Integral Representation of Convex Functionals on a Space of Measures

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In duality pairs such as \((W^b, C_0)\) and \((W^{-1,p}, W^{1,p}_0)\), a convex integral functional on the space of functions has a polar which admits an integral representation. This representation is the sum of a first term involving the absolutely continuous component of the measure and of a second one which is a positively homogeneous function of the singular part. The duality is useful in plasticity theory. In the Sobolev case the study of non-parametric integrands is new. A description of the sub-differential is obtained.


INTRODUCTION

Our motivations arise from two kinds of problems.

FIRST PROBLEM. In the mathematical theory of plasticity the energy can be expressed by

\[ \int_{\Omega} f(x, D\nu(x)) \, dx, \]

where \( f(x, \cdot) \) is convex with linear growth. The function \( u \) can be discontinuous so its gradient (more precisely its deformation) \( D\nu \) has to be taken in the distribution sense. With some appropriate hypotheses (see [36]), \( D\nu \)
belongs to the space $\mathcal{M}^b$ of bounded measures, hence the idea of extending the functional

$$ I_f: v \mapsto \int_{\Omega} f(x, v(x)) \, dx $$

from $L^1$ to $\mathcal{M}^b$ by taking the $\sigma(\mathcal{M}^b, \mathcal{C}_0)$ lower semi-continuous hull

$$ \bar{F}: \lambda \mapsto \lim_{v \rightarrow \lambda} I_f(v). $$

Let us point out that the $\sigma(\mathcal{M}^b, \mathcal{C}_0)$ topology is the one which provides relative compactness of the sequence $D\mu_e$ when $u_e$ approaches the equilibrium.

When $I_f$ is convex and proper one has

$$ \bar{F}(\lambda) = \sup \{ \langle \lambda, \varphi \rangle - I_f(\varphi) \mid \varphi \in \mathcal{C}_0 \}. $$

The problem is to give an integral expression of $\bar{F}(\lambda)$.

**SECOND PROBLEM.** In the variational approach of semi-linear elliptic equations involving measures such as the Thomas–Fermi problem (see Brezis [13, 14] and Attouch, Bouchitté, and Mabrouk [2]), the Euler equation is obtained by computing the sub-differential on the Sobolev space $W_{1,p}$ of an integral functional $\int j(x, u(x)) \, dx$. Usually the domain of the polar functional is contained in $\mathcal{M}^b \cap W^{-1,p}$.

Thus the two problems lead to the calculus on a space of measures of the polar of an integral functional. When $f$ or $j$ do not depend on $x$, the expression of the polar is due to Temam [37] and Demengel and Temam [19] for the first problem (but already in Valadier [40, 41]), and Brezis [11] completed by Grun–Rehomme [23] for the second one.

In the two previous problems it is important to allow $f$ and $j$ to depend on $x$ (non-homogeneous media in the first situation and second member measure in the second one). In this direction the duality $(\mathcal{M}^b, \mathcal{C}_0)$ has been considered by several authors (Rockafellar [32], Olech [28, 29], Valadier [41]). In the same way Giaquinta, Modica, and Soucek [21] and Dal Maso [16], using a result of Resechtlianik [30], obtain the integral representation of $\bar{F}$ under hypotheses implying the continuity of $f$ in $(x, z)$ and its linear growth in $z$. Since 1985 this problem has been intensively studied by Hadhri [24], Valadier [42] (using Tran cao Nguyen [38, 39]), and De Giorgi, Ambrosio, and Buttazzo [17].

Our approach is new. It reduces the calculus of

$$ \sup \left\{ \int \varphi \cdot d\lambda - \int f^*(\cdot, \varphi) \, d\mu \mid \varphi \in \mathcal{C}_0 \right\} $$
to the calculus of
\[
[J + \delta(\cdot | \mathcal{C}_0)]^* \left( \frac{d\lambda}{dm} \right),
\]
where \( m \) is a positive measure such that \( \mu \ll m \) and \( \lambda \ll m \), and \( J (= I_f) \) is an integral functional with respect to \( m \). The basic result (Theorem 1 of Section 2) may seem rather abstract but it contains almost all difficulties. On the whole the proof is shorter than those of all previous paper.

In Section 3 we recover the formula (already in Valadier [40])
\[
\bar{F}(\lambda) = \int g \left( \cdot, \frac{d\lambda_a}{d\mu} \right) d\mu + \int h \left( \cdot, \frac{d\lambda_s}{d|\lambda_s|} d|\lambda_s| \right),
\]
where \( \lambda_a + \lambda_s \) is the Lebesgue decomposition (with respect to \( \mu \)) of \( \lambda \) and the integrands \( h \) and \( g \) derive from \( f \). The situation is quite different from the non-parametric case where \( g = f \) and \( h = f_{\infty} \), the recession function of \( f \). Indeed as shown in the examples of Section 5, \( g \) can be different from \( f \). Nevertheless, under some regularity assumptions which are set in Section 4, the equality \( h = f_{\infty}(x, \cdot) \) may occur \( \mu \)-a.e. (which implies \( g(x, \cdot) = f(x, \cdot) \) a.e.) or everywhere. A comparison is then possible with the results of [1, 16, 21].

