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LIE GROUPS AS FRAMED MANIFOLDS

ERICH OSSA

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§1. INTRODUCTION

IN THIS PAPER we shall study a problem first raised by Gershenson in[8]: Let G be a compact connected Lie group of dimension $d > 0$, and assume an orientation chosen for G. Then we may give G in a canonical way the structure of a framed manifold (G, L): We first chose an oriented basis of the Lie algebra g of G, then we extend this to a basis of the vector space of left invariant tangent fields, and finally we convert this trivialization of the tangent bundle by the standard device (see [16], p. 23 f) to a framing of the stable normal bundle of G. The problem then consists in determining the element $[G, L] \in \pi_d^s$ obtained from (G, L) by the familiar Pontrjagin–Thom-identification of framed bordism with the stable homotopy groups of spheres[8, 16].

It is obvious that the elements $[G, L]$ behave well with respect to products. Since clearly $[SO(2), L] = \eta \in \pi_1^s$, we know $[G, L]$ for any abelian G. Moreover, if G is non-abelian but contains a torus T in its center, there is a diffeomorphism of framed manifolds between (G, L) and $(T, L) \times (G/T, L)$. Thus the general question is reduced to the case of semisimple groups G.

Since its announcement in[8], the problem has received attention in several papers [1, 3, 8, 10, 11, 14, 19] and, perhaps, others. We have gathered the known information on specific homotopy elements represented by Lie groups with left invariant framing in Table 1, which in particular lists all simply connected simple groups of rank ≤ 4 (with the unfortunate consequence of having on the assets side two and a half question marks).

The papers mentioned above usually contain also results on Lie groups with other natural framings which we cannot mention here. The latter statement holds as well for the methods used which include some beautiful geometry and homotopy theory. We refer the interested reader to [3, 14, 19] for two particularly efficacious approaches.

There are a few results of more general nature. Becker and Schultz[3] prove some general relations, concrete examples of which are $2 \times [SO(2n), L] = 0$ and $\eta \times [SU(2n), L] = 0$. They also venture the conjecture that $[G, L] = 0$ for rank $(G) \geq n_0$ where n_0 is hoped to be not bigger than, say, 10. Knapp[10, 11] has shown that the p-primary component of $[G, L]$ has BP–Adams–filtration at least n, and at least $n + 2(p-1)$ if $p > 3$.

In this paper we shall estimate the order of $[G, L]$. The result is

THEOREM 1.1. *Let G be a compact connected Liegroup. Then*

$$72[G, L] = 0.$$

Moreover, if G is not locally isomorphic to a product of groups E_6, E_7, E_8, *then*

$$24[G, L] = 0.$$

Note that in particular the p-primary component of $[G, L]$ is zero for any prime $p > 3$. We conjecture that the stronger result holds as well as in the general case.

Table 1. The homotopy elements represented by some semisimple groups of low rank

rank	type	dimension d	semisimple group G	$[G,L]$	π_d^s
1	A_1	3	$SU(2)$	ν	$\mathbb{Z}_{24}\cdot\nu$
			$SO(3)$	2ν	
2	A_2	8	$SU(3)$	$\bar{\nu}$	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
	B_2	10	$Sp(2)$	$\beta_1(3)$	$\mathbb{Z}_6$
			$SO(5)$	$-\beta_1(3)$	
	G_2	14	G_2	κ	$\mathbb{Z}_2\oplus\mathbb{Z}_2\cdot\kappa$
	$A_1\times A_1$	6	$SO(4)$	0	$\mathbb{Z}_2$
3	A_3	15	$SU(4)$	$\kappa\eta$	$\mathbb{Z}_{480}\oplus\mathbb{Z}_2\cdot\kappa\eta$
			$SO(6)$	0	
	B_3	21	$Spin(7)$	0	$\mathbb{Z}_2\oplus\mathbb{Z}_2$
			$SO(7)$	0	
	C_3		$Sp(3)$	$\sigma^3+\bar{\kappa}\eta$	
4	A_4	24	$SU(5)$	$\eta^*\sigma\eta$ or 0	$\mathbb{Z}_6\oplus\mathbb{Z}_2\cdot\eta^*\sigma\eta$
	B_4	36	$Spin(9)$	0	$\mathbb{Z}_6$
			$SO(9)$	0	
	C_4		$Sp(4)$	?	
	D_4	28	$Spin(8)$	0	$\mathbb{Z}_2$
			$SO(8)$	0	
	F_4	52	F_4	?	$\mathbb{Z}_3\oplus$ 2-primary

