Packings and coverings of $\lambda K_v$ by $k$-circuits with one chord

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Abstract

Let $\lambda K_v$ be the complete multigraph with $v$ vertices, where any two distinct vertices $x$ and $y$ are joined by $\lambda$ edges $\{x, y\}$. Let $G$ be a finite simple graph. A $G$-packing design ($G$-covering design) of $\lambda K_v$, denoted by $(v, G, \lambda)$-PD ($(v, G, \lambda)$-CD) is a pair $(X, \mathcal{B})$, where $X$ is the vertex set of $K_v$ and $\mathcal{B}$ is a collection of subgraphs of $K_v$, called blocks, such that each block is isomorphic to $G$ and any two distinct vertices in $K_v$ are joined in at most (at least) $\lambda$ blocks of $\mathcal{B}$. A packing (covering) design is said to be maximum (minimum) if no other such packing (covering) design has more (fewer) blocks. In this paper, the discussed graphs are $C^{(r)}_k$, i.e., one circle of length $k$ with one chord, where $r$ is the number of vertices between the ends of the chord, $1 \leq r < \lfloor k/2 \rfloor$. We give a unified method to construct maximum $C^{(r)}_k$-packings and minimum $C^{(r)}_k$-coverings. Especially, for $G = C^{(r)}_6(r = 1, 2)$, $C^{(r)}_7(r = 1, 2)$ and $C^{(r)}_8(r = 1, 2, 3)$, we construct the maximum $(v, G, \lambda)$-PD and the minimum $(v, G, \lambda)$-CD.

Keywords: $G$-packing design; $G$-covering design; $G$-holey design; $G$-incomplete design; $G$-incomplete holey design

1. Introduction

A complete multigraph of order $v$ and index $\lambda$, denoted by $\lambda K_v$, is a graph with $v$ vertices, where any two distinct vertices $x$ and $y$ are joined by $\lambda$ edges $\{x, y\}$. A $t$-partite graph is one whose vertex set can be partitioned into $t$ subsets $X_1, X_2, \ldots, X_t$, such that two ends of each edge lie in distinct subsets, respectively. Such a partition

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(X_1, X_2, \ldots, X_t) is a \emph{t-partition} of the graph. A \emph{complete t-partition} graph with replication \( \lambda \) is a t-partition graph with \( t \)-partition \((X_1, X_2, \ldots, X_t)\), in which each vertex of \( X_i \) is joined to each vertex of \( X_j \) by \( \lambda \) times (where \( i \neq j \)). Such a graph is denoted by \( \lambda K_{n_1, n_2, \ldots, n_t} \), if \( |X_i| = n_i \) \((1 \leq i \leq t) \). We denote a path of \( k \) vertices by \( P_k \) and an undirected cycle of length \( m \) by \( C_m \). By \( C_m^r \) we mean one cycle of length \( m \) with one chord, where \( r \) is the number of vertices between the ends of the chord, \( 1 \leq r < [m/2] \).

Let \( G \) be a finite simple graph. A \emph{G-packing design} (\emph{G-covering design}, \emph{G-design}) of \( \lambda K_v \), denoted by \((v, G, \lambda)\)-PD \(((v, G, \lambda)\)-CD, \((v, G, \lambda)\)-GD), is a pair \((X, \mathcal{B})\), where \( X \) is the vertex set of \( K_v \) and \( \mathcal{B} \) is a collection of subgraphs of \( K_v \), called \emph{blocks}, such that each block is isomorphic to \( G \) and any two distinct vertices in \( K_v \) are joined in at most \((\lambda \geq 1)\) \( \lambda \) blocks of \( \mathcal{B} \). The \emph{packing} (\emph{covering}) number \( p(v, G, \lambda) \) \((c(v, G, \lambda))\) is the maximum (minimum) number of blocks in any \((v, G, \lambda)\)-PD \(((v, G, \lambda)\)-CD). It is well known that

\[
p(v, G, \lambda) \leq \left\lfloor \frac{\lambda v(v-1)}{2e(G)} \right\rfloor \leq \left\lfloor \frac{\lambda v(v-1)}{2e(G)} \right\rfloor \leq c(v, G, \lambda),
\]

where \( e(G) \) denotes the number of edges in \( G \), \( \lfloor x \rfloor \) denotes the greatest (least) integer \( y \) such that \( y \leq x \) \((y \geq x) \). A \((v, G, \lambda)\)-PD \(((v, G, \lambda)\)-CD), \((X, \mathcal{B})\), is called \emph{optimal} if \( |\mathcal{B}| = p(v, G, \lambda) \) \((c(v, G, \lambda))\). For convenience, we denote an optimal \((v, G, \lambda)\)-PD \(((v, G, \lambda)\)-CD) satisfying \( p(v, G, \lambda) = \lfloor \lambda v(v-1)/2e(G) \rfloor \) \((c(v, G, \lambda) = \lfloor \lambda v(v-1)/2e(G) \rfloor)\) by \((v, G, \lambda)\)-OPD \(((v, G, \lambda)\)-OCD). Obviously, there exists a \((v, G, \lambda)\)-GD if and only if \( p(v, G, \lambda) = c(v, G, \lambda) \). So a \((v, G, \lambda)\)-GD can be regarded as a \((v, G, \lambda)\)-OPD or a \((v, G, \lambda)\)-OCD. The \emph{leave graph} \( L_j(\mathcal{D}) \) of a packing design \( \mathcal{D} \) is a subgraph of \( \lambda K_v \) and its edges are the complement of \( \mathcal{D} \) in \( \lambda K_v \). The number of edges in \( L_j(\mathcal{D}) \) is denoted by \( |L_j(\mathcal{D})| \). Especially, when \( \mathcal{D} \) is optimal, \(|L_j(\mathcal{D})|\) is called \emph{leave-edge number} and is denoted by \( l_j(v) \). Similarly, the \emph{excess graph} \( R_j(\mathcal{D}) \) of a covering design \( \mathcal{D} \) is a subgraph of \( \lambda K_v \) and its edges are the complement of \( \mathcal{D} \) in \( \lambda K_v \). When \( \mathcal{D} \) is optimal, \(|R_j(\mathcal{D})|\) is called the \emph{repeat-edge number} and denoted by \( r_j(v) \). Generally, the symbols \( L_j(\mathcal{D}), l_j(v), R_j(\mathcal{D}) \) and \( r_j(v) \) can be denoted by \( L_j, l_j, R_j \) and \( r_j \) briefly. For some graphs, which have less vertices and less edges, the problem of their graph designs, packing designs and covering designs has already been researched (see [1–10,12–18]).

Let \((X_1, X_2, \ldots, X_t)\) be the \( t \)-partition of \( \lambda K_{n_1, n_2, \ldots, n_t} \), and \( |X_i| = n_i \). Denote \( v = \sum_{i=1}^t n_i \) and \( \mathcal{B} = \{X_1, X_2, \ldots, X_t\} \). For any given graph \( G \), if the edges of \( \lambda K_{n_1, n_2, \ldots, n_t} \) can be decomposed into edge-disjoint subgraphs \( \mathcal{A} \), each of which is isomorphic to \( G \) and is called \emph{block}, then the system \((X, \mathcal{B}, \mathcal{A})\) is called a \emph{holey G-design} with index \( \lambda \), denoted by \( G-\text{HD}_\lambda(T) \), where \( T = n_1 n_2 \cdots n_t \) is the \emph{type} of the holey \( G \)-design. Usually, the type is denoted by exponential form, for example, the type \( 1^3 2^2 3^1 \cdots \) denotes \( i \) occurrences of 1, \( r \) occurrences of 2, etc. A \( G-\text{HD}_\lambda(1^{r-w}w^1) \) is called an \emph{incomplete G-design}, denoted by \( G-\text{ID}_\lambda(v;w) = (V, W, \mathcal{A}) \), where \(|V| = v, |W| = w \) and \( W \subseteq V \). Obviously, a \((v, G, \lambda)\)-GD is a \( G-\text{HD}_\lambda(1^r) \) or a \( G-\text{ID}_\lambda(v;w) \) with \( w = 0 \) or 1. Let \( H_1, H_2 \) and \( W \) be three disjoint sets. A \( G-\text{ID}_\lambda(h_1, h_2;w) \) is a pair \(((H_1, H_2, W), \mathcal{A})\), where \( \mathcal{A} \) is a collection of subgraphs in \( H_1 \cup H_2 \cup W \), called \emph{blocks}, such that each block is
isomorphic to $G$ and any two distinct vertices $x, y$ are jointed in
\[
\begin{align*}
\text{exactly } \lambda \text{ blocks of } \mathcal{A} & \quad \text{if } x, y \in H_1 \text{ or } x, y \in H_2 \text{ or } x \in H_1 \cup H_2, \; y \in W, \\
\text{none block of } \mathcal{A} & \quad \text{otherwise}.
\end{align*}
\]

For $\text{HD}_{\lambda,1}^G$, $\text{ID}_{\lambda,1}^G$, and $\text{IHD}_{\lambda,1}^G$, the subscript can be omitted when $\lambda = 1$.

In this paper, the discussed graphs are $C_k^{(r)}$. We provide a method to construct optimal packings and optimal coverings. The general structures will be given. Especially, for $k = 6, 7, 8$, and any $r$, we have successfully determined the values of $p(v; C_k^{(r)}; \lambda)$ and $c(v; C_k^{(r)}; \lambda)$, where $v \geq k$. The existence spectrums of these graph-designs with any index have been solved in [11].

2. General structures

Theorem 2.1. Let $G$ be a simple graph. For positive integers $h, m, \lambda$ and nonnegative integer $w$, if there exist $G\text{-HD}_\lambda^G(hm)$, $G\text{-ID}_\lambda^G(h + w; w)$ and $(w, G, \lambda)$-OPD (or $(h + w, G, \lambda)$-OPD), then there exists $(mh + w, G, \lambda)$-OPD with the same leave graph to $(w, G, \lambda)$-OPDs (or $(h + w, G, \lambda)$-OPDs). The conclusion still holds by replacing OPD with OCD.

Proof. Let $X = (Z_h \times Z_m) \cup W$, where $W$ is a $w$-set. Suppose there exist
\[
G\text{-HD}_\lambda^G(hm) = (Z_h \times Z_m, \mathcal{A}),
\]
\[
G\text{-ID}_\lambda^G(h + w; w) = ((Z_h \times \{i\}) \cup W, \mathcal{B}_i), i \in Z_m \text{ or } i \in Z_m \setminus \{0\},
\]
\[
(w, G, \lambda)\text{-OPD} = (W, \mathcal{C}) \quad \text{or} \quad (h + w, G, \lambda)\text{-OPD} = ((Z_h \times \{0\}) \cup W, \mathcal{D}),
\]
then $(X, \Omega)$ is a $(mh + w, G, \lambda)$-OPD, where $\Omega = \mathcal{A} \cup (\bigcup_{i=0}^{m-1} \mathcal{B}_i) \cup \mathcal{C}$ or $\mathcal{A} \cup (\bigcup_{i=1}^{m-1} \mathcal{B}_i) \cup \mathcal{D}$. Note that
\[
|\Omega| = \left\lfloor \frac{\lambda \left( \frac{hm+w}{2} \right)}{e(G)} \right\rfloor
\]
\[
= \begin{cases} \\
\frac{\lambda \left( \frac{m}{2} \right) h^2}{e(G)} + m \times \frac{\lambda \left( \left( \frac{h}{2} \right) + wh \right)}{e(G)} + \left\lceil \frac{\lambda \left( \frac{w}{2} \right)}{e(G)} \right\rceil \\
\frac{\lambda \left( \frac{m}{2} \right) h^2}{e(G)} + (m - 1) \times \frac{\lambda \left( \left( \frac{h}{2} \right) + wh \right)}{e(G)} + \left\lceil \frac{\lambda \left( \frac{w+h}{2} \right)}{e(G)} \right\rceil \\
\end{cases}
\]
\[
= |\mathcal{A}| + \sum_{i=0}^{m-1} |\mathcal{B}_i| + |\mathcal{C}|
\]
\[
= |\mathcal{A}| + \sum_{i=1}^{m-1} |\mathcal{B}_i| + |\mathcal{D}|
\]
if $(W, \mathcal{C})$ is a $(w, G, \lambda)$-OCD, then a $(mh + w, G, \lambda)$-OCD will be obtained, since the above equation still holds by replacing the symbol $\lfloor \rfloor$ by $\lceil \rceil$. \qed
However, the theorem can not be used to construct all orders \( mh + w \). For example, when \( G-\text{HD}_2(h^m) \) exists only for odd \( m \), or \( G-\text{ID}_2(h + w; w) \) merely exists for smaller \( w \). Thus, we have to present other structures, such as IHD, etc.