The application to the duality \( (W^1, W^{-1, p}) \) (second problem) is studied in [5, 7, 8]; the results of Brezis [11] and Grun–Rehme [23] are extended.

The present paper follows and improves in some details on Bouchitté [4, 5, 6]. Sections 2 to 4 include the results of Valadier [42], with new proofs, and some other results (especially in Section 4).

1. Notations

Throughout this paper \( \Omega \) denotes a locally compact metrizable space which is \( \sigma \)-compact, that is, a union of a countable sequence of compact subsets. This allows \( \Omega \) to be compact metrizable (which from the mathematical standpoint would be simpler). This also allows \( \Omega \) to be an open subset of \( \mathbb{R}^N \).

A positive Radon measure \( \mu \) on \( \Omega \) is given. When \( \Omega \) is an open subset of \( \mathbb{R}^N \) it may be the Lebesgue measure. We will denote by \( m \) an auxiliary positive measure.

The space of continuous functions tending to 0 at infinity is denoted by \( \mathcal{C}_0(\Omega) \), abbreviated as \( \mathcal{C}_0 \). The space of \( \mathbb{R}^d \)-valued functions \( \mathcal{C}_0(\Omega; \mathbb{R}^d) \) is also denoted by \( [\mathcal{C}_0]^d \) and \( d \) will often be omitted. By \( \mathcal{C}_0 \), we denote the space of continuous functions with compact supports. When \( \Omega \) is an open
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subset of $\mathbb{R}^N$, $C^\infty$ is the space of infinitely differentiable functions and $C^\infty_c$ or $\mathcal{D}$ is the subspace of functions with compact supports.

By $\mathcal{M}$ and $\mathcal{M}^b$ we denote respectively the spaces of Radon measures on $\Omega$ and of bounded measures. The spaces of $\mathbb{R}^d$-valued measures are denoted by $\mathcal{M}(\Omega; \mathbb{R}^d)$, $\mathcal{M}^b(\Omega; \mathbb{R}^d)$ or $[\mathcal{M}]_d$, $[\mathcal{M}^b]_d$ ($d$ will often be omitted).

Most of the paper uses one of the duality pairs $(\mathcal{M}, C_c)$ or $(\mathcal{M}^b, C_0)$. The bilinear form is denoted with brackets (for example $\langle \lambda, \varphi \rangle$) but the scalar product of $z, z' \in \mathbb{R}^d$ is denoted by $z \cdot z'$. If $F$ is a function on a vector space $E$, $F^*$ denotes its polar

$$F^*(x') = \sup \{ \langle x', x \rangle - F(x) | x \in E \}$$

and $\text{dom } F = \{ x | F(x) < \infty \}$. If $C$ is a subset of $E$, $\delta(\cdot | C)$ denotes its indicator function (taking value 0 on $C$, $+\infty$ outside) and $\delta^*(\cdot | C)$ its support function.

A normal integrand $f$ is a measurable function $f: \Omega \times \mathbb{R}^d \to \mathbb{R}$. We say that $f$ is a convex normal integrand if moreover, $\forall x, F(x, \cdot)$ is convex i.s.c.

Other notation: $\mathbb{N}$ is the set of integers $n \geq 0$, $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$, $B(x, r)$ is the closed ball with center $x$ and radius $r$, and $\delta_a$ is the Dirac measure at $a$.

2. PRELIMINARY RESULTS

2.1. We denote by $\mathcal{L}^0(\Omega, m)$ the vector space of real measurable functions.

DEFINITION. A subset $\mathcal{K}$ of $[\mathcal{L}^0]_d$ is said to be PCU-stable if for any continuous partition of unity $(\alpha_0, ..., \alpha_n)$ such that $\alpha_1, ..., \alpha_n$ belong to $C_c$ (variant, when $\Omega$ is an open subset of $\mathbb{R}^N$, $\alpha_1, ..., \alpha_n \in D(\Omega)$, $\alpha_0 \in C^\infty(\Omega)$), for every $u_0, ..., u_n$ in $\mathcal{K}$, $\sum_{i=0}^n \alpha_i u_i$ belongs to $\mathcal{K}$.

Remark. In the main applications $\mathcal{K}$ will be $[C_0]$ or $[C_c]$ and, in other papers [5, 7, 8], $\{ \tilde{u} | u \in [W_0^{1,p} \cap L^\infty]_d \}$, where $\tilde{u}$ denotes all quasi-continuous elements of the Lebesgue equivalence class of $u$ ([3, 12]).

