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Solitons from a direct point of view: padeons

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Abstract: A systematic approach to soliton interaction is presented in terms of a particular class of solitary waves (padeons) which are linear fractions with respect to the nonlinearity parameter ϵ. A straigthforward generalization of the padeon to higher order rational fractions (multipadeon) yields a natural ansatz for N-soliton solutions. This ansatz produces multisoliton formulas in terms of an 'interaction matrix' A. The structure of the matrix gives some insight into the hidden IST-properties of a familiar set of 'integrable' equations (KdV, Boussinesq, MKdV, sine-Gordon, nonlinear Schrödinger). The analysis suggests a 'padeon' working definition of the soliton, leading to an explicit set of necessary conditions on the padeon equation.

1. Introduction

It is common observation that the term 'soliton' which has become so popular in many areas of the physics concerned with nonlinear phenomena does not refer to a clear and generally accepted definition.

Solitons are generally understood as being 'special solutions to some special nonlinear PDE's'. Though the special character of solitons has become clear for those exceptional equations which enjoy as remarkable hidden properties as the existence of an inverse spectral transform (IST), physicists feel often satisfied with the more phenomenological picture of a solitary wave with 'particle-like interaction properties'.

Yet many equations are known to possess pulsed solutions in the form of a bell or a kink, showing various degrees of stability throughout interaction [1], and it is not always possible to decide from results of numerical experiments whether such excitations should be called 'solitons', or not....

Studying some familiar soliton equations (KdV, Boussinesq, MKdV, Sine–Gordon) from a direct point of view [6] it is easy to realize that the particle-like interaction properties of their solitary waves can be related to their analytic properties with respect to the nonlinearity parameter ϵ. These solitary waves are linearly related to a $[0/1]$ fraction in ϵ of a particular functional form (padeon). The corresponding N-soliton solutions are linearly related to $[N-1/N]$ fractions in ϵ, solving the 'primary' (padeon) equation, each partial fraction of which corresponds to a soliton.

One of the purposes of this paper is to present a 'padeon' approach to soliton interaction which could lead to a working definition of the soliton that would fill the gap between the 'spectral' soliton and the 'phenomenological' one. The analysis is developed from a naive point of view in order to serve the other main purpose of this work. This is to help the soliton hunter to decide whether a given nonlinear equation possesses solitons, or not, and to obtain multisoliton formulas.

In contrast with other direct approaches, such as Hirota's [3] and Rosales' [7], the padeon method does not require the introduction of bilinear forms, or any other clever but not straightforward manipulations.

The paper is organized as follows: We first consider the simplest nonlinear evolution equation (Burgers) that gives rise to solitary waves, and we obtain a two-parameter family of solitary waves (kinks) by scaling the fieldvariable. We then introduce the padeons, solutions of a general 'padeon equation', and the corresponding N-padeons (Section 2). The existence of dipadeons ($N = 2$) and their relation to two-soliton solutions is discussed in Section 3.

We then proceed to the case of $N > 2$-soliton interactions by considering N-padeons which can be expressed with a hermitean (symmetric) $N \times N$ interaction matrix. Their existence and asymptotic properties are discussed in Section 4. The appearance of a condition of a new type (separability) characterizes the transition from $N = 2$ to $N \geqslant 3$. The form of the interaction matrix is also specified.

In Section 5, we discuss the connection between the interaction matrix and the IST approach to multisolitons.

A straightforward generalization to complex padeons is presented in the case of the cubic Schrödinger equation (Section 6). Degenerate N-padeons are illustrated with the Burgers equation (Section 7).

2. Solitary waves, padeons and N-padeons

A straightforward procedure of deriving the solitary wave solutions of a given nonlinear equation has been suggested by several examples given in [6]. To recall the method, we consider Burgers' nonlinear diffusion equation:

$$V_t - V_{xx} + VV_x = 0. \tag{1}$$

Setting $V = \epsilon \overline{V}$ one can look for particular solutions $\overline{V}(x, t, \epsilon)$ of the scaled equation:

$$\overline{V}_t - \overline{V}_{xx} + \epsilon \overline{V}\overline{V}_x = 0, \tag{2}$$

displaying particular properties with respect to the parameter ϵ. Starting with a power series expansion

$$\overline{V}(x, t, \epsilon) = \sum_{n=0}^{\infty} \epsilon^n V_n(x, t), \tag{3}$$

one can set up conditions so as to define the successive V_n's as particular solutions of the iteration hierarchy:

$$\begin{aligned} &V_{0,t} - V_{0,xx} = 0, \\ &V_{n,t} - V_{n,xx} = -\sum_{j=0}^{n-1} V_j V_{n-j-1,x}, \quad n \geqslant 1. \end{aligned} \tag{4}$$

Here, we are looking for progressive wave solutions $V(\xi = x - kt)$ of the Burgers equation which could describe a travelling pulse in the form of a kink, say with:

$$\lim_{\xi \to -\infty} V(\xi) = c \geqslant 0 \quad \text{and} \quad \lim_{\xi \to \infty} V(\xi) = 0. \tag{5}$$

At $n = 0$ we observe that the linearized Burgers equation possesses positive solutions of the form $\exp\theta$, with $\theta = -k\xi$ and $k > 0$, which are totally monotonic functions of ξ. Going through the iteration with this choice for V_0, one can try to end up with a rational fraction in $\exp\theta$ of the appropriate form:

$$\bar{V} = \exp\theta/(1 + (\epsilon/c)\exp\theta) \tag{6}$$

which would indeed produce a kink $V = \epsilon\bar{V}$, with the above properties (5), and which could be regarded as the sum of a geometrical perturbation series.

With this in mind, it is natural to attempt to solve the system (4), with $V_0 = \exp\theta$, under the condition that V_n be proportional to $\exp(n+1)\theta$. This requirement yields the following sequence of particular solutions:

$$V_n = (-)^n (2k)^{-n} \exp(n+1)\theta. \tag{7}$$

The resulting series (3) is the geometrical series:

$$\bar{V}(x, t, \epsilon) = \exp\theta \sum_{n=0}^{\infty} \left(-\epsilon \frac{\exp\theta}{2k}\right)^n. \tag{8}$$

It generates, for arbitrary positive values of ϵ, a regular kink, rational in ϵ as a [0/1] Padé form:

$$[0/1]\,\bar{V} = \exp\theta/(1 + (\epsilon/2k)\exp\theta) \tag{9}$$

which sums the geometrical series (8) and which is an actual solution of (2). This fraction can also be written in the form:

$$[0/1]\,\bar{V} = -2z_x/(1 + \epsilon z) \quad \text{with} \quad z = (1/2k)V_0. \tag{10}$$

The corresponding two parameter family of solitary wave solutions $\epsilon\bar{V}$ of the Burgers equation (1) takes the [1/1] form:

$$[1/1]\,V = -2\partial_x \log(1 + \epsilon z).$$

By applying the same technique to several nonlinear dispersive equations (KdV, Boussinesq, RLW, p-KdV, sine-Gordon, ϕ^4) with lowest nonlinearity of order $p+1$, one obtains [6] closed form solitary wave solutions, generated by exponential solutions of the linearized equation, which exhibit a particular structure with respect to the nonlinearity parameter $\epsilon = \lambda^p$, λ being the scaling parameter.

