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"Lagrangian" Construction for Representations of Hecke Algebras

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In this paper we give the geometric interpretation of Hecke algebras of both ordinary and affine Weyl groups. As a by-product we get the "Springer type" construction for representations of affine Hecke algebras and compute their characters. Our results were strongly motivated by conjectures given by G. Lusztig in [Lu1, Lu2].

1. Let K be a non-archimedean local field with the residue field consisting of q_0 elements. According to "Langlands philosophy" irreducible complex representations of a reductive K-group G_K should roughly speaking correspond to homomorphisms of the Weil–Deligne group into the so-called L-group.

Let $I \subset G_K$ be the Iwahori subgroup. The category of admissible G_K-modules, generated by I-fixed vectors, is known to be equivalent to the category of finite-dimensional representations of the Hecke algebra $H(I \backslash G_K / I)$. "Langlands philosophy" suggests via that equivalence the following classification of irreducible Hecke algebra modules (stated in purely complex terms and not involving any local field):

Conjecture 1 [Lu1]. Let G be an arbitrary complex semisimple Lie group and H the (modified) affine Hecke algebra over $\mathbf{Z}[q, q^{-1}]$ associated to G. Then finite-dimensional irreducible H-modules are in 1–1 correspondence with conjugacy classes of triples:

$$\begin{aligned} (h, n, \chi)\colon\quad & h \in G \text{ is semisimple}, \quad n \in \mathfrak{g} \text{ is nilpotent} \\ & \text{such that:}\quad \operatorname{Ad} h(n) = q_0^{-1} \cdot n, \quad q_0 \in \mathbf{C}, \\ & \text{and} \quad \chi \in A(h, n)\hat{}. \end{aligned} \tag{1.1}$$

Here $\mathfrak{g}$ is the Lie algebra of G, $Z_G(h, n)$ is the centralizer in G of both h and n, and χ is an irreducible character of the finite group $A(h, n) = Z_G(h, n)/Z_G^0(h, n)$.

2. Let X be the Flag manifold (i.e., the variety of all Borel subgroups in G) and T^*X its cotangent bundle. There is a natural action of the group $M = \mathbf{C}^* \times G$ on T^*X and on the nilpotent cone $N \subset \mathfrak{g}$ (in each case

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$\mathbf{C}^*$ acts by multiplication and G by conjugation). For a semisimple element $m=(q,h)\in M$ let $\tilde{\mathfrak{g}}_h$ (resp. N_h) be the subvariety of fixed points of m in T^*X (resp. N). Note that $\tilde{\mathfrak{g}}_h$ is a disjoint union of smooth subvarieties[1] while N_h consists of a finite number of $Z_G(h)$-orbits.

Consider Springer's resolution $\mu\colon T^*X\to N$. Restricting it to m-fixed points and using the decomposition theorem (see [BBD]) we get (as in [BM])

$$R\mu_*(\mathbf{Q}_{\tilde{\mathfrak{g}}_h}) = \bigoplus_{\substack{Z_G(h)\cdot n\subset N_h\\ i\in\mathbf{Z},\ \chi\in A(h,n)^\wedge}} L^i_{h,n,\chi}\otimes {}^\pi\mathcal{L}_{h,n,\chi}[-i]. \tag{2.1}$$

Here $\mathbf{Q}_{\tilde{\mathfrak{g}}_h}$ is the constant sheaf on the *smooth* variety $\tilde{\mathfrak{g}}_h$, ${}^\pi\mathcal{L}_{h,n,\chi}$ is the intersection cohomology complex on $Z(h)\cdot n$ with the monodromy χ, and $L^i_{h,n,\chi}$ are certain vector spaces. In contrast with [BM] these spaces are in general non-trivial not only for one particular $i=i(h,n,\chi)$. Set $L_{h,n,\chi}=\sum(-1)^i\cdot L^i_{h,n,\chi}$.

Conjecture 2. For each triple (h,n,χ) as in (1.1) there is a natural H-action on $L_{h,n,\chi}$. The set $\{L_{h,n,\chi}\}$ is precisely the collection of all irreducible finite dimensional H-modules.

3. Clearly Conjecture 1 is a consequence of our conjecture. In this paper instead of constructing H-modules $L_{h,n,\chi}$ we'll define "standard" H-modules $K_{h,n,\chi}$ such that $L_{h,n,\chi}$ should be their irreducible quotients (cf. [Z, K2]). Namely, for a semisimple element $h\in G$ and $n\in N_h$ consider the fibre $\mu^{-i}(n)=X_{h,n}$ of the map $\mu\colon\tilde{\mathfrak{g}}_h\to N_h$. In other words $X_{h,n}$ is the variety of all Borel subgroups containing both h and $\exp n$. The group $Z_G(h,n)$ acts on $X_{h,n}$ by cojugation, giving rise to the action of $A(h,n)$ on $K^{\mathrm{top}}(X_{h,n})$. Here $K^{\mathrm{top}}(X_{h,n})=K^0(X_{h,n})\oplus K^1(X_{h,n})$ stands for the topological K-theory (with complex coefficients) of the variety $X_{h,n}$.

THEOREM 3. *For each pair* (h,n), $\mathrm{Ad}\,h(n)=q_0^{-1}\cdot n$, $q_0\in\mathbf{C}$, *there is a natural action of the affine Hecke algebra H (over $\mathbf{Z}[q,q^{-1}]$) on* $K^i(X_{h,n})$, $i=0,1$, *such that q acts as multiplication by q_0. The H-action and the $A(h,n)$-action commute with each other.*

This result was conjectured in [K2] and is closely related to [Lu2]. To a certain extent we'll give the geometric explanation of [Lu2].

