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On the complexity of the dominating induced matching problem in hereditary classes of graphs[☆]

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ABSTRACT

The DOMINATING INDUCED MATCHING problem, also known as EFFICIENT EDGE DOMINATION, is the problem of determining whether a graph has an induced matching that dominates every edge of the graph. This problem is known to be NP-complete. We study the computational complexity of the problem in special graph classes. In the present paper, we identify a critical class for this problem (i.e., a class lying on a "boundary" separating difficult instances of the problem from polynomially solvable ones) and derive a number of polynomial-time results. In particular, we develop polynomial-time algorithms to solve the problem for claw-free graphs and convex graphs.

1. Introduction

Let G be a simple graph, i.e, an undirected graph without loops and multiple edges. Given an edge e in G, we say that e dominates itself and every edge sharing a vertex with e. An *induced matching* in G is a subset of edges such that each edge of G is dominated by at most one edge of the subset. In this paper we study the problem of determining whether a graph has a dominating induced matching, i.e., an induced matching that dominates *every* edge of the graph. This problem is also known in the literature as EFFICIENT EDGE DOMINATION. Alternatively, the problem can be viewed as a restricted version of VERTEX 3-COLORABILITY, i.e., the problem of determining whether the vertices of a graph can be partitioned into three independent sets. In the DOMINATING INDUCED MATCHING problem we are looking for a partition of a graph into three independent sets such that two of them induce a 1-regular graph.

One more related problem is that of finding in a graph, an induced matching of maximum cardinality. Recently, it was shown in [8] that an induced matching in a graph is dominating only if it is maximum in terms of its size. Finding a maximum induced matching is a well-studied problem, which is NP-hard in general graphs and in many particular classes such as bipartite graphs of degree at most three [22] or line graphs [19]. On the other hand, the problem is known to be polynomial-time solvable for chordal graphs and interval graphs [10], circular-arc graphs [15], weakly chordal graphs [12], convex graphs [5] and many other special classes (see e.g. [11,13,16,19]).

The complexity of the DOMINATING INDUCED MATCHING problem in special graph classes is less explored. It is known that the problem is NP-complete in general [18] and in some particular classes such as planar bipartite graphs [25] and d-regular graphs [8] (see also [21] for the case $d = 3$). Polynomial-time solutions are available only for bipartite permutation [26] and chordal graphs [25].

[☆] Parts of this paper appeared as extended abstracts in the proceedings of two conferences: "Graph Theory, Computational Intelligence and Thought" dedicated to Martin Charles Golumbic on the occasion of his 60th Birthday (Cardoso and Lozin, 2009 [9]) and the DIMAP Workshop on Algorithmic Graph Theory (Korpelainen, 2009 [20]).

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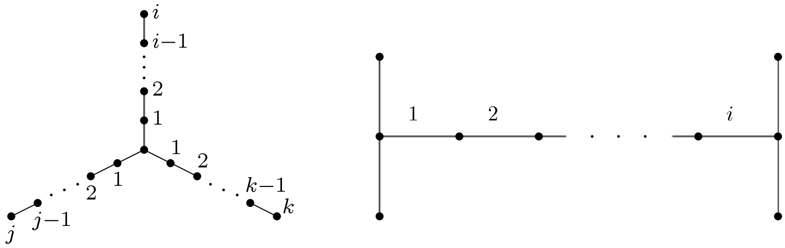

Fig. 1. Graphs $S_{i,j,k}$ (left) and H_i (right).

A helpful tool for a systematic study of an algorithmic problem in special graph classes is the notion of a boundary class. This notion was recently introduced with respect to the MAXIMUM INDEPENDENT SET problem [1] and it was then applied to some other algorithmic problems on graphs [2,3]. In the present paper, we employ this notion for the study of the DOMINATING INDUCED MATCHING problem. In Section 3, we identify the first boundary class for this problem. Whether or not this class is unique remains an open question. Analyzing this question, we derive in Section 4 a number of polynomial-time results. In particular, we show how to solve the problem for convex graphs and claw-free graphs. It is interesting to note that in the abstract of paper [18], the authors mistakenly claimed the NP-completeness of the EFFICIENT EDGE DOMINATION problem in the class of line graphs (a proper subclass of claw-free graphs). Our solution to the problem in the class of claw-free graphs corrects this wrong statement.

All preliminary information related to the topic of the paper, including the notion of a boundary class of graphs, is given in the next section.

2. Preliminaries

All graphs in this paper are finite, and without loops or multiple edges. For a graph G, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. The neighborhood of a vertex $v \in V(G)$ (i.e., the set of vertices adjacent to v) is denoted $N(v)$. The degree of v is the number of its neighbors. A graph is k-regular if the degree of each vertex is k. An independent set in G is a subset of pairwise nonadjacent vertices. As usual, K_n, P_n and C_n are, respectively, the complete graph, chordless path and chordless cycle on n vertices, and $K_{n,m}$ is the complete bipartite graph with parts of size n and m. In particular, $K_{1,3}$ is a claw. By $G + H$ we denote the disjoint union of two graphs G and H. In particular, $mG = G + \cdots + G$ is the disjoint union of m copies of G. Also, $S_{i,j,k}$ and H_i are the two graphs represented in Fig. 1.

For a subset of vertices $U \subseteq V(G)$, we denote by $G[U]$ the subgraph of G induced by U, i.e., the subgraph of G with vertex set U and two vertices being adjacent in $G[U]$ if and only if they are adjacent in G. We say that a graph H is an induced subgraph of G if H is isomorphic to $G[U]$ for some $U \subseteq V(G)$.

A class of graphs is *hereditary* if whenever it contains a graph G, it also contains all the induced subgraphs of G. It is known that a graph class is hereditary if and only if it can be characterized in terms of forbidden induced subgraphs. More formally, given a set of graphs M, we say that a graph G is M-free if G does not contain induced subgraphs from the set M. The class of all M-free graphs will be denoted $Free(M)$. In particular, $Free(C_4, C_5, C_6, \ldots)$ is the class of chordal graphs and $Free(C_3, C_5, C_7, \ldots)$ is the class of bipartite graphs. It is known that a class X of graphs is hereditary if and only if $X = Free(M)$ for some set M. Observe that any set M uniquely defines the class $Free(M)$. On the other hand, for a hereditary class X the set M of forbidden induced subgraphs is generally not unique. For instance, M can be defined as the set of all graphs that are not in X. However, to describe X one can restrict to the set of minimal forbidden induced subgraphs, which exists and is unique for any hereditary class.