**Theorem 2.2.** For given simple graph \( G \) and positive integers \( h, w, t \) and \( \lambda \), if there exist \( G-\text{HD}_2(h^{2t+1}) \), \( G-\text{IHD}_2(h, h; w) \) and \( (h + w, G, \lambda)\)-OPD, then there exists \( ((2t + 1)h + w, G, \lambda)\)-OPD with the same leave graph to \( (h + w, G, \lambda)\)-OPDs. The conclusion still holds by replacing OPD with OCD.

**Proof.** Let \( X = (Z_h × Z_{2t+1}) ∪ W \), where \(|W| = w\). Suppose there exist

\[
G-\text{HD}_2(h^{2t+1}) = (Z_h × Z_{2t+1}, \mathcal{A}),
\]

\[
G-\text{IHD}_2(h, h; w) = ((Z_h × \{2i\}, Z_h × \{2i + 1\}, W), \mathcal{B}_i) \text{ for } 0 ≤ i ≤ t - 1, \text{ and}
\]

\[
(h + w, G, \lambda)\text{-OPD} = ((Z_h × \{2i\} ∪ W, \mathcal{C}), (X, \mathcal{A} ∪ (\bigcup_{i=0}^{t-1} \mathcal{B}_i) ∪ \mathcal{C}) \text{ forms a}
\]

\[
((2t + 1)h + w, G, \lambda)\text{-OPD}. \text{ In fact, we have}
\]

\[
|\mathcal{A}| + t|\mathcal{B}_i| + |\mathcal{C}| = \frac{\lambda \left( \frac{2t+1}{2} \right) h^2}{e(G)} + \lambda t(hw + h(h - 1)) + \left\lceil \frac{\lambda \left( \frac{w + h}{2} \right)}{e(G)} \right\rceil = \left\lceil \frac{\lambda \left( \frac{(2t+1)h+w}{2} \right)}{e(G)} \right\rceil.
\]

And, replacing \( [ ] \) by \( \lceil \rceil \), we can verify the conclusion for OCD. \( \Box \)

**Lemma 2.3** (Qingde Kang et al. [11]). There exist the following holey graph designs:

1. \( C_{2k}^{(r)}\text{-HD}((2k + 1)^r) \) for \( t ≥ 2 \) and even \( r \).
2. \( C_{2k}^{(r)}\text{-HD}((2k + 1)^{2r+1}) \) for \( t ≥ 1 \) and odd \( r \).
3. \( C_{2k-1}^{(r)}\text{-HD}((2k)^{2r+1}) \) for \( k ≥ 3, t ≥ 1 \) and \( 1 ≤ r ≤ k - 2 \).
4. \( C_{2k-1}^{(r)}\text{-HD}((4k)^{2r+1}) \) for \( k ≥ 3, t ≥ 1 \) and \( 1 ≤ r ≤ k - 2 \).

**Lemma 2.4** (Qingde Kang et al. [11]). If there exists a \( C_{2k}^{(r)}\text{-ID}(2k + 1 + w; w) \) for odd \( r \), then there are \( \lceil (k + 2)/3 \rceil \) non-negative integers \( j_0, j_1, \ldots, j_{\lceil (k-1)/3 \rceil} \) such that

\[
\sum_{i=0}^{\lceil \frac{k-1}{2} \rceil} j_i = w \quad \text{and} \quad \sum_{i=0}^{\lceil \frac{k-1}{3} \rceil} ij_i ≤ \min \left\{ \frac{k}{2}, k^2 - w \right\}.
\]

**Corollary 2.5** (Qingde Kang et al. [11]). There exists no \( C_{2k-1}^{(r)}\text{-ID}(2k + 1 + w; w) \) for the parameters: \( (k, r) = (2, 1) \) and \( 5 ≤ w ≤ 9 \); \( (k, r) = (3, 1) \) and \( 10 ≤ w ≤ 13 \); \( (k, r) = (4, 1), (4, 3) \) and \( w = 17 \).

**Lemma 2.6** (Qingde Kang et al. [11]). There exists no \( C_{2k-1}^{(r)}\text{-ID}(2k + w; w) \) for any \( w ≥ 0 \).
Lemma 2.7 (Qingde Kang and Zhihe Liang [10]). Given positive integers \( v, \lambda, \) and \( \mu. \) Let \( X \) be a \( v \)-set.

1) Suppose there exist both a \((v, G, \lambda)\)-OPD \( = (X, \mathcal{D}) \) (with leave graph \( L_j(\mathcal{D}) \)) and a \((v, G, \mu)\)-OPD \( = (X, \mathcal{E}) \) (with leave graph \( L_\mu(\mathcal{E}) \)). If \( |L_j(\mathcal{D})| + |L_\mu(\mathcal{E})| = l_{\lambda+\mu}, \) then there exists a \((v, G, \lambda + \mu)\)-OPD and its leave graph is just \( L_j(\mathcal{D}) \cup L_\mu(\mathcal{E}) \);

2) Suppose there exist both a \((v, G, \lambda)\)-ODC \( = (X, \mathcal{D}) \) (with excess graph \( R_j(\mathcal{D}) \)) and a \((v, G, \mu)\)-OCD \( = (X, \mathcal{E}) \) (with excess graph \( R_\mu(\mathcal{E}) \)). If \( |R_j(\mathcal{D})| + |R_\mu(\mathcal{E})| = r_{\lambda+\mu}, \) then there exists a \((v, G, \lambda + \mu)\)-OCD and its excess graph is just \( R_j(\mathcal{D}) \cup R_\mu(\mathcal{E}) \);

3) Suppose there exist both a \((v, G, \lambda)\)-OPD \( = (X, \mathcal{D}) \) (with leave graph \( L_j(\mathcal{D}) \)) and a \((v, G, \mu)\)-OCD \( = (X, \mathcal{E}) \) (with excess graph \( R_\mu(\mathcal{E}) \)). If \( L_j(\mathcal{D}) \supset R_\mu(\mathcal{E}) \) and \( |L_j(\mathcal{D})| - |R_\mu(\mathcal{E})| = l_{\lambda+\mu}, \) then there exists a \((v, G, \lambda + \mu)\)-OPD and its leave graph is just \( L_j(\mathcal{D}) \setminus R_\mu(\mathcal{E}) \);

4) Suppose there exist both a \((v, G, \lambda)\)-OCD \( = (X, \mathcal{D}) \) (with excess graph \( R_j(\mathcal{D}) \)) and a \((v, G, \mu)\)-OPD \( = (X, \mathcal{E}) \) (with leave graph \( L_\mu(\mathcal{E}) \)). If \( R_j(\mathcal{D}) \supset L_\mu(\mathcal{E}) \) and \( |R_j(\mathcal{D})| - |L_\mu(\mathcal{E})| = r_{\lambda+\mu}, \) then there exists a \((v, G, \lambda + \mu)\)-OCD and its excess graph is just \( R_j(\mathcal{D}) \setminus L_\mu(\mathcal{E}) \).

3. \( C_6^{(1)} \) and \( C_6^{(2)} \)

It is known that there exists a \((v, C_6^{(1)} , \lambda)\)-GD if and only if \( \lambda v(v - 1) \equiv 0(\text{mod} \ 14) \) and \( v \geq 6, \) and there exists a \((v, C_6^{(2)} , \lambda)\)-GD if and only if \( \lambda v(v - 1) \equiv 0(\text{mod} \ 14), \) \( v \geq 6 \) and \( (v, \lambda ) \neq (7, 1) \) (see [11]). For convenience, we denote \( C_6^{(1)} \) (or \( C_6^{(2)} \)) by \((a, b, c, d, e, f)\), where the edges on \( C_6 \) are \( ab, bc, cd, de, ef, fa \) and the chord is \( ac \) (or \( ad \)). First, in order to construct the optimal packing designs and covering designs for \( \lambda = 1, \) by Theorems 2.1 and 2.2 and the following tables, we only need to give the constructions of HD, ID or IHD, and OPD for the pointed orders, where the leave graph of OPD has to be a subgraph of \( C_6^v. \) However, the needed HD, ID, and IHD have been shown in [11], so we only need to construct the OPD listed in the Tables 1(a) and (b).

3.1. Packings and coverings for \( \lambda = 1 \)

Lemma 3.1. There exist \((w, C_6^{(1)}, 1)\)-OPD for \( w = 6, 9, 10, 11, 12, 17, 18, 19 \) and 20.

Proof. Let \((w, C_6^{(1)}, 1)\)-OPD \( = (X, \mathcal{B}) \).

\( w = 6: \) \( X = Z_6 \cup \{a, b\}, (0, a, 1, 3, b, 2), (2, a, 3, 0, b, 1) \).

\( L(\mathcal{B}) = \{(a, b)\}. \)

\( w = 9: \) \( X = Z_7 \cup \{a, b\}, \)

\( (a, 0, 3, 4, 1, 2), (4, 2, 6, b, 1, a), (5, 2, 3, 6, 1, 0), (b, 0, 4, 5, 1, 3), (5, a, 6, 0, 2, b). \)

\( L(\mathcal{B}) = \{(a, b)\}. \)
Table 1

(a) For $C_{6}^{(1)}$

<table>
<thead>
<tr>
<th>v (mod 14)</th>
<th>HD</th>
<th>ID</th>
<th>IHD</th>
<th>OPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7^{2t-1}</td>
<td>(16;9)</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>7^{2t-1}</td>
<td></td>
<td>(7,7;10)</td>
<td>17</td>
</tr>
<tr>
<td>4</td>
<td>7^{2t-1}</td>
<td></td>
<td>(7,7;11)</td>
<td>18</td>
</tr>
<tr>
<td>5</td>
<td>7^{2t-1}</td>
<td></td>
<td>(7,7;12)</td>
<td>19</td>
</tr>
<tr>
<td>6</td>
<td>7^{2t-1}</td>
<td></td>
<td>(7,7;13)</td>
<td>20</td>
</tr>
<tr>
<td>9</td>
<td>7^{2t+1}</td>
<td>(9;2)</td>
<td></td>
<td>9</td>
</tr>
<tr>
<td>10</td>
<td>7^{2t+1}</td>
<td>(10;3)</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>11</td>
<td>7^{2t+1}</td>
<td>(11;4)</td>
<td></td>
<td>11</td>
</tr>
<tr>
<td>12</td>
<td>7^{2t+1}</td>
<td>(12;5)</td>
<td></td>
<td>12</td>
</tr>
<tr>
<td>13</td>
<td>7^{2t+1}</td>
<td>(13;6)</td>
<td></td>
<td>6</td>
</tr>
</tbody>
</table>

(b) For $C_{6}^{(2)}$

<table>
<thead>
<tr>
<th>v (mod 7)</th>
<th>HD</th>
<th>ID</th>
<th>OPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>7^{t}</td>
<td>(9;2)</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>7^{t}</td>
<td>(10;3)</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>7^{t}</td>
<td>(11;4)</td>
<td>11</td>
</tr>
<tr>
<td>5</td>
<td>7^{t}</td>
<td>(12;5)</td>
<td>12</td>
</tr>
<tr>
<td>6</td>
<td>7^{t}</td>
<td>(13;6)</td>
<td>13</td>
</tr>
</tbody>
</table>