Our proof of Theorem 1.1. actually shows more. First of all, the framing on G need not be the left-invariant framing; all that is required, is invariance under suitable subgroups S^1. Secondly, we can estimate the order of the homotopy element $G\to G/T$ in $\pi_d^s(G/T)$. The result is 24, if the Lie algebra of G contains a simple factor of type A_n, B_n, C_n, D_n or G_2, and 72 if it contains a factor F_4, E_6 or E_7, whereas in the case that G is a product of groups E_8, we have only the estimate 360.

Before we comment on the proof, a word on notational conventions: All (co-) homology theories which occur will be taken as reduced theories. Non-reduced groups are obtained by adding a basepoint + to the space in question. Suspension is reduced and is denoted by S; suspension isomorphisms will often be suppressed from the notation. All other conventions will—hopefully—be standard or be explained below.

The method of proof is based on Knapp's approach via the S^1-transfer; it is presented in §2. The important fact is the following: Let $S\subset G$ be a circle subgroup. Then $[G, L]$ can be described as the value on the fundamental class of a canonical element in $J(S^2(G/S^+))$, whose counterimage in $K^0(S^2(G/S^+))$ is obtained from the canonical bundle $x\in K^0(G/S^+)$ by Bott periodicity. Results on the order of this element are derived in §3 by using properties of λ-ideals in the representation ring $R(S^1)$. In fact, Lemmas 3.2. and 3.5. show that, for any representation ρ of G, this element is annihilated by 23 times a power of K_ρ, where K_ρ is the restriction of the Killing form of ρ to S. Thus the fact, that $[G, L]$ can be non-zero only at primes $p<7$, follows rather easily from the tables on Killing forms of representations in ([7], p. 135). However, the results at primes $p<7$ require more work which occupies the main part of §3.

§2. S^1-TRANSFER AND J-HOMOMORPHISM

We denote by ξ the universal line bundle over $\mathbb{C}P^\infty$. Associated to the principal S^1-bundle $S(\xi)\to\mathbb{C}P^\infty$ there is, in the stable category, a well-known transfer morphism $\tau: \mathbb{C}P^{\infty+} \to S^{-1}(S(\xi)^+) \simeq S^{-1}$ (see [5]). τ is best explained by its effect in framed bordism; $\tau_*: \pi_n^s(\mathbb{C}P^{\infty+}) \to \pi_{n+1}^s$ assigns to a principal S^1-bundle over a framed manifold the associated total space equipped with the induced S^1-invariant framing.

This transfer τ is intimately connected to the complex J-homomorphism. Denoting by Q the stabilization functor $Q = \Omega^\infty S^\infty$, we may view J as a map $J: U \to QS^0$, where U is the unitary group, and τ as a map $\tau: S(\mathbb{C}P^{\infty+}) \to QS^0$. Using loop addition with $[-1] \in QS^0$ we obtain the reduced map $\tilde{J} = J_0$ sending U to the 0-component of QS^0. Now there is the well-known reflection map $R: S(\mathbb{C}P^{\infty+}) \to U$ which assigns to a pair $(\alpha, x) \in S^1 \times \mathbb{C}P^\infty$ the unitary map which is the identity on the orthogonal complement of the line $x \subset \mathbb{C}^\infty$ and rotation by α on this line itself. The homotopy class of R, as an element in $K^{-1}(S(\mathbb{C}P^{\infty+})) = K^{-2}(\mathbb{C}P^{\infty+})$, is the image of $[\xi] \in K^0(\mathbb{C}P^{\infty+})$ under the Bott periodicity $\beta: K^0(\mathbb{C}P^{\infty+}) \to K^{-2}(\mathbb{C}P^{\infty+})$.

The following result was established by Knapp:

THEOREM 2.1. ([12]), The diagram

$$\begin{tikzcd} S(\mathbb{C}P^{\infty+}) \arrow[r, "\tau"] & QS^0 \\ U \arrow[u, "R"] \arrow[ur, "\tilde{J}"'] & \end{tikzcd} \tag{2.1}$$

anticommutes.