2.2. Recall the following result [43, Proposition 1.14] (for a more recent paper see Fougeres [20]). For any subset $\mathcal{H}_d$ of $[\mathcal{L}^0]_d$ there exists a smallest closed-valued measurable multifunction $\Gamma$ such that $\forall u \in \mathcal{H}_d$, $u(x) \in \Gamma(x)$ m-a.e. (smallest refers to inclusion a.e.). We write $\Gamma = \text{ess sup}_{u \in \mathcal{H}_d} \{ u(\cdot) \}$ and say that $\Gamma$ is the essential supremum of the multifunctions $x \mapsto \{ u(x) \}$ ($u \in \mathcal{H}_d$). Moreover there exists a sequence $(u_n)$ in $\mathcal{H}_d$ such that a.e. $\Gamma(x) = \text{cl} \{ u_n(x) | n \in \mathbb{N} \}$. If $(v_n)$ is any other sequence in $\mathcal{H}_d$ we can add the $v_n$ to the $u_n$. Thus if $\mathcal{H}_d \subset [C_0(\Omega)]_d$, since $C_0$ is separable (for the uniform convergence norm), we can add a dense sequence and this
proves $\Gamma(x) = \text{cl}\{u(x) | u \in \mathcal{H}_i\}$. If $\mathcal{H}_i$ is convex it is easy to see that $\Gamma$ is (a.e.) convex valued. This remains true if $\mathcal{H}_i$ is PCU stable. Indeed for any compact subset $K$ of $\Omega$ and $r_0, ..., r_n \geq 0$ such that $\sum r_i = 1$, there exists a continuous partition of unity $(\alpha_0, ..., \alpha_n)$ with $\alpha_i \in C_c$ and $\forall i$, $\alpha_i(x) = r_i$ on $K$. Then adding to the $u_n$, all the $\sum \alpha_i u_i$ for $(\alpha_0, ..., \alpha_n)$ corresponding to rational $r_i$ and $K$ running through a countable family of compacts $(K_p)$ such that $\bigcup K_p = \Omega$, one can easily check that $\Gamma(x)$ is convex.

2.3. Let $j: \Omega \times \mathbb{R}^d \to ]-\infty, \infty]$ be a normal convex integrand. For any $u \in [\mathcal{L}^0]^d$, $j(\cdot, u)$ denotes the function $x \mapsto j(x, u(x))$. Denote $J$ the functional

$$
J(u) = \int_\Omega j(\cdot, u) \, dm
$$

where, as usual in convex analysis, $\int j(\cdot, u) \, dm = +\infty$ as soon as $\int j(\cdot, u) \, dm = +\infty$.

**Theorem 1.** Let $\mathcal{H}$ be a PCU-stable subset of $[\mathcal{L}^0]^d$. Suppose $\exists u_0 \in \mathcal{H}$ with $J(u_0) \in \mathbb{R}$. Then $\Gamma = \text{ess sup}_{u \in \mathcal{H} \cap \text{dom} J} \{u(\cdot)\}$ is convex valued,

$$(\inf_{u \in \mathcal{H}} J(u)) = \int_\Omega \left[ \inf_{z \in \Gamma(x)} j(x, z) \right] m(dx)$$

and

$$(\inf_{z \in \Gamma(x)} j(x, z)) = \text{ess inf}_{u \in \mathcal{H} \cap \text{dom} J} j(\cdot, u).$$

**Commentary.** Classical results about commutativity of $\int$ and $\inf$ assume that $\mathcal{H}$ is a decomposable vector space or the set of measurable selectors of a multifunction: see Rockafellar [31, 33], Hiai and Umegaki [25], and Bourass and Valadier [9].

**Remark/Example.** We cannot take $\Gamma = \text{ess sup}_{u \in \mathcal{H}} \{u(\cdot)\}$. Indeed let $\Omega = \mathbb{R}$, $m$ the Lebesgue measure, $d = 1$, $K$ a compact subset of $\mathbb{R}$ such that $\text{int}(K) = \emptyset$ and $m(K) > 0$ (one can construct $K$ analogously to the Cantor set). Let

$$j(x, z) = \begin{cases} z & \text{if } x \in K \\ \delta(z \mid \{0\}) & \text{otherwise}. \end{cases}$$

Let $\mathcal{H} = C_c$. Then $\inf_{u \in \mathcal{H}} J(u) = 0$ because, if $u \neq 0$, the set $\{x | u(x) \neq 0$
and \( x \notin K \) is open and non-empty, so has \( > 0 \) measure and \( J(u) = +\infty \). But ess sup \( u \in \mathbb{R} \) \( \{ u(\cdot) \} \) is the constant multifunction \( x \mapsto \mathbb{R} \) and 

\[
\inf_{z \in \mathbb{R}} j(x, z) = \begin{cases} -\infty & \text{if } x \in K \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** (1) First \( \mathcal{H} \cap \text{dom } J \) is still PCU-stable (because \( j(\cdot, \sum \alpha_i u_i) \leq \sum \alpha_i j(\cdot, u_i)^+ \)), hence \( \Gamma \) is convex valued.

(2) Prove the first equality.