For $w = 10$:

\[ X = Z_6 \cup \{a, b, c, d\}, \]
\[
(a, 2, 0, b, 1, c), \ (a, 3, 1, 0, 5, 4), \ (d, 0, 3, 2, c, 4),
(d, 1, 2, b, 5, a), \ (c, 3, 5, 2, 4, 0), \ (b, 3, 4, 1, 5, d).
\]
\[ L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \]

For $w = 11$:

\[ X = Z_9 \cup \{a, b\}, \]
\[
(a, 8, 0, 2, 7, 3), \ (b, 6, 4, 8, 5, 0), \ (1, 7, 4, 0, 6, 5), \ (1, 3, 0, 7, b, a),
(b, 8, 3, 4, 5, 2), \ (a, 4, 2, 1, 6, 7), \ (2, 6, 3, 5, 7, 8),
L(\mathcal{B}) = \{(a, 6), (6, 8), (8, 1), (1, b), (b, 5), (5, a)\}. \]

For $w = 12$:

\[ X = Z_8 \cup \{a, b, c, d\}, \]
\[
(1, 2, 3, 6, 7, 5), \ (5, 2, 6, 4, 7, 3), \ (d, 6, 1, 7, b, 3),
(a, 7, 0, 3, 4, d), \ (a, 4, 1, c, 0, 6), \ (d, 2, 0, 4, 5, b),
(b, 1, 0, 5, c, 6), \ (c, 2, 7, d, 5, a), \ (2, b, 4, c, 3, a).
\]
\[ L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \]

For $w = 17$:

\[ X = Z_{13} \cup \{A, B, C, D\}, \]
\[
(a, 3, 0, 1, 10, 8), \ (4, 2, 6, 11, 10, 5), \ (D, 5, 1, 8, 7, 9), \ (3, C, 4, 9, 12, 1),
\]
There exist Theorem 3.2. The leave graphs

\[ L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\} \]

\( w = 18 \):

\( X = \mathbb{Z}_{16} \cup \{A,B\} \)

\((A,15,0,12,14,6), (0,9,1,4,6,2), (4,10,9,12,8,3), (6,10,12,2,8,11),
(A,12,3,15,10,7), (2,5,3,0,8,1), (7,5,13,3,9,11), (A,11,B,0,13,4),
(B,14,5,10,13,9), (6,0,7,2,10,8), (2,4,11,15,7,9), (3,1,10,11,5,6),
(A,14,1,12,15,8), (4,12,5,1,7,8), (5,9,8,13,11,0), (B,15,4,14,11,3),
(A,13,2,14,15,9), (0,10,14,3,7,4), (B,13,6,9,14,8), (1,6,15,13,12,11),
(B,12,7,14,13,1),
\]

\( L(\mathcal{B}) = \{(A,10),(10,B),(B,2),(2,15),(15,5),(5,A)\} \).

\( w = 19 \):

\( X = \mathbb{Z}_{15} \cup \{A,B,C,D\} \)

\((D,9,0,12,5,10), (A,7,1,0,8,13), (3,C,7,4,9,11), (5,C,8,9,12,11),
(D,13,2,3,8,11), (A,14,2,5,6,8), (3,1,5,9,14,12), (D,4,B,6,7,12),
(D,8,1,13,3,14), (B,11,1,9,7,8), (4,0,5,13,14,11), (C,13,0,10,12,6),
(B,14,0,2,7,13), (3,A,4,10,6,9), (4,1,6,11,13,12), (A,12,C,4,2,9),
(B,12,2,1,10,9), (A,5,D,7,0,11), (4,14,8,2,10,13), (3,D,6,14,7,10),
(C,11,2,6,13,9), (A,6,0,3,B,10), (5,B,7,11,10,14), (C,14,1,12,8,10),
\]

\( L(\mathcal{B}) = \{(A,B),(B,C),(C,D)\} \).

\( w = 20 \):

\( X = (\mathbb{Z}_9 \times \mathbb{Z}_2) \cup \{a,b\} \)

\((0_0,a,1,1,3_1,5_0,1_0), (0_0,b,2_1,7_0,3_1,6_1), (0_1,6_0,0_0,7_0,6_1,1_1) \mod(9,-)\).

\( L(\mathcal{B}) = \{(a,b)\} \).

**Theorem 3.2.** There exist \((v,C_6^{(1)},1)\)-OPD and \((v,C_6^{(1)},1)\)-OCD for \(v \geq 6\).

**Proof.** By Theorems 2.1 and 2.2 and Lemma 3.1. The leave graphs \(L_1\) for these OPDs are as follows:

\[
\begin{array}{c|c|c|c}
  v \equiv (\text{mod } 7) & 2,6 & 3,5 & 4 \\
  L_1 & P_2 & P_4 & C_6 \\
\end{array}
\]
Obviously, each $L_1$ is a subgraph of the graph $C_6^{(1)}$. So, the optimal covering designs can be obtained by adding a block containing this $L_1$. And their excess graph $R_1$ can be listed in the table:

\[
\begin{array}{c|ccc}
\ell \equiv (\text{mod} \ 7) & 2, 6 & 3, 5 & 4 \\
R_1 & C_6 & P_5 & P_2 \\
\end{array}
\]

Lemma 3.3. $p(6, C_6^{(2)}, 1) = 1$, $c(6, C_6^{(2)}, 1) = 3$,
$p(7, C_6^{(2)}, 1) = 2$, $c(7, C_6^{(2)}, 1) = 4$.

Proof. It is easy to see that $G = K_6 \setminus C_6^{(2)}$ is a graph as follows:

\[
\begin{array}{c}
\end{array}
\]

Obviously, $C_6^{(2)}$ is not a subgraph of $G$. Thus, the packing number $p(6, C_6^{(2)}, 1) = 1$. However, there exists a $(6, C_6^{(2)}, 1)$-OCD $(Z_6, \mathcal{A})$ as follows:

\[
\mathcal{A} = \{(0, 3, 5, 1, 4, 2), (2, 4, 0, 5, 3, 1), (1, 0, 5, 4, 3, 2)\},
\]

\[
R(\mathcal{A}) = \{(0, 1), (0, 5), (1, 2), (1, 4), (2, 4), (3, 5)\}.
\]

We know that there is no $(7, C_6^{(2)}, 1)$-GD (see [11]). Therefore, the packing number $p(7, C_6^{(2)}, 1) < 3$ and the covering number $c(7, C_6^{(2)}, 1) > 3$. But we have

\[
\mathcal{B} = \{(1, 5, 2, 0, 3, 6), (4, 3, 5, 0, 6, 2)\},
\]

\[
L(\mathcal{B}) = \{(1, 3), (1, 2), (2, 3), (4, 5), (4, 6), (5, 6), (1, 4)\},
\]

\[
\mathcal{C} = \mathcal{B} \cup \{(4, 5, 3, 1, 2, 6), (0, 2, 3, 1, 5, 6)\},
\]

\[
R(\mathcal{C}) = \{(3, 5), (2, 6), (0, 1), (0, 2), (0, 6), (1, 3), (1, 5)\}.
\]

So, $p(7, C_6^{(2)}, 1) = 2$ and $c(7, C_6^{(2)}, 1) = 4$. \hfill \Box

Lemma 3.4. There exist $(w, C_6^{(2)}, 1)$-OPD for $9 \leq w \leq 13$.

Proof. Let $(w, C_6^{(2)}, 1)$-OPD $= (X, \mathcal{B})$.

\[
w = 9: \quad X = Z_7 \cup \{a, b\}, (a, 2, 0, 3, 6, 1),
\]

\[
(b, 3, 2, 4, a, 0), (3, 4, 0, 5, b, 1), (4, 5, a, 6, 2, 1), (5, 1, 0, 6, b, 2).
\]

\[
L(\mathcal{B}) = \{a, b\}.
\]
Lemma 3.6. \(\text{Theorem 3.5. There exist}\)

\[ L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \]

\( w = 10: \) \( X = Z_6 \cup \{a, b, c, d\} \)
\( (a, 2, 4, 3, b, 0), \ (c, 0, 2, 5, b, 1), \ (d, 1, 5, 4, c, 2), \)
\( (a, 5, 0, 1, 3, c), \ (b, 4, 1, 2, 3, d), \ (d, 5, 3, 0, 4, a) \).

\( w = 11: \) \( X = Z_9 \cup \{a, b\} \)
\( (a, 7, 4, 0, 2, b), \ (a, 8, 3, 1, 7, 6), \ (4, 8, 7, 2, 5, a), \)
\( (4, 1, 8, 5, 6, b), \ (3, 4, 6, 2, 1, b), \ (5, 0, 6, 3, 7, b), \ (1, 6, 8, 0, 7, 5) \).

\( w = 12: \) \( X = Z_8 \cup \{a, b, c, d\} \)
\( (a, 4, 3, c, 5, 0), \ (a, 5, 3, d, 0, 6), \ (b, 0, 7, d, 4, 1), \)
\( (7, a, 1, 3, b, 4), \ (2, a, 3, 6, c, 0), \ (6, b, 2, 4, c, 1), \)
\( (7, b, 5, 6, d, 2), \ (7, c, 2, 5, d, 1), \ (1, 2, 3, 0, 4, 5) \).

\( w = 13: \) \( X = Z_{11} \cup \{A, B\} \)
\( (A, 8, 5, 0, 9, 7), \ (A, 9, 5, 1, 0, 4), \ (A, 10, 4, 2, 7, 5), \)
\( (4, 8, 3, 5, 6, 9), \ (3, A, 6, 7, 1, 10), \ (B, 8, 7, 4, 6, 3), \)
\( (B, 9, 2, 5, 10, 0), \ (1, 9, 10, 6, 0, 3), \)
\( (B, 10, 2, 6, 8, 1), \ (7, B, 2, 0, 8, 10), \ (3, 9, 8, 2, 1, 4) \).

\( L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \)

\( \square \)

**Theorem 3.5.** There exist \((v, C_6^{(2)}, 1)\)-OPD and \((v, C_6^{(2)}, 1)\)-OCD for \( v \geq 8 \). And, \( p(6, C_6^{(2)}, 1) = 1, \ c(6, C_6^{(2)}, 1) = 3, \ p(7, C_6^{(2)}, 1) = 2, \ c(7, C_6^{(2)}, 1) = 4. \)

**Proof.** It is easy to prove by Theorem 2.1 and Lemmas 3.3 and 3.4. Note that the leave graphs \( L_1 \) for \((v, C_6^{(2)}, 1)\)-OPD are same to \((v, C_6^{(1)}, 1)\)-OPD. The further proof is similar to Theorem 3.2. \( \square \)

### 3.2. Packings and coverings for \( \lambda > 1 \)

**Lemma 3.6.** There exist \((v, C_6^{(r)}, \lambda)\)-OPD and \((v, C_6^{(r)}, \lambda)\)-OCD for \( v \equiv 2, 6 \pmod{7} \) and \( \lambda > 1 \) (where \( r = 1, 2 \)).

