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Nonexistence of positive solutions to a quasilinear elliptic system and blow-up estimates for a quasilinear reaction–diffusion system ☆

Zuodong Yang, Qishao Lu *

Department of Applied Mathematics, Beijing University of Aeronautics and Astronautics, Beijing 100083, China

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Abstract

The prior estimate and decay property of positive solutions are derived for a system of quasilinear elliptic differential equations first. Then, the nonexistence result for radially nonincreasing positive solutions of the system is implied. By using this nonexistence result, blow-up estimates for a class of quasilinear reaction–diffusion systems (non-Newtonian filtration systems) are established to extend the result for semilinear reaction–diffusion systems (Fujita type).

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1. Introduction

The structure of positive solutions for quasilinear reaction–diffusion systems (nonlinear Newtonian filtration systems) and semilinear reaction–diffusion systems (Newtonian filtration systems) is a front topic in the study of static electric fields in dielectric media, in which the potential is described by the boundary value problem of a static non-Newtonian filtration system, called the Poisson–Boltzmann problem. This kind of problems also appears in the study of the non-Newtonian or Newtonian turbulent filtration in porous media and so on, which have extensive engineering background.

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* Corresponding author.
E-mail address: ligang.lu@intel.com (Q. Lu).

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In recent years, the reaction–diffusion systems of Fujita type

$$u_t = \Delta u + u^{m_1} v^{n_1},$$
$$v_t = \Delta v + u^{m_2} v^{n_2}, \quad (x,t) \in \Omega \times (0,T) \tag{A}$$

as well as the related elliptic system

$$-\Delta u = u^{m_1} v^{n_1},$$
$$-\Delta v = u^{m_2} v^{n_2}, \quad x \in \Omega \tag{B}$$

with $\Omega \subseteq \mathbb{R}^{\mathbf{N}}, m_i, n_i \geqslant 0,\ i = 1,2$ were studied by a number of authors [2–6,8,9,11–15,17,18]. The problems concerning system (A) include global existence and global existence numbers, blow-up, blow-up rates, and blow-up sets, uniqueness or nonuniqueness, etc. For system (B) there are problems such as existence or nonexistence, uniqueness or nonuniqueness, and so on. Meanwhile, it seems that very little is known about blow-up rate estimates for quasilinear reaction–diffusion systems.

The aim of this paper is to derive some estimates from the above near the blow-up point for radially symmetric positive solutions of a class of quasilinear reaction–diffusion systems:

$$u_t = \operatorname{div}(|\nabla u|^{p-2}\nabla u) + u^{\alpha_1} v^{\beta_1} w^{\gamma_1},$$
$$v_t = \operatorname{div}(|\nabla v|^{q-2}\nabla v) + u^{\alpha_2} v^{\beta_2} w^{\gamma_2},$$
$$w_t = \operatorname{div}(|\nabla v|^{m-2}\nabla v) + u^{\alpha_3} v^{\beta_3} w^{\gamma_3}, \quad (x,t) \in \Omega \times (0,T) \tag{1.1}$$

as well as the nonexistence of positive solutions of the related elliptic systems:

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = u^{\alpha_1} v^{\beta_1} w^{\gamma_1},$$
$$-\operatorname{div}(|\nabla u|^{q-2}\nabla u) = u^{\alpha_2} v^{\beta_2} w^{\gamma_2},$$
$$-\operatorname{div}(|\nabla u|^{m-2}\nabla u) = u^{\alpha_3} v^{\beta_3} w^{\gamma_3} \quad x \in \Omega, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^{\mathbf{N}}, \alpha_1 > p-1, \beta_2 > q-1, \gamma_3 > m-1$ with $p,q,m > 1, \alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2 \geqslant 0$. For $p = q = m = 2$, (1.1) is the classical reaction–diffusion system of Fujita type. If $p \neq 2, q \neq 2, m \neq 2$, (1.1) appears in the theory of non-Newtonian fluids [1,10] and in nonlinear filtration theory [7]. In the non-Newtonian fluids theory, the pair (p,q,m) is a characteristic quantity of the medium. Media with $(p,q,m) > (2,2,2)$ are called dilatant fluids and those with $(p,q,m) < (2,2,2)$ are called pseudoplastics. If $(p,q,m) = (2,2,2)$, they are Newtonian fluids.

The main result of the present paper is the natural extension of the results given by Weissler et al. [18,2,17], which concern the single equation

$$u_t(x,t) = \Delta u + u^m(x,t), \quad (x,t) \in \mathbf{B}(0,R) \times (0,T)$$

and the system of equations

$$u_t(x,t) = \Delta u(x,t) + v^m(x,t),$$
$$v_t(x,t) = \Delta v(x,t) + u^n(x,t)$$

or

$$u_t(x,t) = \Delta u(x,t) + u^{p_1} v^{q_1},$$
$$v_t(x,t) = \Delta v(x,t) + u^{p_2} v^{q_2}.$$

Throughout this paper let $\Omega = B_R = \{x \in \mathbb{R}^{\mathbf{N}}\colon |x| < R\}(R > 0)$ since we only deal with radially symmetric positive solutions of systems (1.1) and (1.2) here. In Section 2, we give sufficient conditions under which the nonexistence of positive solutions of the elliptic system (1.2) holds in $\mathbb{R}^{\mathbf{N}}$ for $N \geqslant 3$. Then, in Section 3, by using the nonexistence result, we get the desired blow-up estimates for the reaction–diffusion system (1.1) with some additional assumptions.

2. Nonexistence for system (1.2)

Consider radially symmetric solutions of the elliptic system (1.2), i.e., suppose that $u = u(r)$, $v = v(r), w = w(r)$ with $r = |x|$. We have the following theorems.

Theorem 2.1. *Assume that* $\alpha_1 > p - 1$ (*or* $\beta_2 > q - 1$ *or* $\gamma_3 > m - 1$) *with* $p, q, m > 1, \alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2 \geqslant 0$.

If one of the following conditions is satisfied:

(g1) $N > p$ *and* $p = q = m = 2$,

$$N/2 < \max\left\{\frac{\alpha_3+\beta_3+\gamma_3}{\alpha_3+\beta_3+\gamma_3-1}, \frac{\alpha_2+\beta_2+\gamma_2}{\alpha_2+\beta_2+\gamma_2-1}, \frac{\alpha_1+\beta_1+\gamma_1}{\alpha_1+\beta_1+\gamma_1-1}\right\}$$

(g2) $N > p$ *and* $p = q = m$,

$$N/p < \max\left\{\frac{\alpha_3+\beta_3+\gamma_3}{\alpha_3-p+1+\beta_3+\gamma_3}, \frac{\alpha_2+\beta_2+\gamma_2}{\alpha_2-p+1+\beta_2+\gamma_2}, \frac{\alpha_1+\beta_1+\gamma_1}{\alpha_1-p+1+\beta_1+\gamma_1}\right\},$$

(g3) $N > \max\{p, q, m\}$,

$$N < \max\left\{\frac{pd_1+p(\alpha_1-p+1)d_4+\beta_1 q d_3+m\gamma_1 d_2}{(\alpha_1-p+1)d_4+\beta_1 d_3+\gamma_1 d_2}, \frac{qd_1+q(\beta_2-q+1)d_3+\alpha_2 p d_4+m\gamma_2 d_2}{(\beta_2-q+1)d_3+\alpha_2 d_4+\gamma_2 d_2},\right.$$
$$\left.\frac{md_1+m(\gamma_3-m+1)d_2+\alpha_3 p d_4+q\beta_3 d_3}{(\gamma_3-m+1)d_2+\alpha_3 d_4+\beta_3 d_3}\right\},$$

where $d_1 = (p-1)(q-1)(m-1), d_2 = (p-1)(q-1), d_3 = (p-1)(m-1), d_4 = (q-1)(m-1)$, *then system* (1.2) *has no positive radially symmetric solution.*

Theorem 2.2. *Suppose that* $\alpha_1 > p-1$ (*or* $\beta_2 > q-1$ *or* $\gamma_3 > m-1$) *with* $p,q,m > 1, \alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2 \geqslant 0$.