Let \( \gamma(x) = \inf_{z \in \Gamma(x)} j(x, z) \) (\( \gamma \) is \( \mu \)-measurable; Castaing and Valadier [15, Lemma III.39]). First \( \geq \) holds because, \( \forall u \in \mathcal{H} \cap \text{dom } J \), \( u(x) \in \Gamma(x) \) a.e. so

\[ j(x, u(x)) \geq \gamma(x) \quad \text{a.e.} \]

Prove now \( \leq \). Let \( r \in \mathbb{R}, r > \int \gamma \, dm \). Thanks to Bourbaki [10] or Dellacherie and Meyer [18, Théorème 48, pp. 107–108] there exists \( \alpha \) l.s.c. integrable such that \( \forall x, \alpha(x) \geq \gamma(x) \) and \( \int \alpha \, dm < r \) (as \( \gamma^+ \leq j(\cdot, u_0)^+, \gamma^+ \) is integrable and can be approached upper by a l.s.c. function, and \( \gamma^- \) can be approached below by an u.s.c. function). We may modify slightly \( \alpha \) to obtain \( \forall x, \alpha(x) \geq \gamma(x) \).

Let \( (u_n)_{n \geq 1} \) be a sequence in \( \mathcal{H} \cap \text{dom } J \) such that \( \Gamma(x) = \text{cl} \{ u_n(x) \mid n \in \mathbb{N}^* \} \). Let \( N \) be a negligible set such that \( \forall n, \forall x \in \Omega \setminus N, j(x, u_n(x)) \in \mathbb{R} \) (recall that \( u_n \in \text{dom } J \) implies \( j(\cdot, u_n)^+ \) is integrable and that \( j(x, z) > -\infty \)). Let \( \varepsilon > 0 \). There exists \( K \) compact, \( K \subset \Omega \setminus N \) such that \( \int_{\Omega \setminus K} \left[ j(\cdot, u_0) + |\alpha| \right] \, dm < \varepsilon \). There exists \( \eta > 0 \) such that \( m(A) < \eta \) implies \( \int_A \left[ j(\cdot, u_0) + |\alpha| \right] \, dm < \varepsilon \). Let \( K^c \) be a compact such that \( K^c \subset K, m(K \setminus K^c) < \eta \) and \( \forall n, j(\cdot, u_n) \) is continuous on \( K^c \).

Let \( A_n = \{ x \in K^c \mid j(x, u_n(x)) < \alpha(x) \} \). It is an open subset of \( K^c \). From Lemma A1 (see Appendix 1) applied with \( D = \{ u_n(x) \mid n \in \mathbb{N}^* \} \) (so \( D = \Gamma(x) \)), for any \( x \in K^c, \gamma(x) = \inf_{n \geq 1} j(x, u_n(x)) \), hence \( \bigcup_{n \geq 1} A_n = K^c \). By compactness there exists \( p \) such that \( K^c = \bigcup_{n \geq p} A_n \). There exists an open subset \( V^c \) of \( \Omega \) such that \( V^c \supset K^c \) and

\[ \forall n, \quad 0 \leq n \leq p \Rightarrow \int_{\Omega \setminus K^c} j(\cdot, u_n)^+ \, dm < \frac{\varepsilon}{p + 1}. \]

Let \( V_n \) be a relatively compact open subset of \( \Omega \) such that \( V_n \cap K^c = A_n \). We may suppose \( V_n \subset V^c \). There exists a continuous partition of unity \( (\alpha_0, ..., \alpha_p) \), such that \( \forall i = 1, ..., p, \text{ supp } \alpha_i \subset V_i \) and \( \text{ supp } \alpha_0 \subset \Omega \setminus K^c \) (see, for example, Bourbaki [10, Chap. III.1, n° 2, Lemme 1, p. 43]; when \( \Omega \) is an open subset of \( \mathbb{R}^N \) it is possible to get \( \forall i, \alpha_i \in \mathcal{C}^\infty(\Omega), \) see L. Schwartz [35, Chap. I, Théorème II]).
Let \( u = \sum_{n=0}^{p} \alpha_n u_n \). As \( \mathcal{K} \) is PCU-stable, \( u \in \mathcal{K} \). One has

\[
\begin{align*}
j(x, u(x)) & \leq \sum_{n=0}^{p} \alpha_n(x) j(x, u_n(x)) \\
& \leq \begin{cases} 
\alpha(x) & \text{if } x \in K^c \\
\sum_{n=0}^{p} j(x, u_n(x))^+ & \text{if } x \in V^c \setminus K^c
\end{cases}
\end{align*}
\]

Then

\[
\int_{\Omega} j(\cdot, u) \, dm \leq \int_{K^c} \alpha \, dm + \int_{V^c \setminus K^c} \sum_{n=0}^{p} j(\cdot, u_n)^+ \, dm + \int_{\Omega \setminus V^c} |j(\cdot, u_0)| \, dm.
\]

We have

\[
\begin{align*}
\int_{K^c} \alpha \, dm &= \int_{\Omega} \alpha \, dm - \left( \int_{\Omega \setminus K} \alpha \, dm + \int_{K \setminus K^c} \alpha \, dm \right) \\
& \leq \int_{\Omega} \alpha \, dm + 2\varepsilon \leq r + 2\varepsilon
\end{align*}
\]

\[
\int_{V^c \setminus K^c} \sum_{n=0}^{p} j(\cdot, u_n)^+ \, dm \leq \varepsilon
\]

\[
\int_{\Omega \setminus V^c} |j(\cdot, u_0)| \, dm \leq \int_{\Omega \setminus K^c} \cdots
\]

\[
= \int_{\Omega \setminus K^c} \cdots + \int_{K \setminus K^c} \cdots \leq 2\varepsilon.
\]

Finally, \( \int_{\Omega} j(\cdot, u) \, dm \leq r + 5\varepsilon \).

