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# Friedland–Hersonsky problem for matrix algebra ☆

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**Abstract**

In this paper we show that the answer to problem 3.9 in [Duke Math. J. 69 (1993) 593] is positive when $n = 2$ and negative when $n \geqslant 3$.


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*Keywords:* Matrix algebra; Spectral norm; Eigenvalue; Iteration

## 1. Introduction

In [5], Jorgensen obtained a very useful inequality in $SL(2, \mathbf{C})$ which is known as Jorgensen's inequality. As Gilman pointed out in [2] that it is important to obtain Jorgensen's inequality in higher dimensions. Recently, different forms of Jorgensen's inequality in space have been obtained, see [4,6,10] and references therein. They are very useful, see [8,9] and references therein. In [1], Friedland and Hersonsky have generalized Jorgensen's inequality into normed algebras.

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As in [1], let $M_n(\mathbf{C}) = \{A = (a_{ij})_1^n\colon a_{ij} \in \mathbf{C}\}$ and $GL_n(\mathbf{C}) = \{A \in M_n(\mathbf{C})\colon \det(A) \neq 0\}$. Denote by $\Lambda(A) = \{\lambda_1, \ldots, \lambda_n\}$ the $n$ eigenvalues of $A$ counted with their multiplicity and let $\rho(A) = \max_{1\leqslant i\leqslant n} |\lambda_i|$ be the spectral radius of $A$. Denote by $A^{\mathrm{T}}$ the transposed matrix of $A$ and $A^*$ the conjugate transpose of $A$. A matrix $A \in M_n(\mathbf{C})$ is called *normal* if $AA^* = A^*A$.

We shall use [3] as a general reference for basic facts concerning the matrix theory.

For a matrix $A = (a_{ij})_1^n \in GL_n(\mathbf{C})$, we define the following operator on $M_n(\mathbf{C})$,

$$\hat{A}(Z) = AZA^{-1} - Z, \tag{1.1}$$

where $Z = (z_{ij})_1^n \in M_n(\mathbf{C})$.

Let $|\cdot|$ be a vector norm on $M_n(\mathbf{C})$ (not necessarily submultiplicative). The norm of operator $\hat{A}$ is defined by

$$\|\hat{A}\| = \sup_{|Z|\leqslant 1} |\hat{A}(Z)|. \tag{1.2}$$

In order to generalize Jorgenson's inequality into matrix algebra, Friedland and Hersonsky [1] considered the following iterations,

$$X_1 = [A, B], \quad X_{k+1} = [A, X_k] \ (k = 1, 2, \ldots), \tag{1.3}$$

where $A, B \in GL_n(\mathbf{C})$. Then the question of when $X_{k+1} = I$ becomes very important. In particular, they raised the following conjecture.

**Conjecture.** *Let $n$ be a positive integer. Then there exists an integer $s(n)$ so that the following condition holds. Assume that $A, B \in GL_n(\mathbf{C})$ and consider the iteration* (1.3). *Then either $X_{(s(n))} = I$ or $X_m \neq I$ $(m = 0, 1, 2, \ldots)$.*

In [7], we constructed counterexamples to show that the answer to the above conjecture is negative when $n \geqslant 2$.

When they discussed the sharpness of their inequalities, Friedland and Hersonsky proved the following.

**Theorem FH.** *Let $A \in GL_n(\mathbf{C})$. Then*

$$\max_{1\leqslant i,j\leqslant n} \left| \frac{\lambda_i}{\lambda_j} - 1 \right| \leqslant \|\hat{A}\|. \tag{1.4}$$

*If $A$ is diagonable, then there exists an operator norm on $M_n(\mathbf{C})$ such that the equality holds in the inequality* (1.4). *If $A$ is not diagonable then for any $\varepsilon > 0$ there exists an operator norm on $M_n(\mathbf{C})$ such that the right-hand side of* (1.4) *minus its left-hand side is less than $\varepsilon$.*

In this paper we let $\|A\|$ to be the spectral norm $|A|_2 := \sqrt{\rho(AA^*)} = \sqrt{\rho(A^*A)}$. Then $\|\hat{A}\|$ is the operator norm induced by $|\cdot|_2$. Friedland and Hersonsky proposed the following problem in [1].

**Question 1.** For a diagonal matrix $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in GL_n(\mathbf{C})$ does equality in (1.4) holds?

