Discrete Mathematics 87 (1991) 163–174
North-Holland

Inversions, cuts, and orientations

Wei Ping Liu and Ivan Rival
Department of Computer Science, University of Ottawa, Ottawa, Canada

Received 30 October 1987
Revised 3 March 1989

Abstract

Liu, W.P. and I. Rival, Inversions, cuts, and orientations, Discrete Mathematics 87 (1991) 163–174.

Any reorientation of the diagram of an ordered set reverses the direction of some of the edges. However, not all subsets of edges, when reversed, can produce a diagram. We show that reversing the edges of a 'cut' does produce a diagram and that any such reorientation may be constructed by a familiar sequential algorithm. We apply this to the enumeration and complexity of reorientations.

What are the possible reorientations of the covering graph of an ordered set? This fundamental question from the theory of ordered sets is closely related to the problem of characterizing the undirected graphs which are covering graphs. Until recently little was known [2–3, 5, 13, 15–21].

For an ordered set P and elements a and b in P, say that a *covers* b (or b is *covered* by a) and write $a \succ b$ if $a > b$ and, for each x in P, $a > x \geq b$ implies $x = b$. We also call a an *upper cover* of b (and b a *lower cover of* a). The *covering graph* of P has as its vertices the elements of P and as its edges the pairs $\{a, b\}$ such that a covers b. A *diagram* of P is a pictorial representation of it on the plane in which small circles, corresponding to the elements of P, are arranged in such a way that, for a and b in P, the circle corresponding to a is higher than the circle corresponding to b whenever $a > b$ and a straight line segment is drawn to connect the two circles just if a covers b. There is considerable variation possible in the pictorial rendering, but, occasionally (e.g. planar lattices [9]) particular drawings play a central role. Still, any diagram of P determines it. It is common, therefore, to refer to a diagram as the ordered set itself.

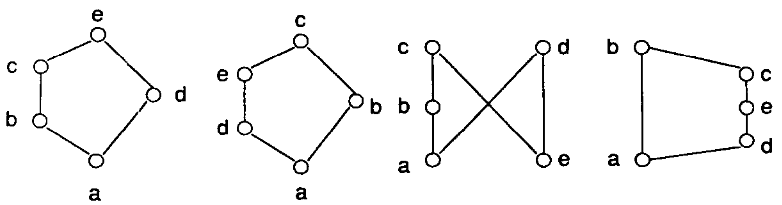
An *orientation* of a covering graph is a diagram with the same (labelled) covering graph. A *reorientation* of an ordered set is an orientation of its covering graph. Of course, a reorientation can differ from the original only in reversing the 'direction' of the edges—if a covers b in P then a reversal makes b an upper cover of a in the reorientation. Therefore, the possible reorientations are precisely the subsets of edges of P which may be 'reversed'. Some subsets of edges may not (Fig. 1).

The problem to describe all subsets of edges which, in this sense, are reversible, has been solved for certain special classes of ordered sets and restricted reorientations (e.g. modular lattices [5–8], cf. [2, 4, 19]). Here we shall consider subsets of edges which are 'cuts', show that such subsets are reversible, and prove that, indeed, each such reorientation can be produced by a familiar sequential algorithm—the 'inversion'. Our main result is this.

Theorem 1. *Let P be a finite ordered set. An ordered set is an inversion of P if and only if the reversed edges can be partitioned into cuts of P.*

Let E be a subset of the edges of P, say E consists of the covering pairs $a_1 \succ b_1, a_2 \succ b_2, \ldots$ and let $P-E$ stand for the diagram obtained from P by removing all of the edges of E. Let U_E denote the subset of all vertices of P connected to some a_i in $P-E$ (that is, the vertices a in P for which there is a sequence $a = x_0, x_1, x_2, \ldots, x_m = a_i$ such that x_j covers x_{j+1} or x_{j+1} covers x_j in $P-E$). U_E is an *upset* of P, that is, if x belongs to U_E and $y > x$ then y belongs to U_E too. Let D_E denote the subset of all vertices of P connected to some b_i in $P-E$. Similarly, D_E is a *downset*, that is, if x belongs to D_E and $y < x$ then y belongs to D_E. Then we call E a *cut* of P if $D_E \cap U_E = \emptyset$ (see Fig. 2). (This formulation differs somewhat from one namesake in graph theory [1], and substantially from another in the theory of ordered sets [21]. For example, if P is the three-element ordered set $\{c < b < a\}$ then its two edges $a_1 = a \succ b = b_1$, $a_2 = b \succ c = b_2$ do not form a cut, although they do disconnect the covering graph of P.)