(3) As shown in (2), \( \gamma(x) = \inf_{\alpha \geq 1} j(x, u_\alpha(x)) \) a.e. Hence \( \gamma \geq \text{ess inf}_{u \in \mathcal{K} \cap \text{dom } J} j(\cdot, u) \). Conversely there exists a sequence \((u_n)\) in \( \mathcal{K} \cap \text{dom } J \) such that \( \text{ess inf}_{u \in \mathcal{K} \cap \text{dom } J} j(\cdot, u) = \inf_{k} j(\cdot, u_k) \).

But \( u_k(x) \in \Gamma(x) \) a.e. so

\( \gamma(x) \leq \inf_{k} j(x, u_k(x)) \).

**Theorem 2.** We keep the hypotheses of Theorem 1. Let \( \mathcal{X} \) and \( \mathcal{Y} \) be vector spaces of \( \mathbb{R}^d \)-valued measurable functions such that \( \forall u \in \mathcal{X}, \forall v \in \mathcal{Y}, u(\cdot) \cdot v(\cdot) \) is m-integrable and \( \mathcal{K} \subset \mathcal{X} \). Then, in the duality \( \langle \mathcal{X}, \mathcal{Y} \rangle \)

\[
\forall v \in \mathcal{Y}, \quad [J + \delta(\cdot | \mathcal{K})]^*(v) = \int_{\Omega} k(\cdot, v) \, dm,
\]

where \( k(x, \cdot) = [j^*(x, \cdot) \nabla \delta^*(\cdot | \Gamma(x))]^{**} \) (here \( \nabla \) denotes the infimum convolution [27]).
Remark. It is possible with a minoration hypothesis to obtain that the
\(\sigma(\mathcal{X}, \mathcal{Y})\) l.s.c. hull of \(J + \delta(. | \mathcal{H})\) is \(u \mapsto J(u) + \int_{\Omega} \delta(u(x) | \Gamma(x)) \, m(dx)\) (see Bouchitté [5, Théorème 2]).

Proof. \[
\left[ J + \delta(. | \mathcal{H}) \right]^*(v) = \sup_{u \in \mathcal{X}} \left[ \langle u, v \rangle - J(u) - \delta(u | \mathcal{H}) \right]
\]
\[
= \sup_{u \in \mathcal{H}} \int [u(\cdot) \cdot v(\cdot) - j(\cdot, u)] \, dm
\]
\[
= - \inf_{u \in \mathcal{H}} \int j'(\cdot, u) \, dm
\]
with \(j'(x, z) = j(x, z) - z \cdot v(x)\). Since \(\text{dom } J' \cap \mathcal{X} = \text{dom } J \cap \mathcal{X}\), the multifunction \(\sup_{u \in \mathcal{X} \cap \text{dom } J} \{u(\cdot)\}\) is still \(J\). Moreover \(J'(u_0) \in \mathcal{R}\).

By Theorem 1,
\[
\left[ J + \delta(. | \mathcal{H}) \right]^*(v) = - \int_{\Gamma(x)} \inf_{z \in \Gamma(x)} \left[ j(x, z) - z \cdot v(x) \right] \, m(dx)
\]
\[
= \int \left[ j(x, \cdot) + \delta(\cdot | \Gamma(x)) \right]^*(v(x)) \, m(dx).
\]
Since \(j(x, \cdot)\) and \(\delta(\cdot | \Gamma(x))\) are l.s.c.
\[
j(x, \cdot) + \delta(\cdot | \Gamma(x)) = \left[ j^*(x, \cdot) \nabla \delta^*(\cdot | \Gamma(x)) \right]^*
\]
(see, for example, Castaing and Valadier [15, Proposition I.19]).

It is possible to choose classical spaces for \(\mathcal{X}\) and \(\mathcal{Y}\).

**Proposition 3.** Let \(j\) be a normal convex integrand. Suppose \(\mathcal{H}\) is a vector subspace of \([L^\infty]^d\) such that \(\forall u \in \mathcal{H}, \forall \alpha \in C_c(\Omega)\) (variant, when \(\Omega\) is an open subset of \(\mathbb{R}^n\), \(\forall \alpha \in \mathcal{D}(\Omega)\)), \(au\) belongs to \(\mathcal{H}\). Suppose \(\exists u_0 \in \mathcal{H}\) such that \(J(u_0) \in \mathbb{R}\). Let \(\Gamma = \text{ess \sup}_{\mathcal{X} \cap \text{dom } J} \{u(\cdot)\}\).

1. Consider the functional on \([L^\infty]^d\), \(J + \delta(\cdot | \mathcal{H})\). Then its polar on \([L^1]^d\) verifies
\[
\left[ J + \delta(\cdot | \mathcal{H}) \right]^*(v) = \int_{\Omega} k(\cdot, v) \, dm,
\]
where \(k(x, \cdot) = \left[ j^*(x, \cdot) \nabla \delta^*(\cdot | \Gamma(x)) \right]^**\).

