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Enumeration of semi-Latin squares

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Abstract

An $(n \times n)/k$ semi-Latin square is an $n \times n$ square in which nk letters are placed so that there are k letters in each row–column intersection and that each letter occurs once per row and once per column. It may be regarded as a family of nk permutations of n objects subject to certain restrictions. Squares of a given size fall into strong isomorphism classes (interchange of rows and columns not permitted), which are grouped into weak isomorphism classes (interchange of rows and columns permitted). We use group theory, graph theory, design theory and computing to find all isomorphism classes of $(4 \times 4)/k$ semi-Latin squares for $k = 2, 3, 4$.

1. Preliminaries

An $(n \times n)/k$ *semi-Latin square* is an $n \times n$ array containing nk letters in such a way that each row–column intersection contains k letters and each letter occurs once in each row and once in each column. General semi-Latin squares were so named by Yates [12]; statistical uses are summarized in [11,2]; optimality properties are given in [1,5,4].

Preece and Freeman [11] enumerated the $(4 \times 4)/2$ semi-Latin squares by ad hoc methods. The purpose of this paper is to give a systematic method of enumerating isomorphism classes of semi-Latin squares of a given size. The method is practicable only for small n and k, but this is sufficient for the practical needs of statisticians.

Let Λ be an $(n \times n)/k$ semi-Latin square. We shall always assume that the rows and columns of Λ are labelled $1, \ldots, n$. Let X^{Λ} be the set of letters in Λ: here, and elsewhere, the superscript will be omitted if there is no ambiguity. For $i, j \in \{1, \ldots, n\}$, let Λ_{ij} be the set of letters in X^{Λ} which occur in row i and column j of Λ. Then each

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letter x in X determines a permutation π_x^Λ in S_n by

$$i\pi_x^\Lambda = j \iff x \in \Lambda_{ij} \quad \text{for } 1 \leqslant i,\ j \leqslant n.$$

Moreover, each permutation σ in S_n determines a subset Y_σ^Λ of X by

$$Y_\sigma^\Lambda = \{x \in X : \pi_x^\Lambda = \sigma\}.$$

Write N_σ^Λ for $|Y_\sigma^\Lambda|$. It is clear that

$$\sum_{\sigma \in S_n : i\sigma = j} N_\sigma^\Lambda = k \quad \text{for } i, j \in \{1, \ldots, n\}. \tag{1}$$

Moreover, if the semi-Latin square M is obtained from Λ simply by relabelling the letters, then $N_\sigma^\Lambda = N_\sigma^M$ for all σ in S_n. Further, given any family of non-negative integers $(N_\sigma : \sigma \in S_n)$ satisfying (1), then there exists a semi-Latin square Λ such that $N_\sigma^\Lambda = N_\sigma$ for all σ in S_n.

Example 1. Let $n = k = 3$ and put

$\Lambda =$

	1	2	3
1	$a\ b\ c$	$d\ e\ f$	$g\ h\ i$
2	$f\ g\ h$	$a\ b\ i$	$c\ d\ e$
3	$d\ e\ i$	$c\ g\ h$	$a\ b\ f$

Then

$$\pi_f = (1\,2), \qquad \pi_g = \pi_h = (1\,3\,2),$$
$$Y_{(1\,2)} = \{f\}, \qquad Y_{(1\,3\,2)} = \{g, h\},$$
$$N_{(1\,2)} = 1, \qquad N_{(1\,3\,2)} = 2.$$

Thus semi-Latin squares may be identified with families of non-negative integers $(N_\sigma\colon \sigma \in S_n)$ satisfying (1). Since N_σ can take only $k+1$ values, the family can be succintly represented by the partition of S_n into subsets $\Lambda_0, \Lambda_1, \ldots, \Lambda_k$, where

$$\Lambda_r = \left\{\sigma \in S_n\colon N_\sigma^\Lambda = r\right\}.$$

2. Isomorphism classes

Following [3], we define a *strong isomorphism* between $(n \times n)/k$ semi-Latin squares Λ and M to be a triple (α, β, γ), where $\alpha, \beta \in S_n$ and γ is a bijection from X^Λ to X^M satisfying

$$\Lambda_{ij}\gamma = M_{i\alpha, j\beta} \quad \text{for all } i,\ j \in \{1, \ldots, n\}.$$