Yet, solitons are not just ordinary solitary waves. Their particle-like interaction properties correspond to the existence of multisoliton solutions which behave asymptotically as a (nonlinear) superposition of separated solitary waves. The fact that the solitary wave solutions of the standard soliton equations (KdV, Boussinesq, MKdV, Sine–Gordon) are linearly related to a rational [0/1] fraction in ϵ for a 'primary' field variable seems to indicate that the existence of N-solitons should be related to the existence of higher order solutions to the 'primary' equations, rational in ϵ of type $[N-1/N]$, each partial fraction of which would account for a soliton.

Furthermore, we remark that the [0/1] solitary wave solutions of the corresponding 'primary' equations have the same functional form, similar to that obtained for the Burgers equation:

$$[0/1]\,\bar{V} = -2z_x/(1 + \epsilon z^p), \qquad z = (1/2k)\exp\theta, \quad \theta = -kx + \omega(k)t, \tag{11}$$

where $p = 1$ or 2 and $\omega(k)$ is given by the linear dispersion relation. Calling such solitary waves 'padeons' we examine under which conditions a given padeon equation (i.e. a nonlinear equation with padeon solitary wave solutions) for a field $\overline{V}(x, t)$ can be regarded as a soliton equation, or as the primary equation associated with a soliton equation. We assume that the linear dispersion relation produces a phasevelocity $\omega(k)/k$ which is a monotonic function of k.

Starting the iteration procedure for the padeon equation with a superposition of N exponential solutions of the linearized equation:

$$V_0 = \sum_{j=1}^{N} a_j^2 \exp \theta_j, \qquad \theta_j = -k_j x + \omega(k_j)t, \quad 0 < k_1 < \cdots < k_N, \quad a_j \in \mathbb{R} \tag{12}$$

we construct a generalized perturbation series $\overline{V} = \sum_{n=0}^{\infty} \epsilon^n V_n$, by requiring that V_n should be a linear combination of the various exponentials which appear as an inhomogeneous term at the r.h. side of the nth iteration equation. The asymptotic properties of the desired N-soliton solutions, and their regularity at $\epsilon > 0$, suggest that one should look for particular solutions of the padeon equation of the form:

$$[N-1/N]\overline{V} = \sum_{j=1}^{N} \frac{-2z_{j,x}}{1+\epsilon z_j^p}, \quad p = 1 \text{ or } 2 \tag{13}$$

involving N positive valued functions $z_j(x, t)$, constrained by the assumption that the nth order term in the expansion of $[N-1/N]\overline{V}$ should coincide with the V_n calculated through iteration:

$$V_n = -2(-)^n \sum_{j=1}^{N} z_j^{pn} z_{j,x} = -2(-)^n \operatorname{Tr}(Z^{pn} Z_x) \tag{14}$$

where Z stands for the diagonal $N \times N$ matrix with element $Z_{ij} = z_i\, \delta_{ij}$. Though the positivity (realness) of the z_j's is not necessary for the regularity of $[N-1/N]\overline{V}$ at $\epsilon > 0$, we impose it here for simplicity. Let us even restrict our search to $[N-1/N]$ solutions in which also the numerators $-2z_{j,x}$ are positive valued, so that their poles interlace with $N-1$ zeros on the negative real ϵ-axis. This additional (Stieltjes) property follows, at $N=2$, from the positivity of z_1 and z_2, and shows up [5] for all N in the KdV-case. At the present stage it seems a reasonable ansatz on account of the asymptotic cancellations that we expect between $N-1$ poles and zeros of $[N-1/N]\overline{V}$. If $[N-1/N]\overline{V}$ solutions of the form (13) can be found with $z_j > 0$ and $z_{j,x} < 0$, we call them 'N-padeons'.

3. Dipadeons ($N = 2$)

That N-padeons have the required asymptotic properties at $N=2$ can be easily checked [6]. As the existence of dipadeons (two-soliton solutions) constitutes a crucial step towards the existence of solitons, we recall the (necessary) conditions it imposes on the padeon equation. Starting with the following solution of the linearized equation:

$$V_0 = \exp \mathscr{S}_1 + \exp \mathscr{S}_2, \quad \mathscr{S}_i = \theta_i + \ln a_i^2, \tag{15}$$

we get after the first two iterations:

$$-V_1 = \frac{\exp(p+1)\mathscr{S}_1}{(2k_1)^p} + \alpha_{12} \exp(p\mathscr{S}_1 + \mathscr{S}_2) + \mathrm{id}(1 \rightleftarrows 2), \tag{16}$$

$$V_2 = \frac{\exp(2p-1)\mathscr{S}_1}{(2k_1)^{2p}} + \sum_{m=1}^{p} \beta_{m,2p+1-m} \exp[(2p+1-m)\mathscr{S}_1 + m\mathscr{S}_2] + \mathrm{id}\,(1 \rightleftarrows 2), \tag{17}$$

with mixing coefficients α_{12} and β_{ij} depending on the particular form of the padeon equation.