For any element a of P let $L(a)$ stand for its lower covers and $U(a)$ for its upper covers. For any maximal element a of P construct an ordered set Q such that $Q\text{-}\{a\} = P\text{-}\{a\}$, a is minimal in Q and its upper covers consist of the elements $L(a)$ in P. The reorientation Q so obtained is variously referred to as 'pushdown', 'pivot', or 'flip'. It was first introduced in a series of articles by Mosesjan [10–12]



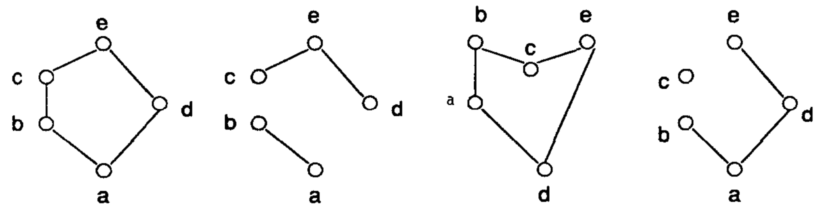
An ordered set (a)

e>-c is reversible (b)

e>-c, e>-d is is reversible (c)

c>-b, e>-c is not reversible (d)

Fig. 1.



P

P   c >- b, d >- a is a cut   A reorientation of P by reversing the cut   e >- c, c >- b is not a cut

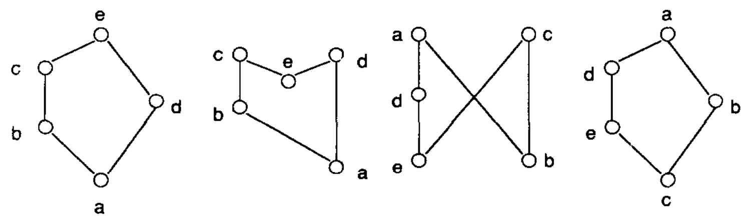
Fig. 2.

(cf. [3, 16–19]). One important result of Mosesjan is that, for a finite, connected ordered set P and any element a in P there is an inversion Q of P in which a is the top element. This is particularly useful in constructing graphs which are not covering graphs [19].

The dual construction ('pullup') is also used: for any minimal element b, construct an ordered set Q with $Q\text{-}\{b\} = P\text{-}\{b\}$, b maximal in Q, and its lower covers consisting of the elements $U(b)$ in P. We call any (mixed) sequence of 'pushdowns' or 'pullups' an *inversion*. See Fig. 3.

Associated with any inversion Q is a (mixed) sequence of pushdowns and pullups. Any pullup may be replaced by a sequence of pushdowns [17]. Therefore, we may associate with any inversion Q a sequence $(a_1, a_2, \ldots)$ of pushdowns. The sequence of pushdowns need not be unique. For example, if P and Q are the ordered sets illustrated in Fig. 4, then both (b, c) and (a, b, c, d, c, b) are sequences of pushdowns producing the same inversion Q of P.

Some elements of the sequence may occur several times. Let $r_a(x)$ stand for the number of occurrences of x in the inversion sequence for Q.



$P=P_0$ P_1: a pushdown of P_0 Pw: a pullup of P_1 P_3: a pushdown of P_2

An inversion

Fig. 3.