In other words, M is obtained from Λ by permuting the rows by α and the columns by β and relabelling the letters by γ. A *weak isomorphism* from Λ to M is either a strong isomorphism or a triple (α, β, δ), where $\alpha, \beta \in S_n$ and δ is a bijection from X^Λ to X^M satisfying

$$\Lambda_{ij}\delta = M_{j\beta, i\alpha} \quad \text{for all } i,\, j \in \{1, \ldots, n\}.$$

In other words, a weak isomorphism may interchange rows and columns. Thus every semi-Latin square is weakly isomorphic to its transpose but may not be strongly isomorphic to it.

Strong isomorphism classes correspond to the notion of *transformation sets* [8,11] or *isotopy classes* [7] for Latin squares, while weak isomorphism classes correspond to the notion of *species* [8,11] or *main class* [7,10].

If Λ is strongly isomorphic to its transpose then the weak isomorphism class $\mathrm{Weak}(\Lambda)$ containing Λ is also a strong isomorphism class; otherwise $\mathrm{Weak}(\Lambda)$ is the union of two strong isomorphism classes.

Theorem 1. *Semi-Latin squares Λ and M of size $(n \times n)/k$ are strongly isomorphic if and only if there exist permutations α and β in S_n such that*

$$\alpha^{-1}\Lambda_r\beta = M_r \quad \textit{for } r = 0, \ldots, k. \tag{2}$$

In particular, if $\Lambda_0, \ldots, \Lambda_k$ can be simultaneously conjugated into $M_0, \ldots, M_k$ (that is, if condition (2) *holds with $\alpha = \beta$) then Λ is strongly isomorphic to M.*

Proof. Suppose that (α, β, γ) is a strong isomorphism from Λ to M. For σ in S_n we have

$$Y_\sigma^\Lambda = \bigcap_{i=1}^{n} \Lambda_{i,i\sigma}$$

so

$$Y_\sigma^\Lambda\gamma = \bigcap_{i=1}^{n} M_{i\alpha, i\sigma\beta} = \bigcap_{j=1}^{n} M_{j, j\alpha^{-1}\sigma\beta} = Y_{\alpha^{-1}\sigma\beta}^M$$

and $N_{\alpha^{-1}\sigma\beta}^M = N_\sigma^\Lambda$. Hence (2) is satisfied.

Conversely, suppose that α and β satisfy (2). Then, for each σ in S_n we have $|Y_\sigma^\Lambda| = |Y_{\alpha^{-1}\sigma\beta}^M|$, so there is a bijection γ_σ between these sets. Since the non-empty sets $\{Y_\sigma^\Lambda\colon \sigma \in S_n\}$ partition X^Λ, and similarly for X^M, the maps γ_σ combine to form a bijection γ from X^Λ to X^M such that, for all σ in S_n, if $x \in Y_\sigma^\Lambda$ then $x\gamma \in Y_{\alpha^{-1}\sigma\beta}^M$. Now, for all i, j in $\{1, \ldots, n\}$ we have

$$\Lambda_{ij}\gamma = \bigcup_{\sigma \in S_n\,:\, i\sigma = j} Y_\sigma^\Lambda\gamma = \bigcup_{\sigma \in S_n\,:\, i\sigma = j} Y_{\alpha^{-1}\sigma\beta}^M = M_{i\alpha, j\beta}.$$

Thus (α, β, γ) is a strong isomorphism from Λ to M. □

Theorem 2. *If $\{\sigma^{-1}\colon \sigma \in \Lambda_r\} = M_r$ for $r = 0, \ldots, k$ then Λ is weakly isomorphic to M.*

Proof. The effect of transposing Λ is to replace each permutation π_x by its inverse. □

We write S^{-1} for $\{s^{-1}: s \in S\}$ for any subset S of S_n.

