \succ 1_{\mathbb{K}}) := (x \succ y) \otimes 1_{\mathbb{K}}$,
- $(x * y) \otimes (1_{\mathbb{K}} \cdot 1_{\mathbb{K}}) := (x \cdot y) \otimes 1_{\mathbb{K}}$,
- $(x * y) \otimes (1_{\mathbb{K}} \prec 1_{\mathbb{K}}) := (x \prec y) \otimes 1_{\mathbb{K}}$, for $x, y \in H$.

Note that if $(H, \prec, \cdot, \succ, \Delta)$ is a q-tridendriform bialgebra, then $(H_+, *, \Delta)$ is a bialgebra in the classical sense.

We describe the bialgebra structure of the q-tridendriform algebras described in Examples **a)** and **b)** of 1.2.

a) Let V be a vector space.

Given elements $x^i = (t^i; v^i_1, \dots, v^i_{n_i}) \in T_{n_i} \otimes V^{\otimes n_i}$, for $1 \leqslant i \leqslant r$ and vectors $w_1, \dots, w_{r-1} \in V$, let $\bigvee_{w_1,\dots,w_{r-1}}(x^1, \dots, x^r) :=$

$$\Big(\bigvee(t^1, \dots, t^r); v^1_1, \dots, v^1_{n_1}, w_1, v^2_1, \dots, v^{r-1}_{n_{r-1}}, w_{r-1}, v^r_1, \dots, v^r_{n_r}\Big),$$

in $T_n \otimes V^{\otimes n}$, where $n = \sum_{i=1}^r n_i + r - 1$.

The coproduct Δ on the free q-tridendriform algebra $Tridend_q(V)$ is the unique linear homomorphism satisfying that:

(1) $\Delta(1_{\mathbb{K}}) = 1_{\mathbb{K}} \otimes 1_{\mathbb{K}}$.
(2) $\Delta(c_n; v_1, \dots, v_n) := (c_n; v_1, \dots, v_n) \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes (c_n; v_1, \dots, v_n)$, for $n \geqslant 1$.

(3) $\Delta(x) := \sum \big(x^1_{(1)} * \cdots * x^r_{(1)}\big) \otimes \bigvee_{w_1,\ldots,w_{r-1}} \big(x^1_{(2)}, \ldots, x^r_{(2)}\big) + x \otimes 1_{\mathbb{K}}$,

for $x = \bigvee_{w_1,\ldots,w_{r-1}}(x^1, \ldots, x^r)$, with $x^i \in T_{n_i} \otimes V^{\otimes n_i}$.

b) For any $\alpha \in \mathbf{ST}_n$, define:

$$\Delta(f) = \sum_j f^r_{(1)} \otimes f^r_{(2)},$$

where the sum is taken over all $0 \leqslant j \leqslant n$, such that there exists $\delta_r \in Sh(j, n-j)^{-1}$ with $f = (f^r_{(1)} \times f^r_{(2)}) \cdot \delta$.

For example

$$\begin{aligned}\Delta(2,1,3,5,3,4,4,1) = 1_{\mathbb{K}} \otimes (2,1,3,5,3,4,4,1) + (1,1) \otimes (1,2,4,2,3,3)\\ + (2,1,1) \otimes (1,3,1,2,2) + (2,1,3,3,1) \otimes (2,1,1)\\ + (2,1,3,3,4,4,1) \otimes (1) + (2,1,3,5,3,4,4,1) \otimes 1_{\mathbb{K}}.\end{aligned}$$

The coproduct may also be described in terms of co-restrictions as follows:

$$\Delta(f) = \sum_{j=1}^{r} f|^{[j]} \otimes std\big(f|^{[n-j]+j}\big).$$

To see that $\mathbf{ST}(q)$ with Δ is a q-tridendriform bialgebra, suppose that $hk \in \mathbf{ST}_n$ are such that $std(h) = f$ and $std(k) = g$. It is easy to check that:

(1) if $max(h) < max(k)$, then

$$\Delta(hk) = \sum_{max(h_{(2)}) < max(k_{(2)})} h_{(1)}k_{(1)} \otimes h_{(2)}k_{(2)},$$

(2) if $max(h) = max(k)$, then

$$\Delta(hk) = \sum_{max(h_{(2)}) = max(k_{(2)})} h_{(1)}k_{(1)} \otimes h_{(2)}k_{(2)},$$

(3) if $max(h) > max(k)$, then

$$\Delta(hk) = \sum_{max(h_{(2)}) > max(k_{(2)})} h_{(1)}k_{(1)} \otimes h_{(2)}k_{(2)},$$

where both $h_{(1)}k_{(1)}$ and $h_{(2)}k_{(2)}$ are surjective.

Moreover, if $h_{(1)} = h|^{[p]\cap Im(h)}$, $h_{(2)} = h|^{[q-p]+p\cap Im(h)}$, $k_{(1)} = k|^{[r]\cap Im(k)}$ and $k_{(2)} = k|^{[s-r]+r\cap Im(k)}$, then $\cap(h,k) = \cap(h_{(1)}, k_{(1)}) + \cap(h_{(2)}, k_{(2)})$.

2. Structure theorem for tridendriform bialgebras

We want to prove that any conilpotent q-tridendriform bialgebra can be reconstructed from the subspace of its primitive elements. In order to do so we need to introduce the notions of brace algebra (see [5]) and of q-Gerstenhaber–Voronov algebra. Our construction mimics previous results obtained for dendriform bialgebras and brace algebras. Whenever the results exposed in the present work are obtained easily by applying the methods developed in [15], we refer to it for the details of the proofs.

2.1. Definition.

(1) A *brace* algebra is a vector space B equipped with $n+1$-ary operations $M_{1n}: B \otimes B^{\otimes n} \longrightarrow B$, for $n \geqslant 0$, which satisfy the following conditions:
 (a) $M_{10} = \mathrm{Id}_B$,
 (b) $$M_{1m}(M_{1n}(x; y_1, \dots, y_n); z_1, \dots, z_m) = \sum_{0 \leqslant i_1 \leqslant j_1 \leqslant \dots \leqslant j_n \leqslant m} M_{1r}\big(x; z_1, \dots, z_{i_1}, M_{1l_1}(y_1; \dots, z_{j_1}), \dots, M_{1l_n}(y_n; \dots, z_{j_n}), \dots, z_m\big),$$
 for $x, y_1, \dots, y_n, z_1, \dots, z_m \in B$, where $l_k = j_k - i_k$, for $1 \leqslant k \leqslant n$, and $r = \sum_{k=1}^{n} i_k + m - j_n + n$.

(2) A *q-Gerstenhaber–Voronov* algebra, GV_q-algebra for short, is a vector space A endowed with a brace structure given by operations M_{1n} and an associative product $\cdot$, satisfying the distributive relation:
$$M_{1n}(x \cdot y; z_1, \dots, z_n) = \sum_{0 \leqslant i \leqslant j \leqslant n} q^{j-i} M_{1i}(x; z_1, \dots, z_i) \cdot z_{i+1} \cdot \dots \cdot z_j \cdot M_{1(n-j)}(y; z_{j+1}, \dots, z_n),$$
 for $x, y, z_1, \dots, z_n \in B$.

In [15] we constructed a functor from the category of dendriform algebras to the category of brace algebras, we recall this construction. Let $(A, \prec, \tilde{\succ})$ be a dendriform algebra, we denote:
$$\omega_{\prec}(y_1, \dots, y_i) := y_1 \prec \big(y_2 \prec \cdots (y_{i-1} \prec y_i)\big)$$
$$\omega_{\tilde{\succ}}(y_{i+1}, \dots y_n) := \big((y_{i+1} \tilde{\succ} y_{i+2}) \tilde{\succ} \cdots\big) \tilde{\succ} y_n.$$

The brace operations M_{1n} are defined as follows:
$$M_{1n}(x; y_1, \dots, y_n) = \sum_{i=0}^{n} (-1)^{n-i} \omega_{\prec}(y_1, \dots, y_i) \tilde{\succ} x \prec \omega_{\tilde{\succ}}(y_{i+1}, \dots, y_n),$$
for $n \geqslant 1$.

Given any q-tridendriform algebra $(A, \prec, \cdot, \succ)$ we associate to it the brace algebra (A, M_{1n}) obtained from the dendriform algebra $(A, \prec, \tilde{\succ} = q \cdot + \succ)$.

2.2. Proposition. *If $(A, \prec, \cdot, \succ)$ is a q-tridendriform algebra, then $(A, M_{1n}, \cdot)$ is a GV_q algebra.*

Proof. We know that (A, M_{1n}) is a brace algebra, therefore it suffices to prove that $\cdot$ and M_{1n} satisfy the distributive relation:
$$M_{1n}(x \cdot y; z_1, \dots, z_n) = \sum_{0 \leqslant i \leqslant j \leqslant n} q^{j-i} M_{1i}(x; z_1, \dots, z_i) \cdot z_{i+1} \cdot \dots \cdot z_j \cdot M_{1(n-j)}(y; z_{j+1}, \dots, z_n),$$
for $x, y, z_1, \dots, z_n \in A$.