We assume the existence of a solution $[1/2]\overline{V}$, of the form (13), which matches the perturbation terms V_n and in particular V_0 and V_1 ($p = 1$ or 2):

$$V_0 = -2 \operatorname{Tr} Z_x, \tag{18}$$

$$V_1 = 2(\operatorname{Tr} Z)^p(\operatorname{Tr} Z_x) - 2\left[(\operatorname{Tr} Z)^{p-1} \det Z\right]_x. \tag{19}$$

It then follows that:

$$\operatorname{Tr} Z = \exp \mathscr{S}_1/2k_1 + \exp \mathscr{S}_2/2k_2, \tag{20}$$

$$\det Z = \frac{\eta}{4k_1k_2} \exp(\mathscr{S}_1 + \mathscr{S}_2), \quad \eta = 1 - 4\frac{k_1k_2}{k_1+k_2}\alpha_{12} \quad \text{if } p = 1, \tag{21}$$

$$\det Z = \tfrac{1}{2}(\operatorname{Tr} Z)^{-1}\left[\left(\frac{1}{4k_1^2k_x} - \frac{\alpha_{12}}{2k_1+k_2}\right) \exp(s\mathscr{S}_1 + \mathscr{S}_2) + \text{id}\,(1 \rightleftarrows 2) \quad \text{if } p = 2.\right. \tag{22}$$

The assumed positivity of $z_{1,2}$ implies the necessary condition on α_{12}

$$0 < \alpha_{12} < (pk_1 + k_2)/4k_1^pk_2. \tag{23}$$

The eigenvalues $z_{1,2}$ obtained from Tr Z and det Z:

$$z_{1,2} = \tfrac{1}{2}\left[\operatorname{Tr} Z \pm \left[(\operatorname{Tr} Z)^2 - 4 \det Z\right]^{1/2}\right] \tag{24}$$

provide the 'eigenvalue' representation of the solution.

The positivity of $-2z_{i,x}$ follows from the condition (23). Thus, if it exists, the solution is a dipadeon. Furthermore, it follows from the explicit expressions for z_1 and z_2 that as $t \to \pm\infty$, with θ fixed, $r = 1$ or 2, only one eigenvalue (z_r) remains finite, while the other one ($z_{i \lessgtr r}$) goes to zero or to $+\infty$:

$$z_i \sim \exp \theta_i \quad \text{as } t \to \pm\infty. \tag{25}$$

This means that at large values of $|t|$ the dipadeon separates into two kinks ($p = 1$) or bells ($p = 2$) which retain their identity through interaction, as one expects from a two-soliton solution.

Once the 'positivity' condition (23) on α_{12} is fulfilled, a further necessary condition for the existence of a dipadeon is that the second-order term in the expansion of $[1/2]\overline{V}$: $-2\operatorname{Tr}(Z^{2p} Z_x)$, calculated from V_0 and V_1, should coincidence with the expression (17) for V_2. Expressing $\operatorname{Tr}(Z^{2p} Z_x)$ in terms of Tr Z and det Z one obtains the following conditions on the β's:

$$\begin{aligned} p = 1:&\quad \beta_{12} = \left[(2k_1 + k_2)/k_1(k_1 + k_2)\right]\alpha_{12}, \\ p = 2:&\quad \beta_{14} = \left[(4k_1 + k_2)/4k_1^2(2k_1 + k_2)\right]\alpha_{12}, \\ &\quad \beta_{23} = \left[(3k_1 + k_2)/4k_1k_2(2k_1 + k_2)\right]\alpha_{12}\left[1 + 4k_1^2k_2/k_1(k_1 + k_2)^2\right]. \end{aligned} \tag{26}$$

4. *N*-padeons and the interaction matrix

Let us now assume that the primary equation produces padeons and dipadeons.

Before deciding that the corresponding solitary waves can be regarded as true solitons one should be able to extend the number of interacting waves to arbitrary N, without loosing the asymptotic property that as $t \to \pm\infty$ with θ_r fixed

$$z_i \sim \alpha_i \exp \theta_i \quad \text{with } \alpha_i = \alpha_i(k_1, \ldots, k_N). \tag{27}$$

For this purpose we need another representation of the N-padeon than the one we have used so far.

In fact, a striking feature of multisoliton interaction is that the number of interacting solitons and their relative size is of no qualitative importance, and that the collisions occur in pairs [8]. Therefore, if an N-padeon is to be understood as an N-soliton solution, one should be able to represent it in terms of an explicit 'interaction' matrix A, of dimension N, such that $A_{ij} = A(k_i, k_j)$.

At $N = 2$ we remark that the positivity of $-2z_{i,x}$ allows us to rewrite the dipadeon in the form:

$$[1/2]\bar{V} = \langle \mu, (I + \epsilon Z^p)^{-1} \mu \rangle \tag{28}$$

where $|\mu\rangle$ denotes a $\mathbb{C}^2$ vector with components μ_i, such that $|\mu_i|^2 = -2z_{i,x}$ and $\langle \mu | = (|\mu\rangle)^+$. This diagonal matrix representation is equivalent with nondiagonal ones involving similar matrices $\tilde{A} = SZ\,S^{-1}$, and more particularly with 'symmetric' representations:

$$[1/2]\bar{V} = \langle \phi, (I + \epsilon A^p)^{-1} \phi \rangle \quad \text{with} A = UZU^+ = A^+ \tag{29}$$

where U is a unitary 2×2 matrix and $|\phi\rangle = U|\mu\rangle$. It means that V_n can also be expressed as a 'sandwich'

$$V_n = (-)^n \langle \phi, A^{np} \phi \rangle. \tag{30}$$

Consistency with the conditions (14) leads to the following 'trace' conditions on A:

$$\mathrm{Tr}[A^{pn}(2A_x + |\phi\rangle\langle\phi|)] = 0. \tag{31}$$

4.1. $p = 1$

If satisfied for $n = 0$ and $n = 1$, it is easy to verify that, in the 2-dimensional case ($N = 2$), the condition (31) is satisfied for all n. When added to the previous conditions (20, 21) which determine Tr A and det A, these trace conditions amount to a constraint between the elements of A and the components of $|\phi\rangle$.

Yet, in order that A be acceptable as a 'two-soliton interaction' matrix it is necessary that, when restricted to $N = 1$, the sandwich (30) reproduces the geometrical form $(-)^n \exp \mathcal{S}_1((1/2k_1) \exp \mathcal{S}_1)^n$ which characterized the padeon. We thus need to have:

$$A_{11} = (1/2k_1) \exp \mathcal{S}_1 \quad \text{and} \quad \phi_1 = \exp(\tfrac{1}{2}\mathcal{S}_1 + \mathrm{i}\delta_1). \tag{32}$$

Together with the conditions (20,21), this requirement fixes ϕ_2 in terms of a phase δ_2: $\phi_2 = \exp(\frac{1}{2}\mathcal{S}_2 + \mathrm{i}\delta_2)$, and determines A in terms of $\mathcal{S}_{1,2}$ and some extra phase χ:

$$A_{22} = (1/2k_2) \exp \mathcal{S}_2, \qquad A_{12} = \sqrt{(1-n)/4k_1k_2} \exp \tfrac{1}{2}(\mathcal{S}_1 + \mathcal{S}_2) + \mathrm{i}\chi). \tag{33}$$