Looking only at the induced maps in framed bordism, the theorem tells us in particular that the total space of a principal S^1-bundle over a framed manifold M is—up to sign—framed bordant to $S^1 \times M$, reframed by the map R (see [9] for the relation between $\tilde{J}$ and reframings). A proof of the theorem along these lines is contained in [15].

There is another interpretation of the theorem, which arises from viewing J as the map assigning to a complex vector bundle its fibre homotopy type. (Here we shall tacitly use the identification $J(SX) \subset Sph^0(SX) \subset \pi_s^1(SX)$.) Thus, by the remark preceding the theorem, we have $\tilde{J}(\beta x) = -\tau$ in $\pi_s^1(S^2(\mathbb{C}P^{\infty+}))$. On the other hand, if $[M^n, f]$ is a framed bordism class in $\mathbb{C}P^{\infty+}$, we may express $\tau_*[M^n, f] \in \pi_{n+1}^s$ as the value of the Kronecker product between $f^*(\tau) \in \pi_s^{-1}(M^{n+})$ and the framed bordism fundamental class $[M^n] \in \pi_n^s(M^{n+})$ (see, e.g. [16]).

Hence we obtain as an immediate corollary of the theorem:

LEMMA 2.2. *Let $\alpha \in \pi_n^s(\mathbb{C}P^{1+})$ be represented by the complex line bundle ξ over the framed manifold M^n. Then*

$$\tau_*(\alpha) = -\langle \tilde{J}(\beta\xi), [M^n] \rangle \text{ in } \pi_{n+1}^s. \tag{2.2}$$

We apply this lemma to our problem. Let G be a compact semisimple Lie group of dimension $d > 0$. Let $S \subset G$ be a circle subgroup, $S \cong S^1$. The left invariant framing L on G induces a framing on G/S. Then we may define a map ϕ_S': $K^0(G/S^+) \to \pi_d^s$ by

$$\phi_S'(y) = \langle \tilde{J}(\beta y), [G/S] \rangle. \tag{2.3}$$

Particular elements of $K^0(G/S^+)$ can be constructed from complex representations of S. There is, however, a more general construction whose usefulness in this context was pointed out to me by M. Crabb. Let $\pi : \tilde{G} \to G$ be the universal covering of G and $\pi : \tilde{S} \to S$ the induced covering of $S \subset G$. Then for a representation ρ of $\tilde{S}$ on the complex vector space V we can construct the bundle $\alpha(\rho) = G \times_{\tilde{S}} V$ over $G/S = \tilde{G}/\tilde{S}$. This defines a homomorphism

$$\alpha : R(\tilde{S}) \longrightarrow K^0(G/S^+),$$

where $R(\tilde{S})$ is the representation ring of $\tilde{S}$.

Composing with ϕ_S' above, we define

$$\phi_S = \phi_S' \circ \alpha : R(\tilde{S}) \longrightarrow K^0(G/S^+) \longrightarrow \pi_d^s. \tag{2.4}$$

To formulate the next lemma, we need a definition. Recall (e.g. from [2]), that exterior powers of representations define on $R(\tilde{S})$ the structure of a special λ-ring. Furthermore, complex conjugation defines a conjugation on $R(\tilde{S})$. We can obtain further λ-rings with conjugation by dividing out an ideal which is invariant under these operations; for brevity, we shall call such an ideal simply a λ-ideal. Finally, if A is any λ-ring with conjugation, there are Adams-operations $\psi^k : A \to A$ for any $k \in \mathbb{Z}$, and we define $J^{-1}(A)$ to be the biggest quotient group of A in which for any $a \in A$ and $k \in \mathbb{Z} - (0)$ the element $k\psi^k(a)$-a is annihilated by a power of k.