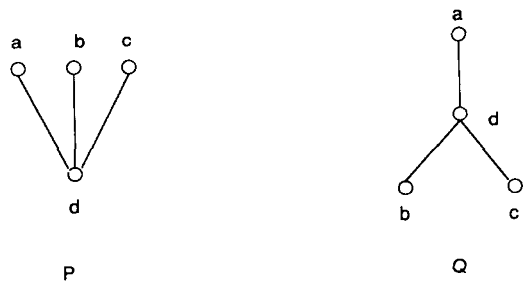


Fig. 4.

Let Q be an inversion of an ordered set P. It is well known that, even if the maximal elements $\max P$ are identical to the maximals $\max Q$ of Q the two ordered sets P and Q need not be identical (see Fig. 5). If, however, each of Q and P has just one maximal element, the same for both, then $Q = P$ (Pretzel [17]). Our next result is a generalization.

Theorem 2. *Let P_1 and P_2 be inversions of a finite connected ordered set. Let $\max P_1 = \max P_2$ and let $r_{P_1}(a) = 0$ for each a in $\max P_1$. Then $P_1 = P_2$ if and only if $r_{P_2}(a) = r_{P_2}(b)$ for every a, b in $\max P_2$.*

We shall apply our results to several items of an algorithmic character.

Let P be an n-element ordered set. Let $p(n)$ be the smallest number such that any inversion of P can be produced with at most $p(n)$ pushdowns. Let $e(n)$ be the smallest number such that any inversion of P can be produced by pushdowns reversing successively at most $e(n)$ edges.

Theorem 3. *$p(n) \leq n(n-1)/2$ and $e(n) \leq n^3/3$.*

The first inequality is attainable quite simply. For the second we have examples of n-element ordered sets which require $\frac{2}{27}n^3$ successive edge reversals.

How many distinct reorientations are there for the covering graph of an

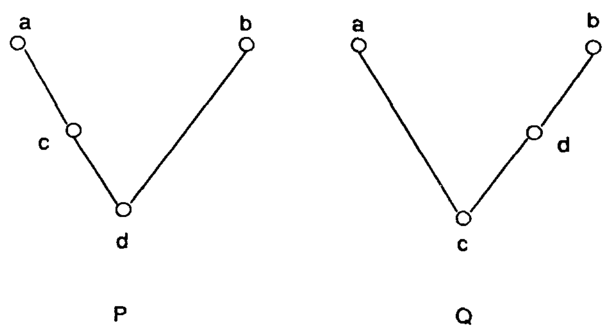


Fig. 5.

ordered set? While we have no general approach to attack this problem at this time, we may estimate a lower bound by computing the minimum number of distinct inversions. Pretzel [17] proved that every ordered set with connected covering graph has at least $n^2/2+n$ reorientations. There is, however, at least one instance in which the number of inversions equals the number of reorientations. This is the case if the covering graph has no cycle at all, that is, it is a tree, for, in this case, any subset of edges is reversible and, as every edge is a minimal cut, any reorientation is an inversion. Thus, *if the covering graph of an ordered set P with $|P|=n$ is a tree then there are precisely 2^{n-1} possible reorientations (or inversions)*. The number of inversions seems to be much smaller. An antichain is, of course, uniquely orderable. Still, any ordered set with a connected covering graph, has a substantial number of distinct inversions. According to Pretzel [17] there are at least $n^2/4+n/2$ inversions.

Theorem 4. *Let P be an n-element ordered set with connected covering graph. Then the number of distinct inversions is at least $(n^2+2n)/2 - n\log_2 n$.*

Although we do not yet know whether this bound is attainable we have examples of n-element ordered sets with precisely $(n^2-n)/2$ inversions.

Proof of Theorem 1

We establish a sequence of lemmas.

Lemma 1. *Let P be a finite ordered set and Q an inversion. Let $a \succ b$ in P.*

(i) *$a \succ b$ in Q if and only if $r_Q(a)=r_Q(b)$* (cf. [17]).