2. If \(\mathcal{H} \subset [C_0]^d\) then \(\Gamma(x) = \text{cl}\{u(x) | u \in \mathcal{H} \cap \text{dom } J\}\) a.e.

Proof. Remark that \(\mathcal{H}\) is PCU-stable because \(\sum_{i=0}^n \alpha_i u_i = u_0 + \sum_{i=1}^n \alpha_i (u_i - u_0)\).
(1) This results from Theorem 2 applied with $\mathcal{X} = [L^\infty]^d$ and $\mathcal{Y} = [L^1]^d$.

(2) This has been said in 2.2.

Remark. It is possible to give a variant with $\mathcal{Y} = [L^1_{\text{loc}}]^d$ and for $\mathcal{X}$ the space of $L^\infty$-functions with compact supports.

3. DESCRIPTION OF $\Phi$

Let $f: \Omega \times \mathbb{R}^d \to [-\infty, \infty]$ be a convex normal integrand. We suppose

(H1) $\exists \phi_0 \in \mathcal{C}_e$, $\exists a \in L^1$ such that $\mu$-a.e. in $x$, $\forall z$, $f(x, z) \geq \phi_0(x) + a(x)$ (equivalently $\exists \phi_0 \in \mathcal{C}_e$ such that $I_f(\phi_0) < \infty$).

(H2) $\exists u_0 \in [L^1_{\text{loc}}(\Omega, \mu)]^d$ such that $I_f(u_0) < \infty$ (equivalently $\exists u_0 \in [L^1_{\text{loc}}]^d$, $\exists b \in L^1$ such that $\mu$-a.e., $\forall z$, $f^*(x, z) \geq z \cdot u_0(x) - b(x)$).

Here, for any $u \in [L^0(\mu)]^d$, $I_f(u) = \int_\Omega f(\cdot, u) \, d\mu$. Let $F: [\mathcal{M}]^d \to [-\infty, \infty]$ be defined as

$$F(\lambda) = \begin{cases} I_f\left(\frac{d\lambda}{d\mu}\right) & \text{if } \lambda \ll \mu \\ +\infty & \text{otherwise.} \end{cases}$$

(Note that $d\lambda/d\mu \in L^1(\mu)$ and, by (H1), $f(\cdot, d\lambda/d\mu) \geq \phi_0(\cdot) \cdot (d\lambda/d\mu)(\cdot) - a$, hence $F(\lambda) > -\infty$.)

**Theorem 4.** Let

$$h(x, z) = \sup \{ \phi(x) \cdot z \mid \phi \in \mathcal{C}_e \cap \text{dom } I_f^* \}$$

$$g(x, \cdot) = [f(x, \cdot) \nabla h(x, \cdot)]^{**},$$

$\lambda \in [\mathcal{M}]^d$, $\lambda_a + \lambda_s$ its Lebesgue decomposition with respect to $\mu$, $\theta$ any positive measure such that $\lambda_s \ll \theta$. Then the $\sigma(\mathcal{M}, \mathcal{C}_e)$ l.s.c. hull of $F$ is

$$F(\lambda) = \int_\Omega g \left(\cdot, \frac{d\lambda_a}{d\mu}\right) \, d\mu + \int_\Omega h \left(\cdot, \frac{d\lambda_s}{d\theta}\right) \, d\theta,$$

and the $\sigma(L^1_{\text{loc}}, \mathcal{C}_e)$ l.s.c. hull of $I_f$ is $I_g$.

With

(H2)' $\exists u_0 \in [L^1]^d$ such that $I_f(u_0) < \infty$, and $F_1: [\mathcal{M}^b]^d \to [-\infty, \infty]$ defined by

$$F_1(\lambda) = \begin{cases} I_f\left(\frac{d\lambda}{d\mu}\right) & \text{if } \lambda \ll \mu \\ +\infty & \text{otherwise} \end{cases}$$

we obtain
Theorem 4'. The $\sigma(\mathcal{M}, \mathcal{C}_0)$ l.s.c. hull $F_1$ of $F_1$ is

$$F_1(\lambda) = \int_\Omega g \left( \cdot, \frac{d\lambda_\alpha}{d\mu} \right) d\mu + \int_\Omega h \left( \cdot, \frac{d\lambda_s}{d\theta} \right) d\theta$$

with $g$ and $h$ defined as in Theorem 4. Moreover the $\sigma(L^1, \mathcal{C}_0)$ l.s.c. hull of $I_f$ is $I_g$.

Remarks. (1) If (H1) were replaced by

(H1)' $\exists \varphi_0 \in \mathcal{C}_0$ such that $I_{f^*}(\varphi_0) < \infty$

one would have to redefine $h$ and $g$.

(2) If $\mu$ is non-atomic one can start from a measurable integrand $f$ not necessarily convex, and the l.s.c. hulls $F$ and $F_1$ are the same as those obtained starting from $f^{**}$; this results from the Liapunov theorem. See Valadier [41] and Bouchitte [5].