3. First stage: overcounting

The first stage of the enumeration is to find all families $(N_\sigma: \sigma \in S_n)$ satisfying (1) but using Theorem 1 to omit any family whose corresponding semi-Latin square is obviously strongly isomorphic to one already listed. We use Theorem 1 in the following way.

Suppose that P is a subset of S_n and that N_σ has been specified for $\sigma \in P$ in such a way that $\mathcal{N}(P)$ holds, where $\mathcal{N}(P)$ means

$$\sum_{\sigma\in P:\ i\sigma=j} N_\sigma \leqslant k \qquad \text{for } i,j \in \{1,\ldots,n\}. \tag{3}$$

We wish to adjoin a subset Q of $S_n \setminus P$ to some Λ_r in such a way that $\mathcal{N}(P \cup Q)$ holds. If Q_1 and Q_2 are candidates for Q, and if there exist α, β in S_n such that $\alpha^{-1}P\beta = P$ and $\alpha^{-1}Q_1\beta = Q_2$ then there is no loss of generality in choosing Q_1 and omitting Q_2. We almost always use this technique in the case that $\alpha = \beta$, because conjugacy is so easy to recognize in S_n.

Note that if $|\Lambda_k| = n$ then Λ is just the k-fold *inflation* of an $n \times n$ Latin square (obtained by replacing each letter of the Latin square by k letters). Thus the number of strong isomorphism classes of semi-Latin squares with $|\Lambda_k| = n$ is the same as the number of isotopy classes of Latin squares. Further, it is impossible to have $|\Lambda_k| = n - 1$. Also, $\sum_r r|\Lambda_r| = nk$.

Example 2. As an example, we show the method on the case $n = 4$, $k = 2$. Although this may be overkill for this small case, this does illustrate the more systematic method which is definitely needed for larger cases.

We have either $|\Lambda_2| = 4$ and $\Lambda_1 = \emptyset$, or $|\Lambda_2| = 2$ and $|\Lambda_1| = 4$, or $|\Lambda_2| = 1$ and $|\Lambda_1| = 6$, or $\Lambda_2 = \emptyset$ and $|\Lambda_1| = 8$. If $|\Lambda_2| = 4$ then Λ is an inflated Latin square so is one of the two squares (a) and (b) in Fig. 1.

If $|\Lambda_2| = 2$ then, by Theorem 1, we may assume that $1 \in \Lambda_2$. Then (1) shows that no other permutation in $\Lambda_1 \cup \Lambda_2$ can have any fixed points. By Theorem 1, we need consider for the other element of Λ_2 only one element of each conjugacy class of S_4 with no fixed points. If the other element of Λ_2 is $(1\,2)(3\,4)$ then (1) forces $\Lambda_1 = \{(1\,3)(2\,4), (1\,3\,2\,4), (1\,4)(2\,3), (1\,4\,2\,3)\}$ and Λ is the square in Fig. 1(c). If $\Lambda_2 = \{1, (1\,2\,3\,4)\}$ then there is no solution to (1).

If $|\Lambda_2| = 1$ then we may take $\Lambda_2 = \{1\}$. Then all elements of Λ_1 have no fixed points, so they have cycle structure 4 or 2^2. The six 4-cycles give one solution to (1) (shown in Fig. 1(d)); while if Λ_1 contains a permutation of cycle type 2^2 then it must

a b	c d	e f	g h
g h	a b	c d	e f
e f	g h	a b	c d
c d	e f	g h	a b

(a) inflated cyclic Latin square

a b	c d	e f	g h
c d	a b	g h	e f
e f	g h	a b	c d
g h	e f	c d	a b

(b) inflated non-cyclic Latin square

a b	c d	e f	g h
c d	a b	g h	e f
e g	f h	a b	c d
f h	e g	c d	a b

(c) $|\Lambda_2| = 2$

a b	c d	e f	g h
f h	a b	c g	d e
d g	e h	a b	c f
c e	f g	d h	a b

(d) $\Lambda_2 = \{1\}$; all 4-cycles

a b	c d	e f	g h
f h	a b	c g	d e
d e	g h	a b	c f
c g	e f	d h	a b

(e) $\Lambda_2 = \{1\}$; two cyclic subgroups

a b	c d	e f	g h
c h	a f	d g	b e
d e	b g	a h	c f
g f	e h	b c	a d

(f) D is (i)