**Proof.** When \((v, r) \neq (6, 2)\), by Lemma 2.7, we have the following table:

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_\lambda )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>( r_\lambda )</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

\( L_\lambda = L_1 \cup L_{\lambda-1} \)

\( R_\lambda = R_{\lambda-1} \setminus L_1 \),

where \( L_1 = P_2 \) and \( R_1 = C_6 \) by Theorems 3.2 and 3.5.
When \((v, r) = (6, 2)\), there is \((v, C_r^{(2)}, 1)\)-OPD, but there is no \((v, C_r^{(2)}, 1)\)-OPD. We may construct \((v, C_r^{(2)}, 2)\)-OPD = \((Z_6, B)\) and \((v, C_r^{(2)}, 2)\)-OCD = \((Z_6, A)\) as follows:

\[
B = (0, 1, 3, 2, 5, 4), (1, 2, 3, 5, 0, 4), (2, 0, 3, 4, 5, 1), (3, 1, 0, 5, 2, 4),
\]

\[
L_2(B) = \{(0, 1), (2, 3)\};
\]

\[
A = B \cup \{(4, 0, 1, 5, 2, 3)\},
\]

\[
R_2(A) = \{(0, 4), (3, 4), (4, 5), (5, 1), (5, 2)\}.
\]

Furthermore, by the leave graph \(L_2(B)\), the excess graph \(R_2(A)\) and the construction of \((6, C_r^{(2)}, 1)\)-OCD and the optimal \((6, C_r^{(2)}, 1)\)-PD listed in Lemma 3.3, we may get the following table:

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_\lambda)</td>
<td>8</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>(L_\lambda)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(r_\lambda)</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(R_\lambda)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Lemma 3.7.** There exist \((v, C_r^{(r)}, \lambda)\)-OPD and \((v, C_r^{(r)}, \lambda)\)-OCD for \(v \equiv 3, 5 \pmod{7}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

**Proof.** By Lemma 2.7, Theorems 3.2 and 3.5, we have the following table:

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_\lambda)</td>
<td>3</td>
<td>6</td>
<td>2</td>
<td>5</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>(L_\lambda)</td>
<td>(P_4)</td>
<td>(L_1 \cup L_1)</td>
<td>(L_1 \setminus R_2)</td>
<td>(L_1 \cup L_3)</td>
<td>(L_3 \setminus R_2)</td>
<td>(L_1 \cup L_5)</td>
</tr>
<tr>
<td>(r_\lambda)</td>
<td>4</td>
<td>1</td>
<td>5</td>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>(R_\lambda)</td>
<td>(P_5)</td>
<td>(R_1 \setminus L_1)</td>
<td>(R_1 \cup R_2)</td>
<td>(R_2 \cup R_2)</td>
<td>(R_2 \cup R_3)</td>
<td>(R_2 \cup R_4)</td>
</tr>
</tbody>
</table>

**Lemma 3.8.** There exist \((v, C_r^{(r)}, \lambda)\)-OPD and \((v, C_r^{(r)}, \lambda)\)-OCD for \(v \equiv 4 \pmod{7}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

**Proof.** By Lemma 2.7, we have the following table:

<table>
<thead>
<tr>
<th>(\lambda)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l_\lambda)</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>(L_\lambda = L_{\lambda-1} \setminus R_1)</td>
<td>(\lambda) (= \lambda_{\lambda-1} \setminus R_1)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(r_\lambda)</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>(R_\lambda = R_1 \cup R_{\lambda-1})</td>
<td>(R_\lambda = R_1 \cup R_{\lambda-1})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

where \(L_1 = C_6\) and \(R_1 = P_2\) by Theorems 3.2 and 3.5.
Theorem 3.9. There exist $(v, C_7^{(r)}), \lambda$)-OPD and $(v, C_7^{(r)}), \lambda$)-OCD for $\lambda > 1$ (where $r = 1, 2$).

Proof. By the results of graph design with any index (see [11]), and Lemmas 3.6, 3.7 and 3.8.

4. $C_7^{(1)}$ and $C_7^{(2)}$

For convenience, we denote $C_7^{(1)}$ and $C_7^{(2)}$ by $(a, b, c, d, e, f, g)$, where the edges on $C_7$ are $ab, bc, cd, de, ef, fg, ga$ and the chord is $ac$ (or $ad$). It is known that there exist $(v, C_7^{(r)}, \lambda)$-GD if and only if $\lambda v(v - 1) \equiv 0 \pmod{16}$ and $v \geq 7$ (see [11]). First, in order to construct the optimal packing designs and covering designs for $\lambda = 1$, by Theorems 2.1 and 2.2 and the following table, we only need to give the constructions of HD, ID or IHD, and OPD for the pointed orders, where the leave graph of OPD has to be a subgraph of $C_7^{(r)}$. However, the needed HD, ID and IHD have been shown in [11], so we only need to construct the OPD listed in Table 2.

4.1. Packings and coverings for $\lambda = 1$

Lemma 4.1. There exist $(w, C_7^{(1)}, 1)$-OPD for $7 \leq w \leq 15$ and $18 \leq w \leq 25$.

Proof. Let $(w, C_7^{(1)}, 1)$-OPD $= (X, \mathcal{B})$.

$w = 7$: $X = Z_7, (6, 2, 0, 5, 3, 1, 4), (6, 1, 5, 2, 4, 0, 3)$.

$L(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$.

Table 2

<table>
<thead>
<tr>
<th>$v \pmod{16}$</th>
<th>HD</th>
<th>ID</th>
<th>IHD</th>
<th>OPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 10)</td>
<td>18</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 11)</td>
<td>19</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 12)</td>
<td>20</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 13)</td>
<td>21</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$16^{2r-1}$</td>
<td>(38; 22), (22, 6)</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$16^{2r-1}$</td>
<td>(39; 23), (23, 7)</td>
<td>7.23</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$16^{2r-1}$</td>
<td>(40; 24), (24, 8)</td>
<td>8.24</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>$16^{2r-1}$</td>
<td>(41; 25), (25, 9)</td>
<td>9.25</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 2)</td>
<td>10</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 3)</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 4)</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$8^{2r-1}$</td>
<td>(8, 8; 5)</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$8^{2r+1}$</td>
<td>(8, 8; 6)</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>$8^{2r+1}$</td>
<td>(8, 8; 7)</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>
\[ w = 8: \quad X = \mathbb{Z}_8, (3, 0, 5, 2, 4, 7, 6), (2, 0, 6, 1, 4, 5, 7), (1, 3, 7, 0, 4, 6, 5). \]
\[ L(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4)\}. \]

\[ w = 9: \quad X = \mathbb{Z}_9, \]
\[ (0, 3, 5, 1, 4, 2, 8), (4, 0, 6, 2, 7, 3, 8), (5, 4, 7, 8, 1, 3, 6), (6, 1, 7, 0, 2, 5, 8). \]
\[ L(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4)\}. \]

\[ w = 10: \quad X = \mathbb{Z}_{10}, \]
\[ (1, 4, 6, 8, 2, 5, 9), (2, 0, 6, 7, 1, 8, 4), (3, 5, 6, 9, 2, 7, 0), (4, 9, 7, 3, 8, 5, 0), (8, 0, 9, 3, 1, 5, 7). \]
\[ L(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}. \]

\[ w = 11: \quad X = \mathbb{Z}_9 \cup \{a, b\}, \]
\[ (a, 1, 5, 6, 4, 0, 2), (a, 3, 6, 7, 0, 5, 8), (b, 2, 7, a, 0, 3, 1), \]
\[ (b, 3, 8, 2, 4, 1, 6), (5, 3, 7, 1, 8, 4, b), (6, 0, 8, 7, 4, 5, 2). \]
\[ L(\mathcal{B}) = \{(a, b), (b, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, a)\}. \]

\[ w = 12: \quad X = \mathbb{Z}_9 \cup \{a, b, c\}, \]
\[ (b, 3, 4, a, 5, 6, 8), (b, 5, 1, a, 3, 2, 7), (1, 7, 3, 5, 8, 2, 4), (5, c, 0, 8, a, 6, 7), \]
\[ (6, 1, 0, 3, 8, 4, c), (7, a, 0, 4, 5, 2, c), (8, a, 1, 2, 6, 4, 7), (b, 0, 2, a, c, 3, 6). \]
\[ L(\mathcal{B}) = \{(a, b), (b, c)\}. \]

\[ w = 13: \quad X = \mathbb{Z}_9 \cup \{a, b, c, d\}, \]
\[ (3, b, 4, 8, d, 2, c), (4, c, 5, 6, 0, b, d), (5, 0, 3, 1, 7, 6, d), \]
\[ (3, d, a, c, 7, 0, 8), (4, 0, a, 6, 8, b, 7), (5, 1, a, 8, c, 6, b), \]
\[ (3, 7, 2, 0, c, 1, 6), (4, 6, 2, a, 7, 8, 1), (5, 8, 2, b, 1, d, 7). \]
\[ L(\mathcal{B}) = \{(a, b), (b, c), (c, d), (d, 0), (0, 1), (1, 2)\}. \]

\[ w = 14: \quad X = \mathbb{Z}_{10} \cup \{a, b, c, d\}, \]
\[ (4, 2, 5, 7, c, 8, 3), (0, 4, 1, a, 6, b, 5), (0, a, 2, 9, 5, c, 6), (0, 7, 3, 5, 8, d, b), \]
\[ (4, 9, 6, 1, d, 5, a), (1, b, 2, 7, 8, 6, 5), (1, 9, 3, a, c, 4, 8), (2, 6, 3, d, 9, a, 8), \]
\[ (4, d, 7, 1, c, 3, b), (8, 9, b, 7, 6, d, 0), (9, 0, c, 2, d, a, 7). \]
\[ L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \]

\[ w = 15: \quad X = \mathbb{Z}_{13} \cup \{A, B\}, \]
\[ (8, 4, 12, 6, 2, 5, 3), (A, 0, 8, 9, 4, 2, 3), (A, 2, 9, 7, 4, 3, 1), (A, 4, 10, 0, 7, 1, 5), \]
\[ (8, 2, 11, 6, 4, 5, 7), (B, 1, 8, 5, 6, 7, 11), (B, 0, 9, 6, 3, 7, 12). \]
\[(9, 1, 11, 4, B, 7, 10), (10, 3, 11, 0, 1, 6, 8), (9, 3, 12, 1, 2, 0, 5),
(11, A, 12, 0, 6, B, 5), (10, 5, 12, 2, 7, A, 6), (B, 2, 10, 1, 4, 0, 3).
\]

\[L(\mathcal{B}) = \{(A, B)\}.\]

\(w = 18: \)
\[X = Z_{16} \cup \{A, B\},\]
\[(0, 2, 15, A, 3, 7, 1), (13, 0, 14, 5, 11, 7, A), (7, 4, 10, 5, 1, B, 0),
(14, 3, 9, 10, 1, 11, 2), (3, B, 15, 6, 5, 4, 0), (14, 11, 15, 5, 4, 4, B),
(11, 0, 12, 5, 7, B, 6), (6, 0, 10, 15, 9, 8, 1), (10, B, 11, 9, 12, A, 2),
(12, 3, 13, 10, 8, 7, 6), (14, 6, 8, 11, A, 9, 7), (13, 5, 8, 4, 10, 12, 2),
(11, 4, 13, 1, 14, 10, 3), (12, 4, 14, A, 6, 13, B), (13, 7, 15, 12, 8, B, 9),
(1, 4, 15, 8, 3, 6, 2), (5, 0, 9, 1, 12, 7, 2), (2, 4, 8, 0, A, 1, 3), (4, 6, 9, 2, B, 5, 3),
\[L(\mathcal{B}) = \{(A, B)\}.\]

\(w = 19: \)
\[X = Z_{15} \cup \{A, B, C, D\},\]
\[(A, 0, D, 12, 5, 8, 1), (A, 2, C, 0, 9, 7, 3), (A, 4, 9, 10, 12, 0, 5),
(A, 6, 10, 0, 2, 4, 7), (A, 8, 12, 7, 0, 6, 13), (D, 8, B, 0, 1, 14, 4),
(D, 3, 9, 8, 4, 12, 1), (D, 5, 14, B, 4, 6, 2), (D, 7, 13, 4, 2, 8, 6),
(B, 2, 13, 10, 14, 8, 3), (B, 5, 9, 2, 11, 6, 1), (B, 7, 10, 1, 4, 5, 6),
(C, 1, 9, 6, 7, 5, 3), (C, 4, 10, 2, 7, 1, 5), (C, 6, 14, 3, 4, 11, 7),
(11, B, 12, 2, 3, 13, 0), (11, 9, 13, 5, 10, 8, C), (11, A, 14, 0, 4, 13, 1),
(11, D, 10, 3, 1, 2, 5), (12, 9, 14, 7, 8, 11, 3), (12, C, 13, 8, 0, 3, 6),
\[L(\mathcal{B}) = \{(A, B), (B, C), (C, D)\}.\]