(3) As $h$ is sublinear the choice of $\theta$ is immaterial as soon as $\lambda_s \ll \theta$. See Goffman and Serrin [22].

Proof of Theorem 4. First, since $L^1_{\text{loc}}$ is decomposable and $I_f(u_0) < \infty$, thanks to a famous theorem by Rockafellar, the polar $F^*$ of $F$ in the duality $(\mathcal{M}, \mathcal{C}_c)$ is

$$F^*(\varphi) = \sup_{u \in L^1_{\text{loc}}} [\langle u, \varphi \rangle - I_f(u)] = I_{f^*}(\varphi).$$

Thanks to minoration (H1) and convexity, $\bar{F} = F^{**}$, hence

$$\bar{F}(\lambda) = \sup_{\varphi \in \mathcal{C}_c} [\langle \lambda, \varphi \rangle - I_{f^*}(\varphi)].$$

Consider now a fixed $\lambda \in [\mathcal{M}]^d$. There exists a Borel set $A$ such that

$$\mu(\Omega \setminus A) = |\lambda_s| (A) = 0.$$

Let $m = \mu + |\lambda_s|$. Then $\lambda \ll m$ and

$$\frac{d\lambda}{dm}(x) = \begin{cases} \frac{d\lambda_\alpha}{d\mu}(x) & \text{if } x \in A \\ \frac{d\lambda_s}{d|\lambda_s|}(x) & \text{if } x \in \Omega \setminus A. \end{cases}$$

Thus $d\lambda/dm \in L^1_{\text{loc}}(m)$. Setting

$$f(x, z) = \begin{cases} f^*(x, z) & \text{if } x \in A \\ 0 & \text{if } x \in \Omega \setminus A \end{cases}$$
one has
\[ \langle \lambda, \varphi \rangle - \int_{\Omega} f^*(\cdot, \varphi) \, d\mu = \int_{\Omega} \frac{d\lambda}{dm} \cdot \varphi \, dm - \int_{\Omega} j(\cdot, \varphi) \, dm. \]

Now we can apply Theorem 2 with \( \mathcal{X} = \mathcal{X} = \mathcal{C}_c \) and \( \mathcal{Y} = [I_{IUC}]^d \). Indeed, by (H1) and (H2), \( J(\varphi_0) \in \mathbb{R} \) (remark \( I = I_{f^*} \)). Thus

\[ \bar{F}(\lambda) = \sup_{\varphi \in \mathcal{C}_c} \int_{\Omega} \left[ \frac{d\lambda}{dm} \cdot \varphi - j(\cdot, \varphi) \right] \, dm \]

\[ = \left[ J + \delta(\cdot | \mathcal{C}_c) \right] \left( \frac{d\lambda}{dm} \right) \]

\[ = \int_{\Omega} k(\cdot, \frac{d\lambda}{dm}) \, dm \]

with \( k(x, \cdot) = [j^*(x, \cdot) \nabla \delta^*(\cdot | I(x))]^* \).

Since \( \Gamma(x) = \text{cl}\{ \varphi(x) | \varphi \in \mathcal{C}_c \cap \text{dom } I_{f^*} \} \) (in fact \( \Gamma \) is defined up to equality \( m \)-a.e. but this expression is independent of \( m \)),

\[ \delta^*(z | \Gamma(x)) = h(x, z). \]

Since

\[ j^*(x, z) = \begin{cases} f(x, z) & \text{if } x \in A \\ \delta(z \setminus \{0\}) & \text{if } x \in \Omega \setminus A \end{cases} \]

\[ k(x, \cdot) = \begin{cases} g(x, \cdot) & \text{if } x \in A \\ h(x, \cdot) & \text{if } x \in \Omega \setminus A \end{cases} \]

Finally,

\[ \bar{F}(\lambda) = \int_{\Omega} g(\cdot, \frac{d\lambda}{dm}) \, dm + \int_{\Omega \setminus A} h(\cdot, \frac{d\lambda}{dm}) \, dm \]

\[ - \int_{\Omega} g(\cdot, \frac{d\lambda_{x}}{dm}) \, dm + \int_{\Omega} h(\cdot, \frac{d\lambda_{x}}{d|\lambda_{x}|}) \, d|\lambda_{x}|. \]

**Proof of Theorem 4'.** We still have, for \( \varphi \in \mathcal{C}_0, F^*_1(\varphi) = I_{f^*}(\varphi) \) and

\[ \bar{F}_1(\lambda) = \sup_{\varphi \in \mathcal{C}_0} \left[ \langle \lambda, \varphi \rangle - I_{f^*}(\varphi) \right]. \]

For a given \( \lambda \in [\mathcal{M}^b]^d \), let \( A, m, \) and \( j \) be as in the proof of Theorem 4. Here \( d\lambda/dm \in L^1(m) \).