LEMMA 2.3. *ϕ_S: $R(\tilde{S}) \to \pi_d^3$ is a homomorphism of abelian groups with the following properties*:

(*i*) *Let $x \in R(\tilde{S})$ stand for the representation $\tilde{S} \xrightarrow{\pi} S \subset \mathbb{C}^*$. Then*

$$\phi_S(x) = -[G, L]$$

(*ii*) *Let $\tilde{I} \subset R(\tilde{S})$ be the ideal generated by zero-dimensional (virtual) representations of $\tilde{G}$. Then ϕ_S factors through $J^{-1}(R(\tilde{S})/\tilde{I})$.*

Proof. It is shown in [9] that on a suspension $\bar{J}$ is a homomorphism. Part (i) of the lemma is just a restatement of Lemma 2.2. For part (ii) observe first that $\alpha : R(\tilde{S}) \longrightarrow K^0(G/S^+)$ factors through $R(\tilde{S})/\tilde{I}$ (e.g. by considering the maps induced in K-theory from the fibration $G/S \longrightarrow B\tilde{S} \longrightarrow B\tilde{G}$). The rest of part (ii) is then the Adams conjecture for $J : K^0(S^2(G/S^+)) \longrightarrow \pi_s^1(S^2(G/S^+))$.

In the next section we shall estimate the order of $[x]$ in $J^{-1}(R(\tilde{S})/I)$ for various choices of $S \subset G$.

§3. PROOF OF THE THEOREM

We keep the notation of the preceding section. First we shall derive some algebraic results on λ-ideals in $R(S^1) = \mathbb{Z}[x, x^{-1}]$.

LEMMA 3.1. *For any $f = \Sigma a_i x^i \in \mathbb{Z}[x, x^{-1}]$ with non-negative coefficients the principal ideal generated by $\lambda_{-1}(f) = \Pi(1 - x^i)^{a_i}$ is a λ-ideal. Conversely, any λ-ideal of $\mathbb{Q}[x, x^{-1}]$ is obtained in this way.*

Proof. The first part is proved in ([4], p. 338). For the converse we observe that any non-zero ideal I of $\mathbb{Q}[x, x^{-1}]$ is generated by some monic polynomial $g(x) \in \mathbb{Q}[x]$. Over $\mathbb{C}$ we can write $g(x) = \prod_{i=1}^{n} (x - \alpha_i)$. Since $\psi^k g(x) = \prod_{i=1}^{n} (x^k - \alpha_i)$ must be a multiple of

$g(x)$, we conclude that $\{\alpha_1^k, \ldots, \alpha_n^k\} \subset \{\alpha_1, \ldots, \alpha_n\}$. This is possible only if $g(x) = (x^{n_1}-1)\ldots(x^{n_r}-1)$.

Obviously, the proof only used the λ-structure and not the conjugation. The next lemma studies λ-ideals generated by a real representation of S^1.

LEMMA 3.2. *Let $k_1, \ldots, k_n$ be positive integers with greatest common divisor equal to* 1. *Put $f = \sum_{i=1}^{n} (x^{k_i} + x^{-k_i} - 2) \in \mathbf{Z}[x, x^{-1}]$ and let $I \subset \mathbf{Z}[x, x^{-1}]$ be the λ-ideal generated by f. Then*

(*i*) *$\mathbf{Q}I \subset \mathbf{Q}[x, x^{-1}]$ is generated by $(x-1)^2$.*

(*ii*) *I contains some power of $(x-1)^2$. If p is a prime not dividing the mth elementary symmetric function of the numbers $k_1^2, k_2^2, \ldots, k_n^2$, we have $(x-1)^{2m} \in \mathbf{Z}_{(p)}I \subset \mathbf{Z}_{(p)}[x, x^{-1}]$.*

Proof. As an ideal I is generated by the coefficients g_m of the power series $g(u) = \Sigma g_m u^m$ defined by

$$\lambda_t(f) = \prod_{i=1}^{n} ((1 + x^{k_i}t)(1 + x^{-k_i}t)(1+t)^{-2}) = \prod_{i=1}^{n} (1 + (x^{k_i} + x^{-k_i} - 2)u) \tag{3.1}$$

where $u = t(1+t)^{-2}$. Since f is divisible by $(x-1)^2$ but not by $(x-1)^3$, we conclude from the preceding lemma that $\mathbf{Q} \times I$ must have a generator of the form $(x^a - 1)(x^b - 1)$. Replacing x in (3.1) by a primitive bth root of unity, we must obtain 1 on the r.h.s. Hence $b = a = 1$, which is (i).