a b	c d	e f	g h
g h	a f	c d	b e
d e	b g	a h	c f
c f	e h	b g	a d

(g) D is (i)

a b	c d	e f	g h
f h	a g	b c	d e
c g	e h	a d	b f
d e	b f	g h	a c

(h) D is (iii)

a b	c d	e f	g h
c d	a b	g h	e f
f h	e g	a c	b d
e g	f h	b d	a c

(i) D is (iv)

a b	c d	e f	g h
c g	a b	d h	e f
f h	e g	a c	b d
d e	f h	b g	a c

(j) D is (iv)

a b	c d	e f	g h
e g	a b	c h	d f
c f	g h	a d	b e
d h	e f	b g	a c

(k) D is (v)

a b	c d	e f	g h
e g	a b	c h	d f
c h	f g	a d	b e
d f	e h	b g	a c

(l) D is (v)

a b	c d	e f	g h
g h	a b	c d	e f
c f	e h	a g	b d
d e	f g	b h	a c

(m) D is (vi)

a b	c d	e f	g h
f g	a b	c h	d e
c d	e h	a g	b f
e h	f g	b d	a c

(n) D is (vi)

a b	c d	e f	g h
d g	a b	c h	e f
c f	e h	a g	b d
e h	f g	b d	a c

(o) D is (vi)

Fig. 1. First stage of counting $(4 \times 4)/2$ semi-Latin squares.

consist of the non-trivial elements of two cyclic subgroups of S_4 of order 4 (as in Fig. 1(e)).

Finally, suppose that $\Lambda_2 = \emptyset$. We may assume that $1 \in \Lambda_1$. Let $D = \{\sigma \in \Lambda_1\colon \sigma \neq 1,$ σ fixes at least one point.$\}$ Then, up to conjugacy, D is one of the following sets.

(i) $\{(1\,2\,3), (2\,1\,4), (4\,1\,3), (3\,2\,4)\}$.

(ii) $\{(1\,2\,3), (1\,2\,4), (2\,3\,4), (3\,4\,1)\}$.

(iii) $\{(1\,2\,3), (1\,2\,4), (2\,3\,4), (3\,1\,4)\}$.

(iv) $\{(1\,2), (3\,4)\}$.

(v) $\{(3\,4),(1\,2\,3),(1\,2\,4)\}$.

(vi) $\{(3\,4),(1\,2\,3),(2\,1\,4)\}$.

In case (i), D contains exactly one element σ such that $i\sigma = j$, for all $i,j \in \{1,\ldots,4\}$. Hence $\Lambda_1 \backslash D$ also defines a Latin square. But $\Lambda_1 \backslash D$ contains the identity, and D is fixed by the whole alternating group A_4. Up to conjugacy by A_4, the only possibilities for $\Lambda_1 \backslash D$ are the Klein subgroup of S_4 and one of the cyclic subgroups of S_4. These possibilities are shown in Figs. 1(f) and (g) respectively.

Case (ii) has no solution to (1).

Case (iii) can be completed in only one way, with the rest of Λ_1 as $(1\,3\,2\,4)$, $(1\,3\,4\,2)$ and $(1\,4\,3\,2)$. See Fig. 1(h).

In case (iv), Λ_1 can be completed in three ways, with any of the following sets of five permutations.

- $\{(1\,2)(3\,4),(1\,3\,2\,4),(1\,3)(2\,4),(1\,4)(2\,3),(1\,4\,2\,3)\}$.
- $\{(1\,2\,3\,4),(1\,3\,2\,4),(1\,3)(2\,4),(1\,4\,3\,2),(1\,4\,2\,3)\}$.
- $\{(1\,2\,4\,3),(1\,3\,2\,4),(1\,3\,4\,2),(1\,4)(2\,3),(1\,4\,2\,3)\}$.

However, D is fixed by $(1\,2)$, which interchanges the second and third of these sets, so we obtain only two further semi-Latin squares. They are shown in Figs. 1(i) and (j).

In case (v), Λ_1 can be completed in two ways.