As $\omega_{\prec}(v_1,\dots,v_r)\cdot v = v_1\cdot(\omega_{\prec}(v_2,\dots,v_r)\succ v)$, for any $v_1,\dots,v_r,v\in A$, we can split the expression

$$\sum q^{j-i}M_{1i}(x;z_1,\dots,z_i)\cdot z_{i+1}\cdot\dots\cdot z_j\cdot M_{1(n-j)}(y;z_{j+1},\dots,z_n),$$

in three types of terms:

$$\begin{aligned}\textbf{a})\quad X_{r,i,j,l} &:= \big(\omega_{\prec}(z_1,\dots,z_r)\tilde{\succ} x\prec\omega_{\succsim}(z_{r+1},\dots,z_i)\big)\\ &\quad\cdot z_{i+1}\cdot\dots\cdot z_j\cdot\big(\omega_{\prec}(z_{j+1},\dots,z_l)\succ y\prec\omega_{\succsim}(z_{l+1},\dots,z_n)\big),\end{aligned}$$

with $j-i\geqslant 1$,

$$\textbf{b})\quad Y_{r,i,l} := \big(\omega_{\prec}(z_1,\dots,z_r)\succ x\prec\omega_{\succsim}(z_{r+1},\dots,z_i)\big)\cdot\big(\omega_{\prec}(z_{i+1},\dots,z_l)\succ y\prec\omega_{\succsim}(z_{l+1},\dots,z_n)\big),$$

$$\begin{aligned}\textbf{c})\quad Z_{r,i,l} &:= \big(z_1\cdot\big(\omega_{\prec}(z_2,\dots,z_r)\succ x\prec\omega_{\succsim}(z_{r+1},\dots,z_i)\big)\big)\\ &\quad\cdot\big(\omega_{\prec}(z_{i+1},\dots,z_l)\succ y\prec\omega_{\succsim}(z_{l+1},\dots,z_n)\big).\end{aligned}$$

For $j-i\geqslant 1$, the term $X_{r,i,j,l}$ appears in:

- $M_{1,i}(x;z_1,\dots,z_i)\cdot z_{i+1}\cdot\dots\cdot z_j\cdot M_{1,n-j}(y;z_{j+1},\dots,z_n)$ with the coefficient $q^{j-i}(-1)^{i+l}$,
- $M_{1,i}(x;z_1,\dots,z_i)\cdot z_{i+1}\cdot\dots\cdot z_{j-1}\cdot M_{1,n-j}(y;z_{j_1},\dots,z_n)$ with the coefficient $q^{j-i-1}\cdot q\cdot(-1)^{i+l+1}$.

So, the coefficient of $X_{r,i,j,l}$ is $q^{j-1}[(-1)^{i+l}+(-1)^{i+l+1}]=0$, and therefore

$$\begin{aligned}&\sum q^{j-i}M_{1,i}(x;z_1,\dots,z_i)\cdot z_{i+1}\cdot\dots\cdot z_j\cdot M_{1,n-j}(y;z_{j+1},\dots,z_n)\\ &\quad=\sum_{0\leqslant r\leqslant i\leqslant l\leqslant n}(-1)^{r+l-i}Y_{r,i,l}+q\sum_{1\leqslant r\leqslant i\leqslant l\leqslant n}(-1)^{r+l-i}Z_{r,i,l}.\end{aligned}$$

For $r<l$, we have that

$$\begin{aligned}Y_{r,i,l} &= \big(\big(\omega_{\prec}(z_1,\dots,z_r)\succ x\big)\prec\big(\omega_{\succsim}(z_{r+1},\dots,z_i)\star\omega_{\prec}(z_{i+1},\dots,z_l)\big)\big)\cdot\big(y\prec\omega_{\succsim}(z_{l+1},\dots,z_n)\big)\\ &= \big(\omega_{\prec}(z_1,\dots,z_r)\succ x\big)\cdot\big(\big(\omega_{\succsim}(z_{r+1},\dots,z_i)\star\omega_{\prec}(z_{i+1},\dots,z_l)\big)\succ\big(y\prec\omega_{\succsim}(z_{l+1},\dots,z_n)\big)\big).\end{aligned}$$

If $r<i$, then

$$\begin{aligned}&\omega_{\succsim}(z_{r+1},\dots,z_i)\star\omega_{\prec}(z_{i+1},\dots,z_l)\\ &\quad=\omega_{\succsim}(z_{r+1},\dots,z_{i+1})\prec\omega_{\prec}(z_{i+2},\dots,z_l)+\omega_{\succsim}(z_{r+1},\dots,z_i)\prec\omega_{\prec}(z_{i+1},\dots,z_l)\end{aligned}$$

which implies that:

$$\begin{aligned}&\sum_{i=r}^{l}(-1)^i\omega_{\succsim}(z_{r+1},\dots,z_i)\star\omega_{\prec}(z_{i+1},\dots,z_l)\\ &\quad=(-1)^r\big(\omega_{\prec}(z_{r+1},\dots,z_l)-z_{r+1}\prec\omega_{\prec}(z_{r+2},\dots,z_l)\big)=0,\end{aligned}$$

Therefore, we get that $\sum_{i=r}^{l}(-1)^{r+l-i}Y_{r,i,l}=0$, for $r<l$. So,

$$\sum_{0\leqslant r\leqslant i\leqslant l\leqslant n}(-1)^{r+l-i}Y_{r,i,l}=\sum_{0\leqslant r\leqslant n}(-1)^{r}Y_{r,r,r}$$
$$=\sum_{0\leqslant r\leqslant n}(-1)^{r}\omega_{\prec}(z_1,\dots,z_r)\succ(x\cdot y)\prec\omega_{\succsim}(z_{r+1},\dots,z_n).$$

Applying an analogous argument we get that

$$\sum_{1\leqslant r\leqslant i\leqslant l\leqslant n}(-1)^{r+l-i}Z_{r,i,l}=\sum_{1\leqslant r\leqslant n}(-1)^{r}Z_{r,r,r}$$
$$=\big(\omega_{\prec}(z_1,\dots,z_r)\cdot(x\cdot y)\prec\omega_{\succsim}(z_{r+1},\dots,z_n)\big).$$

We can conclude that:

$$\sum_{0\leqslant i\leqslant j\leqslant n}q^{j-i}M_{1,i}(x;z_1,\dots,z_i)\cdots z_{i+1}\cdot\dots\cdot z_j\cdot M_{1,n-j}(y;Z_{j+1},\dots,z_n)$$
$$=\sum_{r=0}^{n}\omega_{\prec}(z_1,\dots,z_r)\tilde{\succ}(x\cdot y)\prec\omega_{\succsim}(z_{r+1},\dots,z_n)M_{1n}(x\cdot y;z_1,\dots,z_n),$$

which ends the proof. □

Proposition 2.2 states that there exists a functor $\mathcal{F}$ from the category of q-tridendriform algebras to the category of GV_q algebras. Conversely, for a GV_q-algebra $(B,\tilde{M}_{1n},\cdot)$, let

$$U_{\mathrm{qGV}}(B):=\mathrm{TriDend}(B)/\mathcal{I}$$

where $\mathcal{I}$ is the tridendriform ideal spanned by the elements:

$$\tilde{M}_{1n}(x;y_1,\dots,y_n)-\sum_{i=0}^{n}(-1)^{n-i}\omega_{\prec}(y_1,\dots,y_i)\tilde{\succ}x\prec\omega_{\succsim}(y_{i+1},\dots,y_n),$$

for all $x,y_1,\dots,y_n\in B$. A standard argument shows that U_{qGV} is a left adjoint of $\mathcal{F}$.

The following result shows that the subspace of primitive elements of H is a GV_q-algebra.

2.3. Lemma. *Let $(H,\prec,\cdot,\succ,\Delta)$ a q-tridendriform bialgebra. If the elements $x,y,z_1,\dots,z_n$ of H are primitive, then $M_{1n}(x;z_1,\dots,z_n)$ and $x\cdot y$ are primitive, too.*

Proof. If x and y are primitive, then

$$\Delta(x\cdot y)=x\cdot y\otimes 1_{\mathbb{K}}+x\otimes(1_{\mathbb{K}}\cdot y)+y\otimes(x\cdot 1_{\mathbb{K}})+1_{\mathbb{K}}\otimes x\cdot y=x\cdot y\otimes 1_{\mathbb{K}}+1_{\mathbb{K}}\otimes x\cdot y,$$

because $1_{\mathbb{K}}\cdot y=x\cdot 1_{\mathbb{K}}=0$.

To see that $M_{1n}(x;z_1,\dots,z_n)$ is primitive, it suffices to note that the brace operation M_{1n} on the q-tridendriform algebra $(H,\prec,\cdot,\succ)$ coincides with the brace defined on the dendriform algebra $(H,\prec,\tilde{\succ}:=q\cdot+\succ)$ in [15]. Since $(H,\prec,\tilde{\succ},\Delta)$ is a dendriform bialgebra, it suffices to apply the result of [15]. □

Let $(H,\prec,\cdot,\succ,\Delta)$ be a q-tridendriform bialgebra, we say that H is conilpotent if (H_+,Δ) is a conilpotent coalgebra. For $n\geqslant 1$, define linear maps $\succ^n: H^{\otimes n}\longrightarrow H$ and $\overline{\Delta}^n: H\longrightarrow H^{\otimes n}$ as follows:

$$\succ^1 = \mathrm{Id}, \tag{1}$$

$$\succ^n = \succ^{n-1} \circ \big(\mathrm{Id}^{\otimes n-2} \otimes \succ\big), \tag{2}$$

$$\overline{\Delta}^1 = \mathrm{Id}, \tag{3}$$

$$\overline{\Delta}^n = \big(\mathrm{Id}^{\otimes n-2} \otimes \overline{\Delta}\big) \circ \overline{\Delta}^{n-1}. \tag{4}$$

Note that in this case $H = \overline{H_+}$.

Let $e_{tri} : H \longrightarrow H$ be the linear map given by

$$e(x) := \sum_{n \geqslant 1} (-1)^{n+1} \succ^n \circ \overline{\Delta}^n(x).$$

For any element $x \in H$, we have that $e_{tri}(x) = x - \sum x_{(1)} \succ e_{tri}(x_{(2)})$, for $\overline{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)}$. The previous equality implies that:

(1) If $x \in Prim(H)$, then $e_{tri}(x) = x$.
(2) Whenever $x = y \succ z \in F_n(H)$ for elements $y, z \in F_r(H)$ with $r < n$, a recursive argument on n shows that $e_{tri}(x) = 0$.

So, we may consider e_{tri} as a projection from H to $Prim(H)$. Moreover, the proposition below shows that any element $x \in H$ may be described in terms of the operation $\succ$ and primitive elements.