In Section 2, we construct counterexamples to show that the answer to *Question 1* is negative when $n \geqslant 3$. Also we prove that, when $n = 2$, for any $A \in GL_2(\mathbf{C})$, the equality in inequality (1.4) holds if and only if $A$ is normal (Theorem 2.1). In Section 4, two conditions are obtained to guarantee that the equality or the strict inequality holds for a diagonal matrix $A \in GL_n(\mathbf{C})$ in inequality (1.4) (Theorems 4.1 and 4.2). By using Theorem 4.2 or Corollary 4.1, more counterexamples to *Question 1* can be constructed. The proofs of Theorems 4.1 and 4.2 are relied on the three lemmas proved in Section 3.

## 2. Counterexamples to Question 1

In this section, we will construct counterexamples to *Question 1* for the case $n \geqslant 3$, and then we prove that, for $A \in GL_2(\mathbf{C})$, the equality in inequality (1.4) holds if and only if $A$ is normal.

For $A = (a_{ij})_1^n$, $B = (b_{ij})_1^n \in M_n(\mathbf{C})$, the Hadamard product of $A$ and $B$ is defined as

$$A \odot B = (a_{ij}b_{ij})_1^n. \tag{2.1}$$

For $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in GL_n(\mathbf{C})$, let $\Omega_A = ((\lambda_i/\lambda_j) - 1)_1^n \in M_n(\mathbf{C})$. It follows from (1.1) and (2.1) that

$$\hat{A}(Z) = \Omega_A \odot Z. \tag{2.2}$$

For convenience, we let $\sigma_{ij} = (\lambda_i/\lambda_j) - 1$ and $\sigma = \max_{i,j} |\sigma_{ij}|$. Then, by Theorem FH, it is equivalent to consider the following question.

**Question 2.** For any $Z = (c_{ij})_1^n$, is $|\Omega_A \odot Z|_2 \leqslant \sigma$ if $|Z|_2 \leqslant 1$?

When $n = 3$, we will construct a counterexample to show that the answer to *Question 2* is negative.

**Remark 2.1.** By using Theorem 4.2 or Corollary 4.1 in Section 4, more counterexamples can be constructed.

Let $A = \mathrm{diag}(1, 4, -2)$. Then

$$\Omega_A = \begin{pmatrix} 0 & -\frac{3}{4} & -\frac{3}{2} \\ 3 & 0 & -3 \\ -3 & -\frac{3}{2} & 0 \end{pmatrix}.$$

Obviously, $\sigma = 3$.

Let

$$Z=\begin{pmatrix}\frac{4}{5} & 0 & -\frac{3}{5}\\ \frac{3}{5} & 0 & \frac{4}{5}\\ 0 & 1 & 0\end{pmatrix}.$$

Then

$$\hat{A}(Z)\hat{A}(Z)^*=\begin{pmatrix}\frac{81}{100} & -\frac{54}{25} & 0\\ -\frac{54}{25} & 9 & 0\\ 0 & 0 & \frac{9}{4}\end{pmatrix}.$$

The maximal eigenvalue of $\hat{A}(Z)\hat{A}(Z)^*$ is the maximal solution of the following quadratic equation

$$\left(\lambda-\frac{81}{100}\right)(\lambda-9)=\frac{2916}{625}.$$

A straightforward computation shows that $|Z|_2=1$ and $\|\hat{A}\|\geqslant|\hat{A}(Z)|_2=|\Omega_A\odot Z|_2=3.0878>3$. This shows that when $n=3$, the answer to *Question 2* is negative.

**Lemma 2.1.** *Let* $A=\operatorname{diag}(\lambda_1,\lambda_1,\lambda_2,\ldots,\lambda_n)\in GL_{n+1}(\mathbf{C})$ *and* $B=\operatorname{diag}(\lambda_1,\ldots,\lambda_n)\in GL_n(\mathbf{C})$. *Then* $\|\hat{A}\|\geqslant\|\hat{B}\|$.