For $q=1$ the Hecke algebra degenerates to the group algebra of the Weyl group and our theorem reduces (for $h=1$) to Springer's representations of Weyl groups on cohomologies of $X_n:=X_{1,n}$.

For an irreducible character $\chi\in A(h,n)^\wedge$ let

$$K_{h,n,\chi}=\mathrm{Hom}_{A(h,n)}(\chi, K^0(X_{h,n}))-\mathrm{Hom}_{A(h,n)}(\chi, K^1(X_{h,n}))$$

[1] As fixed points of a maximal compact subgroup in the reductive group generated by m.

be the χ-isotypical component of $K^0(X_{h,n}) - K^1(X_{h,n})$. According to Theorem 3, $K_{h,n,\chi}$ is a virtual H-module. This is the "standard" module mentioned above.

Identifying $K^{\text{top}}(X_{h,n})$ with cohomology we may regard $K_{h,n,\chi}$ as a χ-isotypical component of $H^*(X_{h,n})$. Conjecture 2 combined with (2.1) would imply (cf. [BM]) the following "p-adic" analogue of the Kazhdan–Lusztig formula for the multiplicity of $L_{h,n,\chi}$ in $K_{h,u,\phi}$ conjectured in [Z] (see also [K2]):

COROLLARY 3.1.

$$[K_{h,u,\phi} : L_{h,n,\chi}] = \sum (-1)^i \cdot \dim \mathscr{H}^i_u({}^\pi\mathscr{L}_{h,n,\chi})_\phi .$$

4. Choose a maximal torus $T \subset G$. Let $P = \text{Hom}(T, \mathbf{C}^*)$ be the lattice of weights and W be the Weyl group of (G, T). Consider the affine Weyl group $\tilde{W} = W \ltimes P$ and the corresponding affine Hecke algebra H over $\mathbf{Z}[q, q^{-1}]$. It is known that H contains the commutative subalgebra isomorphic to (and identified with) the group algebra $\mathbf{Z}[P]$.

THEOREM 4. *The character of the restriction to* $\mathbf{Z}[P]$ *of* $K_{h,n,\chi}$ *equals* (cf. [Lu1])

$$\text{tr}(\lambda; K_{n,h,\chi}) = \frac{1}{|A(h,n)|} \cdot \sum_{j=1}^{r} \sum_{\substack{g \in A(h,n) \\ gX_j = X_j}} \chi(g) \cdot l(g, X_j) \cdot \langle \lambda, h \rangle_j .$$

Here $\lambda \in P \subset H$, $\{X_j, j = 1, 2, ..., r\}$ is the collection of connected components of $X_{h,n}$ and $l(g, X_j)$ is the Lefschetz number of the map $g: X_j \to X_j$. The number $\langle \lambda, h \rangle_j$ is defined as follows: choose a Borel subgroup $B \in X_j$ and identify T with $B/[B, B]$. Then λ canonically identifies with the character of B. Since $h \in B$ the value of that character at h is well defined and is denoted $\langle \lambda, h \rangle_j$. The construction does not depend on a choice of $B \in X_j$.

5. We recall a few definitions from [Gi]. The only new element here— the equivariant flavor—is borrowed from [Lu2].

Suppose the algebraic group M acts on a smooth algebraic variety N (over $\mathbf{C}$). For an M-stable subvariety $\Lambda \subset N$ denote by $K_M(\Lambda)$ the group of bounded equivariant complexes of locally free $\mathcal{O}_N$-sheaves exact off Λ. Clearly $K_M(\Lambda)$ is a module over the representation ring $R(M)$. If $M = \{1\}$ then $K_M(\Lambda) = K(\Lambda)$ is just the usual Grothendieck group, generated by coherent $\mathcal{O}_\Lambda$-sheaves.

Let N_i, $i = 1, 2, 3$, be smooth varieties and $p_{ij}: N_1 \times N_2 \times N_3 \to N_i \times N_j$ the natural projections. If M acts on each N_i then it acts on their products and these actions commute with p_{ij}. Suppose $\Lambda \subset N_1 \times N_2$, $\Lambda' \subset N_2 \times N_3$ are M-stable subvarieties such that the map

$$p_{13}: p_{12}^{-1}(\Lambda) \cap p_{23}^{-1}(\Lambda') \to N_1 \times N_3 \tag{5.1}$$

is proper. Then its image is a closed M-stable subvariety in $N_1 \times N_3$ denoted $\Lambda \circ \Lambda'$. Define multiplication $K_M(\Lambda) \otimes K_M(\Lambda') \to K_M(\Lambda \circ \Lambda')$ as follows: for $\mathscr{F}^{\cdot} \in K_M(\Lambda)$, $\tilde{\mathscr{F}}^{\cdot} \in K_M(\Lambda')$ set

$$\mathscr{F}^{\cdot} \circ \tilde{\mathscr{F}}^{\cdot} = (Rp_{13})_* \left(p_{12}^* \mathscr{F}^{\cdot} \overset{L}{\underset{\mathcal{O}_{N_1 \times N_2 \times N_3}}{\otimes}} p_{23}^* \tilde{\mathscr{F}}^{\cdot} \right).$$

Later we'll make use of the similar construction in topological K-theory. In that case suppose M to be a reductive complex Lie group with the maximal compact subgroup M_c. For N as above denote by $K_M^0(N)$ the Grothendieck group, generated by M-equivariant topological vector bundles on N (cf. [Lu2]).