2.4. Proposition. *Let $(H, \prec, \cdot, \succ, \Delta)$ be a conilpotent q-tridendriform bialgebra. Any element $x \in F_n(H)$ satisfies that*:

$$x = e_{tri}(x) + \sum e_{tri}(x_{(1)}) \succ e_{tri}(x_{(2)}) + \cdots + \sum \omega_{\succ}\big(e_{tri}(x_{(1)}), \ldots, e_{tri}(x_{(n)})\big)$$
$$= \sum_{r=1}^{n} \Big(\sum \omega_{\succ}\big(e_{tri}(x_{(1)}), \ldots, e_{tri}(x_{(r)})\big)\Big),$$

where $\overline{\Delta}^r(x) = \sum x_{(1)} \otimes \cdots \otimes x_{(r)}$ and

$$\omega_{\succ}\big(e_{tri}(x_{(1)}), \ldots, e_{tri}(x_{(r)})\big) := \Big(\Big(\big(e_{tri}(x_{(1)}) \succ e_{tri}(x_{(2)})\big) \succ e_{tri}(x_{(3)})\Big) \succ \cdots\Big) \succ e_{tri}(x_{(r)}).$$

Proof. Since H is conilpotent, any element x belongs to $F_n(H)$, for some $n \geqslant 1$. We have also that $x - e_{tri}(x) = \sum x_{(1)} \succ e_{tri}(x_{(2)})$. The result is clear for $n = 1$.

For $n \geqslant 2$, $\overline{\Delta}(x) = \sum x_{(1)} \otimes x_{(2)}$, with $x_{(1)}$ and $x_{(2)}$ in $F_{n-1}(T)$. By a recursive argument, we get that

$$x_{(1)} = \sum_{r=1}^{n-1} \Big(\sum \omega_{\succ}\big(e_{tri}(x_{(1)(1)}), \ldots, e_{tri}(x_{(1)(r)})\big)\Big).$$

So,

$$x = e_{tri}(x) + \sum \left(\sum_{r=1}^{n-1} \Big(\sum \omega_{\succ}\big(e_{tri}(x_{(1)(1)}), \ldots, e_{tri}(x_{(1)(r)})\big) \succ e_{tri}(x_{(2)})\Big)\right)$$
$$= \sum_{r=1}^{n} \Big(\sum \omega_{\succ}\big(e_{tri}(x_{(1)}), \ldots, e_{tri}(x_{(r)})\big)\Big),$$

which ends the proof. $\square$

2.5. Remark. (See [15].) If the elements $x_1, \dots, x_n$ belong to $Prim(H)$, then

$$\Delta\big(\omega_{\succ}(x_1, \dots, x_n)\big) = \sum_{i=0}^{n} \omega_{\succ}(x_1, \dots, x_i) \otimes \omega_{\succ}(x_{i+1}, \dots, x_n),$$

where $\omega_{\succ}(\emptyset) := 1_{\mathbb{K}}$.

Note that Proposition 2.4 and Remark 2.5 imply that for any conilpotent q-tridendriform bialgebra $(H, \prec, \cdot, \succ, \Delta)$, the linear homomorphism from (H_+, Δ) to the cotensor coalgebra $T^c(Prim(H))$ which sends an element $x \in F_n(H)$ to $\sum_{r=1}^{n}(\sum e_{tri}(x_{(1)(1)}) \otimes \cdots \otimes e_{tri}(x_{(1)(r)}))$ is an isomorphism of coalgebras.

We have proved that the subspace of primitive elements of a q-tridendriform bialgebra has a natural structure of GV_q algebra. In fact, there exists an equivalence between the category of conilpotent q-tridendriform bialgebras and the category of GV_q algebras. The last part of the section is devoted to this result.

2.6. Proposition. *Let V be a $\mathbb{K}$-vector space. The primitive part of the free q-tridendriform algebra $Tridend_q(V)$ is the free GV_q algebra over V.*

Proof. To prove the result we may assume that V is a finite dimensional space over $\mathbb{K}$, the general case follows by taking a direct limit.

Suppose that $dim_K(V) = m$ and that $\mathcal{B}$ is a basis of V. We know that a basis for the space $Tridend_q(V)_n$, of homogeneous elements of degree n of $Tridend_q(V)$, is given by the set $T_n \times \mathcal{B}^n$ whose cardinal is $C_n m^n$, where $C_n = |T_n|$ is the super-Catalan number. But the vector space $Tridend_q(V)$ is isomorphic to the tensor space $T(Prim(Tridend_q(V))$, which implies that the dimension of $Prim(Tridend_q(V)_n)$ is $C_{n-1} m^n$.

The paragraph above implies that there exists a bijection between the set $T_{n-1} \times \mathcal{B}^n$ of elements $(t; b_1, \dots, b_n) \in T_n \times \mathcal{B}^n$ such that $t = \bigvee(|, t^2, \dots, t^r)$ and a basis of the space of primitive elements of $Tridend(V)$. Let $T_n^{\succ}$ denote the set of all trees in T_n of the form $\bigvee(t^1, \dots, t^r)$ with $|t^1| \geqslant 1$. We have that for any $t = \bigvee(|, t^2, \dots, t^r)$, $e_{tri}((t; b_1, \dots, b_n) = (t; b_1, \dots, b_n) + z$ where z belongs to the subspace spanned by $T_n^{\succ} \times \mathcal{B}^n$, which implies that the set of elements $e_{tri}((t; b_1, \dots, b_n)$, with $t \in T_n^{\succ}$ form a basis of $Prim(Tridend_q(V))$.

On the other hand, the free GV_q algebra $GV_q(V)$ spanned by V has a basis $GV_q(\mathcal{B})$ whose elements of degree n may be described recursively as follows:

(1) $GV_q(\mathcal{B})_1 = \mathcal{B}$,
(2) $GV_q(\mathcal{B})_n$ is the set of all elements of the form

$$M_{1n_1}\big(b_1; y_1^1, \dots, y_{n_1}^1\big) \cdot \cdots \cdot M_{1n_r}\big(b_r; y_1^r, \dots, y_{n_r}^r\big),$$

where $b_1, \dots, b_r \in \mathcal{B}$, $y_j^i \in GV_q(\mathcal{B})_{n_{ij}}$, with $n_{ij} < n$, and $0 \leqslant n_i$ for $1 \leqslant i \leqslant r$.

To end the proof it suffices to note that there exists a unique bijective map φ from $GV_q(\mathcal{B})$ to $T_n^{\succ} \times \mathcal{B}^n$ such that:

(1) $\varphi_1(b) = (c_1, b)$, for $b \in \mathcal{B}$,
(2) $\varphi_m(M_{1n}(b; y_1, \dots, y_n) = (c_1, b) \prec \omega_{\succ}(\varphi_{m_1}(y_1), \dots, \varphi_{m_n}(y_n))$, where $\omega_{\succ}(x_1, \dots, x_n) = (((x_1 \succ x_2) \succ x_3) \dots) \succ x_n$.
(3) $\varphi_m(y_1, \dots, y_n) = \varphi_{m_1}(y_1) \cdot \cdots \cdot \varphi_{m_n}(y_n)$.

Since $Prim(Tridend_q(V))$ is a GV_q algebra which contains V, it must be isomorphic to $GV_q(V)$. □

Applying the previous results we may show that the category of conilpotent q-tridendriform bialgebras is equivalent to the category of q-Gerstenhaber–Voronov algebras.

2.7. Theorem. *Let $(H, \prec, \cdot, \succ)$ be a q-tridendriform bialgebra.*

(1) *If H is conilpotent then H is isomorphic to the enveloping tridendriform algebra $U_{\mathrm{qGV}}(Prim(H))$.*
(2) *Any GV_q algebra B is isomorphic to the primitive algebra $Prim(U_{\mathrm{qGV}}(B))$ of its enveloping algebra.*

Proof. We give the main line of the proof, for the details we refer to the analogous result for conilpotent dendriform bialgebras proved in [15].

If H is a conilpotent q-tridendriform bialgebra, we know that H is isomorphic as a coalgebra to $T^c(Prim(H))$. To prove the first statement, it suffices to verify that the composition:

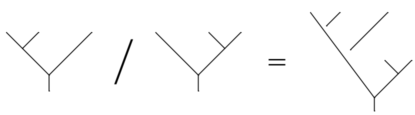
$$H \longrightarrow T^c\big(Prim(H)\big) \cong Tridend_q\big(Prim(H)\big) \longrightarrow U_{\mathrm{qGV}}\big(Prim(H)\big),$$

is an isomorphism of q-tridendriform bialgebras, which is straightforward to check.

For the second point, it is clear that $B \subseteq Prim(U_{\mathrm{qGV}}(B))$. On the other hand, we have that $Prim(Tridend_q(B)) = GV_q(B)$. Since in enveloping algebra $U_{\mathrm{qGV}}(B)$ we identify the elements of $Prim(Tridend_q(B))$ with elements of B, we get the result. $\square$

Proposition 2.6 gives an easy way to compute the free q-Gerstenhaber–Voronov algebra spanned by a vector space V. Let X be a basis of V, we know that $Tridend_q(V)$ is isomorphic, as a coalgebra, to $T^c(GV_q(V))$. We know that the underlying vector space of $Tridend_q(V)$ is the vector space spanned by the set $\bigcup_{n\geqslant 1} T_n \times X^n$ of all pairs $(t, x_1 \times \cdots \times x_n)$, where t is a rooted planar tree and $x_1, \ldots, x_n$ are elements of X.