- $\{(1\,3\,4\,2),(1\,3)(4\,2),(1\,4\,3\,2),(1\,4)(2\,3)\}$.
- $\{(1\,3\,4\,2),(1\,3\,2\,4),(1\,4\,3\,2),(1\,4\,2\,3)\}$.

These give the squares in Figs. 1(k) and (l).

Finally, in case (vi) Λ_1 can be completed in three ways.

- $\{(1\,2\,3\,4),(1\,3\,2\,4),(1\,3)(4\,2),(1\,4\,3\,2)\}$.
- $\{(1\,2\,4\,3),(1\,3\,2\,4),(1\,3\,4\,2),(1\,4)(2\,3)\}$.
- $\{(1\,2)(3\,4),(1\,3\,2\,4),(1\,3)(2\,4),(1\,4)(2\,3)\}$.

These give the squares in Figs. 1(m)–(o).

4. Second stage: pinning down the number of isomorphism classes

The result of the first stage of the search is a list of $(n \times n)/k$ semi-Latin squares which includes at least one representative of each strong isomorphism class. If Theorem 1 has been used effectively, there should not be many representatives of any one class.

The next stage is to separate elements of the list into different isomorphism classes as far as possible. To do this, we can use three progessively weaker criteria.

1. A semi-Latin square Λ defines a *quotient incomplete-block design* $\Delta(\Lambda)$, whose points are the letters in X^Λ and whose blocks are the sets Λ_{ij}. If Λ is weakly isomorphic to M then $\Delta(\Lambda)$ is isomorphic to $\Delta(M)$, but the converse is not true in general.
2. A semi-Latin square Λ defines a graph $G(\Lambda)$ whose vertices are the letters in X^Λ and in which the number of edges between x and y is equal to the number of blocks

in $\Delta(\Lambda)$ in which x and y both occur. If $\Delta(\Lambda)$ is isomorphic to $\Delta(M)$ then $G(\Lambda)$ is isomorphic to $G(M)$, but the converse is not true in general when $k > 2$.

3. If $G(\Lambda)$ is isomorphic to $G(M)$ then these two graphs have the same list of valencies, although the converse is not true in general.

Preece and Freeman [11] used primarily the valency list to distinguish isomorphism classes, but it is too feeble a criterion for larger sizes.

Example 2 revisited. Fig. 2 shows the graphs of the semi-Latin squares in Fig. 1, numbered correspondingly. These are the same as the incomplete-block designs, because $k = 2$. Each of squares (c), (d), (e), (f), (i) has a graph which is not isomorphic to any other graph in the list. Thus each of these is not even weakly isomorphic to any other square in Fig. 1: moreover, each of these squares must be strongly isomorphic to its transpose.

Where two or more squares have isomorphic quotient block designs, we use ad hoc arguments to complete the identification of isomorphism, guided by Theorems 1 and 2 and by explicit isomorphisms between the graphs or the block designs.

Example 2 revisited. The remaining squares in Fig. 1 are partitioned as follows into sets with isomorphic graphs:

$$\{(a),(b)\} \qquad \{(g),(j),(o)\} \qquad \{(h),(k),(l)\} \qquad \{(m),(n)\}.$$

Although squares (a) and (b) have isomorphic graphs, they are inflations of Latin squares which are not themselves isotopic, and each of which is isotopic to its transpose. So (a) and (b) belong to different strong isomorphism classes, each of which is also a weak isomorphism class.

Let Λ, M, N be the squares in Figs. 1(g), (j) and (o). By Theorem 1, if M is strongly isomorphic to Λ then there is some α in S_4 such that $\alpha^{-1}\Lambda_1$ is conjugate to M_1. Since Λ_1 contains six even permutations while M_1 contains six odd permuations, any such α must be an odd permutation. Moreover, M_1 contains the identity, so any such α must be in Λ_1. Now

$$(1\,2\,3\,4)^{-1}\Lambda_1 = \{(1\,4\,3\,2), 1, (1\,2\,3\,4), (1\,3)(2\,4), (1\,4), (1\,2\,4\,3), (2\,3), (1\,3\,4\,2)\},$$

which is conjugate to M_1 by $(1\,2\,3\,4)$. Hence Λ and M are strongly isomorphic. Arguing similarly, we soon find that $N_1 = (1\,2)\Lambda_1(1\,3)$. Thus Λ, M and N belong to a single strong isomorphism class, which is also a weak isomorphism class.