**Proof.** Let $\alpha=(0,\frac{\lambda_1}{\lambda_2}-1,\ldots,\frac{\lambda_1}{\lambda_n}-1)$ and $\beta=(0,\frac{\lambda_2}{\lambda_1}-1,\ldots,\frac{\lambda_n}{\lambda_1}-1)^{\mathrm{T}}$. Then

$$\Omega_A=\begin{pmatrix}0 & \alpha\\ \beta & \Omega_B\end{pmatrix}.$$

For any $Z\in M_n(\mathbf{C})$, let $Z_A=\begin{pmatrix}0 & 0\\ 0 & Z\end{pmatrix}\in M_{n+1}(\mathbf{C})$. Then $|\Omega_A\odot Z_A|_2=|\Omega_B\odot Z|_2$. Lemma 2.1 follows. □

By Lemma 2.1 and the above counterexample for the case $n=3$, we can construct counterexamples for the cases $n>3$. For example, $A=\operatorname{diag}(1,\ldots,1,1,4,-2)\in GL_n(\mathbf{C})$ will satisfy our requirement. This shows that when $n>3$, the answer to *Question 2* is also negative.

Since $|A|_2=|UAU^*|_2$ for any unitary matrix $U$ $(UU^*=I)$ it follows that $\|\hat{A}\|=\|\hat{B}\|$ for $B=UAU^*$ and $U$ unitary. Use Schur's theorem to deduce that we may choose $A$ to be an upper triangular matrix. Furthermore $\|\hat{A}\|=\|\hat{B}\|$, where $A=kB$ and $k$ any nonzero complex number. Thus without loss of generality we assume that $A$ is an upper triangular matrix with having an eigenvalue $\lambda_1=1$. Furthermore for $n=2$ we may assume that $A=\begin{pmatrix}1 & t\\ 0 & \lambda\end{pmatrix}$ where $t$ is a real nonnegative number.

We first assume that $t>0$.

Let

$$Z=\begin{pmatrix}0 & 1\\ \mathrm{e}^{\mathrm{i}(\theta+\pi)} & 0\end{pmatrix}.$$

Then a straightforward computation shows that

$$\hat{A}(Z)\hat{A}(Z)^* = \begin{pmatrix} t^2 + |(\frac{1}{\lambda} - 1) + \frac{t^2}{\lambda}\mathrm{e}^{\mathrm{i}\theta}|^2 & \Delta \\ \overline{\Delta} & |\lambda - 1|^2 + t^2 \end{pmatrix},$$

where $\Delta = t\left[\bar{\lambda} - 1 + \frac{t^2}{\lambda}\right] + t\left(\frac{1}{\lambda} - 1\right)\mathrm{e}^{-\mathrm{i}\theta}$.

Since $t > 0$, we can choose a $\theta$ such that $\left|\left(\frac{1}{\lambda} - 1\right) + \frac{t^2}{\lambda}\mathrm{e}^{\mathrm{i}\theta}\right|^2 = \left[\left|\frac{1}{\lambda} - 1\right| + \frac{t^2}{|\lambda|}\right]^2$. For this $Z$,

$$|\hat{A}(Z)|_2 \geqslant \max\left\{\sqrt{t^2 + \left(\left|\frac{1}{\lambda} - 1\right| + \frac{t^2}{|\lambda|}\right)^2}, \sqrt{|\lambda - 1|^2 + t^2}\right\},$$

i.e., $\|\hat{A}\| > \max\{|\frac{1}{\lambda} - 1|, |\lambda - 1|\}$.

If $t = 0$, obviously, the equality in inequality (1.4) holds. For more discussions in this direction, see Theorem 4.1 and Corollary 4.1.

Since a square matrix is diagonalizable under a unitary matrix if and only if it is normal, we can state the above as follows.

**Theorem 2.1.** *For any matrix $A \in GL_2(\mathbf{C})$, the equality in inequality* (1.4) *holds if and only if $A$ is normal.*

**Remark 2.2.** It follows from Theorem 2.1 that the answer to Question 1 is positive when $n = 2$.

## 3. Three lemmas

For $x = (x_1, \ldots, x_n)^{\mathrm{T}} \in \mathbf{C}^n$, let $|x|_2 = (\sum_{i=1}^n |x_i|^2)^{1/2}$. From the well known characterization:

$$|Z|_2 = \max_{x,y\in\mathbf{C}^n, |x|_2, |y|_2 \leqslant 1} |x^* Z y|,$$

it follows

**Lemma 3.1.** *Let $Z = (z_{ij})_1^n = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix} \in M_n(\mathbf{C})$ and $Z_1 \in M_t(\mathbf{C})$. If $|Z|_2 \leqslant 1$, then*

$$|Z_i|_2 \leqslant 1, \quad i = 1, \ldots, 4.$$

**Lemma 3.2.** *Let $D = \begin{pmatrix} a & 0 \\ c & b \end{pmatrix} \in M_2(\mathbf{C})$ and the set*

$$M = \left\{ Z = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : -\pi \leqslant \theta \leqslant \pi \right\},$$