On the other hand, define the product $/$ on the graded vector space $\mathbb{K}[T_{\infty,X}] = \bigoplus_{n\geqslant 1} \mathbb{K}[T_n \times X^n]$ by setting that $(t, x_1 \times \cdots \times x_n)/(w, y_1 \times \cdots \times y_m)$ is the element $(t/w, x_1 \times \cdots \times x_n \times y_1 \times \cdots \times y_m)$, where t/w is the tree obtained grafting the root of t to the first leaf of w. For example,

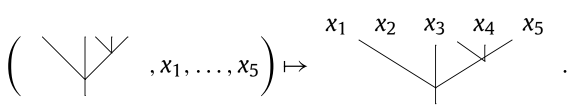


The product $/$ is graded and associative. Moreover, $(\mathbb{K}[T_{\infty,X}], /)$ is the free associative algebra spanned by the colored trees of the form $(t, x_1 \times \cdots \times x_n)$, with $t = \bigvee(|, t^2, \ldots, t^r)$. Given a tree $t = \bigvee(|, t^2, \ldots, t^r)$, the tree $t' \in T_{n-1}$ is defined as follows:

$$t' := \begin{cases} \bigvee(t^2, \ldots, t^r), & \text{for } r > 2, \\ t^2, & \text{for } r = 2. \end{cases}$$

The map $t \mapsto t'$ gives a bijection from T_n to $T_{n-1} \cup T_{n-1}$, where t maps to t' in the first copy of $T_{n-1} \cup T_{n-1}$ for $r > 2$ and t maps to t' in the second copy of $T_{n-1} \cup T_{n-1}$ for $r = 2$. So, the vector spaces $\mathbb{K}[T_{\infty,X}]$ and $\mathbb{K}[X] \oplus \bigoplus_{n\geqslant 2}(\mathbb{K}[T_{n-1} \times X^n] \oplus \mathbb{K}[T_{n-1} \times X^n])$ are isomorphic. Proposition 2.6 states that the set $T_{n-1} \times X^n \cup T_{n-1} \times X^n$ is a basis of the subspace of homogeneous elements of degree n of $GV_q(V)$, for $n \geqslant 2$.

We identify the element $(t, x_1 \times \cdots \times x_n)$ in the first copy of $T_{n-1} \times X^n$ with the tree t, with its leaves colored by the elements $x_1, \ldots, x_n$ from left to right and the root colored with $\cdot$, while the element $(t, x_1 \times \cdots \times x_n)$ in the second copy of $T_{n-1} \times X^n$ is identified with the same colored tree excepted that the root is colored by the letter M. For instance



Let us denote by $\cdot(t, x_1 \times \cdots \times x_n)$ the tree t with its leaves colored by the elements x_i and its root colored by $\cdot$, and by $M(t, x_1 \times \cdots \times x_n)$ the same colored tree but with the rooted colored M instead of $\cdot$. Given planar rooted trees $(t^1, \dots, t^r)$, let $\mathrm{Comb}(t^1, \dots, t^r)$ be the tree $(t^1 \vee t^2) \vee \cdots) \vee t^r$. It is easy to see that for any planar rooted tree t there exist unique integers m, r and unique planar trees $t^1, \dots, t^r$ such that $t = \mathrm{Comb}(c_{m-1}, t^1, \dots, t^r)$, here c_{m-1} is the tree with m leaves and a unique vertex.

We want to define a bijective map α_n from $T_{n-1} \times X^n \cup \mathbb{K}[T_{n-1} \times X^n$ to a basis of the subspace of homogeneous elements of degree n of $GV_q(V)$. For $n = 1$, the set $\mathcal{B}_1 := X$ is a basis of $GV_q(V)_1$. We identify each element of $x \in X$ with the pair (c_0, x), that is the tree with a unique leaf, colored by x, and no vertex.

For $n = 2$, the set $\mathcal{B}_2 := \{x \cdot y \mid x, y \in X\} \cup \{M_{11}(x; y) \mid x, y \in X\}$ is a basis of $GV_q(V)_2$. We define

$\alpha_2(\cdot($  $, x \times y)) := x \cdot y$, and $\alpha_2(M(\cdot($  $, x \times y)) := M_{11}(x; y)$.

For $n > 2$, the definition of GV_q algebra implies that the set

$$\mathcal{B}_n := \Big\{z_1 \cdot \cdots \cdot z_r \;\Big|\; z_i \in \mathcal{B}_{n_i},\ \sum n_i = n\Big\} \cup \Big\{M_{1r}(x; z_1, \dots, z_r) \;\Big|\; x \in X,\ z_i \in \mathcal{B}_{n_i},\ \sum n_i = n - 1\Big\},$$

is a basis of the vector space $GV_q(V)_n$. Define α_n recursively as follows:

$$\begin{aligned}
&\alpha_n\big(\cdot(t; x_1, \dots, x_n)\big) := \alpha_{n_1}\big(M\big(t^1, x_1, \dots, x_{n_1}\big)\big) \cdot \cdots \cdot \alpha_{n_r}\big(M\big(t^r, x_{n-n_r+1}, \dots, x_n\big)\big),\\
&\quad \text{for } t = \bigvee\big(t^1, \dots, t^r\big),\\
&\alpha_n\big(M(t; x_1, \dots, x_n)\big) := M_{1r}\big(x_1; \alpha_{n_1}\big(\cdot\big(t^1, x_2, \dots, x_{n_1+1}\big)\big), \dots, \alpha_{n_r}\big(\cdot\big(t^r, x_{n-n_r+1}, \dots, x_n\big)\big)\big),\\
&\quad \text{for } t = \mathrm{Comb}\big(c_0, t^1, \dots, t^r\big),\\
&\alpha_n\big(M(t; x_1, \dots, x_n)\big) := M_{1r}\big(x_1 \cdot \cdots \cdot x_{m-1}; \alpha_{n_1}\big(\cdot\big(t^1, x_m, \dots, x_{m+n_1-1}\big)\big), \dots,\\
&\qquad\qquad\qquad\qquad\qquad \alpha_{n_r}\big(\cdot\big(t^r, x_{n-n_r+1}, \dots, x_n\big)\big)\big),\\
&\quad \text{for } t = \mathrm{Comb}\big(c_m, t^1, \dots, t^r\big), \quad \text{with } m > 0,
\end{aligned}$$

where $M_{1r}(x_1 \cdot \cdots \cdot x_{m-1}; \alpha_{n_1}(\cdot(t^1, x_m, \dots, x_{m+n_1-1})), \dots, \alpha_{n_r}(\cdot(t^r, x_{n-n_r+1}, \dots, x_n)))$ may be written as a sum of elements of $\mathcal{B}_n$, applying the relationship between the operations M_{1r}'s and $\cdot$.

The construction above gives a simple description of free q-Gerstenhaber–Voronov algebras. This description and Theorem 2.7 will permit us to show that the tridendriform algebra of surjective maps and the tridendriform algebra of parking functions, which we describe in the next section of this paper, are free. These results are the object of a second paper, under redaction.

2.8. Example. For $n \geqslant 1$, consider the subset Irr_n of irreducible elements of $\mathbf{ST}_n$ defined as $Irr_n := \mathbf{ST}_n \setminus \bigcup_{i=1}^{n-1} \mathbf{ST}_i \times \mathbf{ST}_{n-i}$. The product $\times$ defines on the space $\mathbf{ST} := \bigoplus_{n \geqslant 1} \mathbb{K}[\mathbf{ST}_n]$ a structure of free associative algebra spanned by the set $\bigcup_{n \geqslant 1} Irr_n$, which implies that the dimension of the subspace of homogeneous elements of degree n of $Prim\mathbf{ST}(q)$ coincides with $|Irr_n|$.

There exists a natural way to describe a basis of $Prim\mathbf{ST}(q)$, it suffices to observe that for all $f \in \mathbf{ST}_n$, the primitive element $e_{tri}(f) = f + \sum_i f_i$, with $f_i \in \bigcup_{i=1}^{n-1} \mathbf{ST}_i \times \mathbf{ST}_{n-i}$ for all i. So applying the idempotent e_{tri} to the irreducible elements of $\bigoplus_{n \geqslant 1} \mathbf{ST}_n$ we get a basis of $Prim\mathbf{ST}(q)$.

However, there exist another way to describe a basis of $Prim\mathbf{ST}(q)$, which generalizes the construction of a basis of the subspace of primitive elements of the Malvenuto–Reutenauer Hopf algebra given in [2]. Consider on the set $\mathbf{ST}_n$ the partial order spanned by the relation $f < f \cdot s_i$, if $f(i) < f(i+1)$, where s_i is the permutation of S_n which exchanges i and $i+1$. For example

$$\begin{aligned}(2,1,2,3,4) &< (2,2,1,3,4) < (2,2,1,4,3) < (2,2,4,1,3) < (2,2,4,3,1)\\ &< (2,4,2,3,1) < (4,2,2,3,1) < (4,2,3,2,1) < (4,3,2,2,1).\end{aligned}$$

Clearly, the Hasse diagram of the partially ordered set $(\mathbf{ST}_n, <)$ is not connected, two elements f and g are in the same component if, and only if, $|f^{-1}(j)| = |g^{-1}(j)|$ for all $1 \leqslant j \leqslant n$.

For any $f \in \mathbf{ST}_n$, define the element $M_f = \sum_{g \leqslant f} \mu(g; f) g \in \mathbb{K}[\mathbf{ST}_n]$, where μ is the Moebius function of the poset $\mathbf{ST}_n$. Applying the same arguments given in [2], we get that $\Delta(M_f) = \sum_{g \times h = f} M_g \otimes M_h$, so the collection $\{M_f\}_{f \in Irr_n}$ is a basis of the subspace of homogeneous elements of degree n of $Prim\,\mathbf{ST}(q)$.

For instance, consider $f = (3, 2, 1) \in Irr_3$, the primitive elements associated to f are given by:

$$\begin{aligned}e_{tri}(3,2,1) &= (3,2,1) - (1,3,2) - (2,3,1) + (1,2,3) + q\big((1,1,2) - (1,2,1)\big)\\ &= M_{(3,2,1)} q\big((1,1,2) - (1,2,1)\big).\end{aligned}$$

3. Tridendriform structure on the spaces of parking functions and of multipermutations

3.1. Parking functions

In [11], J.-C. Novelli and J.-Y. Thibon defined a 1-tridendriform structure on the space $\mathbf{PQSym}^*$ spanned by parking functions. We show that their result extends naturally to any q, in such a way that the coalgebra structure on the parking functions gives a q-tridendriform bialgebra on $\mathbf{PQSym}^*$. Our main result is that the q-tridendriform bialgebra $\mathbf{ST}(q)$ is a sub-tridendriform bialgebra of $\mathbf{PQSym}^*(q)$. We begin by recalling some basic definitions about parking functions, for a more complete description we refer to [11].

3.2. Definition. A map $f : [n] \to [n]$ is called an *n-non-decreasing parking function* if $f(i) \leqslant i$ for $1 \leqslant i \leqslant n$. The set of n-non-decreasing parking functions is denoted by $NDPF_n$.