The trace condition ($n = 1$) fixes χ in terms of $\delta_1 - \delta_2$:

$$\mathrm{tg}(\chi + \delta_2 - \delta_1) = \frac{\sigma_{12}}{k_1 + k_2} \quad \text{with} \quad \sigma_{12} = \left[\frac{\eta(k_1+k_2)^2 - (k_1-k_2)^2}{1-\eta}\right]^{1/2} \tag{34}$$

though, at the cost of a new condition on α_{12}, in order that σ_{12} be real:

$$\alpha_{12} \leqslant 1/(k_1+k_2). \tag{35}$$

If this additional condition is satisfied, we can find a hermitean interaction matrix A. Rewriting A_{12} in terms of σ_{12}, it takes the simple form:

$$A_{12} = (k_1+k_2-\mathrm{i}\sigma_{12})^{-1}\phi_1\phi_2^*. \tag{36}$$

Let us now go to arbitrary N. In this case $V_0 = \Sigma_{i=1}^N \exp \mathscr{S}_i$ whereas the first order term V_1 takes the form

$$-V_1 = \sum_{i=1}^{N} \frac{\exp 2\mathscr{S}_i}{2k_i} + \sum_{i\neq j}^{N} \alpha_{ij} \exp(\mathscr{S}_i + \mathscr{S}_j). \tag{37}$$

In order that the above 2×2 matrix be extensible to an $N\times N$ interaction matrix with elements

$$A_{mn} = [k_m + k_n - \mathrm{i}\sigma_{mn}]^{-1}\phi_m\phi_n^*, \quad m\neq n \tag{38}$$

which satisfies the generalized trace conditions (31), written in terms of $\mathbb{C}^N$-vectors $|\phi\rangle$ with $\phi_j = \exp(\frac{1}{2}\mathscr{S}_j + \mathrm{i}\delta_j)$, it is necessary (appendix) that:

(i) either the parameters

$$\sigma_{ij} = \left[\frac{\eta_{ij}(k_i+k_j)^2 - (k_i-k_j)^2}{1-\eta_{ij}}\right] \quad \text{with } \eta_{ij} = 1 - 4\frac{k_ik_j}{k_i+k_j}\alpha_{ij},$$

be separable:

$$\sigma_{ij} = \gamma_i - \gamma_j \quad \text{with } \gamma_i = \gamma(k_i),$$

(ii) or that the related parameters $\rho_{ij} = \mathrm{tg}^{-1}\, \sigma_{ij}/(k_i+k_j)$ be separable:

$$\rho_{ij} = \nu_i - \nu_j \quad \text{with } \nu_i = \nu(k_i).$$

The separability of σ_{ij} implies that σ_{ij} must have the form:

$$\sigma_{ij} = \frac{k_i+k_j}{(k_i+k_j)^2 + (\gamma_i-\gamma_j)^2}. \tag{40}$$

When this is the case, i.e. when α_{12} has the particular structure displayed by the formula (40), so that it automatically satisfies the previous conditions (23) and (35), the dipadeon (N-padeon) may be called a 'two soliton' (N-soliton) solution.

Indeed, we may as well particularize the choice of the phases $\delta_{1,2}$ so that the x-derivative of the matrix A satisfies the dyadic relation:

$$-2A_x = |\phi\rangle\langle\phi|. \tag{41}$$

It suffices to take $\delta_{1,2}$ linear in x, and such that $\delta_{i,x} = \frac{1}{2}\gamma_i$. In this form A can be generalized to an $N\times N$ matrix:

$$A = \tfrac{1}{2}\int_x^{\infty} \mathrm{d}x' |\phi(x')\rangle\langle\phi(x')| \tag{42}$$

with elements $A_{mn} = (\chi_m^* + \chi_n)^{-1}\phi_m\phi^*{}_n$, where $\phi_m = \exp\frac{1}{2}[-\chi_m^* x + \omega(k_m)t]$ and $\chi_m = k_m + \mathrm{i}\gamma_m$.

This hermitean matrix is bound to be positive definite for all N. Its principal minors $M(i_1 \cdots i_n)$, $1 < n < N$, are all positive; they can be written in the form:

$$M(i_1 \cdots i_n) = \left[\prod_{m=1}^{n} (2k_{i_m})\right]^{-1} \left[\prod_{l<m,1}^{n} n_{i_l i_m}\right] \exp(\mathscr{S}_{i_1} + \cdots + \mathscr{S}_{i_n}) \tag{43}$$

with

$$\eta_{ij} = \frac{(k_i - k_j)^2 + (\gamma_i - \gamma_j)^2}{(k_i + k_j)^2 + (\gamma_i - \gamma_j)^2}.$$

Using the fact that the eigenvalues $z_j > 0$ of A obey the N equations:

$$\sum_{i_1 < \cdots < i_n = 1}^{N} z_{i_1} \cdots z_{i_n} = \sum_{i_1 < \cdots < i_n = 1}^{N} M(i_1 \cdots i_n), \quad n = 1, \ldots, N \tag{44}$$

it is easy to check (along the same lines as in [5]) that, as $t \to \pm\infty$ with θ_r fixed, the z_j's have the required asymptotic behaviour

$$z_{j \lessgtr r} \to 0, \quad z_{j \gtrless r} \to +\infty \quad \text{and} \quad z_r \to \alpha_r^{(\pm)} \exp \theta_r, \quad \alpha_r^{(\pm)} = \frac{a_r^2}{2k_r} \prod_{i \lessgtr r} \eta_{ir} \tag{45}$$

for phase velocities which increase with k, and the t-reversed situation for phase velocities which decrease with k.

Thus, as we follow the rth solitary wave, the x-derivative of $-\epsilon[N - 1/N]\bar{V}$, which can now be written in the familiar form

$$-\partial_x [N/N] V = -\epsilon \partial_x \langle \phi, (I + \epsilon A)^{-1} \phi \rangle = 2\partial_x^2 \log \det(I + \epsilon A) \tag{46}$$

tends, when $t \to \pm\infty$, to the 'one soliton' limit

$$\tfrac{1}{2} k_r^2 \cosh^{-2} \tfrac{1}{2}(\theta_r + \tau_{r,\pm}) \quad \text{with } \tau_{r,\pm} = \log(a_r^2/2k_r) + \log \alpha_r^{(\pm)}. \tag{17}$$

In the other case (ii) the separability of ρ_{ij} implies that α_{12} must have the particular form:

$$\alpha_{12} = (\cos^2(\nu_1 - \nu_2))/(k_1 + k_2) \tag{48}$$

which, for a first order mixing coefficient (i.e. which arises from the first iteration equation) can hardly be expected from any padeon equation (the usual dispersion relations do not produce transcendental function of k). Nevertheless, we remark that: $0 < \alpha_{12} < (k_1 + k_2)^{-1}$ so that, if it exists, the dipadeon could be expressed in terms of a positive definite hermitean 2×2 interaction matrix with:

$$A_{12} = \frac{|\cos(\nu_1 - \nu_2)|}{k_1 + k_2} \exp \frac{\mathscr{S}_1 + \mathscr{S}_2}{2} + \mathrm{i}(\bar{\delta}_1 - \bar{\delta}_2)$$

and

$$\bar{\delta}_i = \delta_i + \nu_i.$$

However, though in the case the 3×3 interaction matrix which generalizes the above 2×2 matrix can still be shown to be positive definite (it has positive principal minors), it is clear from the form of A_{12}, and its x-derivative, that such interaction matrix can no longer be chosen so as to satisfy the former 'dyadic relation' (41).