\(w = 20: \)
\[X = Z_{16} \cup \{A, B, C, D\},\]
\[(B, 2, 15, 3, D, 4, 0), (C, 4, 15, 6, 12, 1, A), (D, A, 15, 5, 2, 6, B),
(0, 7, 15, 9, A, 11, 2), (1, 8, 15, 10, C, 14, 9), (3, A, 4, 8, D, 11, B),
(3, C, 5, A, 2, 9, 0), (3, 1, 6, 10, D, 13, 7), (4, B, 5, D, 6, C, 1),
(4, 7, 6, 8, 11, 10, 2), (5, 0, 6, 13, A, 14, 1), (7, 14, 8, A, 0, C, 2),
(7, B, 9, 12, C, 11, 1), (7, A, 10, 3, 2, 12, D), (8, 3, 9, 13, B, 12, 0),
(8, B, 10, 13, 1, D, 2), (9, 4, 10, 1, B, 14, D), (11, 15, 12, 5, 10, 14, 0),
(11, 3, 13, 8, C, 7, 5), (11, 4, 14, 5, 8, 12, 7), (12, 4, 13, 5, 9, 6, A),
(12, 3, 14, 2, 13, 0, 10), (13, 15, 14, 6, 11, 9, C),
\[L(\mathcal{B}) = \{(A, B), (B, C), (C, D), (D, 0), (0, 1), (1, 2)\}.\]
By Lemma 4.1, the leave graphs

Proof. $w = 21$: $X = \mathbb{Z}_{18} \cup \{A, B, C\},$

$(B, 0, 12, A, 6, 2, 3), \ (B, 2, 13, 12, 10, 14, 1), \ (B, 4, 14, 11, 15, 10, 5),$
$(B, 6, 15, 0, 1, 4, 7), \ (B, 8, 16, 2, 4, 0, 9), \ (B, 10, 17, 0, 2, 1, 11),$
$(A, 4, 17, 8, 6, 0, 11), \ (A, 0, 13, 1, 5, 6, 10), \ (A, 2, 14, 5, 0, 3, 1),$
$(A, 3, 15, 5, 8, 0, 7), \ (A, 5, 16, 3, 7, 1, 9), \ (C, 1, 12, 9, 4, 8, A),$
$(C, 3, 13, 4, 11, 5, 2), \ (C, 0, 14, 16, 1, 10, 4), \ (C, 6, 17, 3, 9, 7, 5),$
$(C, 7, 16, 6, 9, 8, 10), \ (12, 2, 17, 9, 5, 4, 6), \ (12, 3, 14, 9, 10, 2, 8),$
$(12, 4, 15, 2, 9, 11, 7), \ (13, 5, 17, 11, 2, 7, 6), \ (13, 7, 15, 1, 6, 3, 8),$
$(14, 8, 15, 9, C, 11, 13), \ (16, 11, 12, 5, 3, 10, 0), \ (16, 9, 13, 10, 11, 3, 4),$
(16, 17, 15, C, 8, 7, 10), \ (17, 7, 14, 6, 11, 8, 1).

$L(\mathcal{B}) = \{(A, B), (B, C)\}.$

$w = 22$: $X = \mathbb{Z}_{20} \cup \{A, B\},$

$(A, 3, 0, 8, 19, 15, 5), \ (A, 6, 1, 5, 10, 13, 7), \ (A, 8, 2, 7, 15, 11, 9),$
$(B, 4, 1, 18, 6, 12, 5), \ (B, 6, 2, 15, 17, 14, 7), \ (B, 8, 3, 12, 18, 13, 9),$
$(0, 6, 5, 3, 18, 7, 10), \ (2, 4, 0, 16, 8, 11, 14), \ (3, 9, 6, 8, 14, 19, 11),$
$(3, 7, 1, 11, 16, 18, 10), \ (4, 9, 5, 14, 10, 17, 8), \ (4, 10, 6, 19, 13, B, 14),$
$(5, 7, 8, 13, 17, 11, 2), \ (6, 11, 7, 16, 15, 9, 17), \ (7, 12, 4, 16, 9, 18, 17),$
$(7, 0, 9, 12, 16, 10, 19), \ (8, 1, 9, 19, 16, 13, 15), \ (8, 12, 10, 1, 19, 2, 18),$
(9, 2, 10, B, 15, 3, 14), \ (10, A, 11, 5, 16, 6, 15), \ (11, B, 12, 1, 13, 5, 18),$
(11, 0, 13, 2, 12, 15, 4), \ (12, A, 13, 4, 17, 5, 19), \ (12, 0, 14, 1, 16, 3, 17),$
(13, 6, 14, 18, 4, 19, 3), \ (14, A, 15, 1, 17, 2, 16), \ (16, A, 17, 19, 0, 18, B),$
(18, A, 19, B, 17, 0, 15).

$L(\mathcal{B}) = \{(A, B), (B, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, A)\}.$

$w = 23, 24$ and 25 can be obtained from $(w - 16, C_{17}^{(4)}, 1)$-OPD above and $C_{17}^{(4)}$-ID$(16 + (w - 16); w - 16)$ (see [11]), and its leave graph is as same as that of $(w - 16, C_{17}^{(4)}, 1)$-OPD. □

**Theorem 4.2.** There exist $(v, C_{17}^{(4)}, 1)$-OPD and $(v, C_{17}^{(4)}, 1)$-OCD for $v \geq 7$.

**Proof.** By Lemma 4.1, the leave graphs $L_1$ for these OPDs are:

<table>
<thead>
<tr>
<th>$v \equiv (\text{mod } 16)$</th>
<th>2, 15</th>
<th>3, 14</th>
<th>4, 13</th>
<th>5, 12</th>
<th>6, 11</th>
<th>7, 10</th>
<th>8, 9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_1$</td>
<td>$P_2$</td>
<td>$P_4$</td>
<td>$P_7$</td>
<td>$P_3$</td>
<td>$C_7$</td>
<td>$P_6$</td>
<td>$P_5$</td>
</tr>
</tbody>
</table>
Obviously, each $L_1$ is a subgraph of the graph $C_7^{(1)}$. So, the OCD can be obtained by adding a block containing this $L_1$. And their excess graph $R_1$ are:

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c}
\text{v} \equiv \text{(mod 16)} & 2, 15 & 3, 14 & 4, 13 & 5, 12 & 6, 11 & 7, 10 & 8, 9 \\
\hline
R_1 & C_7 & P_6 & P_3 & P_7 & P_2 & P_4 & P_5.
\end{array}
\]

\[\Box\]

Lemma 4.3. There exist $(w, C_7^{(2)}, 1)$-OPD for $7 \leq w \leq 15$ and $18 \leq w \leq 25$.

Proof.

\begin{itemize}
\item \textbf{w = 7:} \quad X = Z_7, (0, 4, 2, 6, 1, 3, 5), (5, 1, 4, 6, 3, 0, 2).
\quad \text{L}(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}.
\item \textbf{w = 8:} \quad X = Z_8,
\quad (1, 4, 7, 5, 0, 2, 6), (2, 5, 3, 7, 0, 6, 4), (3, 1, 7, 6, 5, 4, 0).
\quad \text{L}(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4)\}.
\item \textbf{w = 9:} \quad X = Z_9,
\quad (5, 1, 3, 0, 2, 4, 8), (6, 3, 7, 4, 0, 8, 1), (7, 8, 3, 5, 2, 6, 0),
\quad (7, 2, 8, 6, 5, 4, 1).
\quad \text{L}(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4)\}.
\item \textbf{w = 10:} \quad X = Z_{10},
\quad (6, 7, 5, 1, 3, 9, 0), (6, 5, 0, 2, 4, 1, 8), (6, 9, 5, 3, 7, 8, 4),
\quad (7, 1, 9, 4, 0, 8, 2), (9, 2, 5, 8, 3, 0, 7).
\quad \text{L}(\mathcal{B}) = \{(0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}.
\item \textbf{w = 11:} \quad X = Z_9 \cup \{a, b\},
\quad (5, 1, 3, a, 2, b, 4), (6, 2, 0, a, 8, 3, 5), (7, a, 1, b, 3, 6, 0),
\quad (8, 7, 6, b, 5, 2, 4), (7, 3, 0, 5, 8, 1, 4), (8, 0, 4, 6, 1, 7, 2).
\quad \text{L}(\mathcal{B}) = \{(a, b), (b, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, a)\}.
\item \textbf{w = 12:} \quad X = Z_9 \cup \{a, b, c\},
\quad (2, 8, 3, b, 0, 1, a), (1, 2, 6, b, 5, a, 7), (4, 5, 8, b, 7, 6, 3),
\quad (0, c, 2, 5, 6, 8, 4), (0, 8, a, 6, c, 3, 2), (0, 3, 5, 7, c, 4, a),
\quad (1, 5, c, 8, 7, 4, 6), (3, a, c, 1, 4, 2, 7).
\quad \text{L}(\mathcal{B}) = \{(a, b), (b, c)\}.
\end{itemize}
\( w = 13: \) \[ X = Z_9 \cup \{a, b, c, d\}, \]
\[(4, 6, 0, 3, b, 1, c), (5, 1, d, 4, 7, c, 0), (3, 8, d, 5, b, 6, c), \]
\[(a, 6, 1, 3, d, 7, 2), (a, 7, 1, 4, 0, b, 8), (a, 0, 7, 5, 6, 8, 1), \]
\[(2, 8, 7, 3, 6, d, b), (2, c, 8, 4, b, 7, 6), (2, 0, 8, 5, c, a, d). \]
\[L(\mathcal{B}) = \{(a, b), (b, c), (c, d), (d, 0), (0, 1), (1, 2)\}. \]

\( w = 14: \) \[ X = Z_{10} \cup \{a, b, c, d\}, \]
\[(0, 4, a, 1, b, d, 5), (0, 8, a, 2, c, 5, 9), (0, a, 7, 3, d, 6, b), \]
\[(1, 4, b, 2, 5, 7, c), (1, 5, b, 3, a, 9, d), (2, 8, c, 3, 6, a, d), \]
\[(4, 9, 6, 5, 8, 7, 2), (4, c, 0, 6, 1, 8, 3), (4, d, 0, 7, 1, 9, 8), \]
\[(8, d, 7, b, 9, 2, 6), (9, 7, 6, c, a, 5, 3). \]
\[L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \]

\( w = 15: \) \[ X = Z_{13} \cup \{A, B\}, \]
\[(A, 0, 2, 8, 3, 5, 1), (A, 2, 6, 9, 4, B, 5), (A, 4, 5, 10, 7, B, 3), \]
\[(B, 1, 0, 8, 9, 5, 6), (B, 0, 3, 9, 7, 4, 2), (B, 11, 0, 10, 1, 4, 12), \]
\[(8, 1, 2, 11, 6, 3, 7), (9, 1, 3, 11, 7, 6, 10), (10, 3, 4, 11, A, 6, 8), \]
\[(8, 5, 7, 12, 6, 0, 4), (9, 2, 5, 12, A, 7, 0), (10, 2, 3, 12, 1, 6, 4), \]
\[(11, 5, 0, 12, 2, 7, 1). \]
\[L(\mathcal{B}) = \{(A, B)\}. \]

\( w = 18: \) \[ X = Z_{16} \cup \{A, B\}, \]
\[(15, B, 2, 0, 1, 3, A), (15, 5, 4, 1, 2, 3, 6), (8, 3, 5, 2, 4, 7, 0), \]
\[(15, 7, 14, 3, 0, 4, 10), (9, 12, 3, 4, 11, A, 8), (9, 4, 1, 5, 0, 6, 2), \]
\[(10, 8, 5, 6, 1, 7, 2), (10, 0, 9, 7, A, 6, B), (11, 7, 5, 10, 1, 12, 8), \]
\[(12, B, 3, 11, 0, 13, 2), (13, A, 2, 11, 5, B, 9), (13, B, 0, 12, 6, 7, 3), \]
\[(14, 2, 15, 12, 10, A, 0), (14, B, 7, 13, 5, A, 4), (15, 11, 1, 13, 4, B, 8), \]
\[(8, 4, 6, 13, 10, 9, 1), (15, 9, 6, 14, 5, 12, 4), (8, 6, 11, 14, A, 12, 7), \]
\[(9, 3, 10, 14, 1, B, 11). \]
\[L(\mathcal{B}) = \{(A, B)\}. \]