A graph class X will be called *DIM-tough* if there is no polynomial-time algorithm to solve the DOMINATING INDUCED MATCHING problem for graphs in X. If $P \neq NP$, the family of *DIM*-tough classes is non-empty, in which case the problem of characterization of the family of graph classes with polynomial-time solvable DOMINATING INDUCED MATCHING problem arises. By analogy with the induced subgraph characterization of hereditary classes, we want to characterize this family in terms of minimal classes that do not belong to it. Unfortunately, a *DIM*-tough class may contain infinitely many *DIM*-tough subclasses, which makes the task of finding minimal *DIM*-tough classes impossible. To overcome this difficulty, we employ the notion of a boundary class, which can be defined as follows.

A class of graphs X will be called a *limit class* for the DOMINATING INDUCED MATCHING problem if $X = \bigcap_{i=1}^{\infty} X_i$, where $X_1 \supseteq X_2 \supseteq \cdots$ is a sequence of *DIM*-tough classes. A minimal limit class will be called a *boundary* class for the problem in question.

3. The boundary class

Throughout the rest of the paper we denote by $\mathcal{S}_k$ the class of $(C_3, \ldots, C_k, H_1, \ldots, H_k)$-free bipartite graphs of vertex degree at most 3 and by $\mathcal{S}$ the intersection $\bigcap_{k\geq 0} \mathcal{S}_k$.

The main result of this section is that the class $\mathcal{S}$ is a boundary class for the DOMINATING INDUCED MATCHING problem. First, in Section 3.1 we show that $\mathcal{S}$ is a limit class for the problem and then in Section 3.2 we prove its minimality.

3.1. Approaching a limit class

From [18] we know that determining if G has a dominating induced matching is an NP-complete problem. Moreover, it is NP-complete even for bipartite graphs [25] and graphs of vertex degree at most three [21]. In this section, we strengthen these results by showing that the problem is NP-complete in the class $\mathcal{S}_k$ for any value of k. To this end, let us first present the following technical lemma.

Lemma 1. *Let G be a graph and e an edge in G. If G' is the graph obtained from G by subdividing the edge e exactly three times, then G has a dominating induced matching if and only if G' has.*

Proof. Denote the endpoints of e by a and b, and the three vertices subdividing the edge e by x, y, z. Assume first that G has a dominating induced matching M. If $e = ab \in M$, then the set $M' = M \cup \{ax, zb\}$ is a dominating induced matching in G'. If $e = ab \notin M$ and e is dominated by a certain edge of M incident to a, then $M' = M \cup \{yz\}$ is a dominating induced matching in G'.

Conversely, suppose G' has a dominating induced matching M'. If neither xy nor yz belong to M', then $ax, zb \in M'$ and hence $M = (M' - \{ax, zb\}) \cup \{ab\}$ is a dominating induced matching in G. Assume now without loss of generality that $yz \in M'$. Then the set $M = M' - \{yz\}$ is a dominating induced matching in G. □

A direct consequence of this lemma is the following result.

Lemma 2. *For any k, the* DOMINATING INDUCED MATCHING *problem is NP-complete in the class $\mathcal{S}_k$.*

Proof. We prove the lemma by reducing the problem from graphs of vertex degree at most three, where the problem is known to be NP-complete.

Let G be a graph of vertex degree at most 3 and G' a graph obtained from G by a triple subdivision of an edge of G. Then G' is also of degree at most three and it has a dominating induced matching if and only if G has. If we subdivide each edge of G three times, then we obtain a bipartite graph, since the length of each edge of G increases 4 times. Applying this operation repeatedly, we can get rid of small induced cycles and small induced graphs of the form H_i. The resulting graph is bipartite, of maximum degree three and it has a dominating induced matching if and only if G has. This proves the lemma. □

Lemma 2 implies that $\mathcal{S}_k$ is a *DIM*-tough class for any k. Therefore, $\mathcal{S} = \bigcap_{k\geq 0} \mathcal{S}_k$ is a limit class for the DOMINATING INDUCED MATCHING problem. In the next section, we show that $\mathcal{S}$ is a minimal limit class for this problem.

3.2. Minimality of the limit class

In general, the proof of minimality is not a trivial task. However, for the class $\mathcal{S}$ some helpful tools have been developed in [2]. In particular, this paper proves the following lemma, where a monotone class is a hereditary class closed under deletion of edges from graphs in the class.

Lemma 3. *Let Π be an NP-hard graph problem polynomial-time solvable for graphs in any monotone class X such that $\mathcal{S} \not\subseteq X$. Then $\mathcal{S}$ is a boundary class for Π whenever it is a limit class for the problem.*

In order to show that the DOMINATING INDUCED MATCHING problem is polynomial-time solvable for graphs in any monotone class X such that $\mathcal{S} \not\subseteq X$, we will use the following result from [4].

Lemma 4. *If X is a monotone graph class such that $\mathcal{S} \not\subseteq X$, then the clique-width of graphs in X is bounded by a constant.*

Now all we have to do to prove the minimality of the class $\mathcal{S}$ for the DOMINATING INDUCED MATCHING problem is to show that the problem is polynomial-time solvable for graphs of bounded clique-width.

Lemma 5. *The* DOMINATING INDUCED MATCHING *problem can be solved in polynomial time in any class of graphs where clique-width is bounded by a constant.*

Proof. In [14], it was shown that any decision problem expressible in MSOL(τ_1) (Monadic Second-Order Logic with quantification over subsets of vertices, but not of edges) can be solved in linear time in any class of graphs of bounded clique-width. The DOMINATING INDUCED MATCHING problem can be expressed in MSOL(τ_1) in the following way:

$$\exists B, W(Partition(B, W) \wedge InducedMatching(B) \wedge IndependentSet(W)),$$

where $Partition(B, W)$, $InducedMatching(B)$ and $IndependentSet(W)$ are defined by

$$Partition(B, W) = \forall v(B(v) \vee W(v)) \wedge \neg\exists u(B(u) \wedge W(u)),$$
$$IndependentSet(W) = \forall u, v((W(u) \wedge W(v)) \rightarrow \neg\exists E(u, v)),$$
$$InducedMatching(B) = \forall u(B(u) \rightarrow \exists! v(B(v) \wedge E(u, v))). \quad \square$$

Summarizing the above discussion we conclude that

Theorem 1. *The class $\mathcal{S}$ is a boundary class for the* DOMINATING INDUCED MATCHING *problem.*

4. Polynomial-time results

In this section, we attack the problem from the polynomial side. Some partial results of this type follow from Lemma 5 proved in the previous section. It is known that the clique-width is bounded for P_4-free graphs and some of their generalizations [27], distance-hereditary graphs [17], and some other classes (see e.g. [24]). Together with Lemma 5, this implies polynomial-time solvability of the problem in all those classes. On the other hand, let us observe that boundedness of the clique-width is sufficient but not necessary for polynomial-time solvability of the problem. Indeed, the clique-width is bounded neither in chordal graphs [27] nor in bipartite permutation graphs [7], the only two previously known classes with polynomial-time solvable DOMINATING INDUCED MATCHING problem. The NP-completeness result proved in the previous section suggests directions for further steps in the search for new classes where the problem is tractable.