From (i) we conclude that $(x-1)^2$ is torsion in $\mathbf{Z}[x, x^{-1}]/I$. By a well-known argument of G. Segal (see e.g. [6], p. 31), $(x-1)^2$ must be nilpotent in $\mathbf{Z}[x, x^{-1}]/I$. In particular, in $\mathbf{Z}_{(p)}[x, x^{-1}]/\mathbf{Z}_{(p)}I$ polynomials of the form $\sum_{i\geq 0} a_i(x-1)^i$ with $a_0 \not\equiv 0 \bmod p$ are invertible. Since the mth elementary symmetric function $\sigma_m(x^{k_1} + x^{-k_1} - 2, \ldots, x^{k_n} + x^{-k_n} - 2)$ represents zero in $\mathbf{Z}_{(p)}[x, x^{-1}]/I$, and since $x^k + x^{-k} - 2 = k^2(x-1)^2 - k^2(x-1)^3 + \text{higher powers of } (x-1)$, the assertion follows.

COROLLARY 3.3. *Let G be a compact semisimple Lie group, $S^1 \subset G$ a circle subgroup not meeting the center of G, and $I \subset R(S^1)$ the ideal generated by the restrictions of zero-dimensional representations of G. Then $\mathbf{Q}I = (x-1)^k\mathbf{Q}[x, x^{-1}]$ where k equals either* 1 *or* 2.

Proof. The (complexified) adjoint representation of G restricts on S^1 to dim $(G) + \Sigma(x^{k_i} + x^{-k_i} - 2)$ where the $\pm k_i$ are the restriction of the rootforms of G to $S^1 \subset G$. Since the chosen S^1 does not meet the center of G, the k_i are relatively prime, and the assertion follows from Lemma 3.2. (i).

COROLLARY 3.4. *Let the assumptions be as in Lemma* 3.2. *Then, if $a = \sigma_1(k_1^2, \ldots, k_n^2)$* and *$\sigma_2(k_1^2, \ldots, k_n^2)$ are relatively prime, I is the ideal generated by $(x-1)^4$ and $a(x-1)^2$.*

Proof. By Lemma 3.2. (ii) we know that I contains $(x-1)^4$. But $\sigma_1(x^{k_1} + x^{-k_1} - 2, \ldots, x^{k_n} + x^{-k_n} - 2)$ is contained in I, and mod $(x-1)^4$ this equals $a(x-1)^2 - a(x-1)^3$. Multiplying with $(x-1)$ we get $a(x-1)^3 \in I$ and finally $a(x-1)^2 \in I$.

We turn now to the groups $J^{-1}(R(S^1)/I)$ under the favourable conditions of Corollary 3.4.

LEMMA 3.5. *Let $I \subset \mathbb{Z}[x, x^{-1}]$ be the ideal generated by $(x-1)^4$, and let x_k be the class of $(x-1)^k$ in $J^{-1}(\mathbb{Z}[x, x^{-1}]/I)$. Then a complete set of relations for the generators x_k, $(0 \leq k \leq 3)$ of $J^{-1}(\mathbb{Z}[x, x^{-1}]/I)$ is*

$$x_3 = x_2, \qquad 2x_0 = 0$$

$$24x_1 = 44x_2, \quad 240x_2 = 0.$$

Proof. Let p be a prime. Then it suffices to show that these relations define $J^{-1}(\mathbb{Z}[x, x^{-1}]/I)_{(p)}$, so we have only to consider the relations $(m\psi^m - 1)\,(x_k)_{(p)} = 0$ for m prime to p. The case $m = -1$ yields $x_3 = x_2$. If $p \neq 2$ we get $(3x_1 + 2x_2)_{(p)} = 0$, $(15x_2)_{(p)} = 0$ and if $p \neq 3$ we get $(8x_1 + 12x_2)_{(p)} = 0$, $(80x_2)_{(p)} = 0$. Thus for $p > 5$ the localised group is zero. But for $p = 2, 3, 5$ it sufficies to check the above values of m prime to p since these generate (topologically) the group of p-adic units.

Similary we obtain Lemma 3.6.