If one of the following conditions is satisfied:

(g1) $N > p$ *and* $p = q = m = 2$,

$$N/2 < \max\left\{\frac{2\alpha_3 + 2\beta_3 + \gamma_3 + 1}{2\alpha_3 + 2\beta_3 + \gamma_3 - 1}, \frac{2\alpha_2 + \beta_2 + 2\gamma_2 + 1}{2\alpha_2 + \beta_2 + 2\gamma_2 - 1}, \frac{\alpha_1 + 2\beta_1 + 2\gamma_1 + 1}{\alpha_1 + 2\beta_1 + 2\gamma_1 - 1}\right\},$$

(g2) $N > p$ *and* $p = q = m$,

$$N/p < \max\left\{\frac{p\alpha_3 + p\beta_3 + (p-1)(\gamma_3 + 1)}{p\alpha_3 + p\beta_3 + (p-1)(\gamma_3 - p + 1)}, \frac{p\alpha_2 + (p-1)(\beta_2 + 1) + p\gamma_2}{p\alpha_2 + (p-1)(\beta_2 - p + 1) + p\gamma_2}, \frac{(p-1)(\alpha_1 + 1) + p\beta_1 + p\gamma_1}{(p-1)(\alpha_1 - p + 1) + p\beta_1 + p\gamma_1}\right\},$$

(g3) $N > \max\{p,q,m\}$,

$$N < \max\left\{\frac{p^2(q-1)(m-1) + p(\alpha_1 - p + 1)(q-1)(m-1) + \beta_1 qp(m-1) + m\gamma_1 p(q-1)}{(\alpha_1 - p + 1)(q-1)(m-1) + \beta_1 p(m-1) + \gamma_1 p(q-1)}, \frac{q^2(p-1)(m-1) + q(\beta_2 - q + 1)(p-1)(m-1) + \alpha_2 pq(m-1) + m\gamma_2(p-1)q}{(\beta_2 - q + 1)(p-1)(m-1) + \alpha_2 q(m-1) + \gamma_2(p-1)q}, \frac{m^2(p-1)(q-1) + m(\gamma_3 - m + 1)(q-1)(p-1) + \alpha_3 p(q-1)m + q\beta_3(p-1)m}{(\gamma_3 - m + 1)(q-1)(p-1) + \alpha_3(q-1)m + \beta_3(p-1)m}\right\},$$

then system (1.2) *has no positive radially symmetric solution.*

To prove Theorems 2.1 and 2.2, system (1.2) can be written in radial coordinates as

$$(\Phi_p(u'))' + \frac{N-1}{r}\Phi_p(u') + u^{\alpha_1}v^{\beta_1}w^{\gamma_1} = 0, \tag{2.1}$$

$$(\Phi_q(v'))' + \frac{N-1}{r}\Phi_q(v') + u^{\alpha_2}v^{\beta_2}w^{\gamma_2} = 0, \tag{2.2}$$

$$(\Phi_m(w'))' + \frac{N-1}{r}\Phi_m(w') + u^{\alpha_3}v^{\beta_3}w^{\gamma_3} = 0, \tag{2.3}$$

$$u(0) > 0, v(0) > 0, w(0) > 0 u'(0) = v'(0) = w'(0) = 0 \tag{2.4}$$

in $\mathbb{R}^{\mathbf{N}}$ with $\mathbf{N} \geqslant \{p,q,m\}$, where $\Phi_p(u) = |u|^{p-2}u, \Phi_q(v) = |v|^{q-2}v, \Phi_m(w) = |w|^{m-2}w$.

Lemma 2.1. *Let* (u,v,w) *be a positive solution of Eqs.* (2.1)–(2.4). *Then for* $r>0$ *we have*

$$u^{\alpha_1-p+1}(r)v^{\beta_1}(r)w^{\gamma_1}(r) \leqslant \left(\frac{p}{p-1}\right)^{p-1} N\left(\frac{p-1}{\alpha_1-p+1}\right)^{p-1} r^{-p}, \tag{2.5}$$

$$u^{\alpha_2}(r)v^{\beta_2-q+1}(r)w^{\gamma_2}(r) \leqslant \left(\frac{q}{q-1}\right)^{q-1} N\left(\frac{q-1}{\beta_2-q+1}\right)^{q-1} r^{-q} \tag{2.6}$$

and

$$u^{\alpha_3}(r)v^{\beta_3}(r)w^{\gamma_3-m+1}(r) \leqslant \left(\frac{m}{m-1}\right)^{m-1} N\left(\frac{m-1}{\gamma_3-m+1}\right)^{m-1} r^{-m}. \tag{2.7}$$

Proof. Systems (2.1)–(2.4) can be written as

$$-(\Phi_p(u')r^{N-1})' = u^{\alpha_1}v^{\beta_1}w^{\gamma_1}r^{N-1}, \tag{2.8}$$

$$-(\Phi_q(v')r^{N-1})' = u^{\alpha_2}v^{\beta_2}w^{\gamma_2}r^{N-1}, \tag{2.9}$$

$$-(\Phi_m(w')r^{N-1})' = u^{\alpha_3}v^{\beta_3}w^{\gamma_3}r^{N-1} \tag{2.10}$$

with

$$u'(0)=v'(0)=w'(0)=0. \tag{2.11}$$

Integrating Eq. (2.8) on $(0,r)$, it follows that

$$-\Phi_p(u')r^{N-1} = \int_0^r u^{\alpha_1}(s)v^{\beta_1}(s)w^{\gamma_1}s^{N-1}\,\mathrm{d}s, \quad -\Phi_q(v')r^{N-1} = \int_0^r u^{\alpha_2}(s)v^{\beta_2}(s)w^{\gamma_2}s^{N-1}\,\mathrm{d}s$$

$$-\Phi_q(w')r^{N-1} = \int_0^r u^{\alpha_3}(s)v^{\beta_3}(s)w^{\gamma_3}s^{N-1}\,\mathrm{d}s.$$

From the above equalities, $u(r)$, $v(r)$ and $w(r)$ are decreasing functions in $(0,\infty)$, which implies

$$u'(r) \leqslant -\left(\frac{1}{r^{N-1}}\right)^{1/(p-1)} u^{\alpha_1/(p-1)}(r)v^{\beta_1/(p-1)}(r)w^{\gamma_1/(p-1)}(r)\left(\int_0^r s^{N-1}\,\mathrm{d}s\right)^{1/(p-1)}$$

$$= -\left(\frac{1}{N}\right)^{1/(p-1)} r^{1/(p-1)}u^{\alpha_1/(p-1)}v^{\beta_1/(p-1)}w^{\gamma_1/(p-1)},$$

$$v'(r) \leqslant -\left(\frac{1}{N}\right)^{1/(q-1)} r^{1/(q-1)}u^{\alpha_2/(q-1)}v^{\beta_2/(q-1)}w^{\gamma_2/(q-1)},$$

$$w'(r) \leqslant -\left(\frac{1}{N}\right)^{1/(m-1)} r^{1/(m-1)}u^{\alpha_3/(m-1)}v^{\beta_3/(m-1)}w^{\gamma_3/(m-1)}.$$

Using these three inequalities leads to

$$\begin{aligned}
&\frac{d}{dr}[u^{(\alpha_1-p+1)/(p-1)}(r)v^{\beta_1/(p-1)}w^{\gamma_1/(p-1)}]\\
&\quad=\frac{\alpha_1-p+1}{p-1}\,u^{(\alpha_1-2(p-1))/(p-1)}(r)v^{\beta_1/(p-1)}(r)w^{\gamma_1/(p-1)}u'(r)\\
&\qquad+\frac{\beta_1}{p-1}\,u^{(\alpha_1-p+1)/(p-1)}(r)v^{(\beta_1-p+1)/(p-1)}(r)w^{(\gamma_1/(p-1)}v'(r)\\
&\qquad+\frac{\gamma_1}{p-1}\,u^{(\alpha_1-p+1)/(p-1)}(r)v^{(\beta_1)/(p-1)}w^{(\gamma_1-p+1)/(p-1)}w'(r)\\
&\quad\leqslant-\frac{\alpha_1-p+1}{p-1}\,u^{(\alpha_1-2(p-1))/(p-1)}v^{\beta_1/(p-1)}w^{\gamma_1/(p-1)}(1/N)^{1/(p-1)}\\
&\qquad\times r^{1/(p-1)}u^{\alpha_1/(p-1)}v^{\alpha_1/(p-1)}w^{\gamma_1/(p-1)}\\
&\quad=-(1/N)^{1/(p-1)}\left(\frac{\alpha_1-p+1}{p-1}\right)r^{1/(p-1)}(u^{(\alpha_1-p+1)/(p-1)}v^{\beta_1/(p-1)}w^{\gamma_1/(p-1)})^2.
\end{aligned}$$