(ii) *$b \succ a$ in Q if and only if $r_Q(a)=r_Q(b)+1$.*

Proof. The sufficiency in both cases follows from the pushdown construction. For the 'necessity' we proceed by induction on $m=r_Q(a)+r_Q(b)$. If $m=1$ then, as Q is constructed by a sequence of pushdowns, $r_Q(a)=1$ and $r_Q(b)=0$, so $b \succ a$ in Q. If $m=2$ then $r_Q(a)=r_Q(b)=1$ so $a \succ b$ in Q. Let Q' be the inversion of P with precisely the same sequence of pushdowns as Q, save for the very last pushdown of a and b. (As $a \succ b$ in P, once a pushdown of a or b is made, the next pushdown of a, respectively b, must be preceded by a pushdown of b, respectively a.) The conclusion now follows by the argument above applied to Q'. $\square$

Here are several simple consequences. Notice that if $a>b$ in P then there is a covering chain $a=a_1 \succ a_2 \succ \cdots \succ b$ to which Lemma 1 may be successively applied.

Lemma 2. *Let P be a finite ordered set and Q an inversion, and let $a > b$ in P. Then $r_Q(a) \geq r_Q(b)$ and, if, moreover, $r_Q(a) = r_Q(b)$ then $a > b$ in Q.*

Lemma 3. *Let P be a finite ordered set, Q an inversion. Let S be a subset of P such that, for each a, b, in S, $r_Q(a) = r_Q(b)$. Then the subset S in Q has the same order as it has in P.*

From the construction of an inversion as a sequence of pushdowns we have this simple fact.

Lemma 4. *Let U be an upset of a finite ordered set P. Then there is an inversion Q of P such that*

$$r_Q(a) = \begin{cases} 1 & \text{if } a \text{ belongs to } U, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 5. *Let P be a finite ordered set. For any cut E of P there is an inversion which reverses precisely the edges of E.*

Proof. Let E consist of the edges $a_1 \succ b_1$, $a_2 \succ b_2, \ldots$ in P. Then U_E is an upset so, according to Lemma 4, there is an inversion Q in which each element a of U_E satisfies $r_Q(a) = 1$. From Lemma 2, only the edges of the cut E are reversed. $\square$

Let b be any minimal element of P and let $c_1 \succ b$, $c_2 \succ b \ldots$ be all of its upper covers. These edges obviously constitute a cut of P, whence, according to Lemma 5, there is an inversion which reverses precisely these edges. As the reversal of these edges corresponds to a pullup of b, we see that any pullup is equivalent to a sequence of pushdowns [17]. In particular, any inversion is 'reversible' in the sense that there is another inversion which will restore the original order.

According to the definition of a cut and Lemma 5 we also have the following.

Lemma 6. *Let P be a finite ordered set and Q an inversion of P. Then any cut of P all of whose edges are either reversed, or all of whose edges are not reversed, in Q, is also a cut of Q.*

We are now ready to prove our principal results.

Theorem 1. *Let P be a finite ordered set. An ordered set is an inversion of P if and only if the reversed edges can be partitioned into cuts of P.*

Proof. First we establish the sufficiency. To this end let $E = E_1 \cup E_2 \cup \cdots \cup E_k$ be a disjoint union of cuts of P. If $k = 1$ the conclusion follows from Lemma 5. Let $k > 1$. In view of Lemma 5 there is an inversion Q_1 of P which reverses precisely the edges of E_1. According to Lemma 6, each of $E_2, E_3, \ldots E_k$ is a cut of Q_1. Then, by induction on the number of cuts, there is an inversion Q of Q_1 which reverses precisely the edges of $E_2 \cup E_3 \cup \cdots \cup E_k$.

We turn now to the necessity. To this end it is convenient to formulate and verify this 'Cut Partition' property. *Every inversion Q of P, induces a partition of P into subsets $A_0, A_1, \ldots$ defined by $A_i = \{x \mid r_Q(x) = r_Q(a_0) + i\}$ where $r_Q(a_0) = \min\{r_Q(x) \mid x \text{ in } P\}$, and the reversed edges are precisely those between successive pairs of A_i's.* (This is illustrated schematically in Fig. 6.) There is no loss of generality in assuming that P is itself connected. Choose a_0 in P such that $r_Q(a_0) = \min\{r_Q(x) \mid x \text{ in } P\}$ ($r_Q(a_0)$ may be 0). Then, let

$$A_0 = \{x \mid r_Q(x) = r_Q(a_0)\}.$$

These properties about A_0 are straightforward to verify.