Unless $P = NP$, according to Lemma 2 the problem is solvable in polynomial time in a class of graphs $X = Free(M)$ only if X excludes graphs from all classes $\mathcal{S}_k$, i.e., only if

$$M \cap \mathcal{S}_k \neq \emptyset \quad \text{for each } k. \tag{1}$$

On the other hand, if the problem is solvable in polynomial time in any class $X = Free(M)$ satisfying (1) then obviously $\mathcal{S}$ is the only boundary class for the problem. Proving or disproving uniqueness of the class $\mathcal{S}$ is a challenging research problem. In this section, we restrict ourselves to distinguishing three major ways to satisfy (1).

One way to satisfy (1) is to include in M a graph G belonging to $\mathcal{S}$, which means G has no induced cycles, no induced graphs of the form H_i and no vertices of degree more than three. In other words, every connected component of G is of the form $S_{i,j,k}$ represented in Fig. 1. In Section 4.1, we study the class of $S_{1,1,1}$-free graphs, also known as the claw-free graphs, and prove that the problem is solvable in polynomial time in this class.

If we do not include in M a graph $G \in \mathcal{S}$, then to satisfy (1) M must contain infinitely many graphs. Two basic ways to satisfy (1) with infinitely many graphs are $M \supseteq \{C_p, C_{p+1}, \ldots\}$ and $M \supseteq \{H_p, H_{p+1}, \ldots\}$ for a constant p. Both polynomially solvable cases mentioned in the introduction (bipartite permutation [26] and chordal graphs [25]) deal with graphs that do not contain large induced cycles. In Section 4.2, we present one more result of this type by extending polynomial-time solvability of the problem from the class of bipartite permutation graphs to the class of convex graphs. Finally, in Section 4.3, we consider classes $Free(M)$ with $M \supseteq \{H_p, H_{p+1}, \ldots\}$ and prove solvability of the problem in such classes whenever the degree of vertices is bounded by a constant.

In our solutions, we will use an alternative definition of the DOMINATING INDUCED MATCHING problem which asks to determine if the vertex set of a graph G admits a partition into two subsets W and B such that W is an independent set and B induces a 1-regular graph. Throughout the section we will call the vertices of W *white* and the vertices of B *black*, and the partition $V(G) = B \cup W$ *black–white partition* of G. In other words, a graph G has a dominating induced matching if and only if G admits a black–white partition. We will use these two notions interchangeably.

An assignment of one of the two possible colors to each vertex of G will be called a *coloring* of G. A coloring is *partial* if only part of the vertices of G are assigned colors, otherwise it is *total*. A partial coloring is *valid* if no two white vertices are adjacent and no black vertex has more than one black neighbor. A total coloring is *valid* if no two white vertices are adjacent and every black vertex has exactly one black neighbor.

Before we proceed to solutions in particular classes of graphs, let us make a few observations valid for arbitrary graphs. First, without loss of generality we will assume that

(A1) all of our graphs are connected, because for a disconnected graph G the problem is solvable if and only if it is solvable for every connected component of G.

We can also assume that

(A2) G has no induced path with three consecutive vertices of degree 2, because any three consecutive vertices of degree 2 can be replaced by an edge and the modified graph has a dominating induced matching if and only if the original one has (Lemma 1).

The assumption A2 implies in particular that any vertex of degree 1 is connected to the nearest vertex of degree more than 2 by a chordless path of length at most 3. Moreover, it is not difficult to see that if the length of the path is 3, we can delete this path and the new graph has a dominating induced matching if and only if the original one has. Therefore, in what follows we assume that

(A3) any vertex of degree 1 is connected to the nearest vertex of degree more than 2 by a chordless path of length at most 2.

Definition 1. A vertex of degree 1 will be called a leaf and the only neighbor of a leaf will be called a preleaf.

It is not difficult to see that

Lemma 6. *In any black–white partition of G, each preleaf is black.*

This simple observation shows that analysis of local properties of a graph G may lead to a partial coloring of G. With a more involved analysis, some stronger conclusions can be made. In particular, we will frequently refer to the following two lemmas.

Lemma 7. *If two triangles share a single vertex, then this vertex must be colored white. If two triangles share two vertices, then both of these vertices must be colored black.*

Proof. Assume two triangles share a single vertex v. If v were to be black, then either v has 2 black neighbors (one in each triangle) or one of the triangles has two adjacent white vertices. Contradiction in both cases shows that v must be white.

Now assume two triangles share two vertices u and v. If u is colored white, then obviously the remaining vertices in both triangles must be colored black. But then the black vertex v has two black neighbors. This contradiction shows that u must be colored black. By symmetry, v also must be colored black. □

Lemma 8. *If a graph G has a dominating induced matching, then the neighborhood of each vertex of G induces a subgraph each connected component of which is a star $K_{1,s}$ for some s.*

Proof. Let v be a vertex in a graph G with a dominating induced matching. Then $G[N(v)]$ is K_3-free, since otherwise G is not 3-colorable (and hence has no dominating induced matching).

Assume $G[N(v)]$ contains an induced $P_4 = (a, b, c, d)$. Then v is not white, since otherwise the vertices a, b, c, d are all black, which is not possible in a valid black–white partition. If v is black, then at most one of the vertices a, b, c, d is black and then at least three are white with two of these three connected by an edge. This contradiction shows that $G[N(v)]$ is P_4-free and hence C_i-free for all $i \geq 5$.

Similarly, we can show that $G[N(v)]$ is C_4-free. Therefore, $G[N(v)]$ is a forest. Since $G[N(v)]$ is P_4-free, each connected component of this forest is a star $K_{1,s}$ for some s. □

Application of the above lemmas may lead either to the conclusion that the input graph has no dominating induced matching or to a partial coloring of the graph. We will assume that any partial coloring is maximal (i.e., cannot be extended to a larger coloring) under some simple rules. The three obvious rules are

R1: each neighbor of a white vertex must be colored black;
R2: all neighbors of two black adjacent vertices must be colored white;
R3: each vertex that has two black neighbors (not necessarily adjacent) must be colored white.

Three other rules that will be used in our solutions are not so obvious, but are also simple:

R4: if a vertex v belongs to a triangle T and has a neighbor w outside T, then v and w must be colored differently;
R5: in any induced C_4, any two adjacent vertices must be colored differently;
R6: if a preleaf v is adjacent to more than one leaf, then all but one leaf adjacent to v can be colored white.