Solving this inequality, we have

$$\begin{aligned}
&\frac{1}{u^{(\alpha_1-p+1)/(p-1)}(0)v^{\beta_1/(p-1)}(0)w^{\gamma_1/(p-1)}(0)}-\frac{1}{u^{(\alpha_1-p+1)/(p-1)}(r)v^{\beta_1/(p-1)}(r)w^{\gamma_1/(p-1)}}\\
&\quad\leqslant-\left(\frac{1}{N}\right)^{1/(p-1)}\left(\frac{m_1-p+1}{p-1}\right)\frac{p-1}{p}\,r^{p/(p-1)}
\end{aligned}$$

or

$$\begin{aligned}
&\frac{1}{u^{(\alpha_1-p+1)/(p-1)}(r)v^{\beta_1/(p-1)}(r)w^{\gamma_1/(p-1)}(r)}\\
&\quad\geqslant\frac{1}{u^{(\alpha_1-p+1)/(p-1)}(0)v^{\beta_1/(p-1)}(0)w^{\gamma_1/(p-1)}(0)}\\
&\qquad+\left(\frac{1}{N}\right)^{1/(p-1)}\left(\frac{(\alpha_1-p+1)(p-1)}{p(p-1)}\right)r^{p/(p-1)},
\end{aligned}$$

then

$$u^{(\alpha_1-p+1)/(p-1)}(r)v^{\beta_1/(p-1)}(r)w^{\gamma_1/(p-1)}(r)\leqslant\left(\frac{p}{p-1}\right)N^{1/(p-1)}\frac{p-1}{\alpha_1-p+1}\,r^{-p/(p-1)}.$$

Similarly, we can prove inequalities of Eqs. (2.6) and (2.7). This completes the proof of Lemma 2.1. □

Lemma 2.2. *If $u(r)$, $v(r)$ and $w(r)$ are positive solutions of Eqs.* (2.1)–(2.4) *in* $(0,\infty)$, *then*

$$(p-1)ru'(r)+(N-p)u(r)\geqslant 0,\quad (q-1)rv'(r)+(N-q)v(r)\geqslant 0\qquad \text{for } r>0$$

and

$$(m-1)rw'(r)+(N-m)w(r)\geqslant 0\quad \text{for } r>0$$

Proof. Systems (2.1)–(2.4) can be rewritten as

$$-(p-1)|u'|^{p-2}u''+(1-N)/r\Phi_p(u')=u^{\alpha_1}v^{\beta_1}w^{\gamma_1}\quad \text{for } r>0,$$
$$-(q-1)|v'|^{q-2}v''+(1-N)/r\Phi_q(v')=u^{\alpha_2}v^{\beta_2}w^{\gamma_2}\quad \text{for } r>0,$$
$$-(m-1)|w'|^{m-2}w''+(1-N)/r\Phi_m(w')=u^{\alpha_3}v^{\beta_3}w^{\gamma_3}\quad \text{for } r>0,$$

hence

$$-ru''+(1-N)u'/(p-1)=\frac{ru^{\alpha_1}v^{\beta_1}w^{\gamma_1}}{(p-1)|u'|^{p-2}},\tag{2.12}$$

$$-rv''+(1-N)v'/(q-1)=\frac{ru^{\alpha_2}v^{\beta_2}w^{\gamma_2}}{(q-1)|v'|^{q-2}},\tag{2.13}$$

$$-rw''+(1-N)w'/(m-1)=\frac{ru^{\alpha_3}v^{\beta_3}w^{\gamma_3}}{(m-1)|w'|^{m-2}}.\tag{2.14}$$

We put

$$M_A(r)=ru'+\frac{N-p}{p-1}u,\quad M_B(r)=rv'+\frac{N-q}{q-1}v,\quad M_C(r)=rw'+\frac{N-m}{m-1}w.$$

From (2.12)–(2.14), we have

$$\frac{\mathrm{d}}{\mathrm{d}r}M_A(r)\leqslant 0,\quad \frac{\mathrm{d}}{\mathrm{d}r}M_B(r)\leqslant 0,\quad \frac{\mathrm{d}}{\mathrm{d}r}M_C(r)\leqslant 0,\tag{2.15}$$

on $(0,+\infty)$. It follows that $M_A(r)$ is nonincreasing on $(0,+\infty)$. Now it is shown that $M_A(r)$ is nonnegative for $r>0$. Otherwise, suppose that $M_A(r_1)<0$ for some $r_1>0$. Then we would have

$$u'(r)+(N-p)/(p-1)r^{-1}u(r)\leqslant r^{-1}M_A(r_1)\quad \text{for } r>r_1.$$

Since u is nonnegative, we obtain

$$u'(r)\leqslant r^{-1}M_A(r_1)\quad \text{for } r>r_1.\tag{2.16}$$

Integrating Eq. (2.16) from r_1 to r, we obtain

$$u(r)-u(r_1)\leqslant M_A(r_1)\ln\left(\frac{r}{r_1}\right)\qquad \text{for } r>r_1,$$

hence $\lim_{r\to+\infty} u(r) = -\infty$, which is a contradiction. Thus the functions $M_A(r)$ is nonnegative. Similarly, we can prove that the functions $M_B(r)$ and $M_C(r)$ are also nonnegative.

Then, we have

$$M_A(r) \geqslant 0, \quad M_B(r) \geqslant 0, \quad M_C(r) \geqslant 0. \tag{2.17}$$

Since $u' < 0$, $v' < 0$ and $w' < 0$, we deduce from (2.17) that

$$-ru' \leqslant \frac{N-p}{p-1} u(r), \quad -rv' \leqslant \frac{N-q}{q-1} v(r), \quad -rw' \leqslant \frac{N-m}{m-1} w(r) \quad \text{for } r > 0. \qquad \square$$

From Pokhozhaev's identity (see [14]), we have

Lemma 2.3. *Let $u(r)$ be a solution of (2.1) in $(r_1, r_2) \subset (0, \infty)$ and a be an arbitrary constant. Then, for each $r \in (r_1, r_2)$ we have*

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}r}\left[r^N\left\{(1-1/p)|u'|^p + F(r,u) + \frac{a}{r} uu'|u'|^{p-2}\right\}\right]\\ &\quad = r^N[NF(r,u) - auf(r,u) + (a+1-N/p)|u'|^p],\end{aligned}$$

where $F(r,u) = \int_0^u f(s,v)\,\mathrm{d}v = \int_0^u (z^{\alpha_1} v^{\beta_1} w^{\gamma_1})\,\mathrm{d}z$.

Proof of Theorem 2.1. Let (u, v, w) be a nontrivial positive and radial solution of Eqs. (2.1)–(2.3). By Lemma 2.2,

$$(r^{N-p} u^{p-1}(r))' = r^{N-p-1} u^{p-2}(r)[(p-1)ru'(r) + (N-p)u(r)] \geqslant 0, \tag{2.18}$$

we have

$$\begin{aligned}&u(r) \geqslant cr^{-(N-p)/(p-1)},\\ &v(r) \geqslant cr^{-(N-q)/(q-1)} \quad \text{and} \quad w(r) \geqslant cr^{-(N-m)/(m-1)} \qquad \text{for } r \geqslant 1.\end{aligned} \tag{2.19}$$

By Lemma 2.1 and Eq. (2.19),

$$cr^{-p} \geqslant u^{\alpha_1 - p+1} v^{\beta_1} w^{\gamma_1} \geqslant c^{\alpha_1+\beta_1+\gamma_1-p+1} r^{-(N-p)(\alpha_1-p+1)/(p-1) - \beta_1(N-q)/(q-1) - \gamma_1(N-m)/(m-1)}$$

and

$$\begin{aligned}&cr^{-q} \geqslant c^{\alpha_2+\beta_2+\gamma_2-q+1} r^{-\alpha_2(N-p)/(p-1)-(N-q)(\beta_2-q+1)/(q-1)-\gamma_2(N-m)/(m-1)},\\ &cr^{-m} \geqslant c^{\alpha_3+\beta_3+\gamma_3-m+1} r^{-\alpha_3(N-p)/(p-1)-(N-q)\beta_3/(q-1)-(\gamma_3-m+1)(N-m)/(m-1)}.\end{aligned}$$

From conditions (g1) or (g2) or (g3) of Theorem 2.1, the above inequalities lead to a contradiction for large r. Hence Eqs. (2.1)–(2.4) have no positive solution in $(0, \infty)$, and thus the theorem follows. $\square$

Proof of Theorem 2.2. Suppose that our conclusion is not true, and $u(r), v(r), w(r)$ are the positive solutions of Eqs. (2.1)–(2.4) in $(0,\infty)$. Eqs. (2.5) and (2.19), lead to

$$u^{\alpha_1-p+1}c^{\beta_1+\gamma_1}r^{-\beta_1(N-q)/(q-1)-\gamma_1(N-m)/(m-1)} \leqslant u^{\alpha_1-p+1}(r)v^{\beta_1}(r)w^{\gamma_1} \leqslant \left(\frac{p}{p-1}\right)^{p-1} N\left(\frac{p-1}{\alpha_1-p+1}\right)^{p-1} r^{-p},$$

which implies that

$$u(r) \leqslant cr^{-p(q-1)(m-1)+\beta_1(m-1)(N-q)+\gamma_1(N-m)/(q-1)(m-1)(\alpha_1-p+1)}. \tag{2.20}$$