The composition $f := f^{\uparrow} \circ \sigma$ of a non-decreasing parking function $f^{\uparrow} \in NDPF_n$ and a permutation $\sigma \in S_n$ is called an *n-parking function*. The set of n-parking functions is denoted by PF_n.

Note that given a parking function $f = f^{\uparrow} \circ \sigma$, the non-decreasing parking function $f^{\uparrow}$ is uniquely determined but σ is not unique. However, if $r_i = |f^{-1}(i)|$, for $1 \leqslant i \leqslant n$, then there exists a unique $(r_1, \ldots, r_n)$-shuffle σ_0 such that $f = f^{\uparrow} \circ \sigma_0^{-1}$.

3.3. Example. In low dimensions, the sets $NDPF_n$ and PF_n are described as follows:

- $NDPF_1 = \{(1)\}$, $NDPF_2 = \{(1,2), (1,1)\}$,
- $NDPF_3 = \{(1,2,3), (1,1,2), (1,1,3), (1,2,2), (1,1,1)\}$,
- $PF_1 = \{(1)\}$, $PF_2 = \{(1,2), (1,1), (2,1)\}$,
- $PF_3 = S_3 \cup (1,2,2) \circ Sh(1,2)^{-1} \cup (1,1,2) \circ Sh(2,1)^{-1} \cup (1,1,3) \circ Sh(2,1)^{-1} \cup \{(1,1,1)\}$.

Recall that the cardinal of $NDPF_n$ is the Catalan number $c_n = \frac{(2n)!}{(n+1)!n!}$, while the number of elements of PF_n is $(n+1)^{n-1}$.

The map $Park : \bigcup_{n \geqslant 1} \mathcal{F}_n \longrightarrow \bigcup_{n \geqslant 1} PF_n$ (see [12]) is defined as follows. Let $f^{\uparrow} : [n] \longrightarrow [r]$ be a non-decreasing function, the element $Park(f^{\uparrow})$ is given by:

$$Park\big(f^{\uparrow}\big)(j) := \begin{cases} 1, & \text{for } j = 1,\\ Min\{Park(f^{\uparrow})(j-1)) + f^{\uparrow}(j) - f^{\uparrow}(j-1), j\}, & \text{for } j > 1.\end{cases}$$

Suppose now that $f = f^{\uparrow} \circ \sigma$, where $f^{\uparrow}$ is a non-decreasing function and σ is a permutation. Define

$$Park(f) := Park\big(f^{\uparrow}\big) \circ \sigma.$$

3.3.1. Remark. Let $f \in PF_n$ be a parking function. It is easy to check that:

(1) $f(i) = f(j)$ if, and only if $Park(f)(i) = Park(f)(j)$,
(2) $f(i) < f(j)$ if, and only if $Park(f)(i) < Park(f)(j)$,

for $1 \leqslant i, j \leqslant n$.

There exists a natural embedding $\times_{\mathcal{P}} : PF_n \times PF_m \hookrightarrow PF_{n+m}$ given by:

$$f \times_{\mathcal{P}} g := \big(f(1), \ldots, f(n), g(1) + n, \ldots, g(m) + n\big), \quad \text{for } f \in PF_n \text{ and } g \in PF_m.$$

Note that it is not the same that the one considered on **ST**, which is denoted $\times$.

Let **PQSym*** denote the vector space spanned by the set $\bigcup_{n\geqslant 1} PF_n$ of parking functions. For any $q \in \mathbb{K}$, we endow **PQSym*** with a structure of q-tridendriform bialgebra, which extends the J.-C. Novelli and J.-Y. Thibon construction of 1-tridendriform bialgebra on this space.

The binary operations $\prec$, $\cdot$ and $\succ$ on **PQSym*** are defined in a similar way that in the case of **ST**:

$$f \prec g := \sum_{max(h) > max(k)} q^{\cap(h,k)} hk,$$

$$f \cdot g := \sum_{max(h) = max(k)} q^{\cap(h,k)-1} hk,$$

$$f \succ g := \sum_{max(h) < max(k)} q^{\cap(h,k)} hk,$$

where the sums are taken over all pairs of maps (h, k) verifying that hk is parking, $Park(h) = f$ and $Park(k) = g$, for $f, g \in \bigcup_{n\geqslant 1} PF_n$.

For example, if $f = (1, 3, 1) \in PF_3$ and $g = (1, 1) \in PF_1$, then

$$\begin{aligned} f \prec g = {} & (2,4,2,1,1) + (2,5,2,1,1) + (3,5,3,1,1) + q\big((1,3,1,1,1) + (1,4,1,1,1) \\ & + (1,5,1,1,1)\big) + (1,3,1,2,2) + (1,4,1,2,2) + (1,4,1,3,3) + (1,5,1,2,2) \\ & + (1,5,1,3,3) + (1,5,2,4,4), \\ f \succ g = {} & (1,3,1,4,4), \\ f \cdot g = {} & (1,3,1,3,3). \end{aligned}$$

Applying the same arguments that in [11] it is easily seen that $(\mathbf{PQSym}^*, \prec, \cdot, \succ)$ is a q-tridendriform algebra. We denote by **PQSym**$^*(q)$ the space **PQSym*** endowed with the structure of q-tridendriform algebra.

Define a coproduct Δ on **PQSym*** by setting for $f \in PF_n$:

$$\Delta(f) = \sum_j f^j_{(1)} \otimes f^j_{(2)},$$

where the sum is taken over all $0 \leqslant j \leqslant n$ such that there exist $f^j_{(1)} \in PF_j$, $f^j_{(2)} \in PF_{n-j}$ and $\delta_j \in Sh(j, n-j)^{-1}$ with $f = (f^j_{(1)} \times_{\mathcal{P}} f^j_{(2)}) \circ \delta_j$. Note that for any $0 \leqslant j \leqslant n$, if the decomposition $f = (f^j_{(1)} \times_{\mathcal{P}} f^j_{(2)}) \circ \delta_j$ exists, then the elements $f^j_{(1)}$, $f^j_{(2)}$ and δ_j are unique.

For example,

$$\Delta\big((1,5,5,3,6,2,3)\big) = (1,5,5,3,6,2,3) \otimes 1_{\mathbb{K}} + (1,3,2,3) \otimes (1,1,2) + (1,2) \otimes (3,3,1,4,1) + (1) \otimes (4,4,2,5,1,2) + 1_{\mathbb{K}} \otimes (1,5,5,3,6,2,3).$$

3.4. Proposition. *The q-tridendriform algebra* $\mathbf{PQSym}^*(q)$, *equipped with* Δ *is a q-tridendriform bialgebra.*

Proof. Let us see that

$$\Delta(f \succ g) = \sum (f_{(1)} * g_{(1)}) \otimes (f_{(2)} \succ g_{(2)}),$$

for $f \in PF_n$ and $g \in PF_m$. The other relations may be verified in a similar way.

Let $h \in \mathcal{F}_n$ and $k \in \mathcal{F}_m$ be such that $hk \in PF_{n+m}$, $Park(h) = f$, $Park(k) = g$ and $max(h) < max(k)$. Suppose that for $0 \leqslant j \leqslant n+m$, the function hk may be written as:

$$hk = \big((hk)^j_{(1)} \times_{\mathcal{P}} (hk)^j_{(2)}\big) \circ \delta_j,$$

with $(hk)^j_{(1)} \in PF_j$, $(hk)^j_{(2)} \in PF_{n+m-j}$ and $\delta_j \in Sh(j, n+m-j)^{-1}$.

Then there exists a unique integer $0 \leqslant r \leqslant j$ such that $(hk)^j_{(1)} = h^r_{(1)} k^{j-r}_{(1)}$, $(hk)^j_{(2)} = h^r_{(2)} k^{j-r}_{(2)}$, and $\delta_j = (\delta^1_j \times \delta^2_j) \cdot \gamma$, with $\delta^1_j \in Sh(r, n-r)^{-1}$, $\delta^2_j \in Sh(j-r, m+r-j)^{-1}$ and $\gamma \in Sh(n,m)^{-1}$. In this case we have that $f = (Park(h^r_{(1)}) \times_{\mathcal{P}} Park(h^r_{(2)})) \circ \delta^1_j$ and $g = (Park(k^{j-r}_{(1)}) \times_{\mathcal{P}} Park(k^{j-r}_{(2)})) \circ \delta^2_j$. Finally, it is easy to see that $max(h^r_{(2)}) < max(k^{j-r}_{(2)})$. So, to any term in $\Delta(f \succ g)$ corresponds a term in $(* \times \succ) \circ (\Delta \times \Delta)(f \otimes g)$.

Conversely, suppose that $f = (f^r_{(1)} \times_{\mathcal{P}} f^r_{(2)}) \circ \delta_r$ and $g = (g^l_{(1)} \times_{\mathcal{P}} g^l_{(2)}) \circ \gamma_l$, for parking functions $f^r_{(1)}$, $f^r_{(2)}$, $g^l_{(1)}$, $g^l_{(2)}$ and permutations $\delta_r \in Sh(r, n-r)^{-1}$ and $\gamma_l \in Sh(l, m-l)^{-1}$. Let $h_1 \in \mathcal{F}_r$, $h_2 \in \mathcal{F}_{n-r}$, $k_1 \in \mathcal{F}_l$ and $k_2 \in \mathcal{F}_{m-l}$ be such that:

(1) $h_1 k_1 \in PF_{r+l}$ and $h_2 k_2 \in PF_{n+m-r-l}$,
(2) $Park(h_i) = f^r_{(i)}$ and $Park(k_i) = g^l_{(i)}$, for $i = 1, 2$,
(3) $max(h_{(2)}) < max(k_{(2)})$.