The main strategy in all our polynomial-time solutions is the following. The algorithm starts by finding an initial partial coloring of the input graph G by analyzing local properties of G. Then the algorithm incrementally extends the partial coloring by application of the above rules and some more specific considerations. At each step of the algorithm, we delete from G those colored vertices that have no neighbors among uncolored ones (as they have no importance for the completion of the procedure) and denote the resulting graph G_0. By Rule R1 any colored vertex of G_0 is black and by Rule R2 the set of colored vertices of G_0 is independent. Application of the above strategy either leads to a conflict (two adjacent vertices colored white or a black vertex with more than one black neighbor) or reduces the problem to a graph G_0 for which the solution is simple.

4.1. Dominating induced matchings in claw-free graphs

Let G be a claw-free graph. If G has no vertices of degree more than 2, the problem is trivial for G: it can be solved either by Lemma 1 (which reduces the problem to connected graphs with at most 5 vertices) or by Lemma 5 (as the clique-width of G is at most 4 in this case). This observation allows us to assume that the maximum vertex degree in G is at least 3. The following lemma shows that we also may assume that the maximum vertex degree in G is at most 4.

Lemma 9. *If a claw-free graph G has a vertex of degree more than 4, then G has no dominating induced matching.*

Proof. Let v be a vertex of degree more than 4 and assume by contradiction that G has a dominating induced matching. From Lemma 8 we know that $G[N(v)]$ is a forest each connected component of which is a star $K_{1,s}$ for some s. Since G is claw-free, the number of components is at most 2 and for each component we have $s \leq 2$. Moreover, to avoid a claw, we conclude that if $G[N(v)]$ has a component $K_{1,2}$, then it has no other components, i.e., the degree of v is 3. If each component has at most 2 vertices, then the degree of v is at most 4. This contradiction completes the proof of the lemma. □

From Lemmas 8 and 9, we conclude that a claw-free graph G has a dominating induced matching only if each vertex v of G is of one of the following six types:

(1) degree 1,
(2) degree 2 with two non-adjacent neighbors,
(3) degree 2 with two adjacent neighbors,
(4) degree 3 with $G[N(v)]$ inducing a $K_1 + K_2$,

(5) degree 3 with $G[N(v)]$ inducing a $K_{1,2}$,
(6) degree 4 with $G[N(v)]$ inducing a $2K_2$.

By Lemma 7, the vertices of types 5 and 6 can be colored from the very beginning. Moreover, the same conclusion can be made with respect to vertices of types 1 and 2. Indeed, keeping in mind assumptions A1, A2 and A3 and remembering that G has at least one vertex of degree more than 2, we prove the following lemma.

Lemma 10. *Let G be a claw-free graph with a black–white partition and v a vertex of G. If*

(10.1) *v is of type 1 with a neighbor of type 4, then v is white.*
(10.2) *v is of type 1 with a neighbor u of type 2, then both v and u are black.*
(10.3) *v is of type 2 with a neighbor u of type 2, then both v and u are black.*
(10.4) *v is of type 2 with both neighbors of type 4, then v is white.*

Proof. For (10.1), we refer the reader to Lemma 6 and Rule R4.

To see (10.2), observe that by assumption A3, vertex u must be adjacent to a vertex w of type 4. By Lemma 6, u is black, and therefore by Rule R4, w is white. As a result, v is black.

Now we prove 10.3. By assumption A2, if v is a vertex of type 2 with a neighbor u of type 2, then the other neighbor of v, say w, is not of type 2. By assumption A3, w cannot be of type 1 and obviously w cannot be of types 3, 5, 6. Therefore, w is of type 4, and by Rule R4, w and v are colored differently. Similarly, the other neighbor of u, say y, must be colored differently with u. Clearly, one of u and v must be black, and to avoid a black vertex with no black neighbor, we conclude that both v and u are black.

Finally, to prove (10.4), we observe that v must be colored differently with both its neighbors, and v cannot be colored black without having a black neighbor. Therefore, v is white. □

Application of Lemmas 7 and 10 and Rules R1–R4 may result in a (partial) coloring of G in which all vertices of types 1, 2, 5, 6 are colored, while vertices of type 3 and 4 may still be uncolored. If this coloring is valid but not yet total, we proceed as follows.

Lemma 11. *Let G be a claw-free graph obtained by application of Lemmas 7 and 10 and Rules R1–R4. If the subgraph of G induced by uncolored vertices contains a chordless cycle C with at least 4 vertices, then C is of even length. Moreover, if G admits a black–white partition, then the vertices of C are colored alternately black and white along the cycle, and furthermore, by switching the colors of vertices of C we again obtain a valid black–white partition of G.*

Proof. Clearly, no vertex of C can be of type 3. Therefore, each vertex of C is of type 4. For any vertex v of type 4, exactly two edges incident to v belong to a triangle and we will call them *heavy* edges, and the remaining edge belongs to no triangle, and we will call it a *light* edge.

Since each vertex of C is of type 4, light edges in C alternate with heavy edges. Therefore, C is of even length. Moreover, since the endpoints of light edges must be colored differently (Rule R4), the colors of vertices of C must alternate. Each vertex u of G that has a neighbor on C must be adjacent to two consecutive vertices of the cycle (otherwise a claw arises). Since one of these neighbors is white, u must be colored black. Therefore, switching the colors along the cycle does not produce any conflicts, and hence leads to another black–white partition of G.

If application of Lemma 11 leaves a graph which still has uncolored vertices, we delete from G those colored vertices that have no neighbors among uncolored ones (as they are of no importance for the completion of the procedure) and denote the resulting graph by G_0.

Lemma 12. *G_0 admits a total valid coloring.*

Proof. The connected components of G_0 can be colored separately and independently of each other, which allows us to assume without loss of generality that G_0 is a connected graph.

According to Rule R1, every colored vertex of G_0 is black, and according to Rule R2, the set of black vertices of G_0 is independent. By Lemmas 7 and 10 and Rule R4, each vertex of G_0 belongs to a triangle and any two triangles of G_0 are disjoint. Let $T_1, T_2, \ldots, T_k$ be the list of all these triangles. By Rules R1 and R2, each triangle T_i has at most one colored vertex, and obviously, each colored vertex of G_0 has a neighbor among deleted vertices of G. In other words, each colored vertex of G_0 has type 4 in G and becomes of type 3 in G_0. This discussion leads to the conclusion that G_0 is chordal, i.e., it contains no chordless cycles of length more than 3. Therefore, by contracting each triangle T_i into a single vertex we obtain a tree, i.e., a connected graph without cycles. A triangle of G_0 that becomes a leaf in this tree will be called a leaf triangle of G_0.

We will prove the lemma by induction on k, i.e., on the number of triangles in G_0. If $k = 1$, G_0 contains exactly one triangle and at most one colored vertex. Obviously, a triangle admits a total valid coloring.