Using Lemma 2.3 for Eq. (2.1) with $a=(N-p)/p$, we have

$$r^N\left[(1-1/p)|u'|^p + \frac{1}{\alpha_1+1}u^{\alpha_1+1}(r)v^{\beta_1}(r)w^{\gamma_1}(r)\right] + (N-p)/pr^{N-1}uu'|u'|^{p-2} = \int_0^r s^{N-1}[(N/(\alpha_1+1)-(N-p)/p)u^{\alpha_1+1}(s)v^{\beta_1}(s)w^{\gamma_1}(s)]\,\mathrm{d}s$$

or

$$\frac{-1}{p}r^{N-1}|u'(r)|^{p-1}((p-1)ru'(r)+(N-p)u(r)) + \frac{1}{\alpha_1+1}r^N u^{\alpha_1+1}(r)v^{\beta_1}w^{\gamma_1}(r) = \frac{Np-(N-p)(\alpha_1+1)}{p(\alpha_1+1)}\int_0^r s^{N-1}u^{\alpha_1+1}(s)v^{\beta_1}(s)w^{\gamma_1}(s)\,\mathrm{d}s. \tag{2.21}$$

It follows from Eqs. (2.19) and (2.20) that

$$r^N u^{\alpha_1+1}(r)v^{\beta_1}(r)w^{\gamma_1}(r) \leqslant cr^N[u^{\alpha_1-p+1}(r)v^{\beta_1}(r)w^{\gamma_1}(r)]u^p(r) \leqslant cr^{N-p+p[\beta_1(m-1)(N-q)+\gamma_1(N-m)(q-1)-p(q-1)(m-1)]/(q-1)(m-1)(\alpha_1-p+1)} \to 0 \quad \text{as } r\to\infty.$$

Thus, by Lemma 2.2, the left-hand side of Eq. (2.21) is less than any $\varepsilon>0$ for large r, but the right-hand side is greater than a positive number. This is a contradiction. Thus our conclusion follows. □

Remark 2.1. From the proof of Theorems 2.1 and 2.2, we know that $\alpha_i, \beta_i, \gamma_i$ $(i=1,2,3)$ must satisfy

$$\alpha_1 > p-1 \text{ or } \beta_2 > q-1 \text{ or } \gamma_3 > m-1 \quad \text{with } \alpha_2,\alpha_3,\beta_1,\beta_3,\gamma_1,\gamma_2 \geqslant 0.$$

Then, from Theorems 2.1 and 2.2, for $p=q=m=2$ and $\alpha_2=\alpha_3, \beta_2=\beta_3, \gamma_1=\gamma_2=\gamma_3=0$, we have

$$N/2 < \max\left\{\frac{\alpha_2+\beta_2}{\alpha_2+\beta_2-1}, \frac{\alpha_1+\beta_1}{\alpha_1+\beta_1-1}\right\} \quad \text{or} \quad N/2 < \max\left\{\frac{2\alpha_2+\beta_2+1}{2\alpha_2+\beta_2-1}, \frac{\alpha_1+2\beta_1+1}{\alpha_1+2\beta_1-1}\right\},$$

or equivalently,

$$N > (N-2)\min(\alpha_1+\beta_1, \alpha_2+\beta_2) \quad \text{or} \quad N+2 > (N-2)\min(\alpha_1+2\beta_1, \beta_2+2\alpha_2),$$

which is just the nonexistence condition obtained by Shaohua Chen and Guozhen Lu [16] for this special case.

From Theorem 2.2, for $p=q=m=2, \beta_1=\gamma_1=0, \alpha_2=\gamma_2=0, \alpha_3=\beta_3=0, \alpha_1=\beta_2=\gamma_3=k>1$, we have

$$N/2 < \frac{k+1}{k-1},$$

which is equivalent to the well-known critical condition

$$k < \frac{N+2}{N-2}.$$

Remark 2.2. Consider the parameters $p=q=m=3, N=4, \alpha_1=\beta_2=\gamma_3=7, \alpha_2=\alpha_3=\beta_1=\beta_3=\gamma_1=\gamma_2=0$, which satisfy inequality (g2) of Theorem 2.1; hence Theorem 2.1 asserts the nonexistence of positive radial solution of system (1.2) in $\mathbb{R}^{\mathbf{N}}$. On the other hand, consider the parameters $p=q=m=3, N=4, \alpha_1=\beta_2=\gamma_3=11$ which do not satisfy condition inequality (g2) of Theorem 2.1. In this case, system (1.2) can be written as

$$\operatorname{div}(|\nabla u|\nabla u)+u^{11}=0, \quad \operatorname{div}(|\nabla v|\nabla v)+v^{11}=0, \quad \operatorname{div}(|\nabla w|\nabla w)+w^{11}=0,$$

which has the following radially symmetric positive solution in $\mathbb{R}^{\mathbf{N}}$:

$$u(r)=v(r)=w(r)=(1+r^{3/2})^{-1/3}.$$

3. Blow-up estimates for system (1.1)

Motivated by Weissler [18], Caristi and Mitidieri [2] and Sining Zheng [17], we use the nonexistence result of the elliptic system (1.2) obtained in Section 2 to establish the blow-up estimates for the quasilinear reaction–diffusion system (1.1). We impose the following initial and boundary value conditions to Eq. (1.1):

$$u(x,0)=u_0(x), \quad v(x,0)=v_0(x), \quad w(x,0)=w_0(x), \quad x\in\Omega=B_R\subset\mathbb{R}^{\mathbf{N}}, \tag{3.1}$$

$$u=v=w=0, \quad (x,t)\in\partial\Omega\times(0,T). \tag{3.2}$$

Theorem 3.1. *Let (u,v,w) be a solution of Eqs.* (1.1), (3.1) *and* (3.2). *Assume that*

(i) $u(\cdot,t)$, $v(\cdot,t)$ *and* $w(\cdot,t)$ *are nonnegative, radially symmetrical and nonincreasing as functions of* $r=|x|$,
(ii) $u_t(\cdot,t)$, $v_t(\cdot,t)$ *and* $w_t(\cdot,t)$ *achieve the maximum at* 0 *for any* $t\in(0,T)$,
(iii) $u,v,w\geqslant 0$ $u_t,v_t,w_t\geqslant 0$ *for* $(x,t)\in Q_T=B_R\times(0,T)$,
(iv) u, v *and* w *have a blow-up time* $T<+\infty$,
(v) $\alpha_i+\beta_i+\gamma_i>\max\{p-1,q-1,m-1\}$ *with* $\alpha_i,\beta_i,\gamma_i\geqslant 0, i=1,2,3, p,q,m>1$,

(vi) $\alpha_1 > p-1$ *or* $\beta_2 > q-1$ *or* $\gamma_3 > m-1$ *with* $p,q,m > 1, \alpha_2, \alpha_3, \beta_1, \beta_3, \gamma_1, \gamma_2 \geqslant 0$,

(vii)

$$\min\{qa_1 + mb_1, (p\alpha_2 + (p-2)q)a_1 + (p\alpha_3 + (p-2)m)b_1\} > \max\{p, p(\alpha_1 - 1)\}c_1,$$

$$\min\{pa_2 + mb_2, (q\beta_1 + (q-2)p)a_2 + (q\beta_3 + (q-2)m)b_2\} > \max\{q, q(\beta_2 - 1)\}c_2$$

and

$$\min\{qa_3 + pb_3, (m\gamma_2 + (m-2)q)a_3 + (m\gamma_1 + p(m-2))b_3\} > \max\{m, m(\gamma_3 - 1)\}c_3$$

or

$$\max\{qa_1 + mb_1, (p\alpha_2 + (p-2)q)a_1 + (p\alpha_3 + (p-2)m)b_1\} < \min\{p, p(\alpha_1 - 1)\}c_1,$$

$$\max\{pa_2 + mb_2, (q\beta_1 + (q-2)p)a_2 + (q\beta_3 + (q-2)m)b_2\} < \min\{q, q(\beta_2 - 1)\}c_2$$

and

$$\max\{qa_3 + pb_3, (m\gamma_2 + (m-2)q)a_3 + (m\gamma_1 + p(m-2))b_3\} < \min\{m, m(\gamma_3 - 1)\}c_3,$$

where $a_1 = \beta_1(\gamma_3 - m + 1) - \gamma_1\beta_3, b_1 = \gamma_1(\beta_2 - q + 1) - \gamma_2\beta_1, c_1 = (\gamma_3 - m + 1)(\beta_2 - q + 1) - \gamma_2\beta_3$, $a_2 = \alpha_2(\gamma_3 - m + 1) - \alpha_3\gamma_2, b_2 = \gamma_2(\alpha_1 - p + 1) - \gamma_1\alpha_2, c_2 = (\alpha_1 - p + 1)(\gamma_3 - m + 1) - \gamma_1\alpha_3$, *and* $a_3 = \beta_3(\alpha_1 - p + 1) - \beta_1\alpha_3, b_3 = \alpha_3(\beta_2 - q + 1) - \alpha_2\beta_3, c_3 = (\beta_2 - q + 1)(\alpha_1 - p + 1) - \beta_1\alpha_2$,