The elements $h = (h_1 \times_{\mathcal{P}} h_2) \circ \delta_r \in \mathcal{F}_n$ and $k = (k_1 \times_{\mathcal{P}} k_2) \circ \gamma_l \in \mathcal{F}_m$ verify that $hk \in PF_{n+m}$, $Park(h) = f$, $Park(k) = g$ and $max(h) < max(k)$. $\Box$

Note that any surjective map from $\{1, \ldots, n\}$ to $\{1, \ldots, r\}$ is a parking function. There exists a natural map from PF_n to $\mathbf{ST}_n$ given by $f \mapsto std(f)$ which is surjective but not injective, and coincides with the identity map on $\mathbf{ST}_n$. The linear map $\alpha_n : \mathbb{K}[\mathbf{ST}_n] \longrightarrow \mathbb{K}[PF_n]$ given by

$$\alpha_n(f) = \sum_{h \in PF_n \mid std(h) = f} h,$$

is a monomorphism, for $n \geqslant 1$.

3.5. Theorem. *The bialgebra* $\mathbf{ST}(q)$ *is a sub-q-tridendriform bialgebra of* $\mathbf{PQSym}^*(q)$.

Proof. Let $f \in \mathbf{ST}_n$ and $g \in \mathbf{ST}_m$. Given $u \in PF_{n+m}$ there exist unique functions $u_1 \in \mathcal{F}_n$ and $u_2 \in \mathcal{F}_m$ such that $u = u_1u_2$, and unique functions $h \in \mathcal{F}_n$ and $k \in \mathcal{F}_m$ such that $std(u) = hk$. Moreover, we have that $std(u_1) = std(h)$ and $std(u_2) = std(k)$.

Note that

$$\alpha_{n+m}(f \succ g) = \sum_{u \in PF_{n+m}} q^{\cap(h,k)} u,$$

where the sum is extended over all the functions u such that $std(u) = hk$, with $std(h) = f$, $std(k) = g$ and $max(h) < max(k)$.

On the other hand,

$$\alpha_n(f) \succ_q \alpha_m(g) = \sum_{u \in PF_{n+m}} q^{\cap(u_1,u_2)} u,$$

where the sum is extended over all the functions $u = u_1u_2$ such that $std(Park(u_1)) = f$, $std(Park(u_2)) = g$ and $max(u_1) < max(u_2)$.

It is immediate to check that:

(1) $std(Park(u_i)) = std(u_i)$, for $i = 1, 2$,
(2) if $u = u_1u_2$ and $std(u) = hk$, then $\cap(h,k) = \cap(u_1,u_2)$,
(3) if $std(u_1) = f$ and $std(u_2) = g$, then $std(u_1u_2) = hk$ with $std(h) = f$ and $std(k) = g$,
(4) if $std(u_1u_2) = hk$, then $max(h) < max(k)$ if, and only if, $max(u_1) < max(u_2)$.

We may conclude that $\alpha_{n+m}(f \succ g) = \alpha_n(f) \succ \alpha_m(g)$.

Similar arguments show that $\alpha_{n+m}(f \cdot g) = \alpha_n(f) \cdot \alpha_m(g)$ and $\alpha_{n+m}(f \prec g) = \alpha_n(f) \prec \alpha_m(g)$.

So, $\mathbf{ST}(q)$ is a q-tridendriform subalgebra of $\mathbf{PQSym}^*(q)$.

To prove that α is a coalgebra homomorphism, suppose that $h \in PF_n$ and $0 \leqslant r \leqslant n$ are such that $std(h) = f$ and

$$h = \left(h^r_{(1)} \times_{\mathcal{P}} h^r_{(2)}\right) \circ \delta_r, \quad \text{for } h^r_{(1)} \in PF_r,\ h^r_{(2)} \in PF_{n-r} \text{ and } \delta_r \in Sh(r, n-r)^{-1}.$$

Let $f^r_{(1)} := std(h^r_{(1)})$ and $f^r_{(2)} := std(h^r_{(2)})$, we get that $f = (f^r_{(1)} \times f^r_{(2)}) \circ \delta_r$.

Conversely, suppose that $f = (f^r_{(1)} \times f^r_{(2)}) \circ \delta_r$, for some $f^r_{(1)} \in \mathbf{ST}_r$, $f^r_{(2)} \in \mathbf{ST}_{n-r}$ and $\delta_r \in Sh(r, n-r)^{-1}$.

Given elements $h^r_{(1)} \in PF_r$ and $h^r_{(2)} \in PF_{n-r}$, the element $h := (h^r_{(1)} \times_{\mathcal{P}} h^r_{(2)}) \circ \delta_r \in PF_n$ verifies that $std(h) = f$.

The arguments above imply that:

$$\begin{aligned}\Delta\left(\alpha_n(f)\right) &= \sum_{std(h)=f} \Delta(h) = \sum_{std(h)=f} \left(\sum_r h^r_{(1)} \otimes h^r_{(2)}\right) \\ &= \sum_r \left(\sum_{std(h^r_{(i)})=f^r_{(i)}} h^r_{(1)} \otimes h^r_{(2)}\right) = \sum_r \alpha_r\left(f^r_{(1)}\right) \otimes \alpha_{n-r}\left(f^r_{(2)}\right),\end{aligned}$$

which proves that α is a coalgebra homomorphism. □

Clearly, since any surjective map is a parking function, there exists the natural inclusion homomorphism $\iota : \mathbf{ST} \hookrightarrow \mathbf{PQSym}^*$, but ι is not a coalgebra homomorphism. For instance, the element $(1,1,2)$ is primitive in $\mathbf{PQSym}^*$. An element $x \in \mathbf{ST}_n$ is such that $\iota_n(x) = \alpha_n(x)$ if, and only if, x is a permutation.

Note that $(\mathbf{PQSym}^*, \times_{\mathcal{P}})$ is an associative algebra, too. If we denote by $PIrr_n$ the subset of $\bigcup_{n\geqslant 1} PF_n$ of all parking functions f such that there do not exist $f_1 \in PF_i$ and $f_2 \in PF_{n-i}$ with $f = f_1 \times_{\mathcal{P}} f_2$ and $1 \leqslant i \leqslant n-1$. So, as a vector space $\mathbf{PQSym}^*$ is isomorphic to $T(\mathbb{K}[\bigcup_{n\geqslant 1} PIrr_n])$, which implies that the space of primitive elements of $\mathbf{PQSym}^*$ of degree n has dimension $PIrr_n$, for $n \geqslant 1$.

3.6. Multipermutations

In [6], T. Lam and P. Pylyavskyy define a *big multi-permutation* or $\mathcal{M}$*-permutation* of n as an ordered partition $(B_1, \dots, B_m)$ of n such that if an element i, $1 \leqslant i \leqslant n-1$, belongs to the block B_j, then $i+1 \notin B_j$. The set of $\mathcal{M}$-permutations of n is denoted $S_n^{\mathcal{M}}$.

The element $B = [(1,4,6),(2,7),(3,5)]$ is an $\mathcal{M}$-permutation of 7, while $D = [(1,6,7),(2,3),5,4]$ is not.

Let $W = (W_1, \dots, W_r)$ be an ordered partition of n, the $\mathcal{M}$-standardization of W is the big multi-permutation $std_{\mathcal{M}}(W)$ obtained by:

(1) delete $i+1$ if both i and $i+1$ belong to the same block W_j,
(2) if i does not appear in any block obtained applying the rule above, then reduce all numbers larger than i in (1).

For example $std_{\mathcal{M}}[(1,6,7),(2,3),5,4] = [(1,5),2,4,3]$.

Let J be a subset of $[n]$ and let $B \in S_n^{\mathcal{M}}$, the *restriction* $B|_J$ of B to J is the intersection B with J. If $B = [(1,4,6),(2,7),(3,5)]$ and $J = \{1,2,4,6\}$, then $B|_J = [(1,4,6),2]$. Let $B = [(i_1^1, \dots, i_{r_1}^1), \dots, (i_1^l, \dots, i_{r_l}^l)]$ be a big multi-permutation, for any integer k we denote by $B+k$ the ordered partition $[(i_1^1+k, \dots, i_{r_1}^1+k), \dots, (i_1^l+k, \dots, i_{r_l}^l+k)]$.

In [6], the authors define an algebra structure on the vector space $\mathcal{M}MR$ spanned by the set of all $\mathcal{M}$-permutations, as follows:

$$B \bullet D = \sum W, \quad \text{for } B \in S_n^{\mathcal{M}} \text{ and } D \in S_m^{\mathcal{M}},$$

where the sum is taken over:

(1) all $W \in S_{n+m}^{\mathcal{M}}$ such that $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n}) = D$,
(2) all $W \in S_{n+m-1}^{\mathcal{M}}$ such that $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n-1}) = D$.

For example,

$$\begin{aligned}\big[(13),2\big] \bullet [2,1] = {} & \big[(1,3),2,5,4\big] + \big[(1,3),(2,5),4\big] + \big[(1,3),5,2,4\big] + \big[(1,3,5),2,4\big] \\ & + \big[5,(1,3),2,4\big] + \big[(1,3,5),(2,4)\big] + \big[5,4,(1,3),2\big] + \big[4,(1,3),2\big] \\ & + \big[(1,3),5,4,2\big] + \big[(1,3,5),4,2\big] + \big[5,(1,3),4,2\big].\end{aligned}$$

For any ordered partition $W = (W_1, \dots, W_l) \in S_{n+m}^{\mathcal{M}}$ such that $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{\{n+1,\dots,n+m\}}) = D$, define the integer $\bigcap_{B,D}^W$ as the number of blocks W_j such that $W_j \cap [n] \neq \emptyset$ and $W_j \cap [m] + n \neq \emptyset$. It is immediate to check that, if $B = (B_1, \dots, B_r)$ and $D = (D_1, \dots, D_s)$, then $\bigcap_{B,D}^W = r + s - l$.

Let $B = (B_1, \dots, B_r)$ and $D = (D_1, \dots, D_s)$. We define binary operations $\succ$, $\cdot$ and $\prec$ on the space $\mathcal{M}MR$ as follows:

(1) $B \succ D := \sum q^{\bigcap_{B,D}^W} W$, where the sum is taken over:
   - all $W = (W_1, \dots, W_l) \in S_{n+m}^{\mathcal{M}}$ with $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n}) = D$,

- all $W \in S^{\mathcal{M}}_{n+m-1}$ with $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n-1}) = D$,

such that $W_l \cap [n] = \emptyset$.