For each element x not in A_0 and for each y in A_0, $x \not> y$ in Q [Lemma 1].

A_0 in Q is identically ordered to A_0 in P [Lemma 3].

There are elements a, a_1 with a in A_0 and a_1 not, such that $a \succ a_1$ in Q and $r_Q(a_1) = r_Q(a) + 1$ [P connected].

Now, let

$$A_1 = \{x \mid r_Q(x) = r_Q(a_1) = r_Q(a_0) + 1\}.$$

Suppose that the subsets $A_j, j = 0, 1, 2, \ldots, m$, have already been defined satisfying the following properties.

$A_j = \{x \mid r_Q(x) = r_Q(a_j) = r_Q(a_0) + j\}$.

A_j in Q is identical to A_j in P.

If x belongs to A_j and y to A_{j-1} then $x \not> y$ in Q.

If y belongs to A_{j-1} and x to $P - A_0 \cup A_1 \cup \cdots \cup A_{j-1}$ with $y \succ x$ in Q then x belongs to A_j and $x \succ y$ in P.

In the light of Lemma 1 again, for each x not in $A_0 \cup A_1 \cup \cdots \cup A_m$, y in A_m, then $x \not> y$ in Q. As P is connected there are elements a, a_{m+1}, with a in $A_0 \cup A_1 \cup \cdots \cup A_m$ and a_{m+1} not, such that $a \succ a_{m+1}$ in Q. Since a_{m+1} is not in

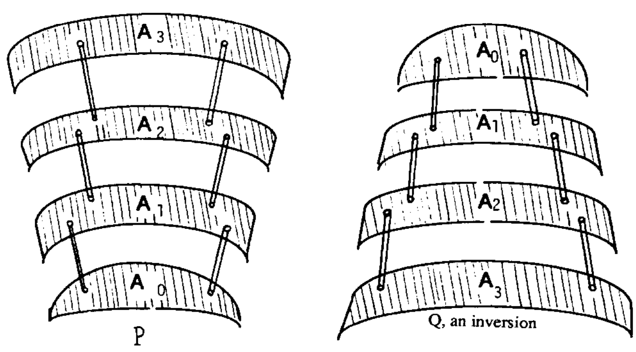


Fig. 6.

$A_0 \cup A_1 \cup \cdots \cup A_m$, a is in A_m. So $r_Q(a_{m+1})) = r_Q(a) + 1 = r_Q(a_0) + m + 1$. Let

$$A_{m+1} = \{x \mid r_Q(x) = r_Q(a_0) + m + 1\}.$$

Then A_{m+1} satisfies these properties, too:

A_{m+1} in Q is identical to A_{m+1} in P;

if x belongs to A_m, and y not, with $x \succ y$ in Q, then y belongs to A_{m+1} and $y \succ x$ in P;

if y belongs to A_{m+1} and z to $P - A_0 \cup A_1 \cup \cdots \cup A_{m+1}$, then $z \not\succ y$ in Q.

In this way we produce a cut partition, that is, a decomposition of P into blocks $A_0, A_1, \ldots$. Let E_i stand for the set of all edges $x \succ y$ in P such that x belongs to A_i and y to A_{i-1}. It is now evident that each E_i is a cut of P, and Q just reverses the edges of $E_1 \cup E_2 \cup \cdots$. This completes the proof of Theorem 1. □