(viii) *There are positive constants* k_1, k_2, k_3, k_4 *and* $\eta < T$ *such that*

$$k_2 u(0,t)^{\delta_2/\delta_1} \leqslant v(0,t) \leqslant k_1 u(0,t)^{\delta_2/\delta_1},$$

$$k_4 u(0,t)^{\delta_3/\delta_1} \leqslant w(0,t) \leqslant k_3 u(0,t)^{\delta_3/\delta_1} \quad \text{for } t \in (\eta, T).$$

If one of the following conditions is satisfied:

(g1) $N = 2$ *and* $p,q,m \geqslant 2, \alpha_i, \beta_i, \gamma_i \geqslant 0$, $i = 1,2,3$.

(g2) $N > p$ *and* $p = q = m = 2$,

$$N/2 < \max\left\{\frac{\alpha_3 + \beta_3 + \gamma_3}{\alpha_3 + \beta_3 + \gamma_3 - 1}, \frac{\alpha_2 + \beta_2 + \gamma_2}{\alpha_2 + \beta_2 + \gamma_2 - 1}, \frac{\alpha_1 + \beta_1 + \gamma_1}{\alpha_1 + \beta_1 + \gamma_1 - 1}\right\}$$

or $N > p$ *and* $p = q = m = 2$,

$$N/2 < \max\left\{\frac{2\alpha_3 + 2\beta_3 + \gamma_3 + 1}{2\alpha_3 + 2\beta_3 + \gamma_3 - 1}, \frac{2\alpha_2 + \beta_2 + 2\gamma_2 + 1}{2\alpha_2 + \beta_2 + 2\gamma_2 - 1}, \frac{\alpha_1 + 2\beta_1 + 2\gamma_1 + 1}{\alpha_1 + 2\beta_1 + 2\gamma_1 - 1}\right\},$$

(g3) $N > p$ *and* $p = q = m$,

$$N/p < \max\left\{\frac{\alpha_3 + \beta_3 + \gamma_3}{\alpha_3 - p + 1 + \beta_3 + \gamma_3}, \frac{\alpha_2 + \beta_2 + \gamma_2}{\alpha_2 - p + 1 + \beta_2 + \gamma_2}, \frac{\alpha_1 + \beta_1 + \gamma_1}{\alpha_1 - p + 1 + \beta_1 + \gamma_1}\right\}$$

or $N > p$ *and* $p = q = m$,

$$N/p < \max\left\{\frac{p\alpha_3 + p\beta_3 + (p-1)(\gamma_3+1)}{p\alpha_3 + p\beta_3 + (p-1)(\gamma_3 - p + 1)}, \frac{p\alpha_2 + (p-1)(\beta_2+1) + p\gamma_2}{p\alpha_2 + (p-1)(\beta_2 - p + 1) + p\gamma_2},\right.$$

$$\left.\frac{(p-1)(\alpha_1+1) + p\beta_1 + p\gamma_1}{(p-1)(\alpha_1 - p + 1) + p\beta_1 + p\gamma_1}\right\},$$

(g4) $N > \max\{p, q, m\} \geqslant 2$,

$$N < \max\left\{\frac{pd_1 + p(\alpha_1 - p + 1)d_4 + \beta_1 qd_3 + m\gamma_1 d_2}{(\alpha_1 - p + 1)d_4 + \beta_1 d_3 + \gamma_1 d_2}, \frac{qd_1 + q(\beta_2 - q + 1)d_3 + \alpha_2 pd_4 + m\gamma_2 d_2}{(\beta_2 - q + 1)d_3 + \alpha_2 d_4 + \gamma_2 d_2},\right.$$

$$\left.\frac{md_1 + m(\gamma_3 - m + 1)d_2 + \alpha_3 pd_4 + q\beta_3 d_3}{(\gamma_3 - m + 1)d_2 + \alpha_3 d_4 + \beta_3 d_4}\right\},$$

where $d_1 = (p-1)(q-1)(m-1), d_2 = (p-1)(q-1), d_3 = (p-1)(m-1), d_4 = (q-1)(m-1)$ *or* $m, p, q > 1$ *and*

$$N < \max\left\{\frac{p^2(q-1)(m-1) + p(\alpha_1 - p + 1)(q-1)(m-1) + \beta_1 qp(m-1) + m\gamma_1 p(q-1)}{(\alpha_1 - p + 1)(q-1)(m-1) + \beta_1 p(m-1) + \gamma_1 p(q-1)},\right.$$

$$\frac{q^2(p-1)(m-1) + q(\beta_2 - q + 1)(p-1)(m-1) + \alpha_2 pq(m-1) + m\gamma_2(p-1)q}{(\beta_2 - q + 1)(p-1)(m-1) + \alpha_2 q(m-1) + \gamma_2(p-1)q},$$

$$\left.\frac{m^2(p-1)(q-1) + m(\gamma_3 - m + 1)(q-1)(p-1) + \alpha_3 p(q-1)m + q\beta_3(p-1)m}{(\gamma_3 - m + 1)(q-1)(p-1) + \alpha_3(q-1)m + \beta_3(p-1)m}\right\}.$$

Then there are positive constants c_1, c_2, c_3 *and* $t_1 \in (0, T)$ *such that*

$$u(x,t) \leqslant u(0,t) \leqslant c_1(T-t)^{-\delta_1},$$

$$v(x,t) \leqslant v(0,t) \leqslant c_2(T-t)^{-\delta_2}, \tag{3.3}$$

$$w(x,t) \leqslant w(0,t) \leqslant c_3(T-t)^{-\delta_3} \tag{3.4}$$

for all $(x,t) \in Q_T \setminus Q_{t_1}$, *where*

$$\delta_1 = \frac{qa_1 + mb_1 - pc_1}{a_1(p\alpha_2 + q(p-2)) + b_1(p\alpha_3 + (p-2)m) - p(\alpha_1 - 1)c_1},$$

$$\delta_2 = \frac{pa_2 + mb_2 - qc_2}{a_2(q\beta_1 + (q-2)p) + b_2(q\beta_3 + (q-2)m) - q(\beta_2 - 1)c_2},$$

$$\delta_3 = \frac{qa_3 + pb_3 - mc_3}{a_3(m\gamma_2 + (m-2)q) + b_3(m\gamma_1 + p(m-2)) - m(\gamma_3 - 1)c_3}.$$

Remark 3.1. Conditions (i)–(iii) in Theorem 3.1 are reasonable if we impose appropriate assumptions on the initial data $u_0(x)$, $v_0(x)$ and $w_0(x)$, such as positivity, radial symmetry, and a suitable decreasing property with

$$\operatorname{div}(|\nabla u_0|^{p-2}\nabla u_0) + u_0^{\alpha_1} v_0^{\beta_1} w_0^{\gamma_1} \geqslant 0, \quad \operatorname{div}(|\nabla v_0|^{p-2}\nabla v_0) + u_0^{\alpha_2} v_0^{\beta_2} w_0^{\gamma_2} \geqslant 0,$$

$$\operatorname{div}(|\nabla w_0|^{p-2}\nabla w_0) + u_0^{\alpha_3} v_0^{\beta_3} w_0^{\gamma_3} \geqslant 0.$$

Remark 3.2. Clearly, condition (viii) seems too strong. If $p=q=m=2, \alpha_2=\alpha_3, \beta_2=\beta_3, \gamma_2=\gamma_3, \alpha_1+\beta_1=\alpha_2+\beta_2$ (in this case, (1.1) reduces to two equations), from Lemma 3.2 in [17], we know that $k_2 u(x,t)^{\delta_2/\delta_1} \leqslant v(x,t) \leqslant k_1 u(x,t)^{\delta_2/\delta_1}$. If $p \neq 2, q \neq 2$ or $\alpha_1+\beta_1 \neq \alpha_2+\beta_2$, we do not know whether or not condition (viii) holds. We hope this condition can be substantially improved in the future. This is an open problem.