\( w = 19: \) \[ X = Z_{15} \cup \{A, B, C, D\}, \]
\[(A, 0, 1, D, 2, C, 3), (A, 4, 0, C, 11, 3, 1), (A, 2, 0, 9, 11, 4, 5), \]
\[(A, 6, 13, 10, 12, B, 7), (A, 13, 0, 12, 9, 3, 14), (D, 3, 0, B, 1, C, 4), \]
\[(D, 5, 1, 9, 4, 7, 10), \ (D, 8, 1, 14, 5, 12, 6), \ (D, 0, 8, 13, 4, 1, 11),
(B, 2, 1, 13, 3, 7, 14), \ (B, 4, 6, 9, 7, 0, 5), \ (B, 6, 0, 10, 2, 7, 8),
(C, 6, 2, 9, 5, 7, 12), \ (C, 7, 1, 10, 4, 2, 8), \ (C, 13, 2, 14, 8, 6, 5),
(11, A, 8, 12, 2, 3, B), \ (11, 0, 14, 13, 9, 10, 6), \ (11, 8, 9, 14, 4, 3, 5),
(11, 2, 5, 10, 3, 6, 7), \ (12, 1, 6, 14, 10, 8, 3), \ (12, D, 7, 13, 5, 8, 4).
\]

\[L(\mathcal{B}) = \{(A, B), (B, C), (C, D)\} \]

\[w = 20: \ X = \mathbb{Z}_{16} \cup \{A, B, C, D\}, \]
\[(B, 4, A, 15, 2, D, 10), \ (C, A, 3, 15, 4, 13, 5), \ (D, 5, 14, 15, 10, 13, 9),
(0, A, 6, 15, 12, 7, 14), \ (1, A, 5, 15, 7, 11, 6), \ (3, B, D, 4, 0, 7, C),
(3, 9, B, 5, 11, A, D), \ (3, 13, B, 6, C, 14, 1), \ (4, 11, 10, 5, 2, C, 1),
(4, 14, D, 6, 0, 9, C), \ (5, 0, 2, 6, 14, B, 1), \ (7, 3, 12, 8, 4, 9, 5),
(7, 13, 15, 9, A, 10, 6), \ (7, 4, 12, 10, 3, 2, B), \ (8, 15, 11, 9, 6, 13, C),
(8, D, 1, 10, 4, 2, A), \ (9, 12, C, 10, 0, 11, 2), \ (11, 8, 6, 12, 0, C),
(11, 3, 0, 13, A, 7, D), \ (11, 1, 9, 14, 2, 8, B), \ (12, 0, 8, 13, 2, 7, 1),
(12, 2, 10, 14, 3, 8, 5), \ (13, 1, 8, 14, A, 12, D).
\]

\[L(\mathcal{B}) = \{(A, B), (B, C), (C, D), (D, 0), (0, 1), (1, 2)\} \]

\[w = 21: \ X = \mathbb{Z}_{18} \cup \{A, B, C\}, \]
\[(B, 2, 0, 12, A, 5, 9), \ (B, 4, 0, 13, 8, 2, 5), \ (B, 6, 0, 14, 9, 2, 7),
(B, 8, 0, 15, 10, 1, 11), \ (B, 10, 0, 16, 5, 11, 3), \ (B, 0, 3, 17, 5, 8, 1),
(A, 1, 0, 17, 4, 10, 8), \ (A, 2, 1, 13, 11, 10, 6), \ (A, 3, 1, 14, 10, 5, 7),
(A, 4, 1, 15, 9, C, 11), \ (A, 0, 7, 16, 10, 3, 9), \ (C, A, 10, 12, 8, 9, 0),
(C, 1, 6, 13, 4, 16, 2), \ (C, 3, 2, 14, 6, 4, 8), \ (C, 5, 1, 17, 9, 11, 4),
(C, 7, 1, 16, 11, 2, 6), \ (12, 6, 8, 17, 7, 4, 2), \ (12, 3, 13, 14, 4, 9, 7),
(12, 4, 5, 15, C, 10, 9), \ (13, 5, 6, 17, 15, 2, 10), \ (13, 12, 11, 15, 3, 6, 9),
(14, 3, 4, 15, 6, 7, 8), \ (16, 9, 1, 12, 5, 3, 8), \ (16, 17, 2, 13, 7, 11, 14),
(16, 3, 7, 15, 8, 11, 6), \ (17, 10, 7, 14, 5, 1, 11).
\]

\[L(\mathcal{B}) = \{(A, B), (B, C)\} \]
Lemma 4.5. There exist \( X = \mathbb{Z}_{20} \cup \{A, B\} \),
\[(4, 19, 3, 0, 4, 2, 18), \ (A, 17, 4, 1, 6, 2, 16), \ (A, 15, 9, 2, 17, 3, 14), \]
\[(B, 4, 9, 1, 7, 2, 5), \ (B, 8, 11, 2, 19, 5, 9), \ (B, 6, 5, 3, 8, 12, 15), \]
\[(0, 19, 1, 5, 7, 8, 6), \ (2, 14, 7, 0, 13, 3, 15), \ (3, 9, 12, 6, 13, 5, 11), \]
\[(3, 16, 8, 1, 10, 4, 18), \ (4, 8, A, 5, 10, 6, 19), \ (4, 11, 15, 6, 14, 5, 12), \]
\[(5, 18, 14, 8, 19, 12, 17), \ (6, 18, 10, 7, 19, 11, 17), \ (7, 17, 13, 4, 16, 6, 11), \]
\[(7, A, 6, 9, 0, 10, 3), \ (8, 17, 14, 9, 19, 15, 18), \ (8, 15, 17, 10, 16, 7, 13), \]
\[(9, 17, 19, 10, B, 13, 18), \ (10, 12, B, 11, 0, 8, 2), \ (11, 9, A, 12, 18, 1, 16), \]
\[(11, A, 10, 13, 2, 12, 1), \ (12, 3, A, 13, 1, 14, 0), \ (12, 7, B, 14, 10, 15, 16), \]
\[(13, 15, 4, 14, 19, 16, 9), \ (14, 16, 5, 15, 7, 18, 11), \ (16, B, 18, 17, 1, 15, 0), \]
\[(18, 16, 13, 19, B, 17, 0). \]
\[L(\mathscr{B}) = \{(A, B), (B, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, A)\}. \]

\( w = 23, 24 \) and 25 can be obtained from \((w - 16, C_7^{(2)}, 1)\)-OPD above and \(C_7^{(2)}\)-ID \((w - 16); w - 16\) (see [11]), and its leave graph is as same as that of \((w - 16, C_7^{(2)}, 1)\)-OPD. \( \Box \)

**Theorem 4.4.** There exist \((v, C_7^{(2)}, 1)\)-OPD and \((v, C_7^{(2)}, 1)\)-OCD for \( v \geq 7 \).

**Proof.** It is easy to prove by Lemma 4.3. Note that the leave graphs \( L_1 \) for \((v, C_7^{(2)}, 1)\)-OPD are as same as that of \((v, C_7^{(1)}, 1)\)-OPD. The further proof is similar to Theorem 4.2. \( \Box \)

4.2. Packings and coverings for \( \lambda > 1 \)

**Lemma 4.5.** There exist \((v, C_7^{(r)}, \lambda)\)-OPD and \((v, C_7^{(r)}, \lambda)\)-OCD for \( v \equiv 2, 15 \pmod{16} \) and \( \lambda > 1 \) (where \( r = 1, 2 \)).

**Proof.** By Lemma 2.7, we have the following table:

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( l_\lambda )</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>( r_\lambda )</td>
<td>7</td>
<td>6</td>
<td>5</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

where \( L_1 = P_2 \) and \( R_1 = C_7 \) by Theorems 4.2 and 4.4. \( \Box \)
Lemma 4.6. There exist \((v, C_7^{(r)}, \lambda)\)-OPD and \((v, C_7^{(r)}, \lambda)\)-OCD for \(v \equiv 3, 14 \pmod{16}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

**Proof.** By Lemma 2.7, and Theorems 4.2, 4.4, we have the following table:

\[
\begin{array}{c|cccccccc}
\lambda & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
l_\lambda & 3 & 6 & 1 & 4 & 7 & 2 & 5 \\
\hline
L_\lambda & P_4 & L_1 \cup L_1 & L_1 \setminus R_2 & L_1 \cup L_3 & L_2 \cup L_3 & L_3 \cup L_3 & L_3 \cup L_4 \\
\hline
r_\lambda & 5 & 2 & 7 & 4 & 1 & 6 & 3 \\
\hline
R_\lambda & P_6 & R_1 \setminus L_1 & R_1 \cup R_2 & R_2 \cup R_2 & R_2 \setminus L_3 & R_2 \cup R_4 & R_2 \cup R_3 \\
\end{array}
\]

Lemma 4.7. There exist \((v, C_7^{(r)}, \lambda)\)-OPD and \((v, C_7^{(r)}, \lambda)\)-OCD for \(v \equiv 4, 13 \pmod{16}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

**Proof.** By Lemma 2.7 and Theorems 4.2, 4.4, we have the following table:

\[
\begin{array}{c|cccc}
\lambda & 1 & 2 & 3 \\
\hline
l_\lambda & 6 & 4 & 2 \\
\hline
L_\lambda & P_7 & L_1 \setminus R_1 & L_2 \setminus R_1 \\
\hline
r_\lambda & 2 & 4 & 6 \\
\hline
R_\lambda & P_3 & R_1 \cup R_1 & R_1 \cup R_2 \\
\end{array}
\]

Lemma 4.8. There exist \((v, C_7^{(r)}, \lambda)\)-OPD and \((v, C_7^{(r)}, \lambda)\)-OCD for \(v \equiv 5, 12 \pmod{16}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

**Proof.** By Lemma 2.7 and Theorems 4.2, 4.4, we have the following table:

\[
\begin{array}{c|cccc}
\lambda & 1 & 2 & 3 \\
\hline
l_\lambda & 2 & 4 & 6 \\
\hline
L_\lambda & P_3 & L_1 \cup L_1 & L_1 \cup L_2 \\
\hline
r_\lambda & 6 & 4 & 2 \\
\hline
R_\lambda & P_7 & R_1 \setminus L_1 & R_2 \setminus L_1 \\
\end{array}
\]
Lemma 4.9. There exist \((v, C^r, \lambda)\)-OPD and \((v, C^r, \lambda)\)-OCD for \(v \equiv 6, 11 \pmod{16}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

Proof. By Lemma 2.7, we have the following table:

\[
\begin{array}{cccccccc}
\lambda & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
l_\lambda & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
& (L_\lambda = L_{\lambda-1} \setminus R_1) \\
r_\lambda & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
& (R_\lambda = R_1 \cup R_{\lambda-1})
\end{array}
\]

where \(L_1 = C_7\) and \(R_1 = P_2\) by Theorems 4.2, 4.4.

Lemma 4.10. There exist \((v, C^r, \lambda)\)-OPD and \((v, C^r, \lambda)\)-OCD for \(v \equiv 7, 10 \pmod{16}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

Proof. By Lemma 2.7 and Theorems 4.2, 4.4, we have the following table:

\[
\begin{array}{cccccccc}
\lambda & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
L_\lambda & P_6 & L_1 \setminus R_1 & L_1 \cup L_2 & L_2 \cup L_2 & L_2 \setminus R_3 & L_2 \cup L_4 & L_2 \cup L_5 \\
\hline
r_\lambda & 3 & 6 & 1 & 4 & 7 & 2 & 5
\end{array}
\]

Lemma 4.11. There exist \((v, C^r, \lambda)\)-OPD and \((v, C^r, \lambda)\)-OCD for \(v \equiv 8, 9 \pmod{16}\) and \(\lambda > 1\) (where \(r = 1, 2\)).

Proof. By Theorems 4.2 and 4.4.

Summarizing the results of graph design with any index (see [11]) and Lemmas 4.5, 4.6, 4.7, 4.8, 4.9, 4.10 and 4.11 we can obtain the following Theorem.