In a similar way one defines $K_M^1(N)$, $K_M^{\mathrm{top}}(N) = K_M^0(N) \oplus K_M^1(N)$ and the multiplication $K_M^{\mathrm{top}}(N_1 \times N_2) \otimes K_M(N_2 \times N_3) \to K_M(N_1 \times N_3)$ compatible with $\mathbf{Z}_2$-gradation.

Since any algebraic vector bundle on N may be regarded as a topological bundle as well there is a natural transformation $K_M(N) \to K_M^0(N)$ commuting with multiplication (see [BFM]). In particular one can define the homomorphism (for $\Lambda \subset N_1 \times N_2$))

$$K_M(\Lambda) \otimes K_M^{\mathrm{top}}(N_2) \to K_M^{\mathrm{top}}(N_1). \tag{5.2}$$

6. *Construction of Hecke Algebras.* Let X be the Flag manifold for G and T^*X its cotangent bundle. Consider G as a diagonal in $G \times G$. Denote by C_w, $w \in W$, the corresponding G-orbit in $X \times X$ and by $T^*_{C_w}(X \times X)$ its conormal bundle. Set $\Lambda = \bigcup_{w \in W} T^*_{C_w}(X \times X)$. It is known that Λ is closed in $T^*(X \times X)$. Let $S \subset W$ be the set of simple reflections. For $s \in S$ the closure $\bar{C}_s$ is the smooth subvariety in $X \times X$ so that $\Lambda_s := T^*_{\bar{C}_s}(X \times X)$ is a component of Λ. The projection $\bar{C}_s \to X$ to the first factor is the fibration with 1-dimensional fibre isomorphic to projective line $\mathbf{P}^1$. Denote by $\Omega^1_{\bar{C}_s/X}$ the sheaf of relative 1-forms on $\bar{C}_s$ and by $\pi_s: \Lambda_s \to \bar{C}_s$ the projection. Set: $\mathcal{O}_s = \pi_s^* \Omega^1_{\bar{C}_s/X} = \mathcal{O}_{\Lambda_s} \otimes_{\pi_s^{\cdot} \mathcal{O}_{\bar{C}_s}} \pi_s^{\cdot} \Omega^1_{\bar{C}_s/X}$.

Further let $\Delta \subset X \times X$ be the diagonal, $\Lambda_\Delta = T^*_\Delta(X \times X)$ its conormal bundle, and $\pi_\Delta: \Lambda_\Delta \to \Delta$ the projection. For any G-equivariant algebraic vector bundle E on Δ let $E_\Delta = \pi_\Delta^{\cdot} E \otimes_{\mathcal{O}_\Delta} \mathcal{O}_{\Lambda_\Delta}$ be the corresponding sheaf on Λ_Δ.

As in Section 2 consider the natural action on $T^*X \times T^*X$ of the group $\mathbf{C}^* \times G$. Clearly Λ is the $\mathbf{C}^* \times G$-stable subvariety so that the groups $K_{\mathbf{C}^*}(\Lambda)$ and $K_{\mathbf{C}^* \times G}(\Lambda)$ are defined. They are modules over the representation ring $\mathbf{Z}[\mathbf{C}^*]=\mathbf{Z}[q,q^{-1}]$. Since $\Lambda\circ\Lambda=\Lambda$ (see [Gi]) these modules actually acquire ring structures with multiplication:

$$K(\Lambda)\otimes K(\Lambda)\to K(\Lambda\circ\Lambda)=K(\Lambda)\quad \text{defined in Section 5.}$$

THEOREM 6. (i) *The subalgebra in* $K_{\mathbf{C}^*\times G}(\Lambda)$ *generated by* $\mathcal{O}_s$, $s\in S$, *is isomorphic to the Hecke algebra* H_W *of the Weyl group* W;

(ii) *the algebra* $K_{\mathbf{C}^*\times G}(\Lambda)$ *is isomorphic to the affine Hecke algebra* H.

EXAMPLE: $G=SL_2$. Let s be the only (simple) reflection in $W\cong\mathbf{Z}/2\mathbf{Z}$, T_s the corresponding generator of H_W satisfying the relation $(T_s+1)(T_s-q)=0$. In terms of $c_s:=T_s+1$ it can be written as $c_s^2=(q+1)\cdot c_s$.

In our case $X=\mathbf{P}^1$ is the projective line. There are two G-orbits in $X\times X$ so that Λ consists of two components: Λ_Δ and $\Lambda_s=$ the zero-section $\simeq \mathbf{P}^1\times\mathbf{P}^1$. Let us verify that $\mathcal{O}_s\circ\mathcal{O}_s=-(q+1)\cdot\mathcal{O}_s$. We have $\mathcal{O}_s=\mathcal{O}_{\mathbf{P}^1}\times\Omega^1_{\mathbf{P}^1}$ is a sheaf on $T^*\mathbf{P}^1\times T^*\mathbf{P}^1$. Using the Koszul complex $0\to\mathcal{O}_{T^*\mathbf{P}^1}\to\pi^*\Omega_{\mathbf{P}^1}\to\Omega_{\mathbf{P}^1}\to 0$ we resolve the sheaf $\mathcal{O}_s$ by means of a complex $0\to\mathcal{O}_{\mathbf{P}^1}\times\mathcal{O}_{T^*\mathbf{P}^1}\to\mathcal{O}_{\mathbf{P}^1}\times\pi^*\Omega_{\mathbf{P}^1}\to\mathcal{O}_s\to 0$. Note that the map $\mathcal{O}_{\mathbf{P}^1}\times\mathcal{O}_{T^*\mathbf{P}^1}\to\mathcal{O}_{\mathbf{P}^1}\times\pi^*\Omega^1_{\mathbf{P}^1}$ in that complex *does not* commute with $\mathbf{C}^*$-action. To make it equivariant we must shift the action on $\pi^*\Omega^1_{\mathbf{P}^1}$. Thus in the Grothendieck group we get $\mathcal{O}_s=q\cdot(\mathcal{O}_{\mathbf{P}^1}\times\pi^*\Omega^1_{\mathbf{P}^1})-\mathcal{O}_{\mathbf{P}^1}\times\mathcal{O}_{T^*\mathbf{P}^1}$. Whence