Remark 3.3. From the definitions of δ_1, δ_2 and δ_3, we see that the conditions $\alpha_i+\beta_i+\gamma_i > \max\{p-1, q-1, m-1\}$ with $\alpha_i, \beta_i, \gamma_i \geqslant 0$, $i=1,2,3$ and

$$\min\{qa_1+mb_1, (p\alpha_2+(p-2)q)a_1+(p\alpha_3+(p-2)m)b_1\} > \max\{p, p(\alpha_1-1)\}c_1,$$

$$\min\{pa_2+mb_2, (q\beta_1+(q-2)p)a_2+(q\beta_3+(q-2)m)b_2\} > \max\{q, q(\beta_2-1)\}c_2,$$

$$\min\{qa_3+pb_3, (m\gamma_2+(m-2)q)a_3+(m\gamma_1+p(m-2))b_3\} > \max\{m, m(\gamma_3-1)\}c_3$$

or

$$\max\{qa_1+mb_1, (p\alpha_2+(p-2)q)a_1+(p\alpha_3+(p-2)m)b_1\} < \min\{p, p(\alpha_1-1)\}c_1,$$

$$\max\{pa_2+mb_2, (q\beta_1+(q-2)p)a_2+(q\beta_3+(q-2)m)b_2\} < \min\{q, q(\beta_2-1)\}c_2,$$

$$\max\{qa_3+pb_3, (m\gamma_2+(m-2)q)a_3+(m\gamma_1+p(m-2))b_3\} < \min\{m, m(\gamma_3-1)\}c_3),$$

where $a_1=\beta_1(\gamma_3-m+1)-\gamma_1\beta_3, b_1=\gamma_1(\beta_2-q+1)-\gamma_2\beta_1, c_1=(\gamma_3-m+1)(\beta_2-q+1)-\gamma_2\beta_3$, $a_2=\alpha_2(\gamma_3-m+1)-\alpha_3\gamma_2, b_2=\gamma_2(\alpha_1-p+1)-\gamma_1\alpha_2, c_2=(\alpha_1-p+1)(\gamma_3-m+1)-\gamma_1\alpha_3$, $a_3=\beta_3(\alpha_1-p+1)-\beta_1\alpha_3, b_3=\alpha_3(\beta_2-q+1)-\alpha_2\beta_3, c_3=(\beta_2-q+1)(\alpha_1-p+1)-\beta_1\alpha_2$, are natural for the discussion of the blow-up rate estimate.

Lemma 3.2. *Assume that condition* (vii) *in Theorem* 3.1 *holds. Then*

$$\min(\delta_1, \delta_2, \delta_3) > 0.$$

Proof. In fact $\delta_1 = (qa_1+mb_1-pc_1)/[a_1(p\alpha_2+q(p-2))+b_1(p\alpha_3+(p-2)m)-p(\alpha_1-1)c_1]$, if $\min\{qa_1+mb_1, (p\alpha_2+(p-2)q)a_1+(p\alpha_3+(p-2)m)b_1\} > \max\{p, p(\alpha_1-1)\}c_1$, then $\delta_1>0$. The others are similar to prove. □

Proof of Theorem 3.1. Define the functions $\mu(t), \theta(t), \delta(t)$ for $t \in (0,T)$ as follows:

$$\mu(t) = u(0,t)^{1/\tau_1}, \quad \theta(t) = v(0,t)^{1/\tau_2}, \quad \delta(t) = w(0,t)^{1/\tau_3},$$

where

$$\tau_1 = \frac{pc_1 - qa_1 - mb_1}{(\gamma_3 - m + 1)c_3 - \gamma_2 a_3 - \gamma_1 b_3},$$

$$\tau_2 = \frac{qc_2 - pa_2 - mb_2}{(\alpha_1 - p + 1)c_1 - \alpha_3 b_1 - \alpha_2 a_1}$$

and

$$\tau_3 = \frac{mc_3 - qa_3 - pb_3}{(\beta_3 - q + 1)c_2 - \beta_3 b_2 - \beta_1 a_2}.$$

By putting

$$h_1(r,t) = \frac{u(r/\rho(t),t)}{\rho(t)^{\tau_1}}, \quad h_2(r,t) = \frac{v(r/\rho(t),t)}{\rho(t)^{\tau_2}}, \quad h_3(r,t) = \frac{w(r/\rho(t),t)}{\rho(t)^{\tau_3}}\, r = |x|,$$

$$\rho(t) = \mu(t) + \theta(t) + \delta(t).$$

Since $u(\cdot,t)$, $v(\cdot,t)$ and $w(\cdot,t)$ achieve their maximum at $r = 0$ by assumption (iv), it is clear that

$$0 \leqslant h_1(r,t) \leqslant \frac{u(0,t)}{\rho(t)^{\tau_1}} \leqslant 1, \tag{3.5}$$

$$0 \leqslant h_2(r,t) \leqslant \frac{v(0,t)}{\rho(t)^{\tau_2}} \leqslant 1, \tag{3.6}$$

$$0 \leqslant h_3(r,t) \leqslant \frac{w(0,t)}{\rho(t)^{\tau_3}} \leqslant 1. \tag{3.7}$$

Since

$$\alpha_1\tau_1 + \beta_1\tau_2 + \gamma_1\tau_3 = p + (p-1)\tau_1, \quad \alpha_2\tau_1 + \beta_2\tau_2 + \gamma_2\tau_3 = q + (q-1)\tau_2,$$

$$\alpha_3\tau_1 + \beta_3\tau_2 + \gamma_3\tau_3 = m + (m-1)\tau_3$$

and taking into account assumptions (i) and (iv), it follows that

$$\begin{aligned} 0 &\leqslant \operatorname{div}(|\nabla h_1(r,t)|^{p-2}\nabla h_1(r,t)) + h_1^{\alpha_1} h_2^{\beta_1} h_3^{\gamma_1}(r,t) \\ &\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}, \end{aligned} \tag{3.8}$$

$$\begin{aligned} 0 &\leqslant \operatorname{div}(|\nabla h_2(r,t)|^{q-2}\nabla h_2(r,t)) + h_1^{\alpha_2} h_2^{\beta_2} h_3^{\gamma_2}(r,t) \\ &\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}, \end{aligned} \tag{3.9}$$

$$0 \leqslant \operatorname{div}(|\nabla h_3(r,t)|^{p-2}\nabla h_3(r,t)) + h_1^{\alpha_3} h_2^{\beta_3} h_3^{\gamma_3}(r,t)$$
$$\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}} \tag{3.10}$$

for any $t \in (0,T)$ and $r \in [0, R\rho(t))$.

Using the symmetry assumption (i), we can rewrite inequalities (3.8)–(3.10) in radial coordinates and get

$$0 \leqslant (\Phi_p(h_1'))' + \frac{N-1}{r}\,\Phi_p(h_1') + h_1^{\alpha_1} h_2^{\beta_1} h_3^{\gamma_1}$$
$$\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}, \tag{3.11}$$

$$0 \leqslant (\Phi_q(h_2'))' + \frac{N-1}{r}\,\Phi_q(h_2') + h_1^{\alpha_2} h_2^{\beta_2} h_3^{\gamma_2}$$
$$\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}, \tag{3.12}$$

$$0 \leqslant (\Phi_m(h_3'))' + \frac{N-1}{r}\,\Phi_m(h_3') + h_1^{\alpha_3} h_2^{\beta_3} h_3^{\gamma_3}$$
$$\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}. \tag{3.13}$$

Multiplying (3.11) by $h_{1,r}$, we get

$$(\Phi_p(h_{1,r}))_r h_{1,r} + \frac{N-1}{r}\,|h_{1,r}|^p + h_1^{\alpha_1} h_2^{\beta_1} h_3^{\gamma_1} h_{1,r} \leqslant 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}r}((p-1)/p|h_{1,r}|^p) + h_1^{\alpha_1} h_2^{\beta_1} h_3^{\gamma_1} h_{1,r} \leqslant 0. \tag{3.14}$$

Integrating (3.14) on $(0,r)$, we obtain

$$\frac{p-1}{p}\,|h_{1,r}|^p + \frac{1}{\alpha_1+1}\,h_1^{\alpha_1+1}(r,t)h_2^{\beta_1}(r,t)h_3^{\gamma_1}(r,t)$$
$$-\frac{1}{\alpha_1+1}\,h_1^{\alpha_1+1}(0,t)h_2^{\beta_1}(0,t)h_3^{\gamma_1}(0,t) - \beta_1/(\alpha_1+1)\int_0^r h^{\alpha_1+1} h_2^{\beta_1-1} h_{2,r}' h_3^{\gamma_1}\,\mathrm{d}r$$
$$-\gamma_1/(\alpha_1+1)\int_0^r h_1^{\alpha_1+1} h_2^{\beta_1} h_3^{\gamma_1-1} h_{3,r}'\,\mathrm{d}r \leqslant 0. \tag{3.15}$$

From (3.15), and $h_{2,r}(r,t), h_{3,r}(r,t) \leqslant 0$, it follows that

$$|h_{1,r}| \leqslant \left(\frac{2p}{(p-1)(\alpha_1+1)}\right)^{1/p} \tag{3.16}$$

for and $t \in (0,T), r \in [0, R\rho(t))$. Similarly, we get

$$|h_{2,r}| \leqslant \left(\frac{2q}{(q-1)(\beta_2+1)}\right)^{1/q}, \quad |h_{3,r}| \leqslant \left(\frac{2m}{(m-1)(\gamma_3+1)}\right)^{1/m} \tag{3.17}$$

for any $t \in (0,T), r \in [0, R\rho(t))$.