Assume the lemma is true for any number of triangles less then k and let T_i be a leaf triangle of G_0. By deleting T_i we obtain a subgraph of G_0 which, by the induction hypothesis, admits a total valid coloring ϕ. This subgraph contains a unique vertex x that has a neighbor y in T_i. Observe that y is necessarily uncolored in G_0, as every colored vertex of G_0 has degree 2 in this graph. By Rule R4, if x is black in ϕ, then we color y white, and vice versa. The rest of T_i is colored arbitrarily according to Rules R1 and R2. □

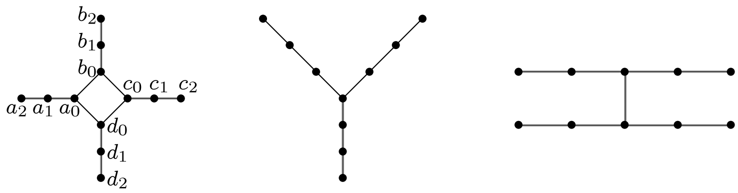

Fig. 2. Graphs X (left), Y (middle), and Z (right).

We summarize the above discussion in Algorithm $\mathcal{A}$ below. This algorithm is robust in the sense that it does not require the input graph G to be claw-free. The algorithm either finds a black–white partition of G or reports that G has no such partition or G is not claw-free. As before, we assume without loss of generality that G has at least one vertex of degree at least 3 and satisfies (A1), (A2) and (A3). We also repeat that Rules R1–R6 are used in the algorithm whenever they are applicable.

Algorithm $\mathcal{A}$
Input: a graph G
Output: a black–white partition of G or report "G has no black–white partition or G is not claw-free"

1. If at least one vertex of G is not of type 1, 2, 3, 4, 5 or 6, then STOP and output "G has no black–white partition or G is not claw-free".
2. If G has no vertices of type 1, 2, 5, 6, then $A := \emptyset$, otherwise, color the vertices of type 1, 2, 5, 6 according to Lemmas 7 and 10, and denote the set of colored vertices by A.
3. If the coloring of vertices of A is not valid, then STOP and output "G has no black–white partition or G is not claw-free".
4. If $A = V(G)$, then STOP and output the black–white partition of $V(G)$, otherwise, $U := V(G) - A$.
5. As long as $G[U]$ has a cycle C of length more than 3, do
   5.1. color the vertices of C alternately black and white (starting with an arbitrary vertex), add the newly colored vertices to A and delete them from U;
   5.2. if the coloring of vertices of A is not valid, then STOP and output "G has no black–white partition or G is not claw-free";
   5.3. if $A = V(G)$, then STOP and output the black–white partition of $V(G)$.
6. Extend the coloring of A to a total coloring according to Lemma 12 and output the black–white partition of G.

Theorem 2. *Algorithm $\mathcal{A}$ correctly solves the* DOMINATING INDUCED MATCHING *problem for any claw-free graph G with n vertices in time $O(n^2)$.*

Proof. Correctness of the algorithm follows from Lemmas 3–9. The most time-consuming steps of the algorithm are 5 and 6. In the analysis of step 5, it is helpful to consider the subgraph H of G obtained by deleting those colored vertices that have no neighbors among uncolored ones. Similarly as in Lemma 12, every vertex of H belongs to exactly one triangle and any two triangles of H are disjoint. By contracting each triangle of H into a single vertex, we obtain an auxiliary (multi)graph H' that has a cycle if and only if H has a cycle of length more than 3. Finding a cycle in H' is a linearly solvable problem, hence step 5 can be implemented in quadratic time. Obviously, this time is also sufficient to execute step 6. □

4.2. Dominating induced matchings in convex graphs

A *convex graph* is a bipartite graph $G = (V_1, V_2, E)$ in which at least one of the parts, V_1 or V_2, has the adjacency property, i.e., the vertices in that part can be ordered so that for any vertex v in the opposite part, $N(v)$ forms an interval (the vertices of $N(v)$ appear consecutively in the order).

The class of convex graphs generalizes several important subclasses such as bi-convex graphs and bipartite permutation graphs (see e.g. [6]). In the latter class, the DOMINATING INDUCED MATCHING problem has a polynomial-time solution [25]. In the present section, we extend this result to convex graphs.

It is known (and can be easily seen) that no cycle of length more than 4 is convex. Three other non-convex graphs that play an important role in our solution are X, Y and Z, represented in Fig. 2.

Lemma 13. *The graphs X, Y and Z are not convex.*

Proof. To prove the lemma for the graph X, assume by symmetry that the part of X containing a_0 has the adjacency property. Then both triples a_0, b_1, c_0 and a_0, d_1, c_0 must create intervals, which means the vertices b_1, a_0, c_0, d_1 create an interval with a_0, c_0 being in the middle. But then a_0, a_2 cannot create an interval. Therefore, X is not convex.

Let v be the vertex of degree 3 in Y. The part of Y containing v cannot have the adjacency property, since otherwise v would be consecutive with three different vertices in its part. Suppose the other part of Y has the adjacency property. Then the three vertices adjacent to v must create an interval, and the middle vertex of this interval must be also consecutive with one more vertex, which is impossible. Therefore, Y is not convex.

Let v be a vertex of degree 3 in Z. By symmetry, we may assume that the part of Z containing v has the adjacency property. But then v must be consecutive with three different vertices in its part, which is impossible. Hence Z is not convex. □

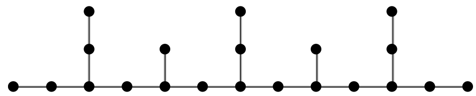

Fig. 3. An example of a τ-caterpillar.

To solve the problem for a convex graph G, we start by coloring the vertices of each C_4 in G. According to Rule R5, the colors must alternate along the cycle in any valid coloring of a C_4. So, in general, an induced C_4 admits two possible colorings. However, as we prove below, in a convex graph only one coloring is possible, and this coloring can be determined in a polynomial time.

Lemma 14. *In a convex graph any C_4 is uniquely colorable, and the only possible coloring of a C_4 can be determined in polynomial time.*

Proof. Let G be a convex graph and let vertices a_0, b_0, c_0, d_0 induce a C_4. We will illustrate the proof with the help of the picture of the graph X in Fig. 2. The algorithm that determines a coloring of the $C_4 = G[a_0, b_0, c_0, d_0]$ can be described as follows.

Algorithm C_4

1. If $G[a_0, b_0, c_0, d_0]$ cannot be extended to an induced subgraph of G isomorphic to $X[a_0, b_0, c_0, d_0, a_1, c_1]$, then color a_0 white.
2. If $G[a_0, b_0, c_0, d_0, a_1, c_1]$ cannot be extended to an induced subgraph of G isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$, then color b_0 white.
3. If $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ cannot be extended to an induced subgraph of G isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$, then color a_0 black, otherwise color a_0 white.