As we shall see in the next section, such property of the interaction matrix, as the one displayed by formula (42), is closely related to the IST-interpretation of solitons.

4.2. $p = 2$

Looking at $N = 2$ for a symmetric representation of the dipadeon in the form (29) in terms of a hermitean matrix A with $A_{ij} = A(k_i, k_j)$ and $A_{ii} = (2k_i)^{-1} \exp \mathscr{S}_i$, and a $\mathbb{C}^2$ vector $|\phi\rangle$ with components $\phi_j = \exp(\frac{1}{2}\mathscr{S}_i + \mathrm{i}\delta_j)$, it follows from identification of the first order term

$$V_1 = -\langle \phi, A^2\phi \rangle \tag{49}$$

with the expression (16) that $|A_{12}|$ must be proportional to $\exp\frac{1}{2}(\mathscr{S}_1 + \mathscr{S}_2)$. Hence, $\det A = \det Z$ must be proportional to $\exp(\mathscr{S}_1 + \mathscr{S}_2)$. According to the formula (22) this means that $k_1\sigma_{12}/(2k_1 + k_2)$ must be symmetric for the interchange of the indices 1 and 2. Under this condition, we see that $\det Z$ takes a simpler form, analogous to (21):

$$\det Z = \frac{\eta}{4k_1k_2} \exp(\mathscr{S}_1 + \mathscr{S}_2), \qquad \eta = 1 - \frac{4k_1{}^2k_2}{2k_1 + k_2}\alpha_{12}. \tag{50}$$

Repeating now the reasoning of case (a) one easily concludes that if $\alpha_{12}(\alpha_{21})$ satisfies the condition:

$$k_1\alpha_{12}/(2k_1 + k_2) = k_2\alpha_{21}/(k_1 + 2k_1) \tag{51}$$

and belongs to the smaller interval:

$$0 < \alpha_{12} < (2k_1 + k_2)/k_1(k_1 + k_2)^2 \tag{52}$$

there exists a hermitean 2×2 interaction matrix A for representing the dipadeon and that its non-diagonal elements take a form analogous to (36). Going to arbitrary N, with $V_0 = \Sigma_{i=1}^N \exp \mathscr{S}_i$, the first order term V_1 takes the form

$$-V_1 = \sum_{i=1}^{N} \frac{\exp 3\mathscr{S}_i}{4k_1^2} + \sum_{i \neq j}^{N} \alpha_{ij} \exp(2\mathscr{S}_i + \mathscr{S}_j) + \sum_{i<j<l}^{N} \alpha_{ijl} \exp(\mathscr{S}_i + \mathscr{S}_j + \mathscr{S}_1). \tag{53}$$

Setting

$$\eta_{ij} = 1 - 4\frac{k_i^2k_j}{2k_i + k_j}\alpha_{ij} \quad \text{and} \quad \sigma_{ij} = \left[\frac{\eta_{ij}(k_i + k_j)^2 - (k_i - k_j)^2}{1 - \eta_{ij}}\right]^{1/2},$$

it follows from what was shown in part (a) that, in order that A be extensible to a hermitaen $N \times N$ interaction matrix with elements $(m \neq n)$:

$$A_{mn} = (k_m + k_n - i\sigma_{mn})^{-1}\phi_m\phi_n^*, \qquad \phi_m = \exp(\tfrac{1}{2}\mathscr{S}_m + \mathrm{i}\delta_m) \tag{54}$$

which can be represented by the dyadic:

$$A = \tfrac{1}{2}\int_x^\infty \mathrm{d}z\, |\phi(z, t)\rangle\langle \phi(z, t)|$$

it is necessary that σ_{ij} be separable: $\sigma_{ij} = \gamma_i - \gamma_j$ with $\gamma_i = \gamma(k_i)$, or that the α_{ij}'s be of the form:

$$\alpha_{ij} = (2k_i + k_j)/k_i\left[(k_i + k_j)^2 + (\gamma_i - \gamma_j)^2\right]. \tag{55}$$

Furthermore, in order that the expression (53) be consistent with the N-dimensional generalization of formula (49) the coefficients α_{ijl} must have the form:

$$\alpha_{ijl} = 2(c_{ijl} + c_{jli} + c_{lij}) \quad \text{with} \quad c_{ijl} = \frac{(k_i + k_j)(k_j + k_l) - \sigma_{ij}}{\left[(k_i + k_j)^2 + \sigma_{ij}^2\right]\left[(k_j + k_l)^2 + \sigma_{jl}^2\right]}. \tag{56}$$

5. Solitons and their interaction matrix

Let us consider the case of the KdV equation (p = 1):

$$q_t + q_{xxx} + 6qq_x = 0. \tag{57}$$

Taken as a reflectionless Schrödinger potential [4] the KdV-soliton is obtained through solution of the Marchenko equation:

$$K(x, y; t) + F(x+y; t) + \epsilon\int_x^\infty \mathrm{d}z K(x, z; t) F(z+y; t) = 0 \tag{58}$$

with

$$F(x+y; t) = \tfrac{1}{2}\sum_{j=1}^{N} a_j^2 \exp\left[-\tfrac{1}{2}k_j(x+y) + k_j^3 t\right] = \tfrac{1}{2}\langle\phi(y, t), \phi(x, t)\rangle,$$

and $\phi_j(x, t) = a_j \ \exp\frac{1}{2}(-k_j x + k_j^3 t)$, by taking $K(x, y; t)$ on the diagonal $x = y$ and by considering: $q(x, t) = 2\epsilon\partial_x K(x, x; t)$. Yet, the solution $K(x, y; t)$ has the form: $K(x, y; t) = \frac{1}{2}\langle\phi(y, t), h(x, t)\rangle$ with

$$|h(x, t)\ \rangle = -\left[I + \tfrac{1}{2}\epsilon\int_x^\infty \mathrm{d}z\,|\phi(z, t)\ \rangle\langle\phi(z, t)|\right]^{-1}|\phi(x, t)\rangle,$$

so that

$$K(x, x; t) = -\tfrac{1}{2}\langle\phi(x, t), (I + \epsilon A_N)^{-1}\phi(x, t)\rangle,$$
$$A_N = \tfrac{1}{2}\int_x^\infty \mathrm{d}z\,|\phi(z, t)\ \rangle\langle\ \phi(z, t)|. \tag{59}$$