$$\begin{aligned}\mathcal{O}_s\circ\mathcal{O}_s&=(q\cdot\mathcal{O}_{\mathbf{P}^1}\times\pi^*\Omega_{\mathbf{P}^1}-\mathcal{O}_{\mathbf{P}^1}\times\mathcal{O}_{T^*\mathbf{P}^1})\circ\mathcal{O}_{\mathbf{P}^1}\times\Omega_{\mathbf{P}^1}\\&=q\cdot\chi(\Omega_{\mathbf{P}^1})\cdot\mathcal{O}_{\mathbf{P}^1}\times\Omega_{\mathbf{P}^1}-\chi(\mathcal{O}_{\mathbf{P}^1})\cdot\mathcal{O}_{\mathbf{P}^1}\times\Omega_{\mathbf{P}^1}\\&=-(q+1)\cdot\mathcal{O}_{\mathbf{P}^1}\times\Omega_{\mathbf{P}^1}=-(q+1)\cdot\mathcal{O}_s,\end{aligned}$$

Here $\chi(\cdot)$ is the alternating sum of cohomology groups.

Remark. The group of G-equivariant vector bundles on Δ is isomorphic to the representation ring $R(T)$. Hence the subalgebra in $K_{\mathbf{C}^*\times G}(\Lambda_\Delta)$ generated by various sheaves E_Δ identifies with $R(T)\simeq\mathbf{Z}[P]$. We see that H has two subalgebras, H_W and $\mathbf{Z}[P]$, and as a vector space

$$H\simeq H_W\otimes\mathbf{Z}[P].$$

7. *Construction of representations of H.* Keeping to the notations of Section 6 consider the Lagrangian subvariety $\Lambda\subset T^*X\times T^*X$ and the moment map $\mu\times\mu$: $T^*X\times T^*X\to\mathfrak{g}\times\mathfrak{g}$.

LEMMA. *The image of* Λ *is the nilpotent cone in the subalgebra* $\mathfrak{g}_\Delta:=$ the diagonal in $\mathfrak{g}\times\mathfrak{g}$.

COROLLARY. *For any subset $U \subset \mathfrak{g}$ we have*

$$\Lambda \circ \mu^{-1}(U) = \mu^{-1}(U). \tag{7.1}$$

Set $M = \mathbf{C}^* \times G$. Suppose $n \in \mathfrak{g}$ is a nilpotent element. Let M_n be the reductive part of the stabilizer of n in M. The restriction map $K_M(\Lambda) \to K_{M_n}(\Lambda)$ is clearly the ring-homomorphism. According to Theorem 6 we get the homomorphism

$$H \to K_{M_n}(\Lambda). \tag{7.2}$$

Let U_n be a small open neighborhood of $n \in \mathfrak{g}$ stable under the action of a maximal compact subgroup in M_n. In view of (5.2) and (7.1) there is a map

$$K_{M_n}(\Lambda) \otimes K^{\mathrm{top}}_{M_n}(\mu^{-1}(U_n)) \to K^{\mathrm{top}}_{M_n}(\mu^{-1}(U_n)). \tag{7.3}$$

Without loss of generality we may choose U_n to be equivariantly contractible to n so that $\mu^{-1}(U_n)$ contracts to $\mu^{-1}(n) = X_n$. Then $K^{\mathrm{top}}_{M_n}((\mu^{-1}(U_n)) \simeq K^{\mathrm{top}}_{M_n(X_n)}$. Combining this isomorphism with (7.2), (7.3) we obtain the following result, conjectured in [Lu2]:

PROPOSITION 7. *There is a natural H-module structure on $K^{\mathrm{top}}_{M_n}(X_n)$.*

Proof of Theorem 3. Suppose $m = (q, h) \in M_n$ where h is semisimple. Let M_h be the smallest complex reductive subgroup in M_n containing m. It is clearly commutative. Identify elements of $R(M_h)$ with their characters, i.e., with functions on M_h. Let $\tilde{R}$ be the localization of the ring $R(M_h)$ with respect to all characters that do not vanish at m. Consider the action of $\phi \in R(M_h)$ (resp. $\phi \in \tilde{R}$) on $\mathbf{C}$ via the multiplication by $\phi(m)$. Denote by $\mathbf{C}_h$ the $R(M_h)$- (resp. $\tilde{R}$-) module obtained in that way. Obviously

$$\mathbf{C}_h \underset{R(M_h)}{\otimes} R(M_h) \simeq \mathbf{C}_h \underset{\tilde{R}}{\otimes} \tilde{R}.$$