Now, we proceed by contradiction as in [2]. If

$$\liminf_{t\to T}\left(\frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}\right) = 0, \tag{3.18}$$

then there exists a sequence $\{t_m\} \subseteq (0,T)$ with $t_m \to T$ such that

$$\liminf_{t_m\to T}\left(\frac{u_t(0,t_m)}{\rho(t_m)^{p+(p-1)\tau_1}} + \frac{v_t(0,t_m)}{\rho(t_m)^{q+(q-1)\tau_2}} + \frac{w_t(0,t_m)}{\rho(t_m)^{m+(m-1)\tau_3}}\right) = 0.$$

Since inequalities (3.5)–(3.7) and (3.16)–(3.17), $\{h_1(\cdot,t_m)\}$, $\{h_2(\cdot,t_m)\}$ and $\{h_3(\cdot,t_m)\}$ are equi-bounded and Lipschitz continuous with the Lipschitz constant less than or equal to $(2p/(p-1))^{1/p}$, $(2q/(q-1))^{1/q}$ and $(2m/(m-1))^{1/m}$, it follows from the Ascoli–Arzela theorem that there exists a subsequence (still denoted by $\{t_m\}$) such that

$$h_1(\cdot,t_m) \to \bar{h}_1(\cdot) \quad \text{as } m \to +\infty, \tag{3.19}$$

$$h_2(\cdot,t_m) \to \bar{h}_2(\cdot) \quad \text{as } m \to +\infty, \tag{3.20}$$

$$h_3(\cdot,t_m) \to \bar{h}_3(\cdot) \quad \text{as } m \to +\infty \tag{3.21}$$

uniformly on compact subsets of $[0,+\infty)$. Moreover, $\bar{h}_1, \bar{h}_2, \bar{h}_3 \in \mathbf{C}([0,+\infty), \mathbb{R}^+)$, $\bar{h}_1(0) = \bar{h}_2(0) = \bar{h}_3(0) = 1$, and $\bar{h}_1, \bar{h}_2, \bar{h}_3$ are decreasing on $[0,+\infty)$. Further, taking into account that $\bar{h}_1, \bar{h}_2, \bar{h}_3$ are Lipschitz continuous, we conclude that they are absolutely continuous on $[0,+\infty)$. Considering that $\bar{h}_1, \bar{h}_2, \bar{h}_3$ as distributions, it also follows that (3.16) and (3.17) holds in the sense of distributions, and hence, in the distributional sense we have

$$h_{1,r}(\cdot,t_m) \to \bar{h}_{1,r}(\cdot), \quad (\Phi_p(h_{1,r})(\cdot,t_m))_r \to (\Phi_p(\bar{h}_{1,r}(\cdot)))_r \quad \text{as } m \to +\infty, \tag{3.22}$$

$$h_{2,r}(\cdot,t_m) \to \bar{h}_{2,r}(\cdot), \quad (\Phi_q(h_{2,r})(\cdot,t_m))_r \to (\Phi_q(\bar{h}_{2,r}(\cdot)))_r \quad \text{as } m \to +\infty, \tag{3.23}$$

$$h_{3,r}(\cdot,t_m) \to \bar{h}_{3,r}(\cdot), \quad (\Phi_m(h_{3,r})(\cdot,t_m))_r \to (\Phi_m(\bar{h}_{3,r}(\cdot)))_r \quad \text{as } m \to +\infty. \tag{3.24}$$

Now, (3.22)–(3.24) imply that

$$(\Phi_p(\bar{h}_1'))' + \frac{N-1}{r}\Phi_p(\bar{h}_1') + \bar{h}_1^{\alpha_1}\bar{h}_2^{\beta_1}\bar{h}_3^{\gamma_1} = 0, \tag{3.25}$$

$$(\Phi_q(\bar{h}_2'))' + \frac{N-1}{r}\,\Phi_q(\bar{h}_2') + \bar{h}_1^{\alpha_2}\bar{h}_2^{\beta_2}\bar{h}_3^{\gamma_2} = 0, \tag{3.26}$$

$$(\Phi_m(\bar{h}_3'))' + \frac{N-1}{r}\,\Phi_m(\bar{h}_3') + \bar{h}_1^{\alpha_3}\bar{h}_2^{\beta_3}\bar{h}_3^{\gamma_3} = 0 \tag{3.27}$$

on $(0,+\infty)$ in the sense of distributions. From Eqs. (3.25)–(3.27), it also follows that $\bar{h}_1, \bar{h}_2, \bar{h}_3$ are $\mathbf{C^1}(0,+\infty)$ and, by local existence and uniqueness of the initial value problem for Eqs. (3.25)–(3.27), we conclude that $\bar{h}_1, \bar{h}_2, \bar{h}_3 > 0$ on $(0,+\infty)$ with $\bar{h}_1'(0) = \bar{h}_2'(0) = \bar{h}_3'(0) = 0$.

If $N = 2, p > 2$, we proceed as follows: from Eqs. (3.25)–(3.27), it is inferred that $r\Phi_p(\bar{h}_1')$, $r\Phi_q(\bar{h}_2')$ and $r\Phi_m(\bar{h}_3')$ are decreasing, and that there exist $M < 0$ and $r_0 > 0$ such that

$$r\Phi_p(\bar{h}_1') < M \quad \text{for } r \in (r_0,+\infty).$$

The last inequality implies that

$$\begin{aligned}\bar{h}_1(s) &> \bar{h}_1(s) - \bar{h}_1(t)\\ &= (-M)^{1/(p-1)}\int_s^t r^{-1/(p-1)}\,\mathrm{d}r = (-M)^{1/(p-1)}(t^{(p-2)/(p-1)} - s^{(p-2)/(p-1)})\end{aligned} \tag{3.28}$$

for $r_0 \leqslant s \leqslant t$. Letting $t \to +\infty$ in (3.28), we obtain a contradiction.

If $N = 2, p = 2$ similar with the above method implies that

$$\bar{h}_1(s) > \bar{h}_1(s) - \bar{h}_1(t) > (-M)[\ln(t) - \ln(s)]$$

for $r_0 \leqslant s \leqslant t$. Letting $t \to +\infty$ in the last inequality, we obtain a contradiction.

Finally, if $N > \max\{p,q\} \geqslant 2$ holds, we know from Theorem 2.1 or 2.2 that system (3.25)–(3.27) has no positive solutions. We conclude that Eq. (3.18) cannot hold: hence

$$\liminf_{t\to T}\left(\frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}}\right) = c > 0. \tag{3.29}$$

It follows from Eq. (3.29) that there exists $t_1 \in (0,T)$ such that for any $t \in (t_1,T)$ we have

$$\begin{aligned} c &\leqslant \frac{u_t(0,t)}{\rho(t)^{p+(p-1)\tau_1}} + \frac{v_t(0,t)}{\rho(t)^{q+(q-1)\tau_2}} + \frac{w_t(0,t)}{\rho(t)^{m+(m-1)\tau_3}},\\ &\leqslant \frac{u_t(0,t)}{u(0,t)^{(1+\delta_1)/\delta_1}} + \frac{v_t(0,t)}{v(0,t)^{(1+\delta_2)/\delta_2}} + \frac{w_t(0,t)}{w(0,t)^{(1+\delta_3)/\delta_3}}.\end{aligned} \tag{3.30}$$

Integrating inequality (3.30) on $(t,s) \subseteq (t_1,T)$ and then letting $s \to T$, we obtain

$$c(T-t) \leqslant \delta_1 u(0,t)^{-1/\delta_1} + \delta_2 v(0,t)^{-1/\delta_2} + \delta_3 w(0,t)^{-1/\delta_3}. \tag{3.31}$$

By using condition (viii) in (3.31) we have

$$c(T-t) \leqslant \delta_1 u(0,t)^{-1/\delta_1} + \delta_2 k^{-1/\delta_2} u(0,t)^{-1/\delta_1} + \delta_3 k_3^{-1/\delta_3} u(0,t)^{-1/\delta_1}$$

and hence

$$u(x,t) \leqslant u(0,t) \leqslant c_1(T-t)^{-\delta_1}$$

for any $(x,t) \in \mathbf{B}(0,R) \times (0,T)$.