We use a different technique for the squares in Figs. 1(h), (k) and (l). Here the automorphism group of the graph fixes a unique edge: for Fig. 1(h) it is ac in cell $(4,4)$; for Fig. 1(k) it is cd in cell $(1,2)$; for Fig. 1(l) it is eg in cell $(2,1)$. Any strong isomorphism between these squares must map these cells to each other. We easily find that

$$((1\,4)(2)(3),\ (1)(2\,4)(3),\ (a\,d)(b\,h\,e)(c)(f\,g))$$

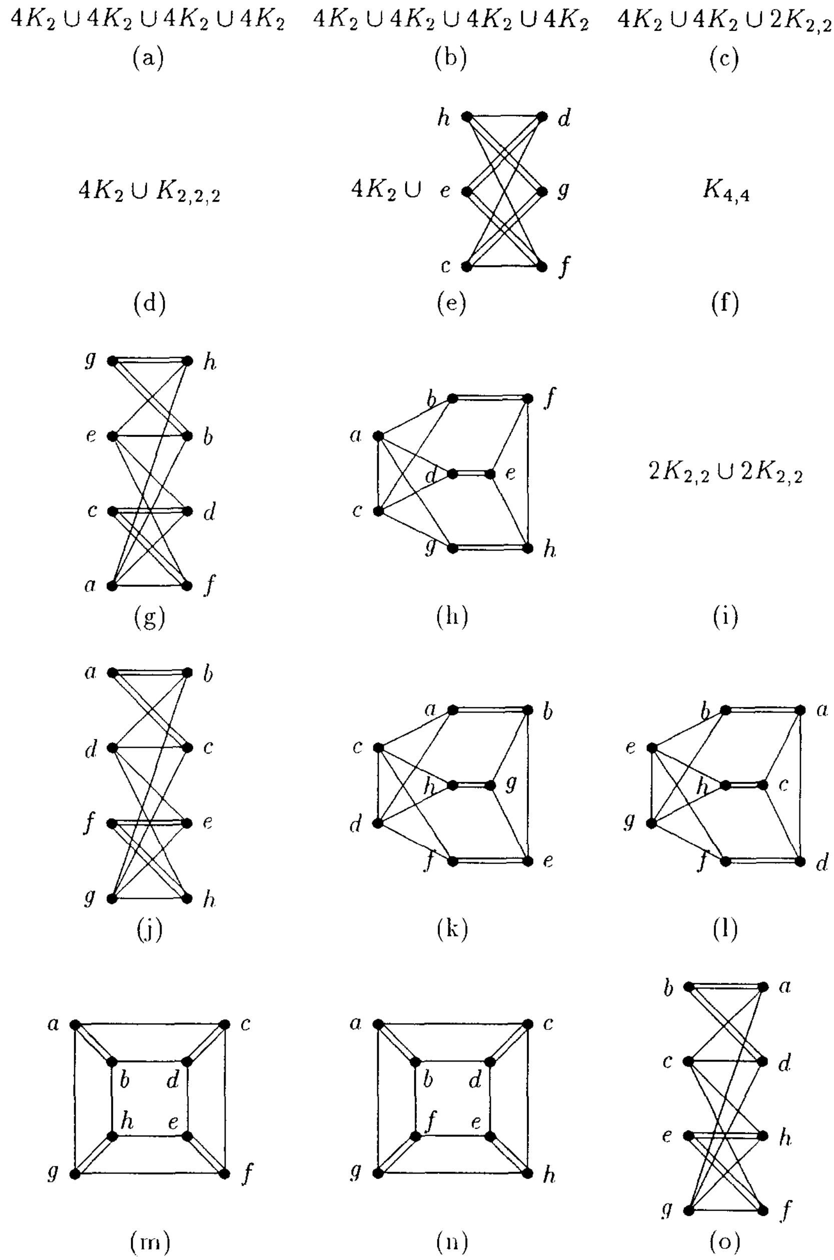

Fig. 2. Graphs of the semi-Latin squares in Fig. 1.

is a strong isomorphism from square 1(h) to square 1(k); while

$$((2\,4)(1)(3),\ (1\,4)(2\,3),\ (a\,g\,b\,h)(c\,e\,d\,f))$$

is a strong isomorphism from square 1(h) to square 1(l).