The same argument as above with M_n replaced by M_h shows that there is a natural H-module structure on $K^{\mathrm{top}}_{M_h}(X_n)$. Consider the restriction homomorphism $K^{\mathrm{top}}_{M_h}(X_n) \to K^{\mathrm{top}}_{M_h}(X_{h,n})$ and note that $X_{h,n}$ is the set of fixed points of M_h in X_n. According to the localization theorem in equivariant topological K-theory [Se] the induced homomorphism

$$\tilde{R} \underset{R(M_h)}{\otimes} K^{\mathrm{top}}_{M_h}(X_n) \to \tilde{R} \underset{R(M_h)}{\otimes} K^{\mathrm{top}}_{M_h}(X_{h,n})$$

is an isomorphism. Hence $\mathbf{C}_h \otimes_{R(M_h)} K^{\mathrm{top}}_{M_h}(X_n) \simeq \mathbf{C}_h \otimes_{R(M_h)} K^{\mathrm{top}}_{M_h}(X_{h,n})$ so that we get an H-module structure on the right-hand side. It remains to note that

$$\begin{aligned}\mathbf{C}_h &\underset{R(M_h)}{\otimes} K^{\mathrm{top}}_{M_h}(X_{h,n})\\ &= \mathbf{C}_h \underset{R(M_h)}{\otimes} (R(M_h)\otimes K^{\mathrm{top}}(X_{h,n}))\\ &= \mathbf{C}_h \otimes K^{\mathrm{top}}(X_{h,n}) \cong K^{\mathrm{top}}(X_{h,n}).\end{aligned} \qquad \text{Q.E.D.}$$

Remarks. (i) If $q=1$ so that $h\cdot n\cdot h^{-1}=n$ our construction gives the action of the affine Weyl group $\tilde{W}=W\ltimes P$ on $K^{\mathrm{top}}(X_{h,n})$. When transferred to the cohomology $H^*(X_{h,n})$ by means of the Chern character it presumably coincides with representations of $\tilde{W}$ defined in [K1].

(ii) For $q=1$ and $h=1$ the construction of Sections 6 and 7 is very close to that of [Gi]. Note, however, that our present approach gives rise to a Weyl group representation on the whole cohomology group $H^*(X_n)$ while in [Gi] we were only able to define the action on the top cohomologies $H^d(X_n)$, $d=\dim X_n$.

8. *Proof of Theorem* 4. For $\lambda\in\mathrm{Hom}(T,\mathbf{C}^*)$ let E^λ be the line bundle on X associated with λ (i.e., $E^\lambda=\mathbf{C}^\lambda\times_B G$ where B is a Borel subgroup, containing T, and $\mathbf{C}^\lambda$ the 1-dimensional B-module, corresponding to the character $B\to T\to^\lambda \mathbf{C}^*$). Then E^λ_Δ is the line bundle on Λ_Δ.

Choose $(q,h)\in M_n$ and consider the action of the sheaf E^λ_Δ on $K^{\mathrm{top}}_{M_h}(X_{h,n})$. Regarding $X_{h,n}$ as a subvariety in X we see that this action coincides with multiplication by $E^\lambda|_{X_{h,n}}$.

Set $A=A(h,n)$. The collection $\{X_j\}$ of connected components of $X_{h,n}$ is the disjoint union of A-orbits $A\cdot X_j$. Accordingly, the A-module $K^{\mathrm{top}}(X_{h,n})$ breaks up into a direct sum

$$K^{\mathrm{top}}(X_{h,n}) = \bigoplus_{\{A\cdot X_j\}} \mathrm{Ind}^A_{A_j} K^{\mathrm{top}}(X_j) \tag{8.1}$$

where A_j is the stabilizer of X_j in A. For any $\chi\in A^\wedge$ we have $\mathrm{Hom}_A(\chi,\mathrm{Ind}^A_{A_j}K^{\mathrm{top}}(X_j))\simeq \mathrm{Hom}_{A_j}(\chi|_{A_j}, K^{\mathrm{top}}(X_j))$. Thus

$$\mathrm{tr}(\lambda; K_{h,n,\chi}) = \sum_{\{A\cdot X_j\}} \mathrm{tr}(E^\lambda|_{X_j}; \mathrm{Hom}_{A_j}(\chi|_{A_j}, K^{\mathrm{top}}(X_j))). \tag{8.2}$$

Let $G(h)$ be the centralizer of h in G. The subvariety in X of all Borel subgroups containing h is a disjoint union of connected manifolds Y_i, $i=1,2,\ldots$. Each Y_i is a $G(h)$-orbit. Since M_h commutes with $G(h)$ it acts on each fibre of $E^\lambda|_{Y_i}$ as a multiplication by a certain character $\lambda_i\colon M_h\to\mathbf{C}^*$. This character does not depend on the choice of the fibre and is defined in the same way as numbers $\langle\lambda, h\rangle_j$: pick up a Borel subgroup $B \in Y_i$, identify T with $B/[B, B]$ and λ with the character of B. Since the projection of M_h to G belongs to T that character λ gives rise to the character of M_h denoted λ_i. With that understood we may identify the element $E^\lambda|_{Y_i}$ of $K^{\mathrm{top}}_{M_h}(Y_i) \simeq R(M_h) \otimes K^{\mathrm{top}}(Y_i)$ with $\lambda_i \otimes E^\lambda_{Y_i}$ where $E^\lambda_{Y_i} \in K^{\mathrm{top}}(Y_i)$ is the same topological bundle as $E^\lambda|_{Y_i}$ but with the trivial action of M_h.