LEMMA 3.6. *Let $I \subset \mathbb{Z}[x, x^{-1}]$ be the ideal generated by $(x^2-1)^2$, and let x_k be the class of $(x-1)^k$ in $J^{-1}(\mathbb{Z}[x, x^{-1}]/I)$. Then a complete set of relations for the generators x_k, $(0 \leq k \leq 3)$, of $J^{-1}(\mathbb{Z}[x, x^{-1}]/I)$ is*

$$24x_1 = 0, \qquad 2x_0 = 0$$

$$8x_2 = 0, \quad 3x_2 + x_3 = 0.$$

We proceed now to apply these results to the situation of §2, which we can picture in the diagram

$$\begin{tikzcd} \tilde{S} \arrow[r] \arrow[d] & \tilde{G} \rightleftarrows H \arrow[d] \\ S \arrow[r, dash] & G. \end{tikzcd} \tag{3.2}$$

Recall that G is a compact semisimple liegroup, $\tilde{G}$ its universal covering, $S \subset G$ a circle subgroup and $\tilde{S} \longrightarrow S$ the covering induced from $\tilde{G} \longrightarrow G$. The map $\tilde{G} \longrightarrow H$ is supposed to be the projection onto a simple factor of $\tilde{G}$.

Whenever possible, we shall choose $S \subset G$ is such a way that the following condition is fulfilled:

CONDITION 3.7 *The identity component of $\tilde{S}$ has only trivial intersection with the center of G.*

If this is the case, we may identify the identity component of $\tilde{S}$ with S. Thus $\tilde{S} = S \times C$ where C is a subgroup of the center of $\tilde{G}$.

Denote by $\tilde{I}_S$ the ideal of $R(\tilde{S})$ generated by zero-dimensional representations of $\tilde{G}$, and let I_S be the intersection of $\tilde{I}_S$ with $R(S) = \mathbb{Z}[x, x^{-1}]$. By Corollary 3.3 we have $\mathbb{Q}I_S \supset (x-1)^2\mathbb{Q}[x, x^{-1}]$. We shall use the preceding lemmas to estimate the order of x in $J^{-1}(R(\tilde{S})/\tilde{I}_S)$. By Lemma 2.3. we have then obtained an estimate for the order of $[G, L]$ in π_d^s.

We shall distinguish several cases as to the type of H. In most cases S will be a subgroup of H, and representations of H will be used to get a hold on $\tilde{I}_S$.

Case 1. H is of the type A_n.

If $n = 1$ we assume for the moment that the center of $H = SU(n+1)$ maps non-trivially to G. Let ρ be the standard representation of H. Clearly there is a subgroup $S \subset H$ such that ρ restricts on S to $x + x^{-1} + (n-1)$. Let $\tilde{S} = S \times C$ be defined by Diagram 3.2. Then ρ restricts on C to $(n+1)y^{-1}$ where $y^{n+1} = 1$. Multiplying $\rho - (n+1)$ by y we obtain

$$f = (x + x^{-1} - 2) - (n+1)(y-1) \in \tilde{I}_S.$$

Now, if we subtract f from its conjugate we obtain $(n+1)\,(y^2 - 1) \in \tilde{I}_S$, and hence $x^2 + x^{-2} - 2 = \psi^2 f + (n+1)\,(y^2-1) \in \tilde{I}_S$. By Lemma 3.6. we conclude that $24[G, L] = 0$.

If $n = 1$ and the center of H does map trivially to G, denote the circle subgroup of H by S'. The subgroup S of Diagram 3.2 is then doubly covered by S'. Reasoning as above we obtain $x'^2 + x'^{-2} - 2 \in \tilde{I}_S$. But now $x'^2 = x$ and hence even $x + x^{-1} - 2 \in \tilde{I}_S$.

Case 2. H is of type B_n, C_n or D_n.

Let T be the standard maximal torus of $SO(2n+1)$, $Sp(n)$, $SO(2n)$, respectively, and let ρ be the defining representation of these groups. In terms of coordinates t_i on T define $S \subset T \subset H$ by $t_i = x^{k_i}$, where $k_1, \ldots, k_n$ are relatively prime, and in case of B_n or D_n satisfy $\sum_i k_i \equiv 0(2)$. Condition 3.7. is satisfied if in case of D_n we have $k_i - k_j \not\equiv 0(2)$ for some i, j.