Proof of Theorem 2

Let $P_1 = P_2$ be inversions of P. In terms of the cut partition property (as in the proof of Theorem 1) each inversion P_i induces a partition $A_0^i, A_1^i, \cdots$ $(i = 1, 2)$. Suppose there are maximal elements a, b of P_2 such that $r_{P_2}(a) < r_{P_2}(b)$. Then a and b lie in different A_j^2 sets, although both a and b lie in A_0^i. As P is connected there is a 'zig-zag' $Z = \{a = x_1, x_2, \ldots, x_m = b\}$ such that, for each j, $x_j \succ x_{j+1}$ or $x_{j+1} \succ x_j$. We can count the number of edge reversals, in P_1 and in P_2, along this zig-zag, which, in view of the hypothesis $P_1 = P_2$, must be identical, too. Thus, let

$$r_i = |\{x_{j+1} \succ x_j \text{ in } Z \mid x_j \succ x_{j+1} \text{ in } P_i\}| - |\{x_j \succ x_{j+1} \text{ in } Z \mid x_{j+1} \succ x_j \text{ in } P_i\}|.$$

As a, b both belong to A_0 it follows from the cut partition property that $r_1 = 0$. On the other hand, $r_2 \neq 0$. This is a contradiction.

To prove the converse, let us suppose that $r_{P_2}(a) = r_{P_2}(b)$ for all maximal elements a, b of P_2. Let x belong to A_0^2, say $x \leqslant a$ in P_2. By Theorem 2, $x \leqslant a$ in P. By hypothesis, $r_{P_1}(a) = 0$ so, again, $r_{P_1}(x) = 0$, too. Thus, x belongs to A_0^1. Now let x belong to A_0^1. Then for some maximal element a, $x \leqslant a$ in A_0^1, so $x < a$ in P, too. Again $r_{P_2}(x) = r_{P_2}(a)$, whence x belongs to A_0^2. Therefore, $A_0^1 = A_0^2$. We may then apply the same argument to conclude that $A_1^1 = A_1^2$, $A_2^1 = A_2^2$, etc. In particular, $P_1 = P_2$.

Remark. We know that the sequence of pushdowns associated with an inversion of an ordered set need not be unique. Suppose the sequences S_1, and S_2 produce the same inversion Q of an ordered set P. Let $r_{S_i}(x)$ stand for the number of occurrences of x in the sequence S_i and

$$A_j^i = \{x \text{ in } P \mid r_{S_i}(x) = \min\{y_{S_i}(y) \mid y \text{ in } P\} + j\}$$

$i = 1, 2$. According to the argument of the first part of the proof of Theorem 2,

$A_j^1 = A_j^2$ for all j, although $r_{S_1}(x)$ need not be equal to $r_{S_2}(x)$. That is the reason that we use $r_Q(x)$ instead of $r_S(x)$ for an inversion Q of an ordered set.

Proof of Theorem 3

Let deg a stand for the *degree* of the element a in P, that is, the total number of upper and lower covers of a in P. Let Q be an inversion of P and let

$$r_Q(a_0) = \min\{r_Q(a) \mid a \text{ in } P\}.$$

Then, according to the cut partition property

$$p(n) = \sum_{i=1}^{k} (r_Q(a_0) + i)\, |A_i|$$

and

$$e(n) = \sum_{i=1}^{k} (r_Q(a_0) + i) \sum_{a \text{ in } A_i} \deg a.$$

As any inversion amounts, according to Theorem 1, to reversing the edges of a disjoint union of cuts, we may suppose, by Lemma 4, that $r_Q(a_0) = 0$. Therefore,

$$p(n) = 1\,|A_1| + 2\,|A_2| + 3\,|A_3| + \cdots + k\,|A_k|$$

and

$$e(n) = \sum_{i=1}^{k} i \sum_{a \text{ in } A_i} \deg a.$$

To obtain an upper bound for $p(n)$ observe that $1 \leqslant |A_i| \leqslant n - i$, for each $i = 1, 2, 3, \ldots, k$, so

$$p(n) \leqslant 1 + 2 + 3 + \cdots + k - 1 + k(n-k) = nk - \frac{k^2 + k}{2} \leqslant \frac{n^2 - n}{2}.$$

In fact, to see that this upper bound can be attained it is enough to take an n-element chain $a_0 < a_1 < \cdots < a_{n-1}$. It is easy to verify that the inversion in which a_0 is the top requires $(n^2 - n)/2$ individual pushdowns. (Each element a_i occurs i times in the inversion sequence.)