*where* $|b| \leqslant |c|$ *and* $|a| \leqslant |c|$. *Then,*

(i) *if* $|a|^2 + |b|^2 \leqslant |c|^2$, *we have, for any* $Z \in M$, $|D \odot Z|_2 \leqslant |c|$. *The equality holds if and only if* $\cos\theta = 0$;

(ii) *if* $|a|^2 + |b|^2 > |c|^2$, *we have*

$$\max_{Z \in M} |D \odot Z|_2 \geqslant \frac{2|ab|}{\sqrt{4|ab|^2 - (|a|^2 + |b|^2 - |c|^2)^2}} |c| > |c|.$$

*Furthermore, if* $|c|^2(|a|^2 + |b|^2 - |c|^2) \geqslant |ab|^2$, *then for any* $Z \in M$, $|D \odot Z|_2 \geqslant |c|$, *the equality holds if and only if* $\cos\theta = 0$, *and if* $|c|^2(|a|^2 + |b|^2 - |c|^2) < |ab|^2$, *then for all* $\theta$ $(\in [-\pi, \pi])$ *which satisfy*

$$0 \neq \cos^2\theta < \frac{|c|^2(|a|^2 + |b|^2 - |c|^2)}{|ab|^2},$$

*we have* $|D \odot Z|_2 > |c|$.

**Proof.** For any $Z = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, a simple computation shows that

$$B = (D \odot Z)(D \odot Z)^* = \begin{pmatrix} |a|^2\cos^2\theta & a\bar{c}\sin\theta\cos\theta \\ \bar{a}c\sin\theta\cos\theta & |b|^2\cos^2\theta + |c|^2\sin^2\theta \end{pmatrix}.$$

The characteristic polynomial of $B$ is

$$\lambda^2 - (|a|^2\cos^2\theta + |b|^2\cos^2\theta + |c|^2\sin^2\theta)\lambda + |ab|^2\cos^4\theta = 0.$$

Obviously, the maximal eigenvalue of $B$ is

$$\lambda_{\max} = \tfrac{1}{2}|c|^2 + \tfrac{1}{2}(|a|^2 + |b|^2 - |c|^2)\cos^2\theta + \tfrac{1}{2}\sqrt{[|c|^2 + (|a|^2 + |b|^2 - |c|^2)\cos^2\theta]^2 - 4|ab|^2\cos^4\theta}.$$

If $|a|^2 + |b|^2 \leqslant |c|^2$, then

$$\sqrt{[|c|^2 + (|a|^2 + |b|^2 - |c|^2)\cos^2\theta]^2 - 4|ab|^2\cos^4\theta} \leqslant |c|^2.$$

It follows that

$$\lambda_{\max} \leqslant \tfrac{1}{2}|c|^2 + \tfrac{1}{2}(|a|^2 + |b|^2 - |c|^2)\cos^2\theta + \tfrac{1}{2}|c|^2 \leqslant |c|^2.$$

This implies that $|D \odot Z|_2 \leqslant |c|$ holds for any $Z \in M_2(\mathbf{C})$. This proves (i).

We now come to consider the second case $|a|^2 + |b|^2 > |c|^2$.

Let $T^2 = |a|^2 + |b|^2 - |c|^2$ and

$$L = (|c|^2 + T^2\cos^2\theta)^2 - 4|ab|^2\cos^4\theta - (|c|^2 - T^2\cos^2\theta)^2.$$

Then

$$L = 4\cos^2\theta(|c|^2T^2 - |ab|^2\cos^2\theta).$$

If $|c|^2T^2 \geqslant |ab|^2$, then for any $Z \in M$, $L \geqslant 0$, i.e.,

$$\lambda_{\max} \geqslant \tfrac{1}{2}|c|^2 + \tfrac{1}{2}T^2\cos^2\theta + \tfrac{1}{2}(|c|^2 - T^2\cos^2\theta) = |c|^2.$$

Hence $|D \odot Z|_2 \geqslant |c|$.