Clearly, the algorithm has a polynomial running time. Now let us prove the correctness of the algorithm.

Suppose $G[a_0, b_0, c_0, d_0]$ cannot be extended to $X[a_0, b_0, c_0, d_0, a_1, c_1]$ and assume by contradiction that there is a valid coloring of G in which a_0, c_0 are black and b_0, c_0 are white. Denoting by a_1 the unique black neighbor of a_0 and by c_1 the unique black neighbor of c_0, we conclude that $G[a_0, b_0, c_0, d_0, a_1, c_1]$ is isomorphic to $X[a_0, b_0, c_0, d_0, a_1, c_1]$, which contradicts the assumption. This contradiction proves the correctness of Step 1 of the algorithm.

Suppose $G[a_0, b_0, c_0, d_0, a_1, c_1]$ cannot be extended to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ and assume by contradiction that there is a valid coloring of G in which b_0, d_0 are black and a_0, c_0 are white. Then a_1, c_1 are black (Rule R1). Denoting by b_1 the unique black neighbor of b_0 and by d_1 the unique black neighbor of d_0 and remembering that a black vertex cannot have more than one black neighbor, we conclude that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ is isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$, which contradicts the assumption. This contradiction proves the correctness of Step 2 of the algorithm.

To show the correctness of Step 3, suppose $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ cannot be extended to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$ and assume by contradiction that there is a valid coloring of G in which a_0, c_0 are white and b_0, d_0 are black. Then a_1, c_1 are black (Rule R1). Denoting by a_2 the unique black neighbor of a_1 and by c_2 the unique black neighbor of c_1 and remembering that a black vertex cannot have more than one black neighbor and that G has no induced cycles except C_4, we conclude that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$ is isomorphic to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$, which contradicts the assumption. This contradiction proves the correctness of the first part of Step 3 of the algorithm.

To prove the second part of Step 3, suppose that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1]$ admits an extension to $X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, c_2]$ and assume by contradiction that there is a valid coloring of G in which a_0, c_0 are black and b_0, d_0 are white. Then b_1, d_1 are black (Rule R1). Denoting by b_2 the unique black neighbor of b_1 and by d_2 the unique black neighbor of d_1 and remembering that a black vertex cannot have more than one black neighbor and that G has no induced cycles except C_4, we conclude that $G[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2] = X[a_0, b_0, c_0, d_0, a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2]$, which is not possible because the latter graph is not convex. This contradiction completes the proof of the lemma. $\square$

Lemma 14 and assumption A1 reduce the problem from convex graphs to connected graphs without cycles, i.e., trees. Moreover, we will show that with the help of Lemma 6 and rules R1–R6 the problem further reduces to trees of a special form which we call *τ-caterpillars* (Fig. 3).

Definition 2. A τ-caterpillar is a tree of vertex degree at most 3 in which

- all vertices of degree 3 lie on a single path,
- no two vertices of degree 3 are adjacent,
- the distance between any vertex of degree 3 and a nearest leaf is at most 2.

As before, we denote by G_0 the subgraph of G obtained by deletion of those colored vertices that have no neighbors among uncolored ones.

Claim 1. *Let v be a vertex of degree at least 3 in G_0. Then*

- *v has degree 3,*

- *each neighbor of v has degree at most 2,*
- *either v is a preleaf or v is adjacent to a preleaf.*

Proof. Assume first that v is not adjacent to a leaf. To avoid an induced Y (Fig. 2), at least one of the neighbors of v, say w, is a preleaf. By Rule R6 w has degree 2 and by Lemma 6 w is colored black. Therefore, by Rule R3 no other neighbor of v is a preleaf (otherwise v is white and hence does not belong to G_0). This implies that no neighbor of v has degree more than 2 (since otherwise an induced Z arises) and the degree of v is exactly 3 (since otherwise an induced Y arises).

Suppose now that v is adjacent to a leaf u. Then, by Lemma 6, v is black and by Rule $R6$, u is the only leaf adjacent to v. No neighbor x of v is a preleaf, since otherwise neither x nor v belong to G_0 (Rule $R2$). This implies that the degree of v is exactly 3 (since otherwise an induced Y arises) and no neighbor x of v has degree more than 2 (since otherwise we are in the conditions of the previous paragraph with respect to x, in which case x cannot be adjacent to a vertex of degree at least 3). □

Lemma 15. *G_0 is a τ-caterpillar.*

Proof. The lemma is obviously true if G_0 has at most 2 vertices of degree 3. Assume now that G_0 has at least three vertices of degree 3 and suppose by contradiction that there is no path containing all of them. Then G_0 must contain three vertices u, v, w of degree 3 with no path containing them. Denote by P be the unique path connecting u to w in G_0 and by P' a shortest path connecting v to a vertex x of P. By assumption $x \neq u, v$ (otherwise $P \cup P'$ is a path containing all three vertices). Then x is also a vertex of degree 3. By Claim 1 x is adjacent to none of the vertices u, v, w, but then G_0 contains Y as an induced subgraph. This contradiction proves the lemma. □

We denote by $P = (v_0, v_1, \ldots, v_p)$ a maximal path containing all vertices of degree three of G_0. The maximality implies that both v_0 and v_p have degree 1 in G_0. According to the definition of a τ-caterpillar, there are two types of vertices of degree 3 in G_0: preleaves (type 1) and vertices adjacent to a preleaf (type 2). No vertex v of degree three can be simultaneously of type 1 and type 2, since otherwise v must be colored black and one of its neighbors must be colored black, in which case neither v nor its back neighbor belong to G_0.

If v_i is of type 1, we denote by $v_{i,1}$ the leaf adjacent to v_i, and if v_i is of type 2, we denote by $v_{i,1}$ and $v_{i,2}$, respectively, the preleaf adjacent to v_i and the leaf adjacent to $v_{i,1}$.

To complete the procedure of coloring of G_0, we will use, in addition to rules R1–R6, one more rule:

R7: if v_i is of type 2, then color v_{i-2} and v_{i+2} black. To prove correctness of this rule, assume that v_{i+2} is colored white. Then v_{i+1} must be black. Remembering that v_{i+1} has degree 2, we conclude that v_i must be black as well, since otherwise v_{i+1} has no black neighbor. But now the black vertex v_i has two black neighbors $v_{i,1}$ and v_{i+1}. This contradiction shows that black is the only possible color for v_{i+2}, and similarly for v_{i-2}.