(2) $B \cdot D := \sum q^{\bigcap^W_{B,D} - 1} W$, where the sum is taken over:
- all $W = (W_1, \dots, W_l) \in S^{\mathcal{M}}_{n+m}$ with $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n}) = D$, such that $W_l \cap [n] \neq \emptyset$ and $W_l \cap [m] + n \neq \emptyset$,
- all $W \in S^{\mathcal{M}}_{n+m-1}$ with $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n-1}) = D$, such that $W_l \cap [n] \neq \emptyset$ and $W_l \cap [m] + n - 1 \neq \emptyset$.

(3) $B \prec D := \sum q^{\bigcap^W_{B,D}} W$, where the sum is taken over:
- all $W = (W_1, \dots, W_l) \in S^{\mathcal{M}}_{n+m}$ with $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n}) = D$, such that $W_l \cap [m] + n = \emptyset$,
- all $W \in S^{\mathcal{M}}_{n+m-1}$ with $W|_{[n]} = B$ and $std_{\mathcal{M}}(W|_{[m]+n-1}) = D$, such that $W_l \cap [m] + n - 1 = \emptyset$.

Let us denote by $\mathcal{M}MR(q)$ the space $\mathcal{M}MR$ equipped with the products $\succ$, $\cdot$ and $\prec$.

3.7. Proposition. *The data $(\mathcal{M}MR(q), \succ, \cdot, \prec)$ is a q-tridendriform algebra.*

Proof. Let $B = (B_1, \dots, B_r) \in S^{\mathcal{M}}_n$, $D = (D_1, \dots, D_s) \in S^{\mathcal{M}}_m$ and $E = (E_1, \dots, E_t) \in S^{\mathcal{M}}_p$. We prove that $B \succ (D \succ E) = (B * D) \succ E$, and that $B \cdot (D \succ E) = (B \prec D) \cdot E$. The other relationships can be verified in a similar way.

We have that $B \succ (D \succ E) = \sum_W \delta(W) W$, while $(B * D) \succ E = \sum_W \alpha(W) W$, where the both sums are taken over all the $\mathcal{M}$-permutations $W = (W_1, \dots, W_l)$ satisfying that:

- $W \in S^{\mathcal{M}}_{n+m+r}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{\{n+1,\dots,n+m\}} = D$, $W^3 := W|_{\{n+m+1,\dots,n+m+r\}} = E$ and $W_l = E_t + n + m$,
- $W \in S^{\mathcal{M}}_{n+m+r-1}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{\{n+1,\dots,n+m\}} = D$, $W^3 := W|_{\{n+m,\dots,n+m+r-1\}} = E$ and $W_l = E_t + n + m - 1$,
- $W \in S^{\mathcal{M}}_{n+m+r-1}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{\{n,\dots,n+m-1\}} = D$, $W^3 := W|_{\{n+m,\dots,n+m+r-1\}} = E$ and $W_l = E_t + n + m - 1$,
- $W \in S^{\mathcal{M}}_{n+m+r-2}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{\{n,\dots,n+m-1\}} = D$, $W^3 := W|_{\{n+m-1,\dots,n+m+r-2\}} = E$ and $W_l = E_t + n + m - 2$.

We need to prove that $\alpha(W) = \delta(W)$. We give a detailed proof of it for the case $W \in S^{\mathcal{M}}_{n+m+r}$, the other cases are analogous.

Let $V \in S^{\mathcal{M}}_{n+m}$ be such that $W|_{[n+m]} = V$ and let $R \in S^{\mathcal{M}}_{m+r}$ be such that $W|_{[m+r]+n} = R + n$. We have that $\alpha(W) = \bigcap^V_{B,D} + \bigcap^W_{V,E}$, where $\bigcap^V_{B,D}$ is the number of blocks of V which have both elements in $[n]$ and elements in $[m] + n$, while $\bigcap^W_{V,E}$ is the number of blocks of W which have both elements in $[n+m]$ and elements in $[r] + n + m$. So, $\alpha(W) = \sum_{i=1}^l \alpha(W_i)$, where $\alpha(W_i) =$

$$\begin{cases} 0, & \text{if } W_i \subseteq [n] \text{ or } W_i \subseteq [m] + n \text{ or } W_i \subseteq [r] + n + m, \\ 1, & \text{if } W_i \text{ contains integers in exactly two sets of } [n],\ [m] + n \text{ and } [r] + n + m, \\ 2, & \text{if } W_i \text{ contains integers in all the sets } [n],\ [m] + n \text{ and } [r] + n + m. \end{cases}$$

On the other hand, $\delta(W) = \bigcap^R_{D,E} + \bigcap^W_{B,R}$, where $\bigcap^R_{D,E}$ is the number of blocks of R which have both elements in $[m]$ and elements in $[r] + m$, while $\bigcap^W_{B,R}$ is the number of blocks of W which have both elements in $[n]$ and elements in $[m + r] + n$, which implies that $\delta(W) = \sum_{i=1}^l \alpha(W_i) = \alpha(W)$.

For the second equality, we have that $B \cdot (D \succ E) = \sum q^{\beta(W)} W$ and $(B \prec D) \cdot E = \sum q^{\gamma(W)} W$, where both sums are taken over all $\mathcal{M}$-permutations $W = (W_1, \dots, W_l)$ such that:

- $W \in S^{\mathcal{M}}_{n+m+r}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{[m]+n} = D$, $W^3 := W|_{[r]+n+m} = E$, and $W_l = B_r \cup E_t + n + m$,

- $W \in S^{\mathcal{M}}_{n+m+r-1}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{[m]+n} = D$, $W^3 := W|_{[r]+n+m-1} = E$, and $W_l = B_r \cup E_t + n + m - 1$,
- $W \in S^{\mathcal{M}}_{n+m+r-1}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{[m]+n-1} = D$, $W^3 := W|_{[r]+n+m-1} = E$, and $W_l = B_r \cup E_t + n + m - 1$,
- $W \in S^{\mathcal{M}}_{n+m+r-2}$, $W^1 := W|_{[n]} = B$, $W^2 := W|_{[m]+n-1} = D$, $W^3 := W|_{[r]+n+m-2} = E$, and $W_l = B_r \cup E_t + n + m - 2$.

To check that $\beta(W) = \gamma(W)$, for all W, is suffices to do a similar computation that the one in the previous case. $\square$

The coproduct on the space $\mathcal{M}\mathrm{M}R_+$ is defined by T. Lam and P. Pylyavskyy (see [6]) as follows:

$$\Delta(B) = \sum_{[W,R]=B} std_{\mathcal{M}}(W) \otimes std_{\mathcal{M}}(R),$$

where $[W, R]$ is the union of two ordered partitions W and R, such that W is a partition of J and R is a partition of K with $[n] = J \cup K$ and $J \cap K = \emptyset$. In other words, for $B = (B_1, \dots, B_r)$ and $0 \leqslant j \leqslant r$ define $B_{\leqslant j} := (B_1, \dots, B_j)$ and $B_{>j} := (B_{j+1}, \dots, B_r)$. The coproduct Δ on B is given by:

$$\Delta(B) = \sum_{i=0}^{l} std_{\mathcal{M}} B_{\leqslant i} \otimes std_{\mathcal{M}} B_{>i}.$$

We have, for example, that:

$$\begin{aligned}
\Delta\big([1]\big) &= [1] \otimes 1_{\mathbb{K}} + 1_{\mathbb{K}} \otimes [1], \\
\Delta\big(\big[(2)(1)\big]\big) &= \big[(2)(1)\big] \otimes 1_{\mathbb{K}} + [1] \otimes [1] + 1_{\mathbb{K}} \otimes \big[(2)(1)\big], \\
\Delta\big(\big[(13)(2)\big]\big) &= \big[(13)(2)\big] \otimes 1_{\mathbb{K}} + [1] \otimes [1] + 1_{\mathbb{K}} \otimes \big[(13)(2)\big],
\end{aligned}$$

in the last example, note that $[(13)(2)] = [[(13)], [2]]$ and $std_{\mathcal{M}}[(13)] = [1] = std_{\mathcal{M}}[2]$.

In [6] the authors prove that $(\mathcal{M}MR(1), *, \Delta)$ is a bialgebra. We want to show that $\mathcal{M}MR(q)$ equipped with the coproduct Δ is a quotient of the q-tridendriform bialgebra $\mathbf{ST}(q)$.

Let φ be the map from the set $\bigcup_{n \geqslant 1} \mathbf{ST}_n$ of all surjections to the set $\bigcup_{n \geqslant 1} S^{\mathcal{M}}_n$ of $\mathcal{M}$-permutations, which sends $f \in \mathbf{ST}_n$ to the element $std_{\mathcal{M}}[(f^{-1}(1)), \dots, (f^{-1}(n))]$. For example, if $f = (2, 3, 3, 6, 1, 5, 1, 2, 4)$ then

$$\varphi(f) = std_{\mathcal{M}}\big[(5,7), (1,8), (2,3), 9, 6, 4\big] = \big[(4,6), (1,7), 2, 8, 5, 3\big].$$

Note that φ is surjective and does not respect the graduation.

3.8. Remark. Let $f \in \mathbf{ST}_n$ be a surjection, and let $1 \leqslant l \leqslant n$ be such that $std_{\mathcal{M}}[(f^{-1}(1)), \dots, (f^{-1}(n))] \in S^{\mathcal{M}}_l$, then there exists a unique $\overline{f} \in \mathbf{ST}_l$ such that $std_{\mathcal{M}}[(f^{-1}(1)), \dots, (f^{-1}(n))] = std_{\mathcal{M}}[(\overline{f}^{-1}(1)), \dots, (\overline{f}^{-1}(l))]$. Moreover, for any map $\bar{h} : \{1, \dots, l\} \longrightarrow \{1, \dots, r\}$ such that $std(\bar{h}) = \overline{f}$, there exist a unique $h \in \mathcal{F}_n$ such that:

(1) $(\bar{h}(1), \dots, \bar{h}(l))$ is obtained from $(h(1), \dots, h(n))$ by eliminating all integers $h(i)$ which are equal to $h(i-1)$, for $1 < i \leqslant n$,
(2) $std(h) = f$.