The N-soliton solution of the KdV equation takes than the familiar form:

$$[N/N]q = -\epsilon\partial_x \operatorname{Tr}\left[(I + \epsilon A_N)^{-1}|\phi(x, t)\ \rangle\langle\ \phi(x, t)|\right] = 2\partial_x^2 \log\det(I + \epsilon A_N). \tag{60}$$

Hence, the existence of an interaction matrix A, as it arises in the form (42) with the padeon approach, hints at the existence of a Marchenko equation of type (58) underlying the KdV dynamics.

On the other hand, we remark that for soliton equations with a higher nonlinearity ($p = 2$), such as the MKdV equation and the Sine–Gordon equation, the [1/2] fraction which results from taking ϵ times the squared padeon has the particular form [6]:

$$[1/2]V^2 = \epsilon\{[0/1]\overline{V}\}^2 = \partial_x^2 \log(1 + \epsilon z^2), \quad z = (1/2k)V_0. \tag{61}$$

In fact it turns out [7] that also their N-padeons $[N-1/N]\overline{V}$ are such that the related

$[2N-1/2N]$ fractions: $[2N-1/2N]V^2=\epsilon\{[N-1/N]\bar{V}\}^2$ can be expressed in the remarkable form

$$[2N-1/2N]\,V^2=\partial_x^2\sum_{i=1}^{N}\log(1+\epsilon z_i^2). \tag{62}$$

For dipadeons ($N=2$), with $p=2$, which can be expressed with a 2×2 interaction matrix A, a formula of type (62) can only hold if α_{12} takes the special form

$$\alpha_{12}=(2k_1+k_1)/k_1(k_1+k_2)^2. \tag{63}$$

Indeed, identification of the first two orders in ϵ of:

$$[3/4]\,V^2=\partial_x^2\sum_{i=1}^{2}\log(1+\epsilon z_i^2)\quad\text{with}\quad V^2=\epsilon\left[V_0^2+2\epsilon V_0V_1+\epsilon^2(V_1^2+2V_0V_2)+\cdots\right]$$

leads to the relations: $V_0^2=\partial_x^2\,\mathrm{Tr}(Z^2)$

$$2V_0V_1=\tfrac{1}{2}\partial_x^2\left\{2\det(Z^2)-\left[\mathrm{Tr}(Z^2)\right]^2\right\}. \tag{64}$$

According to the expression $V_0=\exp\mathscr{S}_1+\exp\mathscr{S}_2$ and the form (16) of V_1, we get for $\det(Z^2)$ the expression:

$$\begin{aligned}\det(Z^2)=&\left[\frac{1}{(k_1+k_2)^4}+\frac{1}{32k_1^2k_2^2}-\frac{\alpha_{12}}{2(k_1+k_2)^2}\right]\exp(2\mathscr{S}_1+2\mathscr{S}_2)\\&+\left[\frac{1}{2k_1^2(k_1+k_2)^2}-\frac{(1+4k_1^2\alpha_{12})}{2k_1^2(3k_1+k_2)^2}\right]\exp(3\mathscr{S}_1+\mathscr{S}_2)+\mathrm{id}(1\pm 2).\end{aligned} \tag{65}$$

Comparison with the former expression (50) of $\det Z$ shows that the coefficient of $\exp(3\mathscr{S}_1+\mathscr{S}_2)$ should vanish. This determines α_{12} as it is given by formula (63).

This means that the interaction matrix for expressing N-soliton solutions which correspond to N-padeons with the property (62) should be real symmetric.

Finally, it is worth noticing that when α_{12} takes this particular form, the N-padeons:

$$[N-1/N]\,\bar{V}=\langle\phi,(I+\epsilon A^2)^{-1}\phi\rangle,$$

where ϕ and A are given by the relations (54), correspond precisely to the form of the reflectionless Schrödinger potentials which arise from the two-component inverse method [4]

6. Complex padeons and the cubic Schrödinger equations

Setting $\psi=\lambda\bar{\psi}$ in the nonlinear Schrödinger equation

$$\psi_{xx}+\mathrm{i}\psi_t+2|\psi|^2\psi=0 \tag{66}$$

we obtain the scaled version

$$\bar{\psi}_{xx}+\mathrm{i}\bar{\psi}_t+2\epsilon|\bar{\psi}|^2\bar{\psi}=0\quad\text{with}\quad\epsilon=|\lambda|^2. \tag{67}$$

A complex iteration series $\bar{\psi} = \sum_{n=0}^{\infty} \epsilon^n \psi_n$ can now be constructed by taking the following solution of the linearized equation:

$$\psi_0 = \exp\theta, \quad \theta = -kx + \mathrm{i}k^2 t, \qquad k \in \mathbb{C} \tag{68}$$

and by choosing for ψ_n the particular solution of the nth iteration equation:

$$\psi_{n,xx} + \psi_{n,t} = -2 \sum_{j,l,m=0,\, j+m+l=n-1}^{n-1} \psi_j \psi_l \psi_m^* \tag{69}$$

which is proportional to the exponential in the r.h. side: $\exp(\theta + 2n \operatorname{Re}\theta)$. The resulting series:

$$\bar{\psi} = \exp\theta \sum_{n=0}^{\infty} \left[-\frac{\epsilon}{4(\operatorname{Re} k)^2} \exp(2 \operatorname{Re}\theta) \right] \tag{70}$$

is geometrical and generates the regular [0/1] solitary wave solution of (67)

$$[0/1]\,\bar{\psi} = \frac{\exp\theta}{1 + \epsilon/(2\operatorname{Re} k)^2 \exp(2\operatorname{Re}\theta)}. \tag{71}$$