Returning to the trace computation consider a component X_j of $X_{h,n}$. If $X_j \subset Y_i$ for some i then obviously $\langle\lambda, h\rangle_j = \lambda_i(h)$. It is clear that multiplication by $E^\lambda|_{X_j}$ identifies via the isomorphisms

$$\mathbf{C}_h \underset{R(M_h)}{\otimes} K^{\mathrm{top}}_{M_h}(X_j) \simeq K^{\mathrm{top}}(X_j) \simeq H^*(X_j)$$

with multiplication by $\langle\lambda, h\rangle_j \cdot \mathrm{ch}(E^\lambda_{X_j})$ where "ch" denotes the Chern character. Note that $\mathrm{ch} = 1 + \mathrm{ch}^1 + \mathrm{ch}^2 + \cdots$, $\mathrm{ch}^i \in H^i(X_j)$. Thus the operator on $H^*(X_j)$ corresponding to $\lambda \in \mathrm{Hom}(T, \mathbf{C}^*)$ is of the form $\langle\lambda, h\rangle_j \cdot (1 + N)$ where N is a nilpotent operator. We see that for any λ-stable subspace $V \subset K^{\mathrm{top}}(X_j)$

$$\mathrm{tr}(\lambda; V) = \langle\lambda, h\rangle_j \cdot \dim V. \tag{8.3}$$

It follows from (8.2) and (8.3) that

$$\mathrm{tr}(\lambda; K_{h,n,\chi}) = \sum_{\{A \cdot X_j\}} \langle\lambda, h\rangle_j \cdot \dim \mathrm{Hom}_{A_j}(\chi|_{A_j}, K^{\mathrm{top}}(X_j)). \tag{8.4}$$

Using the well-known equality

$$\dim \mathrm{Hom}_{A_j}(V_1, V_2) = |A_j|^{-1} \sum_{g \in A_j} \mathrm{tr}(g; V_1) \cdot \overline{\mathrm{tr}(g; V_2)}$$

the right-hand side of (8.4) can be rewritten as

$$|A_j|^{-1} \sum_j \Big(\langle\lambda, h\rangle_j \cdot \sum_{g \in A_j} \chi(g) \cdot \mathrm{tr}(g; K^{\mathrm{top}}(X_j)) \Big).$$

It remains to note that

$$\mathrm{tr}(g; K^{\mathrm{top}}(X_j)) = l(g; X_j). \qquad \text{Q.E.D.}$$

9. *$\mathscr{D}$-Modules and Chern Classes.* Let us briefly discuss our construction of Hecke algebras from a more general point of view. Suppose X is a complex manifold and $\Lambda \subset T^*X$ is a homogeneous Lagrangian subvariety. Consider the natural $\mathbf{C}^*$-action on T^*X and the group $K_{\mathbf{C}^*}(\Lambda)$. This is a module over the representation ring $R(\mathbf{C}^*) \simeq \mathbf{Z}[q, q^{-1}]$. Further consider the projection $\pi: T^*X \to X$ and the graded algebra $\pi_* \mathcal{O}_{T^*X}$. The category of

$\mathbf{C}^*$-equivariant $\mathcal{O}_{T^*X}$-sheaves is equivalent to the category of graded $\pi.\mathcal{O}_{T^*X}$-modules. Therefore $K_{\mathbf{C}^*}(\Lambda)$ may be regarded as a group generated by graded $\pi.\mathcal{O}_{T^*X}$-modules supported at Λ. Multiplication by $q \in \mathbf{Z}[q, q^{-1}]$ corresponds to the shift of gradation.

Let $\mathcal{D}_X$ be the sheaf of linear differential operators on X and $\mathcal{M}$ a filtered holonomic $\mathcal{D}_X$-module whose characteristic variety is contained in Λ. The associated graded module $\operatorname{gr} \mathcal{M}$ is by definition the graded $\pi.\mathcal{O}_{T^*X}$-module, hence the element of $K_{\mathbf{C}^*}(\Lambda)$. In this way we get an additive homomorphism "gr" from the Grothendieck group $\mathrm{FHol}(X)$ of filtered holonomic $\mathcal{D}_X$-modules to an appropriate $K_{\mathbf{C}^*}$-group (here $\mathrm{FHol}(X)$ is a quotient of the free abelian group generated by filtered holonomic modules modulo relations $[\mathcal{M}] = [\mathcal{M}'] + [\mathcal{M}'']$ for any exact sequence $0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0$ strictly compatible with filtrations).

Let us now turn to bivariant theories (see, e.g., [FM, Gi]). Suppose X_1 and X_2 are smooth projective varieties and $\bar{p}\colon T^*(X_1 \times X_2) \to (T^*X_1) \times X_2$ is the natural projection. Consider the following bivariant groups:

$K(X_1 \times X_2)$	is the Grothendieck group of coherent $\mathcal{O}_{X_1 \times X_2}$-sheaves;
$\mathrm{FHol}^T(X_1 \times X_2)$	is the subgroup in $\mathrm{FHol}(X_1 \times X_2)$ generated by all $\mathcal{D}_{X_1 \times X_2}$-modules which are coherent over $\mathcal{D}_{X_1} \otimes \mathcal{O}_{X_2}$;
$\mathcal{L}_{\mathbf{C}^*}(X_1 \times X_2)$	is the Grothendieck group generated by $\mathbf{C}^*$-equivariant coherent $\mathcal{O}_{T^*(X_1 \times X_2)}$-sheaves $\mathcal{F}$ such that:

(a) $\operatorname{supp} \mathcal{F}$ is an isotropic subvariety in $T^*(X_1 \times X_2)$;

(b) the map $\bar{p}\colon \operatorname{supp} \mathcal{F} \to (T^*X_1) \times X_2$ is finite.