If $|c|^2T^2 < |ab|^2$, we can choose a $\theta$ in the interval $[-\pi, \pi]$ such that

$$0 \neq \cos^2\theta < \frac{|c|^2T^2}{|ab|^2}.$$

For this $\theta$, we know $L > 0$. This implies that $\lambda_{\max} > |c|^2$.

We now come to find the maximal value of $\lambda_{\max}$ in the set $M$.

Let

$$f(\theta) = \tfrac{1}{2}|c|^2 + \tfrac{1}{2}T^2\cos^2\theta + \tfrac{1}{2}\sqrt{(|c|^2 + T^2\cos^2\theta)^2 - 4|ab|^2\cos^4\theta}.$$

Then

$$\frac{\partial f}{\partial\theta} = -\frac{1}{2}\left[T^2 + \frac{T^2(|c|^2 + T^2\cos^2\theta) - 4|ab|^2\cos^2\theta}{\sqrt{(|c|^2 + T^2\cos^2\theta)^2 - 4|ab|^2\cos^4\theta}}\right]\sin 2\theta.$$

A straightforward computation shows that all solutions to the equation

$$\frac{\partial f}{\partial\theta} = 0$$

are

$$\cos\theta = 0,\ \cos^2\theta = 1,\ \cos^2\theta = \frac{2|c|^2T^2}{4|ab|^2 - T^4}.$$

These imply that the maximal value of $\lambda_{\max}$ is

$$\begin{aligned}\lambda_{\max} &= \tfrac{1}{2}|c|^2 + \tfrac{1}{2}T^2\cos^2\theta + \tfrac{1}{2}\sqrt{(|c|^2 + T^2\cos^2\theta)^2 - 4|ab|^2\cos^4\theta}\\ &= \frac{4|abc|^2}{4|ab|^2 - T^4}\\ &= \frac{4|abc|^2}{4|ab|^2 - (|a|^2 + |b|^2 - |c|^2)^2}.\end{aligned}$$

The proof of (ii) is completed. □

The following lemma can be established in a similar manner as above.

**Lemma 3.3.** *Let* $D = \begin{pmatrix} 0 & a \\ c & b \end{pmatrix} \in M_2(\mathbf{C})$ *and the set*

$$M = \left\{ Z = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}:\ -\pi \leqslant \theta \leqslant \pi \right\},$$

*where* $|b| \leqslant |c|, \quad |a| \leqslant |c|$ *and* $\quad |a|^2 + |b|^2 > |c|^2$. *Then*

$$\begin{aligned}\max_{Z\in M} |D \odot Z|_2 &\geqslant \frac{2|ab|}{\sqrt{4|ac|^2 - (|a|^2 + |c|^2 - |b|^2)^2}}|c| \\ &= \frac{2|ab|}{\sqrt{4|ab|^2 - (|a|^2 + |b|^2 - |c|^2)^2}}|c| > |c|.\end{aligned}$$

*Furthermore, if* $|b|^2(|a|^2 + |c|^2 - |b|^2) \geqslant |ac|^2$, *then for any* $Z \in M$, $|D \odot Z|_2 \geqslant |c|$, *the equality holds if and only if* $\sin\theta = 0$, *and if* $|b|^2(|a|^2 + |c|^2 - |b|^2) < |ac|^2$, *then for all* $\theta$ *which satisfy*

$$0 \neq \sin^2\theta < \frac{|b|^2(|a|^2 + |c|^2 - |b|^2)}{|ac|^2},$$

*we have* $|D \odot Z|_2 > |c|$.

## 4. Main theorems

The counterexamples in Section 2 show that the equality in inequality (1.4) does not always hold $n \geqslant 3$. In this section, first, we investigate the condition under which the equality in inequality (1.4) holds; then we find some condition under which the strict inequality in inequality (1.4) holds.

For $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in GL_n(\mathbf{C})$, if there are $t$ different elements among $\lambda_i$'s ($i = 1, \ldots, n$), without loss of generality, we assume that they are $\lambda_1, \ldots, \lambda_t$. Then we denote $t$ by $\mathrm{Card}(A)$ and define $\mathrm{Max}(A) \in GL_t(\mathbf{C})$ as follows

$$\mathrm{Max}(A) = \mathrm{diag}(\lambda_1, \ldots, \lambda_t).$$

When $\mathrm{Card}(A) \leqslant 2$, we will prove the following result which is a partly generalization of Theorem 2.1.