The restriction of ρ-dim(ρ) to $\tilde{S}$ is then of the form

$$f = \sum (x^{k_i} + x^{-k_i} - 2)y + 2n(y-1) \in \tilde{I}_S \tag{3.3}$$

where $y^2 = 1$ and in the case of B_n or C_n even $y = 1$. Thus in the case of C_n we may choose $k_1 = 1$, $k_i = 0$ for $i > 1$, obtaining $x + x^{-1} - 2 \in I_S$ and $24 \times [G, L] = 0$.

Now as in the proof of Lemma 3.2., the ideal $\tilde{I}_S$ contains the coefficients g_m of u^m in

$$g(u) = \prod_{i=1}^{n} (1 + (x^{k_i} + x^{-k_i} - 2)u) - (1 + 2(y-1)u)^n \tag{3.4}$$

Hence in the case of B_n we conclude from $y = 1$ that I_S contains the symmetric functions of the $(x^{k_i} + x^{-k_i} - 2)$. Choosing $k_i = k_2 = 1$, $k_i = 0$ for $i > 2$, we obtain $(x-1)^4 \in I_S$ and $2(x-1)^2 \in I_S$. By Lemma 3.5. the case B_n is also finished.

In the remaining case of D_n we localize at a given prime p. If we put $k_1 = k_2 = 1$ and $k_i = 0$ for $i > 2$, we have $\psi^2 g_1 = 8(x-1)^2 +$ higher powers of $(x-1)$. In view of Lemmas 3.2 and 3.5 this suffices for $p > 2$.

The case of $p = 2$ is more difficult. Put $A = \sum_i k_i^2$ and $B = \sum_{i<j} k_i^2 k_j^2$. We assume that $B \equiv 1(2)$ and $A \equiv 2(4)$ which is compatible with the condition $\Sigma k_i \equiv 0(2)$. Since the ideal generated by $\psi^2(\tilde{I}_S)$ in $Z_{(2)}[x, x^{-1}]$ is already a λ-ideal, and since up to higher

powers of (x^2-1) we have $\psi^2 g_1 = A(x^2-1)^2 + \ldots$ and $\psi^2 g_2 = B(x^2-1)^4 + \ldots$, we conclude from Lemma 3.2. that $(x^2-1)^4$ and $2(x^2-1)^2$ are elements of I_S.

Now we have to separate the cases of odd and even n. Let us assume first that n is odd. Choose the k_i in such a way that $A \equiv 2n \bmod 8$. Let $g_m(1)$ be the coefficient of u^m in (3.4) with y replaced by 1. Thus $g_1 = g_1(1) - 2n(y-1)$ and $g_2 = g_2(1) + 4n(n-1)(y-1)$. Since $g_1, g_2 \in \tilde{I}_S$ we obtain $g_2(1) + (n-1/2)g_1(1)^2 \in \tilde{I}_S$. But up to higher powers of $(x-1)$ this element is $B(x-1)^4$, and we conclude $(x-1)^4 \in I_S$.

Let $R = Z_{(2)}[x, x^{-1}, y]/(y^2-1)$ and $M := J^{-1}(R/\tilde{I}_S)$. Since $(x-1)^4$ and $2(x^2-1)^2$ are in $\tilde{I}_S$, by Lemma 3.5. we have in M the relations $(x-1)^3 \equiv (x-1)^2$, $8(x-1) \equiv 4(x-1)^2$, $8(x-1)^2 \equiv 0$ and $2 \equiv 0$. But M is a module over $Z_{(2)}[Y]/(y^2-1)$, and from $(y+1)(x-1)g_1 \in \tilde{I}_S$ and $(x-1)^2 g_1 \in \tilde{I}_S$ we obtain the relations $A(y+1)(x-1)^3 \equiv 0$, $-2n(y-1)(x-1)^3 \equiv 0$. These add up to $4(x-1)^3 \equiv 0$, which finishes the proof for the case of odd n.