For three varieties X_1, X_2, X_3 one has the multiplication

$$\mathrm{FHol}^T(X_1 \times X_2) \otimes \mathrm{FHol}^T(X_2 \times X_3) \to \mathrm{FHol}^T(X_1 \times X_3)$$

and similar operations for K-groups and $\mathcal{L}_{\mathbf{C}^*}$-groups. It's also easy to verify (see [Gi]) that for any module $\mathcal{M} \in \mathrm{FHol}^T(X_1 \times X_2)$ the associated graded module $\operatorname{gr}(\mathcal{M} \otimes \Omega_{X_2})$ belongs to $\mathcal{L}_{\mathbf{C}^*}(X_1 \times X_2)$ so that we get the map $\mathrm{FHol}^T(X_1 \times X_2) \to \mathcal{L}_{\mathbf{C}^*}(X_1 \times X_2)$

$$\mathrm{ch}\colon \mathcal{M} \mapsto \operatorname{gr}(\mathcal{M} \otimes \Omega_{X_2}^{\dim X_2}).$$

For any manifold X denote by π_X the projection $\pi_X\colon T^*X \to X$ and by i_X the zero-section inclusion $i_X\colon X \hookrightarrow T^*X$. There are Thom isomorphisms

$$\pi_X^*\colon K_{\mathbf{C}^*}(X) \xrightarrow{\sim} K_{\mathbf{C}^*}(T^*X), \qquad i_X^*\colon K_{\mathbf{C}^*}(T^*X) \xrightarrow{\sim} K_{\mathbf{C}^*}(X)$$

inverse to each other. With this understood note that an element

$\mathscr{F} \in \mathscr{L}_{\mathbf{C}^*}(X_1 \times X_2)$ gives rise to the homomorphism $\hat{\mathscr{F}}: \mathbf{Z}[q, q^{-1}] \otimes K(X_2) \to \mathbf{Z}[q, q^{-1}] \otimes K(X_1)$ defined as a composite

$$\begin{tikzcd}
\mathbf{Z}[q, q^{-1}] \otimes K(X_2) \simeq K_{\mathbf{C}^*}(X_2) \arrow[r, "\pi_{X_2}^*", "\sim"'] \arrow[rr, bend left, "\hat{\mathscr{F}}"] & K_{\mathbf{C}^*}(T^*X_2) \arrow[r, "\circ \mathscr{F}"] & K_{\mathbf{C}^*}(T^*X_1) \\
& \arrow[r, "i_{X_1}^*", "\sim"'] & K_{\mathbf{C}^*}(X_1) \simeq \mathbf{Z}[q, q^{-1}] \otimes K(X_1).
\end{tikzcd}$$

In other words if $p_1: X_1 \times X_2 \to X_1$ and $\tilde{p}_1: T^*(X_1 \times X_2) \to T^*X_1$ are the natural projections then for $E \in K_{\mathbf{C}^*}(X_2)$ we have $\hat{\mathscr{F}}(E) = i_{X_1}^*(\tilde{p}_1)_*(\mathscr{F} \otimes \pi_{X_2}^* E)$. It follows from the diagram

$$\begin{tikzcd}
T^*(X_1 \times X_2) \arrow[r, "\tilde{p}"] \arrow[dr, "\tilde{p}_1"'] & (T^*X_1) \times X_2 \arrow[d, "p_1"] & X_1 \times X_2 \arrow[l, hook', "\tilde{i}"'] \arrow[d, "p_1"] \\
& T^*X_1 & X_1 \arrow[l, hook', "i_{X_1}"']
\end{tikzcd}$$

that $i_{X_1}^* \circ \tilde{p}_{1*} = (p_1)_* \circ \tilde{i}^* \circ \tilde{p}_*$. Hence $\hat{\mathscr{F}}(E) = i_{X_1}^*(\tilde{p}_1)_*(\mathscr{F} \otimes \pi_{X_2}^*) = (p_1)_* \tilde{i}^* \tilde{p}_*(\mathscr{F} \otimes \pi_{X_2}^* E) = p_{1*} \tilde{i}^*(\tilde{p}_* E \otimes E) = p_{1*}(\tilde{i}^* \tilde{p}_* \mathscr{F} \otimes p_1^* E)$. Thus we see that the operator $\hat{\mathscr{F}}$ is represented by the bivariant class $\tilde{i}^* \tilde{p}_* \mathscr{F} \in K_{\mathbf{C}^*}(X_1 \times X_2)$. So we get a homomorphism

$$R: \mathscr{L}_{\mathbf{C}^*}(X_1 \times X_2) \to \mathbf{Z}[q, q^{-1}] \otimes K(X_1 \times X_2), \qquad R(\mathscr{F}) = \tilde{i}^* \tilde{p}_* \mathscr{F}.$$

THEOREM 9 (cf. [Gi]). *The sequence*

$$\mathrm{FHol}^T(X_1 \times X_2) \xrightarrow{\mathrm{ch}} \mathscr{L}_{\mathbf{C}^*}(X_1 \times X_2) \xrightarrow{R} \mathbf{Z}[q, q^{-1}] \otimes K(X_1 \times X_2)$$

is a sequence of Grothendieck transformations (*i.e.*, ch *and* R *commute with multiplication*).

Proof. For "ch" this was essentially done in [Gi] (for "direct images" see [La]). In order to prove it for R suppose $\mathscr{F}_1 \in \mathscr{L}_{\mathbf{C}^*}(X_1 \times X_2)$, $\mathscr{F}_2 \in \mathscr{L}_{\mathbf{C}^*}(X_2 \times X_3)$. It's obvious that the operator $\widehat{\mathscr{F}_1 \circ \mathscr{F}_2}$ equals $\hat{\mathscr{F}}_1 \circ \hat{\mathscr{F}}_2$. Since $\hat{\mathscr{F}}_i$ is represented by $R(\mathscr{F}_i)$ it follows that $R(\mathscr{F}_1 \circ \mathscr{F}_2) = R(\mathscr{F}_1) \circ R(\mathscr{F}_2)$. Q.E.D.