Since (67) has a complex [0/1] solitary wave whose absolute value has the typical padeon structure:

$$|\bar{\psi}| = -2z_x/(1 + \epsilon z^2), \qquad z = (1/2\operatorname{Re} k)\exp(\operatorname{Re}\theta) \tag{72}$$

it is reasonable to look for complex dipadeons in the form

$$[1/2]\,\bar{\psi} = \frac{\mu_1^2}{1 + \epsilon z_1^2} + \frac{\mu_2^2}{1 + \epsilon z_2^2} = \tilde{\mu}(I + \epsilon Z^2)^{-1}\mu \tag{73}$$

where μ denotes a $\mathbb{C}$-vector with components $\mu_{1,2}$, such that $|\mu_i|^2 = -2z_{i,x}$, and Z^2 stands for the real diagonal matrix

$$Z^2 = \begin{pmatrix} z_1^2 & 0 \\ 0 & z_2^2 \end{pmatrix}. \tag{74}$$

Let O be a complex orthogonal 2×2 matrix and $\phi = O\mu$; it provides an equivalent representation:

$$[1/2]\,\bar{\psi} = \phi^{\tau}(I + \epsilon S^2)^{-1}\phi \quad \text{with} \quad S^2 = OZ^2O^{\tau}. \tag{75}$$

According to the polar decomposition [2] of complex symmetric matrices. $S^2 = A\tilde{A}$, where A denotes a regular complex 2×2 matric. Now, in order to be interpretable as a (complex) two-soliton solution, $[1/2]\bar{\psi}$ should be expressible in terms of an 'interaction matrix' A, such that, when restricted to the one-component case ($N = 1$), the expression (75) reduces to the former (complex) padeon-expression (71). This requires:

$$\phi = \begin{pmatrix} \exp \frac{1}{2}\mathscr{S}_1 \\ \exp \frac{1}{2}\mathscr{S}_2 \end{pmatrix} \quad \text{and} \quad A_{ii} = (2\operatorname{Re} k_i)^{-1}\exp(\operatorname{Re}\mathscr{S}_i). \tag{76}$$

Yet, det $S^2 > 0$ and det A is real, so that $A = A^+$.

In order to determine A_{12} we must go back to the iteration series which arises from taking $\psi_0 = \exp\mathscr{S}_1 + \exp\mathscr{S}_2$. Solving the first iteration equation (69)

$$\mathrm{i}\psi_{1,t} + \psi_{1,xx} = 2\psi_0^2\psi_0^* \tag{77}$$

with

$$\psi_1 = \sum_{i,j,l=1}^{2} \alpha_{ijl} \exp(\mathscr{S}_i + \mathscr{S}_j + \mathscr{S}_l^*), \tag{78}$$

we get

$$-\psi_1 = (2 \operatorname{Re} k_1)^{-2} \exp(2\mathscr{S}_1 + \mathscr{S}_1^*) + c_{12} \exp(\mathscr{S}_1 + \mathscr{S}_1^* + \mathscr{S}_2) + \bar{c}_{12} \exp(2\mathscr{S}_1 + \mathscr{S}_2^*) + \mathrm{id}(1 \rightleftarrows 2), \tag{79}$$

where

$$c_{12} = \alpha_{121} + \alpha_{211} = (\operatorname{Re} k_1)^{-1}(k_1^* + k_2)^{-1},$$

$$\bar{c}_{12} = \alpha_{112} = (k_1 + k_2^*)^{-2}.$$

Let us now compare the expression (79) with the first-order 'sandwich' that arises from formula (75)

$$-\psi_1 = \phi^{\tau} A A^* \phi. \tag{80}$$

Setting $A_{12} = \Lambda_{12}\phi_1\phi_2^*$, with $\Lambda_{12} = \Lambda_{21}^*$, one readily obtains

$$\Lambda_{12} = (k_1 + k_2^*)^{-1}$$

and thus

$$A_{12} = (k_1 + k_2^*) \exp \tfrac{1}{2}(\mathscr{S}_1 + \mathscr{S}_2^*). \tag{81}$$

Generalizing to an $N \times N$ hermitean interaction matrix $A_{ij} = \phi_i \Lambda_{ij} \phi_j^*$ with $\phi_i = \exp \frac{1}{2}\mathscr{S}_i$ and $\Lambda_{ij} = (k_i + k_j^*)^{-1}$, we recover the N-soliton solutions of the scaled cubic Schrödinger equation [7] in the form of N-padeons:

$$[N-1/N]\,\bar{\psi} = \langle \phi, (I + \epsilon A A^*)^{-1} \phi \rangle \tag{82}$$

where ϕ denotes the $\mathbb{C}^N$-vector with components: $\phi_i = \exp \frac{1}{2}\mathscr{S}_i$.

7. Degenerate case: the Burgers equation

With a first order mixing coefficient $\alpha_{12} = (k_1 + k_2)/4k_1k_2$, the condition (23) is no more satisfied: the Burgers equation fails to produce ($N \geqslant 2$)-padeons. Yet, β_{12} still satisfies the condition (26). In fact, starting from the linearized Burgers equation with a solution $V_0 = a_1^2 \exp \theta_1 + \cdots + a_N^2 \exp \theta_N$, the iteration series remains purely geometrical: the corresponding solution of the Burgers equation is a [1/1] fraction:

$$[1/1]\, V = -2\partial_x \log \psi, \quad \psi = 1 + \epsilon \sum_{i=1}^{N} \frac{a_i^2}{2k_i} \exp \theta_i. \tag{83}$$

This particular relation between solutions ψ of the linearized equation and [1/1] solutions of the full equation indicates the existence of the Cole–Hopf transformation $V = -2\partial_x \log \psi$ which linearizes the Burgers equation [7]. Clearly, the 1-pole solutions (83) cannot describe the interaction of N solitons: the Burgers solitary wave (kink) is no soliton. However, it is easy to check that for $N = 2$ such 1-pole solution splits up, at large negative times, into two separated kinks, only one of which (the smaller one determined by k_1) corresponds to the pole. The other one is not the solitary wave determined by k_2: it travels with velocity $k_1 + k_2$. After some time the latter will catch up the former and absorb it: the remaining kink is the solitary wave determined by k_2.

The Burgers equation corresponds to a degenerate case for which the rank of the $N \times N$ interaction matrix equals 1: it possesses degenerate N-padeons, $N - 1$ poles of which coincide with $N - 1$ zeros.

8. Conclusion

The present analysis shows that if N-padeons are expressible in terms of a regular hermitean $N \times N$ interaction matrix they possess the particular asymptotic properties which characterize an N-soliton solution; i.e. the existence of such padeons guarantees the existence of solitons. Such padeon-solitons are easily detected: necessary conditions for their existence can be readily checked on the first two iterated solutions that arise from a superposition of two exponential solutions of the linearized equation (positivity condition, consistency conditions, separability condition). These conditions can be used to discuss the precise status of candidate soliton equations and of non-integrable padeon equations whose solitary waves do not enjoy the full set of stability properties which characterize the soliton.