For example, if $f = (1,2,2,3,1,4)$, then $\overline{f} = (1,2,3,1,4)$. Take $\bar{h} = (4,6,7,4,9)$, we get that $h = (4,6,6,7,4,9)$.

Applying Remark 3.8 we are able to prove the following result.

3.9. Theorem. *For any pair of elements $f \in \mathbf{ST}_n$ and $g \in \mathbf{ST}_m$ we have that*:

(1) $\varphi(f \prec g) = \varphi(f) \prec \varphi(g)$,
(2) $\varphi(f \cdot g) = \varphi(f) \cdot \varphi(g)$,
(3) $\varphi(f \succ g) = \varphi(f) \succ \varphi(g)$,
(4) $\Delta(\varphi(f)) = (\varphi \otimes \varphi)(\Delta(f))$.

Proof. If h and k are two maps such that $hk \in \mathbf{ST}_{n+m}$, $std(h) = f$ and $std(k) = g$, then

$$std_{\mathcal{M}}\big[\big(h^{-1}(1)\big), \ldots, \big(h^{-1}(r)\big)\big] = std_{\mathcal{M}}\big[\big(f^{-1}(1)\big), \ldots, \big(f^{-1}(n)\big)\big],$$
$$std_{\mathcal{M}}\big[\big(k^{-1}(1)\big), \ldots, (k^{-1}(r)\big] = std_{\mathcal{M}}\big[\big(g^{-1}(1)\big), \ldots, (g^{-1}(m)\big],$$

where $max(hk) = r \leqslant n+m$.

Suppose that $max(h) > max(k)$, we have that:

$$\begin{aligned}&\big[\big((hk)^{-1}(1)\big), \ldots, \big((hk)^{-1}(r)\big)\big]\\ &\quad = \big[\big(h^{-1}(1) \cup \big(k^{-1}(1)+n\big)\big), \ldots, \big(h^{-1}(r-1) \cup \big(k^{-1}(r-1)+n\big)\big), \big(h^{-1}(r)\big)\big],\end{aligned}$$

where $(h^{-1}(i) \cup (k^{-1}(i)+n))$ denotes the disjoint union of the sets $h^{-1}(i)$ and $k^{-1}(i)+n$, for $1 \leqslant i \leqslant r-1$. The standardization $std_{\mathcal{M}}[((hk)^{-1}(1)), \ldots, ((hk)^{-1}(r))]$ is an $\mathcal{M}$-permutation $W = (W_1, \ldots, W_r)$ satisfying that:

(1) if $h(n) \neq k(1)$, then $W|_{[n]} = std_{\mathcal{M}}[(f^{-1}(1)), \ldots, (f^{-1}(n))]$, $W|_{[m]+n} = std_{\mathcal{M}}[(g^{-1}(1)+n), \ldots, (g^{-1}(m)+n)]$ and $W_r \cap ([m]+n) = \emptyset$,
(2) if $h(n) = k(1)$, then $W|_{[n]} = std_{\mathcal{M}}[(f^{-1}(1)), \ldots, (f^{-1}(n))]$, $W|_{[m]+n-1} = std_{\mathcal{M}}[(g^{-1}(1)+n-1), \ldots, (g^{-1}(m)+n-1)]$ and $W_r \cap ([m]+n-1) = \emptyset$.

Conversely, let $W = (W_1, \ldots, W_r)$ be an $\mathcal{M}$-permutation such that $W|_{[n]} = \varphi(f) = std_{\mathcal{M}}[(f^{-1}(1)), \ldots, (f^{-1}(n))]$ and $W_r \subseteq [n]$, we have that

(1) if $W|_{[m]+n} = std_{\mathcal{M}}[(g^{-1}(1)+n), \ldots, (g^{-1}(m)+n)]$, then there exist maps $\bar{h}$ and $\bar{k}$ defined as follows:
   (a) $\bar{h}(i)$ is the unique integer such that $i \in W_{\bar{h}(i)}$.
   (b) $\bar{k}(j)$ is the unique integer such that $j+n \in W_{\bar{k}(j)}$.
   By Remark 3.8, there exist unique elements $h \in \mathcal{F}_n$ and $k \in \mathcal{F}_m$ such that $std(h) = f$, $std(k) = g$ and $std_{\mathcal{M}}[((hk)^{-1}(1)), \ldots, ((hk)^{-1}(r))] = W$.
(2) if $W|_{[m]+n-1} = std_{\mathcal{M}}[(g^{-1}(1)+n-1), \ldots, (g^{-1}(m)+n-1)]$, then the maps $\bar{h}$ and $\bar{k}$ are defined as follows:
   (a) $\bar{h}(i)$ is the unique integer such that $i \in W_{\bar{h}(i)}$.
   (b) $\bar{k}(j)$ is the unique integer such that $j+n-1 \in W_{\bar{k}(j)}$.
   Again, there exist unique elements $h \in \mathcal{F}_n$ and $k \in \mathcal{F}_m$ such that $std(h) = f$, $std(k) = g$ and $std_{\mathcal{M}}[((hk)^{-1}(1)), \ldots, ((hk)^{-1}(r))] = W$.

Moreover, since $W_r \cap ([m]+n) = \emptyset$, we get that $max(k) < max(h) = r$ in both cases.

We get then that $\varphi(f \prec g) = \varphi(f) \prec \varphi(g)$, the proofs of the second and third statements follow from similar arguments.

To end the proof of the theorem we need to show that φ is a coalgebra homomorphism. For $f \in \mathbf{ST}_n$, let $\overline{f} \in \mathbf{ST}_l$ be the unique surjection such that $std_{\mathcal{M}}[(f^{-1}(1)), \dots, (f^{-1}(r))] = [(\overline{f}^{-1}(1)), \dots, (\overline{f}^{-1}(l))]$. It is easy to see that there exists a bijection between the set of elements $(r, f_{(1)}^k, f_{(2)}^k)$, with $0 \leqslant k \leqslant n$, $f_{(1)}^k \in \mathbf{ST}_k$ and $f_{(2)}^k \in \mathbf{ST}_{n-k}$, such that

$$f = \left(f_{(1)}^k \times f_{(2)}^k\right) \circ \delta_k, \quad \text{for some } \delta_r \in Sh(k, n-k),$$

and the set of elements $(j, \overline{f}_{(1)}^j, \overline{f}_{(2)}^j)$, with $0 \leqslant j \leqslant l$, $\overline{f}_{(1)}^j \in \mathbf{ST}_j$ and $\overline{f}_{(2)}^j \in \mathbf{ST}_{l-j}$, such that

$$\overline{f} = \left(\overline{f}_{(1)}^j \times \overline{f}_{(2)}^j\right) \circ \tau_j, \quad \text{for some } \tau_j \in Sh(j, l-j).$$

So, it suffices to verify that $\Delta(\varphi(f)) = (\varphi \otimes \varphi)(\Delta(f))$ for f satisfying that $std_{\mathcal{M}}[(f^{-1}(1)), \dots, (f^{-1}(n))] = [(f^{-1}(1)), \dots, (f^{-1}(r))]$, that is when $f(i) \neq f(i+1)$ for $1 \leqslant i \leqslant n-1$. If for some $0 \leqslant k \leqslant n$ there exist $f_{(1)}^k \in \mathbf{ST}_k$, $f_{(2)}^k \in \mathbf{ST}_{n-k}$ and $\delta_k \in Sh(k, n-k)$ such that $f = (f_{(1)}^k \times f_{(2)}^k) \circ \delta_k$ and $W := [((f_{(2)}^k)^{-1}(1) + s), \dots, ((f_{(2)}^k)^{-1}(n-k) + s)]$.

Conversely, let $[(f^{-1}(1)), \dots, (f^{-1}(r))] = [R, W]$, with $R = [R_1, \dots, R_s]$ and $W = [W_1, \dots, W_{r-s}]$, and suppose that R is a partition of $\{i_1 < \dots < i_k\}$ and W is a partition of $\{j_1 < \dots < j_{n-k}\}$. Define $f_{(1)}^k$ and $f_{(2)}^k$ as follows:

(1) $f_{(1)}^k(l)$ is the unique integer $1 \leqslant f_{(1)}^k(l) \leqslant s$ such that $i_l \in R_{f_{(1)}^k(l)}$, for $1 \leqslant l \leqslant k$,
(2) $f_{(2)}^k(l)$ is the unique integer $1 \leqslant f_{(2)}^k(l) \leqslant r-s$ such that $j_l \in R_{f_{(2)}^k(l)}$, for $1 \leqslant l \leqslant n-k$.

It is clear that there exists a shuffle $\delta_k \in Sh(k, n-k)$ such that $f = (f_{(1)}^k \times f_{(2)}^k) \circ \delta_k$. We get then that

$$\begin{aligned}
(\varphi \otimes \varphi)\big(\Delta(f)\big) &= \sum_k \varphi\big(f_{(1)}^k\big) \otimes \varphi\big(f_{(2)}^k\big) \\
&= \sum_k std_{\mathcal{M}}\big[\big(\big(f_{(1)}^k\big)^{-1}(1)\big), \dots, \big(\big(f_{(1)}^k\big)^{-1}(k)\big)\big] \\
&\quad \otimes std_{\mathcal{M}}\big[\big(\big(f_{(2)}^k\big)^{-1}(1)\big), \dots, \big(\big(f_{(2)}^k\big)^{-1}(n-k)\big)\big] \\
&= \sum_{\varphi(f)=[R,W]} std_{\mathcal{M}}(R) \otimes std_{\mathcal{M}}(W) = \Delta\big(\varphi(f)\big). \qquad \square
\end{aligned}$$

As a consequence of Theorem 3.9, we can assert that $\mathcal{M}MR(q)$ is a q-tridendriform bialgebra.

3.10. Corollary. *The q-tridendriform algebra $\mathcal{M}MR(q)$ equipped with the coproduct Δ is a q-tridendriform bialgebra which is a quotient of $\mathbf{ST}(q)$.*

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