Comments. (a) Suppose X is the Flag manifold and $\mathscr{F} \in K_{\mathbf{C}^* \times G}(\Lambda)$ (notations of Section 6). The above construction (with $K_{\mathbf{C}^*}(X)$ replaced by $K_{\mathbf{C}^* \times G}^{\mathrm{top}}(X)$) gives rise to the operator

$$\hat{\mathscr{F}}: K_{\mathbf{C}^* \times G}^{\mathrm{top}}(X) \to K_{\mathbf{C}^* \times G}^{\mathrm{top}}(X).$$

Note that this is nothing but the general construction of the $K_{\mathbf{C}^* \times G}(\Lambda)$-

action on $K^{\mathrm{top}}_{M_n}(X_n)$ (see Section 7) in the special case $n=0$. For $\mathscr{F}=\mathscr{O}_s$ (s a simple reflection) the operators $\hat{\mathscr{O}}_s$ are exactly those considered in [Lu2]. That observation in fact motivated the writing of these notes.

(b) Combining Theorem 9 with the Riemann–Roch theorem (see [BFM, FM]) we get the Grothendieck transformation

$$c\colon \mathscr{L}_{\mathbf{C}^*}(X_1\times X_2)\to \mathbf{Z}[q,q^{-1}]\otimes H^*(X_1\times X_2).$$

If the $\mathbf{C}^*$-structure is disregarded (i.e., $q=1$) the map c specializes to the top-dimensional bivariant Chern class $\mathscr{F}\to c(\mathscr{F})$ (see [Gi, Sab]). In general, however, the class $c(\mathscr{F})$ is a polynomial in q rich enough to produce all other Chern classes as well. It is possible to adapt the present approach in order to get the simple definition of the total bivariant Chern class in the non-equivariant theory. (See appendix to [Gi2].)

(c) It is more or less generally assumed after [Br, Sai] that the category of "pure" or "mixed" objects over $\mathbf{C}$ is related somehow to the category of filtered regular holonomic $\mathscr{D}$-modules. As explained above the $\mathbf{C}^*$-equivariant objects such as $\mathscr{L}_{\mathbf{C}^*}(X)$ are geometric (or perhaps micro-local) counterparts for filtered $\mathscr{D}$-modules. The existence of the "Lagrangian" definition of the Hecke algebra instead of the definition based on mixed Weil sheaves in finite characteristics also confirms these relations. So the present paper may be regarded as an infinitesimal step towards the understanding of the role of Frobenius in characteristic zero.

10. Recall the notations of Sections 2 and 3.

The pair (h,n) is called an L^2-pair (cf. [Lu1]) if $Z_G(h,n)$ contains no torus. Write $h=h_c\cdot h_v$ for the decomposition of h into "compact" and "vector" parts. For L^2-pairs it was shown in [Lu1] that:

(i) h_c has finite order; the group $Z_G^0(h_c)$ is semisimple and h_s, $\exp n\in Z_G^0(h_c)$;

(ii) the pair (h_v,n) can be complemented in $Z_G^0(h_c)$ to an "sl_2-triple" (n,h_v,n_-);

(iii) the centralizer of n in $Z_G^0(h_c)$ contains no torus.

It easily follows from (ii) that the variety $X_{h,n}$ is the fixed-point set of a reductive group generated by $(q,h)\in\mathbf{C}^*\times G$ acting on a smooth manifold. Hence (see footnote 1) $X_{h,n}$ is a disjoint union of smooth subvarieties. Further, one can deduce from (iii) that the orbit $Z_G(h)\cdot n$ is open in N_h. So according to Corollary 3.1 the module $K_{h,n,\chi}$ is expected to be irreducible (this is known for $G=GL_n$). Casselman's criterion combined with Theorem 4 and the arguments in [Lu1] show that $K_{h,n,\chi}$ corresponds to the square-integrable representation of a p-adic group with its character given by Theorem 4 (as conjectured in [Lu1]).

For an arbitrary pair h, n ($\operatorname{Ad} h(n) = q^{-1} \cdot n$) choose a maximal torus $S \subset Z_G(h, n)$ and denote by L the semisimple part of the Levi subgroup $Z_G(S) = S \cdot L$. Then (h, n) is the L^2-pair in L. Let X^L be the Flag manifold for L. The inclusion $i: X^L_{h,n} \to X_{h,n}$ gives rise to the homomorphism $i_!: K^{\text{top}}(X^L_{h,n}) \to K^{\text{top}}(X_{h,n})$. It is likely that $i_!$ is compatible with the action of the Hecke algebra $H^L \subset H$ associated with L and that its image generates $K^{\text{top}}(X_{h,n})$ as an H-module so that

$$K^{\text{top}}(X_{h,n}) \simeq \operatorname{Ind}^H_{H^L} K^{\text{top}}(X^L_{h,n}).$$

Note added in proof. (i) Results similar to those of the present paper were simultaneously obtained by D. Kazhdan and G. Lusztig (see Equivariant K-theory and representations of Hecke algebras II, *Invent. Math.* **80** (1985), 209–231).

(ii) The proof of Theorem 6 is based on the following three facts:

(1) the map $\mathscr{F} \to \hat{\mathscr{F}}$ of the comment (a) after Theorem 9 is an algebra-homomorphism according to Theorem 9;

(2) the map in (1) can be shown to be injective;

(3) its image is isomorphic to the Hecke algebra by a theorem of [Lu2].

See the references given in (iii) for more details.

(iii) Conjectures 1 and 2 and the multiplicity formulae (3.1) are now proved (see the author's paper "Deligne–Langlands' conjecture and representations of Hecke algebras" and the paper of Kazhdan and Lusztig "The proof of Deligne–Langlands' conjecture for Hecke algebras." The isomorphism conjectured at the end of the present paper is also proved there).

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