We turn to the case of even n. Let $I' \subset Z[x, x^{-1}]$ be the ideal generated by $2(x^2-1)^2$ and $(x^2-1)^4$. Then we know already that $I' \subset I_S$. Put $c_2 = g_2 + 2(n-1)g_1$; then also $c_2 \in I_S$. Let I be generated by I' and c_2. Modulo elements of I' we can write $c \equiv 8q(x-1) - 4q(x^2-1) - 4q(x-1)(x^2-1) + (x^2-1)^2 + (x^2-1)^3$ where the constant q is even. Now a straightforward computation (which we omit) shows that x is of order 8 in $J^{-1}(Z_{(2)}[x, x^{-1}]/I)$. This finishes the proof for the case of even n.

Case 3. H is of type F_4 or G_2

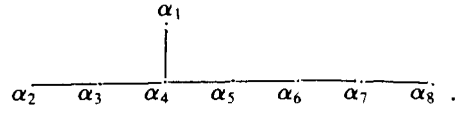
Assume first that H is of type G_2. Let α_1 resp. α_2 be the long resp. short simple root, and define $S \subset H$ by the condition that α_v restricts to the weight w_v of S, where the integers w_1, w_2 are relatively prime.

Let *ad* be the adjoint representation of H. If $\pm k_i$ are the weights of *ad* restricted to S, we have $\sum_i k_i^2 = 8(w_1^2 + 3w_1w_2 + 3w_2^2)$. By Lemma 3.2. we obtain the p-primary part of the theorem for any $p > 2$. For the 2-primary part let ρ be the representation with highest weight $\alpha_1 + 2\alpha_2$. Then the weights of ρ restricted to S are $\pm w_2$, $\pm(w_1 + w_2)$, $\pm(w_1 + 2w_2)$, and the 2-primary part follows from Lemma 3.2. (by taking, e.g. $w_2 = 0$, $w_1 = 1$).

Now let H be of type F_4, and α_1, α_2 the long simple roots, α_3, α_4 the short simple roots, numbered in the standard way with $(\alpha_2, \alpha_3) \neq 0$. For $S \subset H$ let w_v the restriction of α_v to S. If $\pm k_i$ are the weights of the adjoint representation of H restricted to S, we see that $\sum_i k_i^2 = 18w_1^2 + \ldots$ and $\sum_{i<j} k_i^2 k_j^2 = 21w_1^4 + \ldots$ where the terms omitted involve w_2, w_3, w_4. Choosing $w_1 = 1$, $w_2 = w_3 = w_4 = 0$ we conclude from Lemmas 3.2. and 3.5. that the p-primary part of the theorem holds for all primes $p \neq 3$. However, since $H = F_4$ is centerfree, H is actually a direct factor of G. But the 3-primary component of $[F_4, L]$ in π_{52}^s is zero because of filtration reasons[10].

Case 4. H is of type E_n, $n = 6, 7, 8$.

Let $\alpha_1, \ldots, \alpha_n$ be the simple roots of E_n, numbered in such a way that the Dynkin diagram of E_8 becomes



Let $\chi_1, \ldots, \chi_n$ be the corresponding fundamental weights, and define $S \subset H$ by the condition that all χ_i except χ_1 vanish on S. If we restrict the adjoint representation to S we obtain the character $f = (x^2 + x^{-2} - 2) + b(x + x^{-1} - 2)$ with $b = 20 + 6(n-6)(n-5)$. By Lemmas 3.2 and 3.5. the critical primes are $p = 2$, 3, 5. The prime $p = 5$ occurs only for E_8, where it can be eliminated since E_8 is center-free and, by the results of [10], has vanishing 5-primary component. For $p = 2$, 3, our earlier lemmas do not apply. Instead we calculate directly the order of x in the J-group corresponding to the λ-ideal generated by f. The result is 72 except for E_6, where it is 9×32. Fortunately, E_6 has a low-dimensional representation, namely the representation of dimension 27 with highest weight χ_2, which we can use to cut down the 2-primary part of the order. With the same circle subgroup as above we obtain at $p = 2$ the character $g = 6(x + x^{-1} - 2)$. By Lemma 3.2. we are done.

Thus, finally, the proof of Theorem 1.1 is finished.

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Fachbereich Mathematik der
Gesamthochschule Wuppertal
Gaußstraße 20
D 5600 *Wuppertal* 1
West Germany