We have in the same way the blow-up estimate for v, w:

$$v(x,t) \leqslant v(0,t) \leqslant c_2(T-t)^{-\delta_2},$$

$$w(x,t) \leqslant w(0,t) \leqslant c_3(T-t)^{-\delta_3}.$$

The proof is completed. □

Remark 3.4. For the special parabolic system

$$u_t = \Delta u + u^{p_1} v^{q_1},$$

$$v_t = \Delta v + u^{p_2} v^{q_2}$$

with $p_i + q_i > 1, p_i, q_i \geqslant 0$, Zheng [17] obtained the blow-up estimates

$$u(x,t) \leqslant u(x,0) \leqslant c_1(T-t)^{-\alpha}, \tag{3.32}$$

$$v(x,t) \leqslant v(x,0) \leqslant c_2(T-t)^{-\beta}, \tag{3.33}$$

where

$$\alpha = \frac{1+q_1-q_2}{p_2 q_1 - (p_1-1)(q_2-1)},$$

$$\beta = \frac{1+p_2-p_1}{p_2 q_1 - (p_1-1)(q_2-1)}.$$

Besides, for the special variational parabolic system

$$u_t = \Delta u + v^{\mu},$$

$$v_t = \Delta v + u^{\delta}$$

with $\mu, \delta > 1$, Caristi and Mitidieri [2] obtained the blow-up estimates

$$u(x,t) \leqslant u(x,0) \leqslant c(T-t)^{-(\mu+1)/(\mu\delta-1)}, \tag{3.34}$$

$$v(x,t) \leqslant v(x,0) \leqslant c(T-t)^{-(\delta+1)/(\mu\delta-1)}. \tag{3.35}$$

The single equation case was treated by Weissler [18] with

$$u(x,t) \leqslant u(x,0) \leqslant c(T-t)^{-1/(\mu-1)}. \tag{3.36}$$

Clearly, inequalities (3.32)–(3.36) agree with Theorem 3.1 if one takes $p=q=m=2, \alpha_1 = p_1, \beta_1 = q_1, \alpha_2 = p_2, \beta_2 = q_2, \gamma_1 = \gamma_2 = \gamma_3 = 0, \alpha_2 = \alpha_3, \beta_2 = \beta_3$ or $p=q=m=2, \alpha_1 = \beta_2 = 0, \beta_1 = \mu, \alpha_2 = \delta$,

$\gamma_1=\gamma_2=\gamma_3=0, \alpha_2=\alpha_3, \beta_2=\beta_3$ or $p=q=m=2, \alpha_2=\beta_1=0, \alpha_1=\beta_2=\mu, \gamma_1=\gamma_2=\gamma_3=0, \alpha_2=\alpha_3, \beta_2=\beta_3$, respectively. Therefore, this paper extends their results essentially.

Remark 3.5. For the special variational parabolic system

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u) + v^{\mu},$$

$$v_t = \mathrm{div}(|\nabla v|^{q-2}\nabla v) + u^{\theta}$$

with $\mu > p-1, \theta > q-1$, we [19] obtained the blow-up estimates

$$u(x,t) \leqslant u(0,t) \leqslant c_1(T-t)^{-\delta_1}, \quad v(x,t) \leqslant v(0,t) \leqslant c_2(T-t)^{-\delta_2} \tag{3.37}$$

for $(x,t) \in \mathbf{B}(0,R) \times (0,T)$, where

$$\delta_1 = \frac{\mu q + (q-1)p}{\mu(p\theta + q(p-2)) - p(q-1)}, \quad \delta_2 = \frac{\theta p + (p-1)q}{\theta(q\mu + p(q-2)) - q(p-1)}$$

and $T \in (0,\infty)$ is the blow-up time.

The single Eq. (1.1) was treated in [20] with (3.36). Clearly, inequalities (3.37) and (3.36) agree with Theorem 3.1 if one takes $p,q=m>1, \alpha_1=\beta_2=0, \alpha_2=\theta, \beta_1=\mu, \gamma_1=\gamma_2=\gamma_3=0, \alpha_2=\alpha_3, \beta_2=\beta_3$ or $p,q=m>1, \beta_1=\gamma_1=\alpha_2=\gamma_2=\alpha_3=\beta_3=0, \alpha_1=\beta_2=\gamma_3=\mu$, respectively. Therefore, this paper is also an extension of the above results.

References

[1] G. Astrita, G. Marrucci, Principles of Non-Newtonian Fluid Mechanics, McGraw-Hill, New York, 1974.

[2] G. Caristi, E. Mitidieri, Blow-up estimates of positive solutions of a parabolic system, J. Differential Equations 113 (1994) 265–271.

[3] Ph. Clement, D.G. de Figueiredo, E. Mitidieri, Positive solutions of semilinear elliptic systems, Comm. Partial Differential Equations 17 (5/6) (1992) 923–940.

[4] Ph. Clement, R, Manasevich, E. Mitidieri, Positive solutions for a quasilinear system via blow up, Comm. Partial Differential Equations 18 (12) (1993) 2071–2106.

[5] M. Escobedo, M.M. Herrero, Boundedness and blow up for a semilinear reaction–diffusion system, J. Differential Equations 89 (1991) 176–202.

[6] M. Escobedo, H.A. Levine, Critical blow-up and global existence numbers for a weakly coupled system of reaction–diffusion equations, Arch. Rational Mech. Anal. 129 (1995) 47–100.

[7] J.R. Esteban, J.L. Vazquez, On the equation of turbulent filtration in one-dimensional porous media, Nonlinear Anal. 10 (1982) 1303–1325.

[8] Y. Giga, R.V. Kohn, Asymptotically self-similar blow up of semilinear heat equations, Comm. Pure Appl. Math. 38 (1985) 297–319.

[9] A. Haraux, F.B. Weissler, Non-uniqueness for a semilinear initial value problem, Indiana Univ. Math. J. 31 (1982) 167–189.

[10] L.K. Martinson, K.B. Pavlov, Unsteady shear flows of a conducting fluid with a rheological power law, Magnit. Gidrodinamika 2 (1971) 50–58.

[11] E. Mitidieri, A Rellich type identity and applications, Comm. Partial Differential Equations 18 (1993) 125–171.

[12] E. Mitidieri, Nonexistence of positive solutions of semilinear elliptic system in $\mathbb{R}^N$, Differential Integral Equations 9 (1996) 465–479.

[13] E. Mitidieri, G. Sweers, R. vander Vorst, Nonexistence theorems for systems of quasilinear partial differential equations, Differential Integral Equations 8 (1995) 1331–1354.

[14] W.M. Ni, J. Serrin, Nonexistence theorems for singular solutions of quasilinear partial differential equations, Comm. Pure Appl. Math. 39 (1986) 379–399.
[15] L.A. Peletier, R.C.A.M. van der Vorst, Existence and non-existence of positive solutions of non-linear elliptic systems and the biharmonic equations, Differential Integral Equations 54 (1992) 747–767.
[16] S. Chen, G. Lu, Existence and nonexistence of positive radial solutions for a class of semilinear elliptic system, Nonlinear Anal. 38 (1999) 919–932.
[17] S. Zheng, Nonexistence of positive solutions to a semilinear elliptic system and blow-up estimates for a reaction–diffusion system, J. Math. Anal. Appl. 232 (1999) 293–311.
[18] F.B. Weissler, An L^∞ blow-up estimate for a nonlinear heat equation, Comm. Pure Appl. Math. 38 (1985) 291–295.
[19] Z. Yang, Q. Lu, Blow-up estimates for a non-Newtonian filtration system, Appl. Math. Mech. 22 (3) (2001) 332–339.
[20] Z. Yang, Q. Lu, Blow-up estimates for a non-Newtonian filtration, J. Math. Res. Expositive, to appear.