We turn now to the other inequality, for $e(n)$. Of course, $i \leqslant k \leqslant n$ and $\deg a < n$ so, at any rate, $e(n) = 0(n^3)$. Now, it is fairly straightforward to analyse some of the local structure of the order, in view of Theorem 1, to produce $e(n) \leqslant n^3/3$. We can actually even do much better but with considerably more tedious calculations. The example illustrated in Fig. 7 gives $e(n) \geqslant \frac{1}{27}(2n^3 + n^2 + 31n + 50)$.

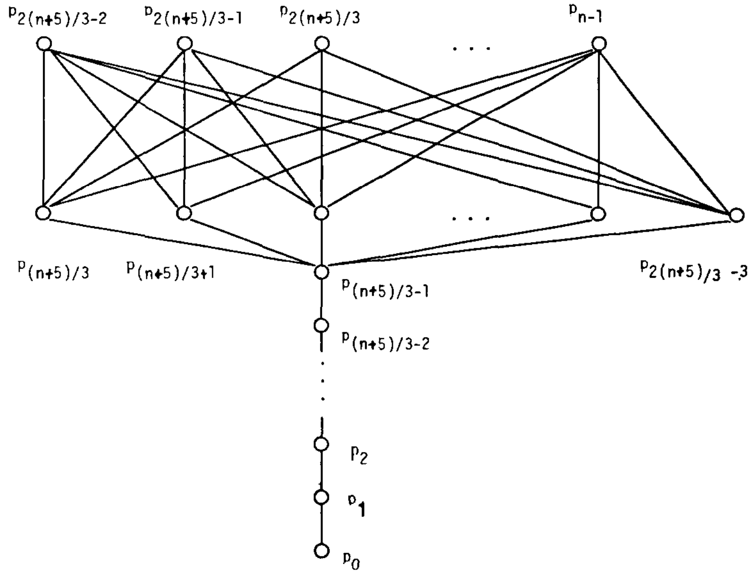


Fig. 7.

Proof of Theorem 4

First we consider an example, the n-element, connected, ordered set illustrated in Fig. 8. Using Theorem 1 it is to easy verify that this ordered set has precisely

$$\tfrac{1}{2}(n-2)(n-3)+2(n-2)+1=\frac{n^2-n}{2}$$

distinct inversions.

We shall now show that any connected, n-element ordered set has at least

$$\frac{n^2+2n}{2}-n\log_2 n$$

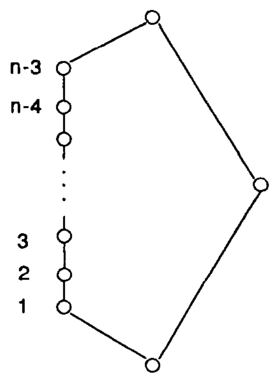


Fig. 8.

inversions. (At this writing we know of no example which attains this lower bound.)

Lemma 7 [17]. *For any two elements, in a finite ordered set, neither of which covers the other, there is an inversion with just these two as the maximal elements.*

Proof. Let a, b be distinct elements of a finite ordered set P such that neither a covers b nor b covers a. (It may be, however, that $a > b$ or $b > a$.) Let $U = \{x \mid x \leqslant a\}$. As U is an upset of P there is, by Lemma 4, an inversion P_1 of P in which $\max P_1 = \{a\}$. Now, consider the pushdown P_2 of P_1 obtained by the pushdown of a. In P_2, a and b are noncomparable. Let U_2 be the upset of P_2 consisting of all elements x such that $x \leqslant a$ and $x \leqslant b$. Applying Lemma 4 again will produce an inversion P_3 of P_2 (whence of P, too) in which $\max P_3 = \{a, b\}$.

We are ready to give the proof of Theorem 4.