Lemma 16. *G_0 admits a total valid coloring.*

Proof. According to Rules R2 and R3, between any two black vertices v_i and v_j $(i < j)$ of P there are at least 2 uncolored vertices. According to assumption A2, the number of uncolored vertices between v_i and v_j is exactly 2, unless one of the uncolored vertices is of type 2 in which case $j = i + 4$ (Rule R7).

We prove the lemma by induction on the number of vertices of type 2 in G_0. If there are no vertices of type 2, then $p = 3k + 2$ for some k and vertices v_{3i+1} $(i = 0, \ldots, k)$ are black. There are two possible ways to extend this partial coloring to a total valid coloring:

W1: vertices v_{3i} $(i = 0, \ldots, k)$ are colored black and all the other vertices of G_0 are colored white,
W2: vertices v_{3i+2} $(i = 0, \ldots, k)$ are colored black and all the other vertices of G_0 are colored white.

Assume now that G_0 has at least one vertex of type 2, and let v_t be such a vertex with minimum index t. Let G_0' be the subgraph of G_0 induced by vertices $v_0, \ldots, v_{t-1}$, and G_0'' the subgraph of G_0 induced by the remaining vertices. By the inductive hypothesis, G_0'' admits a total valid coloring ϕ, and G_0' has no vertices of type 2. If v_t is colored black in ϕ, apply coloring $W1$ to G_0', otherwise apply coloring $W2$ to G_0'. It is not difficult to see that in both cases we obtain a total valid coloring of G_0. □

We now summarize the above discussion in Algorithm $\mathcal{B}$ below. This algorithm is robust in the sense that it does not require the input graph G to be convex. The algorithm either finds a black–white partition of G or reports that G has no such partition or G is not convex.

Algorithm $\mathcal{B}$
Input: a graph G
Output: a black–white partition of G or report "G has no black–white partition or G is not convex"

1. As long as G has an induced C_4, apply *Algorithm C_4* to color the vertices of the C_4. If the partial coloring obtained in this way is not valid or the subgraph G_0 of G is not a τ-caterpillar, then STOP and output "G has no black–white partition or G is not convex".
2. Apply Rule R7 to G_0. If the partial coloring obtained by this application is not valid, then STOP and output "G has no black–white partition or G is not convex".
3. Extend the partial coloring of G_0 to a total coloring according to Lemma 16 and output the black–white partition of G.

Theorem 3. *Algorithm $\mathcal{B}$ correctly solves the* DOMINATING INDUCED MATCHING *problem for convex graphs in polynomial time.*

Correctness of the algorithm and its polynomial running time follow directly from the results preceding the algorithm.

4.3. Graphs without large copies of H_i

In this section, we deal with $(H_k, H_{k+1}, \ldots)$-free graphs. Notice that $(H_k, H_{k+1}, \ldots)$-free graphs generalize claw-free graphs. We will show that graphs of bounded vertex degree in this class are "not too different" from claw-free graphs.

Lemma 17. *For every fixed positive integers k and Δ, there is a constant $\rho = \rho(k, \Delta)$ such that any connected $(H_k, H_{k+1}, \ldots)$-free graph G of maximum vertex degree at most Δ contains an induced subgraph with at most ρ vertices that properly contains all induced claws of G.*

Proof. To prove the lemma, we will show that for any two induced copies of a claw in G, the distance between them does not exceed $k+1$. Suppose by contradiction that a shortest path P joining a claw $K = (x; a, b, c)$ to another claw $K' = (x'; a', b'c')$ consists of $r \geq k + 2$ edges. Let us write $P = (v_0, v_1, \ldots, v_{r-1}, v_r)$ where $v_0 \in V(K)$, $v_r \in V(K')$, and the only edges of P are $v_i v_{i+1}$ for $0 \leq i \leq r - 1$.

Observe that vertex v_1 may belong to another claw induced by some vertices of $V(K) \cup \{v_1, v_2\}$, in which case we denote this claw by $\widetilde{K}$; otherwise let $\widetilde{K} := K$. Analogously, by $\widetilde{K}'$ we denote either a claw containing vertex v_{r-1} and induced by some vertices of $V(K') \cup \{v_{r-1}, v_{r-2}\}$ (if such a claw exists) or K' otherwise. But now the two claws $\widetilde{K}$ and $\widetilde{K}'$ together with the vertices of P connecting them induce a graph H_l with $l \geq k$, a contradiction.

To conclude the proof, assume that G contains an induced claw K. According to the above discussion, the distance from the center of K to the center of any other claw in G (if any) is at most $k+3$. Since G is a connected graph of maximum degree at most Δ, there is a constant $\rho = \rho(k, \Delta)$ bounding the number of vertices of G of distance at most $k+4$ from the center of K. Clearly, the set of vertices of distance at most $k + 4$ from the center of K induces as subgraph of G that properly contains all induced claws of G. □

With the help of the above lemma, we now prove the main result of this section.

Theorem 4. *For every fixed positive integers k and Δ,* DOMINATING INDUCED MATCHING *is solvable in polynomial time for $(H_k, H_{k+1}, \ldots)$-free graphs of maximum vertex degree at most Δ.*

Proof. Let G be an $(H_k, H_{k+1}, \ldots)$-free graph of maximum vertex degree at most Δ. Without loss of generality we assume that G is connected. Then by Lemma 17 there is a constant $\rho = \rho(k, \Delta)$ such that the vertices of G can be partitioned into two subsets U and C so that $|U| \leq \rho$ and $G[C]$ is claw-free. Moreover, C has no intersection with any claw in G. It is clear from the proof of Lemma 17 that the partition $V(G) = C \cup U$ can be found in polynomial time.

Since the number of vertices of U is bounded by a constant, there are constantly many ways to color the vertices of U in black and white. For each such coloring, we first extend it, if possible, according to Rules R1–R4, and then delete those colored vertices that have no neighbors among uncolored ones. The graph G_0 obtained in this way is claw-free, because the vertices of any claw must all be colored, while in G_0 the set of colored vertices is independent. Therefore, to solve the problem for G we have to apply, constantly many times, the algorithm for claw-free graphs to induced subgraphs of G. □

To conclude this section, let us observe that although the class of $(H_k, H_{k+1}, \ldots)$-free graphs generalizes claw-free graphs (for any fixed $k \geq 1$), Theorem 4 is not a generalization of Theorem 2, because in Theorem 2 we do not assume any restriction on the vertex degree. On the other hand, since for the problem in claw-free graphs vertices of degree more than 4 are useless, Theorem 4 can be viewed as a partial generalization of Theorem 2. Whether Theorem 2 admits a full generalization to the class of $(H_k, H_{k+1}, \ldots)$-free graphs is an interesting open problem. Some other open problems are discussed in the concluding section.