Finally, let Φ and Ψ be the squares in Figs. 1(m) and (n). The double edges of $G(\Phi)$ occur in all the cells of the first two rows of Φ, while the double edges of $G(\Psi)$ occur in all the cells of the first two columns of Ψ, so Φ cannot be strongly isomorphic to Ψ. However, this observation suggests that Φ might be strongly isomorphic to the transpose of Ψ. We use Theorem 2 and find that

$$\begin{aligned}\Psi_1^{-1} &= \{1,(3\,4),(2\,1\,3),(1\,2\,4),(1\,3\,4\,2),(1\,4\,2\,3),(1\,2\,4\,3),(1\,4)(2\,3)\}\\ &= (3\,4)\Phi_1(3\,4)\end{aligned}$$

and so Ψ is weakly isomorphic to Φ.

In summary, we have proved the following:

Theorem 3. *The* $(4 \times 4)/2$ *semi-Latin squares lie in* 11 *strong isomorphism classes, of which two merge into a single weak isomorphism class.*

This is consistent with the findings of Preece and Freeman [11], who called weak isomorphism classes *species* and strong isomorphism classes *transformation sets*.

5. Results for larger sizes

Isomorphism classes have also been found for $n = 4$ with $k = 3$ and $k = 4$ [6].

By using the same kind of systematic method and overcounting techniques for $n = 4$ and $k = 2$ in Section 3, we obtained one list of squares for $n = 4$ with $k = 3$, and one for $n = k = 4$. The list for $n = 4$ with $k = 3$ consists of 49 squares while that of $n = k = 4$ consists of 245 squares.

All the three progressively weaker criteria discussed in Section 4 for pinning down the overcounted squares into isomorphism classes are also applicable here. However, the part which involves checking the isomorphism of squares based on their graphs needed some more help since this becomes more complicated with larger sizes. As mentioned earlier, the criterion of distinguishing one isomorphism class of a square from the other based on their valency lists is too feeble for larger sizes. So, to ascertain the strong and weak isomorphism classes of larger sizes of the $(n \times n)/k$ semi-Latin squares we made use of NAUTY [9].

The NAUTY package compares a set of graphs pairwise for isomorphism. Its output is a list indexing each set according to isomorphism classes. To be able to use this package for our purpose we need to convert the semi-Latin squares into graphs in two ways. In the first way the graphs should be isomorphic if and only if the semi-Latin squares are weakly isomorphic; in the second way the graphs should be isomorphic if and only if the semi-Latin squares are strongly isomorphic. Thus given an $(n \times n)/k$ semi-Latin square Λ we convert it into two new graphs $H(\Lambda)$ and $H'(\Lambda)$ as follows.

The first stage of this conversion involves identifying five types of vertex for each square, namely: row-type, column-type, letter-type, position-type and extra-type vertices

	1	2	3	4
1	$a\ b\ c$	$d\ e\ f$	$g\ h\ i$	$j\ k\ l$
2	$d\ e\ f$	$a\ b\ c$	$j\ k\ l$	$g\ h\ i$
3	$j\ k\ l$	$g\ h\ i$	$a\ b\ c$	$d\ e\ f$
4	$g\ h\ i$	$j\ k\ l$	$d\ e\ f$	$a\ b\ c$

Fig. 3. A $(4 \times 4)/3$ semi-Latin square

(see below). The number n of rows, the number k of entries per row–column intersection and the number nk of letters are determined. Then, the number n^2k of positions in a square and the total number v of its vertices are calculated. Squares with different values of v cannot be compared for isomorphism.