Suppose that P contains an element a satisfying

$$\deg a \geqslant \log_2 \frac{n^2}{2}.$$

Consider the inversion P_1 of P in which a is the top element. Then, in P_1, a has at least $\log_2 (n^2/2)$ lower covers. Evidently, for every distinct subset S of the $\log_2 (n^2/2)$ lower covers of a there is an inversion in which the set of maximal elements is precisely S. This gives, in all, at least

$$2^{\log_2(n^2/2)} = \frac{n^2}{2} \geqslant \frac{n^2 + 2n}{2} - n \log_2 n$$

inversions.

Let us now suppose that, for each a in P,

$$\deg a < \log_2 \frac{n^2}{2}.$$

If P_1 is an inversion of P in which a is the top element, then there are at least $n - 1 - \deg a$ elements not covered by a. According to Lemma 7 a together with any one of these $n - 1 - \deg a$ elements is a subset for which there is an inversion with just these two as the maximal elements. This holds for every element a belonging to P. There are also n inversions with a top element. Therefore, in all, there are at least

$$\begin{aligned} \tfrac{1}{2} \sum_{a \text{ in } P} (n - 1 - \deg a) + n &= \frac{n^2 + n}{2} - \tfrac{1}{2} \sum_{a \text{ in } P} \deg a \geqslant \frac{n^2 + n}{2} - \frac{n}{2} \log_2 \frac{n^2}{2} \\ &= \frac{n^2 + 2n}{2} - n \log_2 n \end{aligned}$$

inversions.

References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1973).
[2] H.J. Bandelt, Retracts of hypercubes, J. Graph Theory 8 (1984) 501–510.
[3] H.J. Bandelt and I. Rival, Diagrams, orientations and varieties (1987), Order 6 (1989) 119–132.
[4] D. Duffus and I. Rival, Path length in the covering graph of a lattice, Discrete Math. 19 (1979) 139–158.
[5] D. Duffus and I. Rival, Graphs orientable as distributive lattices, Proc. Amer. Math. Soc. 88 (1983) 197–200.
[6] J. Jakubik, On lattices whose graphs are isomorphic [in Russian], Czechoslovak Math. J. 4 (1954) 131–140.
[7] J. Jakubik, Unoriented graphs of modular lattices, Czechoslovak Math. J. 25 (1975) 240–246.
[8] J. Jakubik, Modular lattices of locally finite length, Acta Sci. Math. 37 (1975) 79–82.
[9] D. Kelly and I. Rival, Planar lattices, Canad. J. Math. 27 (1975) 636–665.
[10] K.M. Mosesjan, Strongly basable graphs [in Russian], Akad. Nauk Armjan. SSR Dokl. 54 (1972) 134–138.
[11] K.M. Mosesjan, Certain theorems on strongly basable graphs [in Russian], Akad. Nauk. Armjan SSR Dokl. 54 (1972) 241–245.
[12] K.M. Mosesjan, Basable and strongly basable graphs [in Russian], Akad. Nank. Armjan. SSR Dokl. 55 (1972) 83–86.
[13] J. Nesetril and V. Rödl, Complexity of diagrams, Order 3 (1987) 321–330.
[14] O. Ore, Theory of Graphs (Amer. Mathematical Soc, Providence, RI, 1962).
[15] M. Pouzet and I. Rival, Is there a diagram invariant?, Discrete Math. 73 (1988/89) 181–188.
[16] O. Pretzel, On graphs that can be oriented as diagrams of ordered sets, Order 2 (1985) 25–40.
[17] O. Pretzel, On reorienting graphs by pushing down maximal vertices, Order 3 (1986) 135–153.
[18] O. Pretzel, Orientations and reorientations of graphs, in: I. Rival, ed., Combinatorics and Ordered Sets, Contemp. Math. 57 (1986) 103–125.
[19] I. Rival, The diagram, in: I. Rival, ed., Graphs and Order (Reidel, Dordrecht, 1985) 103–133.
[20] I. Rival, The diagram, Order 2 (1985) 101–104.
[21] I. Rival and N. Zaguia, Antichain cutsets, Order 1 (1985) 235–247.