5. Concluding remarks and open problems

In this paper, we studied computational complexity of the DOMINATING INDUCED MATCHING problem in special graph classes. This problem is also known in the literature under the name EFFICIENT EDGE DOMINATION and has interesting connections to some other problems, such as VERTEX 3-COLORABILITY and MAXIMUM INDUCED MATCHING. We proved a number of results of both negative (NP-complete) and positive (polynomial) type. In particular, we proved polynomial-time solvability of the problem in the class of claw-free graphs. Observe that the MAXIMUM INDUCED MATCHING problem in this class is NP-hard, which suggests the idea that MAXIMUM INDUCED MATCHING is, in a sense, harder than DOMINATING INDUCED MATCHING. It is known [12] that finding a maximum induced matching is polynomial-time solvable in the class of weakly chordal graphs. This class generalizes simultaneously two polynomially solvable cases for the DOMINATING INDUCED MATCHING problem, namely, chordal graphs and convex graphs. It would be interesting to investigate whether these two cases can be extended to the larger class of weakly chordal graphs.

In general, further narrowing the gap between NP-complete and polynomially solvable cases of the DOMINATING INDUCED MATCHING problem is an interesting direction for future research. In this respect, classes of graphs without long induced paths are of particular interest. Indeed, by forbidding a path P_k we simultaneously exclude a graph from the class $\mathcal{S}$, long cycles, and long graphs of the form H_k, which are the three major ways to satisfy condition (1) stated in the beginning of Section 4. It is known that the clique-width of P_4-free graphs is at most 2, which implies polynomial-time solvability of many algorithmic graph problems in this class, including DOMINATING INDUCED MATCHING and MAXIMUM INDUCED MATCHING. However, for $k \geq 5$, the complexity of both problems in the class of P_k-free graphs is unknown. Both of them are solvable in polynomial time for $(P_k, K_{1,s})$-free graphs for any fixed k and s. For the MAXIMUM INDUCED MATCHING problem this was proved in [23], while for the DOMINATING INDUCED MATCHING this trivially follows from Lemma 8 because the vertex degree in the input graph must be bounded by a constant (depending on s), in which case the number of vertices of the graph is bounded by a constant (assuming the graph is connected). Finding the complexity of the problem in the entire class of P_k-free graphs (without additional restrictions) is a challenging open problem.

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References

[1] V.E. Alekseev, On easy and hard hereditary classes of graphs with respect to the independent set problem, Discrete Appl. Math. 132 (2003) 17–26.
[2] V.E. Alekseev, R. Boliac, D.V. Korobitsyn, V.V. Lozin, NP-hard graph problems and boundary classes of graphs, Theoret. Comput. Sci. 389 (2007) 219–236.
[3] V.E. Alekseev, D.V. Korobitsyn, V.V. Lozin, Boundary classes of graphs for the dominating set problem, Discrete Math. 285 (2004) 1–6.
[4] R. Boliac, V. Lozin, On the clique-width of graphs in hereditary classes, Lecture Notes in Comput. Sci. 2518 (2002) 44–54.
[5] A. Brandstädt, E.M. Eschen, R. Sritharan, The induced matching and chain subgraph cover problems for convex bipartite graphs, Theoret. Comput. Sci. 381 (2007) 260–265.
[6] A. Brandstädt, V.B. Le, J.P. Spinrad, Graph Classes: a Survey, in: SIAM Monographs on Discrete Math. Appl., vol. 3, SIAM, Philadelphia, 1999.
[7] A. Brandstädt, V.V. Lozin, On the linear structure and clique-width of bipartite permutation graphs, Ars Combin. 67 (2003) 273–281.
[8] D.M. Cardoso, J.O. Cerdeira, C. Delorme, P.C. Silva, Efficient edge domination in regular graphs, Discrete Appl. Math. 156 (2008) 3060–3065.
[9] D.M. Cardoso, V.V. Lozin, Dominating induced matchings, Lecture Notes in Comput. Sci. 5420 (2009) 77–86.
[10] K. Cameron, Induced matchings, Discrete Appl. Math. 24 (1989) 97–102.
[11] K. Cameron, Induced matchings in intersection graphs, Discrete Math. 278 (2004) 1–9.
[12] K. Cameron, R. Sritharan, Y. Tang, Finding a maximum induced matching in weakly chordal graphs, Discrete Math. 266 (2003) 133–142.
[13] J.-M. Chang, Induced matchings in asteroidal triple-free graphs, Discrete Appl. Math. 132 (2004) 67–78.
[14] B. Courcelle, J.A. Makowsky, U. Rotics, Linear time solvable optimization problems on graphs of bounded clique-width, Theory Comput. Syst. 33 (2000) 125–150.
[15] M.C. Golumbic, R.C. Laskar, Irredundancy in circular arc graphs, Discrete Appl. Math. 44 (1993) 79–89.
[16] M.C. Golumbic, M. Lewenstein, New results on induced matchings, Discrete Appl. Math. 101 (2000) 157–165.
[17] M.C. Golumbic, U. Rotics, On the clique-width of some perfect graph classes, Int. J. Foundations Comput. Sci. 11 (2000) 423–443.
[18] D.L. Grinstead, P.J. Slater, N.A. Sherwani, N.D. Holmes, Efficient edge domination problems in graphs, Inform. Process. Lett. 48 (1993) 221–228.
[19] D. Kobler, U. Rotics, Finding maximum induced matchings in subclasses of claw-free and P_5-free graphs, and in graphs with matching and induced matching of equal maximum size, Algorithmica 37 (2003) 327–346.
[20] N. Korpelainen, A polynomial-time algorithm for the dominating induced matching problem in the class of convex graphs, Electron. Notes Discrete Math. 32 (2009) 133–140.
[21] J. Kratochvíl, Regular codes in regular graphs are difficult, Discrete Math. 133 (1994) 191–205.
[22] V.V. Lozin, On maximum induced matchings in bipartite graphs, Inform. Process. Lett. 81 (2002) 7–11.
[23] V. Lozin, D. Rautenbach, Some results on graphs without long induced paths, Inform. Process. Lett. 88 (2003) 167–171.
[24] V. Lozin, D. Rautenbach, On the band-, tree- and clique-width of graphs with bounded vertex degree, SIAM J. Discrete Math. 18 (2004) 195–206.
[25] C.L. Lu, M.-T. Ko, C.Y. Tang, Perfect edge domination and efficient edge domination in graphs, Discrete Appl. Math. 119 (2002) 227–250.
[26] C.L. Lu, C.Y. Tang, Solving the weighted efficient edge domination problem on bipartite permutation graphs, Discrete Appl. Math. 87 (1998) 203–211.
[27] J.A. Makowsky, U. Rotics, On the clique-width of graphs with few P_4's, Int. J. Foundations Comput. Sci. 10 (1999) 329–348.