Theorem 4.12. There exist \((v, C^r, \lambda)\)-OPD and \((v, C^r, \lambda)\)-OCD for \(\lambda > 1\) and any \(v \geq 7\) (where \(r = 1, 2\)).

5. \(C^{(1)}_8, C^{(2)}_8\) and \(C^{(3)}_8\)

It is known that there exists a \((v, C^r, \lambda)\)-GD, \(r = 1, 2, 3\), if and only if \(\lambda v(v - 1) \equiv 0 \pmod{18}\) and \(v \geq 8\), with a unique exception \((v, r, \lambda) = (9, 3, 1)\) (see [11]).
Table 3

(a) For $C_{8}^{(1)}$ and $C_{8}^{(3)}$

<table>
<thead>
<tr>
<th>$v \pmod{18}$</th>
<th>HD</th>
<th>ID</th>
<th>IHD</th>
<th>OPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$g^{2i-1}$</td>
<td>$(20;11)$</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>$g^{2i-1}$</td>
<td>$(21;12)$</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>$g^{2i-1}$</td>
<td>$(22;13)$</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>$g^{2i-1}$</td>
<td>$(23;14)$</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>$g^{2i-1}$</td>
<td>$(24;15)$</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>$g^{2i-1}$</td>
<td>$(25;16)$</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>$g^{2i-1}$</td>
<td>$(9,9;17)$</td>
<td>26</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>$g^{2i+1}$</td>
<td>$(11;2)$</td>
<td>11</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>$g^{2i+1}$</td>
<td>$(12;3)$</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>$g^{2i+1}$</td>
<td>$(13;4)$</td>
<td>13</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>$g^{2i+1}$</td>
<td>$(14;5)$</td>
<td>14</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>$g^{2i+1}$</td>
<td>$(15;6)$</td>
<td>15</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>$g^{2i+1}$</td>
<td>$(16;7)$</td>
<td>16</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>$g^{2i+1}$</td>
<td>$(17;8)$</td>
<td>8,17</td>
<td></td>
</tr>
</tbody>
</table>

(b) For $C_{8}^{(2)}$

<table>
<thead>
<tr>
<th>$v \pmod{9}$</th>
<th>HD</th>
<th>ID</th>
<th>OPD</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$g^r$</td>
<td>$(11;2)$</td>
<td>11</td>
</tr>
<tr>
<td>3</td>
<td>$g^r$</td>
<td>$(12;3)$</td>
<td>12</td>
</tr>
<tr>
<td>4</td>
<td>$g^r$</td>
<td>$(13;4)$</td>
<td>13</td>
</tr>
<tr>
<td>5</td>
<td>$g^r$</td>
<td>$(14;5)$</td>
<td>14</td>
</tr>
<tr>
<td>6</td>
<td>$g^r$</td>
<td>$(15;6)$</td>
<td>15</td>
</tr>
<tr>
<td>7</td>
<td>$g^r$</td>
<td>$(16;7)$</td>
<td>16</td>
</tr>
<tr>
<td>8</td>
<td>$g^r$</td>
<td>$(17;8)$</td>
<td>8</td>
</tr>
</tbody>
</table>

For convenience, we denote $C_{8}^{(1)}$ (or $C_{8}^{(2)}$, or $C_{8}^{(3)}$) by $(a,b,c,d,e,f,g,h)$, where the edges on $C_{8}$ are $ab$, $bc$, $cd$, $de$, $ef$, $fg$, $gh$, $ha$ and the chord is $ac$ (or $ad$, or $ae$). First, in order to construct the optimal packing designs and covering designs for $\lambda = 1$, by Theorems 2.1 and 2.2 and the following tables, we only need to give the constructions of HD, ID or IHD, and OPD for the pointed orders, where the leave graph of OPD has to be a subgraph of $C_{8}^{(r)}$. However, the needed HD, ID and IHD have been shown in [11], so we only need to construct the OPD listed in Table 3(a) and (b).

5.1. Packings and coverings for $\lambda = 1$

**Lemma 5.1.** There exist $(w,C_{8}^{(1)},1)$-OPD and $(w,C_{8}^{(1)},1)$-OCD for $w = 8, 11, 12, 13, 14, 15, 16, 17$ and 26.

**Proof.** Let $(w,C_{8}^{(1)},1)$-OPD = $(X,\emptyset)$. 
$w = 8$: \( X = (Z_3 \times Z_2) \cup \{a, b\} \)
\( (1, 1, a, 1, 2, b, 2, 0, 0, 1, 2, 1, 0, 0, 2, 0, 0, 1) \mod (3, -). \)
\( L(\mathcal{B}) = \{(a, b)\}. \)

$w = 11$: \( X = (Z_3 \times Z_2) \cup \{a, b\} \)
\( (a, 1, 0, 0, 0, 1, 0, 0, 0, 2, 1, 0, 2, 1, 1, 0, 2, 1) \mod (3, -). \)
\( L(\mathcal{B}) = \{(a, b)\}. \)

$w = 12$: \( X = Z_8 \cup \{a, b, c, d\} \)
\( (a, 1, 0, c, 6, 5, 2, 3), (a, 4, c, 1, 2, 6, b, 5), (2, a, 7, 6, 4, 1, 5, 0), \)
\( (d, 6, 1, b, 3, 4, 0, 7), (d, 0, b, 2, c, 7, 5, 3), (4, d, 5, c, 3, 1, 7, b), \)
\( (3, 0, 6, a, d, 2, 4, 7). \)
\( L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \)

$w = 13$: \( X = Z_9 \cup \{a, b, c, d\} \)
\( (c, 1, 4, 0, 6, 7, 5, b), (b, 1, 0, a, 6, c, 5, 8), (b, 6, 3, a, 5, 4, 8, 7), \)
\( (d, 8, 1, 7, 4, 6, 2, c), (3, 0, 2, a, 4, b, d, 7), (5, 3, 1, 2, 7, a, d, 6), \)
\( (4, d, 2, 8, 0, 7, c, 3), (5, d, 0, c, 8, a, b, 2). \)
\( L(\mathcal{B}) = \{(c, a), (a, 1), (1, 6), (6, 8), (8, 3), (3, d)\}. \)

$w = 14$: \( X = Z_{12} \cup \{A, B\} \)
\( (A, 3, 4, 6, 10, 7, 0, 5), (A, 9, 10, 1, 7, 4, 5, 11), (B, 1, 2, 5, 8, 9, 4, 0), \)
\( (A, 6, 7, 9, 3, 11, 2, 8), (A, 0, 1, 3, 10, 11, 4, 2), (0, 2, 3, 6, 5, 7, 8, 11), \)
\( (B, 6, 8, 1, 9, 2, 10, 4), (B, 9, 11, 7, 3, 8, 0, 10), (6, 0, 9, 5, 10, 8, 4, 1), \)
\( (B, 3, 5, 1, 11, 6, 2, 7). \)
\( L(\mathcal{B}) = \{(A, B)\}. \)

$w = 15$: \( X = Z_{11} \cup \{A, B, C, D\} \)
\( (A, 8, D, 5, C, 4, 2, 6), (3, 0, 6, 9, 8, 10, 7, 2), (1, A, 3, 9, 5, 10, 4, B), \)
\( (C, 3, D, 7, B, 8, 1, 2), (4, 3, 5, 8, 7, 9, 10, 6), (A, 9, B, 5, 0, 8, C, 10), \)
\( (0, A, 4, 9, 1, 7, C, B), (5, A, 2, 10, 3, 8, 4, 7), (B, 10, D, 9, C, 6, 7, 3), \)
\( (2, D, 0, C, 1, 5, 6, B), (6, D, 1, 10, 0, 9, 2, 8). \)
\( L(\mathcal{B}) = \{(C, A), (A, 7), (7, 0), (0, 1), (1, 4), (4, D)\}. \)

$w = 16$: \( X = Z_{12} \cup \{A, B, C, D\} \)
\( (A, 5, 11, 3, 4, 9, D, 7), (A, 4, 10, 9, 2, 7, 0, 8), (2, A, 3, 0, 9, 1, 8, 6), \)
There exist Lemma 5.3.

Let \((B, 3, 7, C, 11, 1, 5, 9), (B, 8, 4, 1, 3, 5, 6, 11), (A, 1, C, 0, 2, 4, 7, 9), (0, B, 1, 7, 6, 10, 2, 11), (C, 2, 8, 10, 1, 6, 0, 5), (A, 0, D, 1, 2, 5, B, 6), (D, 10, 5, 7, 11, 9, 6, 3), (C, 3, 9, 8, 7, 10, 0, 4), (D, 2, B, 10, 11, 4, 5, 8), (D, 4, 6, C, 10, 3, 8, 11). \]

\[ L(B) = \{(A, B), (B, C), (C, D)\}. \]

\[ w = 17: \quad X = (Z_5 \times Z_3) \cup \{a, b\} \]
\[ (a, 3_0, 01, 1_1, 2_0, 02, 4_1, 2_2), (b, 4_0, 01, 02, 1_2, 4_1, 1_0, 3_2), \]
\[ (4_0, 3_2, 3_0, 1_0, 2_2, 02, 1_1, 4_1) \mod (5, -). \]
\[ L(\mathcal{B}) = \{(a, b)\}. \]

\[ w = 26: \quad X = (Z_{12} \times Z_2) \cup \{A, B\} \]
\[ (0_1, 5_0, 0_0, 11_0, 7_0, 9_0, 8_1, 6_0), (5_1, B, 0_0, 3_0, 1_1, 3_1, 8_1, 4_1) \mod (12, -); \]
\[ \left\{ \begin{array}{l}
(6 + i)_0, A, i_0, (1 + i)_1, (10 + i)_1, (7 + i)_0, (4 + i)_1, (7 + i)_1 \quad 0 \leq i \leq 5.
\end{array} \right. \]
\[ L(\mathcal{B}) = \{(A, B)\}. \]

\textbf{Theorem 5.2.} There exist \((v, e_8^{(1)}, 1)\)-OPD and \((v, e_8^{(1)}, 1)\)-OCD for \(v \geq 8\).

\textbf{Proof.} By Lemma 5.1. The leave graphs \(L_1\) for these OPDs are as follows:

\[
\begin{array}{c|c|c|c|c}
\equiv \text{ (mod 9)} & 2, 5, 8 & 3, 7 & 4, 6 \\
L_1 & P_2 & P_4 & P_7 \\
\end{array}
\]

Obviously, each \(L_1\) is a subgraph of the graph \(C_8^{(r)}\). So, the optimal covering designs can be obtained by adding a block containing this \(L_1\). And their excess graphs \(R_1\) are listed in the following table:

\[
\begin{array}{c|c|c|c|c}
\equiv \text{ (mod 9)} & 2, 5, 8 & 3, 7 & 4, 6 \\
R_1 & C_8 & P_7 & P_4 \\
\end{array}
\]

\textbf{Lemma 5.3.} There exist \((w, e_8^{(2)}, 1)\)-OPD and \((w, e_8^{(2)}, 1)\)-OCD for \(w = 8, 11, 12, 13, 14, 15\) and \(16\).