**Theorem 4.1.** *Let* $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in GL_n(\mathbf{C})$. *If* $\mathrm{Card}(A) \leqslant 2$, *then*

$$\|\hat{A}\| = \max_{1\leqslant i,j\leqslant n} \left|\frac{\lambda_i}{\lambda_j} - 1\right|.$$

**Proof.** If $\mathrm{Card}(A) = 1$, then $\hat{A}(Z) = 0$ and $\|\hat{A}\| = \max_{1\leqslant i,j\leqslant n} \left|(\lambda_i/\lambda_j) - 1\right| = 0$. Our equality follows.

If $\mathrm{Card}(A) = 2$, then, without loss of generality, we may assume that the first $r$ diagonal entries of $A$ are $\lambda_1$ and the rest are $\lambda_2$. Then

$$\Omega_A = \begin{pmatrix} 0 & \Omega_1 \\ \Omega_2 & 0 \end{pmatrix},$$

where

$$\Omega_1 = \left(\frac{\lambda_1}{\lambda_2} - 1\right)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

is a $r$-by-$(n-r)$ matrix and

$$\Omega_2 = \left(\frac{\lambda_2}{\lambda_1} - 1\right)\begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \cdots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}$$

is an $(n-r)$-by-$r$ matrix.

Let $Z = (z_{ij})_1^n = \begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}$, where $Z_1 \in M_r(\mathbf{C})$. Then

$$\hat{A}(Z) = \begin{pmatrix} 0 & \left(\frac{\lambda_1}{\lambda_2} - 1\right) Z_2 \\ \left(\frac{\lambda_2}{\lambda_1} - 1\right) Z_3 & 0 \end{pmatrix}$$

and

$$\hat{A}(Z)\hat{A}(Z)^* = \begin{pmatrix} \left|\frac{\lambda_1}{\lambda_2} - 1\right|^2 Z_2 Z_2^* & 0 \\ 0 & \left|\frac{\lambda_2}{\lambda_1} - 1\right|^2 Z_3 Z_3^* \end{pmatrix}.$$

It follows from Lemma 3.1 that

$$\|\hat{A}\| = \sup_{|Z|_2 \leqslant 1} \left\{ \left|\frac{\lambda_2}{\lambda_1} - 1\right| \sqrt{\rho(Z_2 Z_2^*)}, \right. \left. \left|\frac{\lambda_1}{\lambda_2} - 1\right| \sqrt{\rho(Z_3 Z_3^*)} \right\} \leqslant \max\left\{ \left|\frac{\lambda_2}{\lambda_1} - 1\right|, \left|\frac{\lambda_1}{\lambda_2} - 1\right| \right\}.$$

By theorem FH, $\|\hat{A}\| = \max_{1\leqslant i,j\leqslant n} \left|\frac{\lambda_i}{\lambda_j} - 1\right|$.

This completes the proof. □

**Remark 4.1.** When $n = 2$, Theorem 4.1 implies that the answer to Question 1 is positive.

For $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in GL_n(\mathbf{C})$, we assume that $\operatorname{Card}(A) \geqslant 3$ in the following.

From the definitions of $\Omega_A$ and $\sigma$ it follows that there exists a pair $(p, q)$ such that $\sigma = |\sigma_{pq}| = |(\lambda_p/\lambda_q) - 1|$. We denote the set of all such pair(s) by $P(A)$. For a pair $(p, q) \in P(A)$, then there exists $\lambda_t \neq \lambda_p,\ \lambda_q$ since Card(A)$\geqslant 3$.

Let

$$A(t)_{pq} = \operatorname{diag}(\lambda_p,\ \lambda_t\ \lambda_q\ \lambda_{i_1}\ \cdots\ \lambda_{i_{n-3}})$$

be obtained from $A$ by reordering its diagonal entries.

In $\Omega_{A(t)_{pq}} = (\sigma_{ij}(t, p, q))_1^n$, let

$$\sigma(t, p, q) = \max\{|\sigma_{21}(t, p, q)|^2 + |\sigma_{32}(t, p, q)|^2, |\sigma_{12}(t, p, q)|^2 + |\sigma_{32}(t, p, q)|^2\}.$$