Though the present framework represents only a first attempt to characterize solitons from a direct point of view (generalizations of the padeon should also be considered) it produces the solitons of the familiar IST-solvable equations as well as the corresponding multisoliton formulas. By considering only symmetric (hermitean) interaction matrices we did not cover the entire positivity interval to which the first mixing coefficients α_{12} must belong for the existence of dipadeons. We restricted our analysis to padeon equations for which α_{12} belongs to a sub-interval, the end-point of which coincides with the KdV (MKdV)-case. Extensions will be treated elsewhere.

By producing multisoliton interaction matrices the padeon approach can give some insight into the hidden IST-properties of the equation. In particular, it can tell whether a given soliton equation can be associated with a familiar scattering problem.

Appendix

The $N \times N$ interaction matrix A has nondiagonal elements: $A_{mn} = \Phi_m \Lambda_{mn} \Phi_n^*$ with $\Lambda_{mn} = (k_m + k_n - \mathrm{i}\sigma_{mn})^{-1}$, and $\Phi_m = \exp(\frac{1}{2}\varphi_m + \mathrm{i}\delta_m)$. From the form of its diagonal elements $A_{mm} = (1/2k_m)k_m \exp \varphi_m$ it follows that the first generalized trace condition (31), with $n = 0$, is automatically satisfied. The second trace condition ($n = 1$) implies

$$2(k_m + k_n) = (\Lambda_{mn})^{-1} + (\Lambda_{nm})^{-1} \quad \text{and thus} \quad \sigma_{nm} = -\sigma_{mn}. \tag{A1}$$

The third trace condition ($n = 2$) can be written:

$$\sum_{l,m,n} \Lambda_{lm}\Lambda_{mn}\left[(k_l - k_n - 2\mathrm{i}\delta_{n,x} + 2\mathrm{i}\delta_{l,x})\Lambda_{nl} - 1\right] \exp(\varphi_l + \varphi_m + \varphi_n) = 0. \tag{A2}$$

According to the property (A1) the contributions with two equal indices vanish identically. Thus, the equation (A2) reduces to:

$$\sum_{l<m<n} \exp(\varphi_l + \varphi_m + \varphi_n)(\sigma_{lm} + \sigma_{mn} + \sigma_{nl}) \cdot \mathrm{Im}(\Lambda_{lm}\Lambda_{mn}\Lambda_{nl}) = 0. \tag{A3}$$

This implies that either of the two following conditions must be satisfied:

(i) $\sigma_{lm} + \sigma_{mn} + \sigma_{nl} = 0$, which, together with the property (A1), requires the separability of $\sigma_{ij} = \sigma(k_i, k_j)$:

$$\sigma_{ij} = \gamma_i - \gamma_j \quad \text{with} \quad \gamma_i = \gamma(k_i). \tag{A4}$$

(ii) $\mathrm{Im}(\Lambda_{lm}\Lambda_{mn}\Lambda_{nl}) = 0$, which means that the related $\bar{\sigma}_{ij} = (k_i + k_j)^{-1}\sigma_{ij}$ must satisfy the condition:

$$\bar{\sigma}_{ij} + \bar{\sigma}_{jl} + \bar{\sigma}_{li} = \bar{\sigma}_{ij}\bar{\sigma}_{jl}\bar{\sigma}_{li}, \tag{A5}$$

or that

$$\bar{\sigma}_{ij} = \mathrm{tg}(\nu_i - \nu_j) \quad \text{with} \quad \nu_i = \nu(k_i), \tag{A6}$$

requiring thus the separability of the parameters $\rho_{ij} = \mathrm{tg}^{-1}(\sigma_{ij}/k_i + k_j)$.

In both cases one verifies that the further trace conditions are automatically satisfied for all n. In particular, it is easy to see that the fourth trace condition ($n = 3$) amounts to the equation:

$$\begin{aligned}\sum_{i<j<l<m} & \exp(\varphi_i + \varphi_j + \varphi_l + \varphi_m)\{\mathrm{Im}(\Lambda_{ij}\Lambda_{jl}\Lambda_{lm}\Lambda_{mi})(\sigma_{ij} + \sigma_{jl} + \sigma_{lm} + \sigma_{mi}) \\ & + \mathrm{Im}(\Lambda_{il}\Lambda_{lm}\Lambda_{mj}\Lambda_{ji})(\sigma_{il} + \sigma_{lm} + \sigma_{mj} + \sigma_{ji}) \\ & + \mathrm{Im}(\Lambda_{il}\Lambda_{lj}\Lambda_{jm}\Lambda_{mi})(\sigma_{il} + \sigma_{lj} + \sigma_{jm} + \sigma_{mi})\} \\ & + 4\sum_{i<l<m} \exp(2\varphi_i + \varphi_l + \varphi_m) \cdot \frac{1}{2k_i}\mathrm{Im}(\Lambda_{il}\Lambda_{lm}\Lambda_{mi})(\sigma_{il} + \sigma_{lm} + \sigma_{mi}) = 0.\end{aligned} \tag{A7}$$

In the first case, this condition is clearly satisfied as a result of condition (A4). In the second case, $\mathrm{Im}(\Lambda_{ij}\Lambda_{jl}\Lambda_{lm}\Lambda_{mi})$ vanishes as a result of the relation

$$\bar{\sigma}_{ij} + \bar{\sigma}_{jl} + \bar{\sigma}_{lm} + \bar{\sigma}_{mi} = \bar{\sigma}_{ij}\bar{\sigma}_{jl}\bar{\sigma}_{lm} + \bar{\sigma}_{jl}\bar{\sigma}_{lm}\bar{\sigma}_{mi} + \bar{\sigma}_{lm}\bar{\sigma}_{mi}\bar{\sigma}_{ij} + \bar{\sigma}_{mi}\bar{\sigma}_{ij}\bar{\sigma}_{jl}$$

which is itself a consequence of (A6) and the property that as $A + B + C + D = 0$,

$$\begin{aligned}\mathrm{tg}\,A + \mathrm{tg}\,B + \mathrm{tg}\,C + \mathrm{tg}\,D = {} & \mathrm{tg}\,A \cdot \mathrm{tg}\,B \cdot \mathrm{tg}\,C + \mathrm{tg}\,B \cdot \mathrm{tg}\,C \cdot \mathrm{tg}\,D + \mathrm{tg}\,C \cdot \mathrm{tg}\,D \cdot \mathrm{tg}\,A \\ & + \mathrm{tg}\,D \cdot \mathrm{tg}\,A \cdot \mathrm{tg}\,B.\end{aligned}$$

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