\textbf{Proof.} Let \((w, e_8^{(2)}, 1)\)-OPD = \((X, \mathcal{B})\).

\[ w = 8: \quad X = (Z_3 \times Z_2) \cup \{a, b\} \]
\[ (0_0, a, 2_1, 1_1, 2_0, 1_0, b, 0_1) \mod (3, -). \]
\[ L(\mathcal{B}) = \{(a, b)\}. \]
\[ w = 11: \quad X = (Z_3 \times Z_3) \cup \{a, b\} \]
\[
(a, 1, 0, 1, 0, 0, 1, 2, 0, 2), \quad (b, 1, 0, 1, 0, 2, 0, 1, 2) \mod (3, -). \]
\[ L(\mathcal{B}) = \{(a, b)\}. \]

\[ w = 12: \quad X = Z_8 \cup \{a, b, c, d\} \]
\[
(a, c, 1, 0, d, 5, 7, 4), \quad (d, b, 7, 2, 0, 6, 4, 1), \quad (b, 4, 0, 3, 2, 6, 1, 5), \]
\[
(a, d, 7, 1, b, 2, 4, 3), \quad (5, 0, 7, c, 2, a, 6, 3), \quad (4, c, 6, d, 3, 1, 2, 5), \]
\[
(6, 5, a, 7, 3, c, 0, b). \]
\[ L(\mathcal{B}) = \{(a, b), (b, c), (c, d)\}. \]

\[ w = 13: \quad X = Z_9 \cup \{a, b, c, d\} \]
\[
(1, a, 5, 2, 4, 3, 8, c), \quad (d, 2, b, a, 8, 5, 1, 6), \quad (b, 0, 3, 1, 8, 4, 7, 6), \]
\[
(c, 4, 6, a, 3, 7, 0, 5), \quad (c, 0, 4, b, 8, 7, d, 3), \quad (d, 4, 1, 0, 8, 2, 6, 5), \]
\[
(0, 6, c, 2, 7, 5, 4, a), \quad (d, c, 7, b, 5, 3, 6, 8). \]
\[ L(B) = \{(b, 3), (3, 2), (2, a), (a, 7), (7, 1), (1, d)\}. \]

\[ w = 14: \quad X = Z_{12} \cup \{A, B\} \]
\[
(A, 5, 0, 2, 1, 4, 6, 9), \quad (A, 11, 0, 8, 1, 7, 10, 3), \quad (A, 1, 3, 0, 9, 11, 8, 7), \]
\[
(B, 0, 6, 3, 4, 5, 2, 8), \quad (A, 10, 11, 6, 2, 9, 7, 4), \quad (B, 10, 0, 1, 5, 3, 7, 2), \]
\[
(4, 2, 3, 9, 5, 7, 6, 10), \quad (B, 9, 1, 11, 2, 10, 8, 5), \quad (B, 4, 0, 7, 11, 3, 8, 6), \]
\[
(5, 6, 1, 10, 9, 8, 4, 11), \quad (A, 1, 3, 0, 9, 11, 8, 7). \]
\[ L(\mathcal{B}) = \{(A, B)\}. \]

\[ w = 15: \quad X = Z_{11} \cup \{A, B, C, D\} \]
\[
(2, 0, 10, 1, 7, C, B), \quad (A, 4, 8, D, 2, 7, 0, 9), \quad (6, 3, 7, 5, 1, 2, 4, D), \]
\[
(1, A, 6, 4, B, 9, 2, 10), \quad (A, 3, 2, C, 6, 9, 1, 8), \quad (D, 5, 4, 0, 8, 7, 9, 10), \]
\[
(3, 5, C, 4, 10, B, 8, 9), \quad (C, 1, 3, D, B, 7, 6, 8), \quad (5, 10, 8, 2, A, 7, 4, 9), \]
\[
(0, C, 10, 3, 8, 5, B, 6), \quad (B, A, 0, 1, D, 9, C, 3). \]
\[ L(\mathcal{B}) = \{(B, 0), (0, 5), (5, A), (A, 10), (10, 7), (7, D)\}. \]

\[ w = 16: \quad X = Z_{12} \cup \{A, B, C, D\} \]
\[
(A, 7, B, 2, 9, 8, 0, 10), \quad (A, 0, 1, D, 5, 8, 7, 4), \quad (D, 2, 3, 9, 5, 1, 6, 7), \]
\[
(B, 0, 6, D, 11, 9, 4, 3), \quad (0, 2, 4, 11, 10, 1, 7, 5), \quad (A, 5, 11, 3, 7, 2, 6, 8), \]
\[
(C, 3, 5, 6, B, 1, 2, 11), \quad (A, 6, 11, 1, C, 2, 10, 9), \quad (B, 11, 8, 4, 0, 3, 1, 9), \]
\[
(C, 4, D, 8, 10, 7, 0, 9), \quad (C, A, 11, 7, 9, 6, 10, 5), \quad (B, 8, 2, 5, 4, 6, 3, 10). \]
There exist Theorem 5.4. There exist

By Lemma 5.3. Note that the leave graphs $L_1$ for $(v, c_8^{(2)}, 1)$-OPD are same to $(v, c_8^{(1)}, 1)$-OPD. The further proof is similar to Theorem 5.2.

Lemma 5.5. There exist $(w, c_8^{(3)}, 1)$-OPD and $(w, c_8^{(3)}, 1)$-OCD for $w = 8, 11, 12, 13, 14, 15, 16, 17, 26$. The further proof is similar to Theorem 5.2.

Proof. Let $(w, c_8^{(3)}, 1)$-OPD = $(X, B)$.

\begin{align*}
w = 8: & \quad X = (Z_2 \times Z_2) \cup \{a, b\} \\
& \quad (0_0, 1_0, a, 1_1, 0_1, 2_0, b, 2_1) \mod(3, -) \\
& \quad L(B) = \{(a, b)\}
\end{align*}

\begin{align*}
w = 11: & \quad X = (Z_2 \times Z_2) \cup \{a, b\} \\
& \quad (a, 0_0, 0_1, 0_2, 1_2, 2_0, 1_1, 2_1), (b, 1_0, 2_0, 0_1, 1_2, 0_0, 0_2, 1_1) \mod(3, -) \\
& \quad L(B) = \{(a, b)\}
\end{align*}

\begin{align*}
w = 12: & \quad X = Z_8 \cup \{a, b, c, d\} \\
& \quad (a, 1, 3, b, 0, 6, d, 5), (a, 2, 0, 7, c, 5, 3, 4), (4, 7, 6, 1, 5, 2, 3, c), \\
& \quad (d, 0, c, 1, b, 5, 6, 4), (2, c, 6, a, 7, 5, 0, 4), (3, d, 7, b, 6, 2, 1, 0), \\
& \quad (d, a, 3, 7, 1, 4, b, 2). \\
& \quad L(B) = \{(a, b), (b, c), (c, d)\}
\end{align*}

\begin{align*}
w = 13: & \quad X = Z_9 \cup \{a, b, c, d\} \\
& \quad (d, 5, 7, 6, a, 3, 0, 8), (1, a, 5, 6, 2, 4, 7, 3), (c, 4, 5, 0, b, 6, d, 7), \\
& \quad (b, 4, 6, 0, 1, c, 3, 8), (c, 2, 8, 4, a, 7, 1, 6), (d, c, 8, 5, b, 2, 3, 4), \\
& \quad (d, 1, 8, 7, 0, c, 5, 3), (0, 4, 1, 5, 2, 7, b, a). \\
& \quad L(B) = \{(b, 3), (3, 6), (6, 8), (8, a), (a, 2), (2, d)\}
\end{align*}

\begin{align*}
w = 14: & \quad X = Z_{12} \cup \{A, B\} \\
& \quad (A, 8, 4, 6, 0, 11, 1, 10), (A, 2, 6, 7, 4, 0, 5, 3), (6, 8, 7, 3, 10, 0, 9, 5), \\
& \quad (A, 6, 3, 8, 5, 10, 4, 11), (B, 0, 1, 2, 3, 4, 9, 6), (A, 1, 5, 2, 9, 3, 11, 7), \\
& \quad (B, 5, 11, 6, 1, 7, 9, 10), (B, 2, 4, 5, 7, 0, 8, 9), (2, 0, 3, 1, 8, 11, 10, 7), \\
& \quad \square
\end{align*}
Lemma 5.6. \(p(9, c_{8}^{(3)}, 1) = 3, c(9, c_{8}^{(3)}, 1) = 5.\)

Proof. We know that there is no \((9, c_{8}^{(3)}, 1)\)-GD (see [11]). Therefore, there exist no \((9, c_{8}^{(3)}, 1)\)-OPD and \((9, c_{8}^{(3)}, 1)\)-OCD. But we have

\[\mathcal{B} = \{(5, 0, 2, 3, 6, 4, 1, 7), (6, 2, 1, 3, 7, 4, 8, 0), (7, 0, 4, 3, 8, 1, 5, 2)\},\]

\[L(\mathcal{B}) = \{(5, 8), (3, 5), (4, 5), (6, 8), (1, 6), (2, 8), (0, 1), (0, 3), (2, 4)\},\]
\[ \mathcal{C} = \mathcal{B} \cup \{(8, 6, 1, 3, 5, 4, 0, 2), (5, 1, 0, 3, 6, 4, 2, 7)\}, \]

\[ R(\mathcal{C}) = \{(1, 3), (0, 2), (0, 4), (1, 5), (5, 7), (5, 6), (3, 6), (4, 6), (2, 7)\}. \]

So, \( p(9, C_8^{(3)}, 1) = 3 \) and \( c(9, C_8^{(3)}, 1) = 5. \)

**Theorem 5.7.** There exist \((v, C_8^{(3)}, 1)\)-OPD and \((v, C_8^{(3)}, 1)\)-OCD for any \(v \geq 8\) and \(v \neq 9\). And, \( p(9, C_8^{(3)}, 1) = 3 \) and \( c(9, C_8^{(3)}, 1) = 5. \)

**Proof.** By Lemmas 5.5 and 5.6. Note that the leave graphs \(L_1\) for \((v, C_8^{(3)}, 1)\)-OPD are same to \((v, C_8^{(1)}, 1)\)-OPD. The further proof is similar to Theorem 5.2.

### 5.2. Packings and Coverings for \( \lambda > 1 \)

**Lemma 5.8.** There exist \((v, C_8^{(r)}, \lambda)\)-OPD and \((v, C_8^{(r)}, \lambda)\)-OCD for \(v \equiv 2, 5, 8 \pmod{9}\) and \(\lambda > 1\) (where \(r = 1, 2, 3\)).

**Proof.** By Theorems 5.2, 5.4 and 5.7, we have the following table:

\[
\begin{array}{cccccccc}
\lambda & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
L_\lambda & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 (L_\lambda = L_1 \cup L_{\lambda-1}), \\
r_\lambda & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 (R_\lambda = R_{\lambda-1} \setminus L_1) \\
\end{array}
\]

where \(L_1 = P_2\) and \(R_1 = C_8\).

**Lemma 5.9.** There exist \((v, C_8^{(r)}, \lambda)\)-OPD and \((v, C_8^{(r)}, \lambda)\)-OCD for \(v \equiv 3, 7 \pmod{9}\) and \(\lambda > 1\) (where \(r = 1, 2, 3\)).

**Proof.** By Theorems 5.2, 5.4 and 5.7, we have the following table:

\[
\begin{array}{cc}
\lambda & 1 & 2 \\
L_\lambda & 3 & 6 \\
R_\lambda & P_4 & L_1 \cup L_1. \\
r_\lambda & 6 & 3 \\
\end{array}
\]

**Lemma 5.10.** There exist \((v, C_8^{(r)}, \lambda)\)-OPD and \((v, C_8^{(r)}, \lambda)\)-OCD for \(v \equiv 4, 6 \pmod{9}\) and \(\lambda > 1\) (where \(r = 1, 2, 3\)).
Proof. By Theorems 5.2, 5.4 and 5.7, we have the following table:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$l_{2\lambda}$</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>$L_{2\lambda}$</td>
<td>$P_7$</td>
<td>$L_1 \setminus R_1$</td>
</tr>
<tr>
<td>$r_{2\lambda}$</td>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>$R_{2\lambda}$</td>
<td>$P_4$</td>
<td>$R_1 \cup R_1$</td>
</tr>
</tbody>
</table>

Theorem 5.11. There exist $(v, C_8^{(r)}(\lambda))$-OPD and $(v, C_8^{(r)}(\lambda))$-OCD for $\lambda > 1$ and any $v \geq 8$ (where $r = 1, 2, 3$).

Proof. By the results of graph design with any index (see [11]) and Lemmas 5.8–5.10. Note that there are $(9, C_8^{(3)}(\lambda))$-GD for $\lambda > 1$ (see [11]), even though there is no $(9, C_8^{(1)}(1))$-GD.

References

[11] Qingde Kang, Huijuan Zuo, Yanfang Zhang, Decompositions of $lK_n$ into $k$-circuits with one chord, manuscript.