The five types of vertices are labelled from 0 to $v-1$ as follows: $0, \ldots, n-1$ for the row-type vertices, $n, \ldots, 2n-1$ for the column-type vertices, $2n, \ldots, 2n+nk-1$ for the letter-type vertices, $2n+nk, \ldots, 2n+nk+n^2k-1$ for the position-type vertices and $2n+nk+n^2k$, $2n+nk+n^2k+1$ for the extra-type vertices. Each position-vertex is joined to the row-vertex, the column-vertex and the letter-vertex corresponding to the row, column and letter at that position. The row-type vertices are differentiated from the column-type vertices by putting one of the extra-type vertices adjacent to every row-vertex and the other adjacent to every column vertex.

Then the extra-type vertices may or may not be allowed to interchange depending on whether interest is on strong or weak isomorphism. We make the graphs $H(\Lambda)$ for determining weak isomorphism classes by allowing the extra-type vertices of each square to interchange. On the other hand, the extra-type vertices of each square are disallowed from interchanging if we want to make graphs $H'(\Lambda)$ for determining the strong isomorphism classes. In this case, a loop is added to the row extra-type vertices so that no isomorphism can take a row extra-type vertex to a column extra-type vertex. Now it is clear that $H(\Lambda)$ and $H(M)$ are isomorphic graphs if and only if the semi-Latin squares Λ and M are weakly isomorphic; while $H'(\Lambda)$ is isomorphic to $H'(M)$ if and only if Λ is strongly isomorphic to M.

Example 3. The vertex-labels and adjacencies below are determined from the $(4\times 4)/3$ semi-Latin square in Fig. 3.

- $n=4$, $k=3$, $nk=12$, $v=70$.

Labels of vertices:

- row type: $0, \ldots, 3$,
- column type: $4, \ldots, 7$,
- letter type (for letters $a, b, \ldots, k, l$): $8, 9, \ldots, 18, 19$,
- position type (for all the 48 entries of the square): $20, \ldots, 67$,
- extra type: 68, 69.

Some adjacencies:

- first position (vertex 20): $\{0, 4, 8\}$,
- first row (vertex 0): $\{20, 21, 22, \ldots, 29, 30, 31, 68\}$,

- first column (vertex 4): $\{20, 21, 22, 32, 33, 34, 44, 45, 46, 56, 57, 58, 69\}$,
- first letter (vertex 8): $\{20, 35, 50, 65\}$,
- first extra (vertex 68): $\{0, 1, 2, 3\}$,
- second extra (vertex 69): $\{4, 5, 6, 7\}$.

NAUTY classifies the graphs into their isomorphism classes. If the graphs are labelled $1, \ldots, m$ then the output from the program is a list $C_1, \ldots, C_m$, where C_i is the smallest value of j with $1 \leqslant j \leqslant i$ such that graph i is isomorphic to graph j. The details of the classification procedure and results of the strong and weak isomorphism classifications for $n = 4$ with $k = 3$ and $k = 4$ are given in [6].

We summarize our results for these larger sizes in the following theorems:

Theorem 4. *The* $(4 \times 4)/3$ *semi-Latin squares lie in* 43 *strong isomorphism classes, of which* 10 *merge into five weak isomorphism classes, giving* 38 *weak isomorphism classes in total.*

Theorem 5. *The* $(4 \times 4)/4$ *semi-Latin squares lie in* 157 *strong isomorphism classes, of which* 46 *merge into* 23 *weak isomorphism classes, giving* 134 *weak isomorphism classes in total.*

(A referee has pointed out that the graphs $H(\Lambda)$ and $H'(\Lambda)$ can be made into smaller graphs $K(\Lambda)$ and $K'(\Lambda)$ by amalgamating all the position-type vertices in each row–column intersection. Isomorphism of the graphs $K(\Lambda)$ and $K(M)$ is still equivalent to weak isomorphism of the semi-Latin squares Λ and M, and isomorphism of the graphs $K'(\Lambda)$ and $K'(M)$ is still equivalent to strong isomorphism of Λ and M. Although NAUTY had no difficulty in classifying the larger graphs for our values of n and k, it would be sensible to use the smaller graphs for larger values.)

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