**Condition CW**. For $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, we say that $A$ satisfies *Condition CW* if

$$\max_{t \neq p,q,(p,q)\in P(A)} \sigma(t, p, q) > \sigma^2.$$

If $A$ satisfies *Condition CW*, let

$$\sigma(A)_1 = \max_{t \neq p,q,(p,q)\in P(A)} \frac{2|\sigma_{21}(t, p, q)\sigma_{32}(t, p, q)|\sigma}{\sqrt{4|\sigma_{21}(t, p, q)\sigma_{32}(t, p, q)|^2 - (|\sigma_{21}(t, p, q)|^2 + |\sigma_{32}(t, p, q)|^2 - \sigma^2)^2}},$$

if $|\sigma_{21}(t, p, q)|^2 + |\sigma_{32}(t, p, q)|^2 > \sigma^2$, and

$$\sigma(A)_2 = \max_{t \neq p,q,(p,q)\in P(A)} \frac{2|\sigma_{12}(t, p, q)\sigma_{32}(t, p, q)|\sigma}{\sqrt{4|\sigma_{12}(t, p, q)\sigma_{32}(t, p, q)|^2 - (|\sigma_{12}(t, p, q)|^2 + |\sigma_{32}(t, p, q)|^2 - \sigma^2)^2}},$$

if $|\sigma_{12}(t, p, q)|^2 + |\sigma_{32}(t, p, q)|^2 > \sigma^2$.

Let $\sigma(A) = \max\{\sigma(A)_1, \quad \sigma(A)_2\}$.

**Theorem 4.2.** *Let* $A = \mathrm{diag}(\lambda_1, \ldots, \lambda_n) \in GL_n(\mathbf{C})$. *If* $\mathrm{Card}(A) \geqslant 3$ *and* $A$ *satisfies Condition CW*, *then*

$$\|\hat{A}\| > \max_{1\leqslant i,j\leqslant n} \left|\frac{\lambda_i}{\lambda_j} - 1\right|.$$

**Proof.** Since $A$ satisfies *Condition CW*, without loss of generality, we may assume that $|\sigma_{21}|^2 + |\sigma_{32}|^2 > |\sigma_{31}|^2 = \sigma^2$ or $|\sigma_{12}|^2 + |\sigma_{32}|^2 > |\sigma_{31}|^2 = \sigma^2$.

For the first case, let

$$Z = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \cos\theta & -\sin\theta & 0 & \cdots & 0 \\ \sin\theta & \cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Then

$$\hat{A}(Z) = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ \sigma_{21}\cos\theta & 0 & 0 & \cdots & 0 \\ \sigma_{31}\sin\theta & \sigma_{32}\cos\theta & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

For this $Z$, the spectral norm of matrix $\hat{A}(Z)$ is equal to the spectral norm of matrix

$$\begin{pmatrix} \sigma_{21}\cos\theta & 0 \\ \sigma_{31}\sin\theta & \sigma_{32}\cos\theta \end{pmatrix}.$$

By Lemma 3.2, we know

$$|\hat{A}(Z)|_2 \geqslant \frac{2|\sigma_{21}\sigma_{32}|\sigma}{\sqrt{4|\sigma_{21}\sigma_{32}|^2 - (|\sigma_{21}|^2 + |\sigma_{32}|^2 - \sigma^2)^2}} > \sigma.$$

For the second case, by Lemma 3.3, we know

$$|\hat{A}(Z)|_2 \geqslant \frac{2|\sigma_{12}\sigma_{32}|\sigma}{\sqrt{4|\sigma_{12}\sigma_{32}|^2 - (|\sigma_{12}|^2 + |\sigma_{32}|^2 - \sigma^2)^2}} > \sigma.$$

The discussions as stated above imply that

$$\|\hat{A}\| \geqslant \sigma(A) > \max_{1\leqslant i,j\leqslant n} \left|\frac{\lambda_i}{\lambda_j} - 1\right|.$$

The proof is completed. □

For any $B = (b_{ij})_1^n \in M_n(\mathbf{C})$, we define $|B| = (|b_{ij}|)_1^n$.
From Lemma 2.1 and Theorem 4.2 we deduce:

**Corollary 4.1.** *Let* $A = \operatorname{diag}(\lambda_1, \ldots, \lambda_n) \in M_n(\mathbf{C})$. *If* $t = \operatorname{Card}(A) \geqslant 3$ *and* $\sigma$ *appears at least two times in some column or row of the matrix* $|\Omega_{\max(A)}|$, *then*

$$\|\hat{A}\| > \max_{1\leqslant i,j\leqslant t} \left|\frac{\lambda_i}{\lambda_j} - 1\right|.$$

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