We apply Theorem 2 with \( \mathcal{Y} = \mathcal{L}^1, \mathcal{X} = \mathcal{C}_0 \), and \( \mathcal{E} = \mathcal{C}_0 \) (or \( \mathcal{L}^\infty \)) (we may also apply Proposition 3). We get \( \bar{F}_1(\lambda) = \int_{\Omega} k(\cdot, d\lambda/dm) \, dm \). Here the only difference is that

\[ \Gamma(x) = \text{cl}\{ \varphi(x) | \varphi \in \mathcal{C}_0 \cap \text{dom } I_{f^*} \}. \]
A priori, using \( \mathcal{C}_0 \) in place of \( \mathcal{C}_c \) should give a greater function \( h \). But let \( \phi \in \mathcal{C}_0 \cap \text{dom } I_{\star} \). There exists \( \beta_n \in \mathcal{C}_c, \beta_n \geq 0, \beta_n \not	o \chi_{\Omega}, \) then \( \psi_n = \beta_n \phi + (1 - \beta_n) \phi_0 \) (where \( \phi_0 \) satisfies (H1)) belongs to \( \mathcal{C}_c \cap \text{dom } I_{\star} \). Hence, for any \( x, \psi_n(x) \to \phi(x) \) and the function \( \delta^*(z \mid I(x)) \) is the same \( h \) as in Theorem 4.

**Theorem 5.** Under (H1), with \( h \) and \( g \) defined in Theorem 4 one has, for any bounded positive Borel function \( \psi, \forall \lambda \in [\mathcal{M}]^d \) (or \( [\mathcal{M}^b]^d \)),

\[
\int_{\Omega} \psi g \left( \cdot, \frac{d\lambda}{d\mu} \right) d\mu + \int_{\Omega} \psi h \left( \cdot, \frac{d\lambda}{d\theta} \right) d\theta
\]

\[
= \sup \left\{ \int_{\Omega} \psi \phi \cdot d\lambda - \int_{\Omega} \psi f^*(\cdot, \phi) \, d\mu \mid \phi \in \text{dom } I_{\star} \cap \mathcal{C}_c \ (\text{resp. } \mathcal{C}_0) \right\}.
\]

Moreover, if \( \psi \) is continuous, the supremum can be taken on the whole space \( \mathcal{C}_c \) or \( \mathcal{C}_0 \).

**Comment.** Consider the measure \( G(\lambda) \) with values in \( ] - \infty, \infty \) defined by, \( \forall B \) Borel set,

\[
[G(\lambda)](B) = \int_B g \left( \cdot, \frac{d\lambda}{d\mu} \right) d\mu + \int_B h \left( \cdot, \frac{d\lambda}{d\theta} \right) d\theta.
\]

The first member in the statement is \( \int \psi \, dG(\lambda) \). When \( G(\lambda) \) is a Radon measure (equivalently takes finite values on compact sets) it is characterized by the knowledge of the values \( \int \psi \, dG(\lambda) \), \( \psi \) continuous. The formula has been given by Temam [36, 37], Demengel and Temam [19], Hadhri [24], and Valadier [42, 45]. The continuity of \( \psi \) is necessary to take the supremum on \( \mathcal{C}_0 \).

**Proof.** (a) Consider for a fixed \( \lambda, \lambda' = \psi \lambda \) and \( m = \psi \mu + \psi \mid \lambda_s \). Then \( \lambda' \ll m \) and, if \( A \) is a Borel set such that \( \mu(\Omega \setminus A) = |\lambda_s| (A) = 0 \), one has

\[
\frac{d\lambda'}{dm} (x) = \begin{cases} \frac{d\lambda_a}{d\mu} (x) & \text{if } x \in A \\ \frac{d\lambda_s}{d|\lambda_s|} (x) & \text{if } x \in \Omega \setminus A, \end{cases}
\]

and, since \( \psi \) is bounded, \( d\lambda'/dm \in L^1_{\text{loc}}(m) \) (resp. \( L^1(m) \)). Set also

\[
j(x, z) = \begin{cases} f^*(x, z) & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}
\]
Then \( \int_{\Omega} \psi f^*(\cdot, \varphi) \, d\mu = \int_{\Omega} j(\cdot, \varphi) \, dm \), which will be denoted by \( J(\varphi) \). Thus the right-hand side of the formula of Theorem 5 equals

\[
\sup_{\varphi \in X} \left[ \int_{\Omega} \frac{d\lambda'}{dm} \cdot \varphi \, dm - J(\varphi) \right],
\]

where \( \mathcal{H} = \mathcal{E}_c \cap \text{dom} \, I_{f^*} \) (or \( \mathcal{E}_0 \cap \text{dom} \, I_{f^*} \)). Since \( \psi \) is bounded one has \( \mathcal{H} \subseteq \text{dom} \, J \), hence \( \mathcal{H} \cap \text{dom} \, J = \mathcal{H} \) and \( \varphi_0 \in \mathcal{H} \cap \text{dom} \, J \). Moreover \( \mathcal{H} \) is PCU-stable. We can apply Theorem 2 with \( \mathcal{Y} = [L^1_{\text{loc}}]^d \) (or \( [L^1]^d \)), \( \mathcal{X} = \mathcal{E}_c \) or \( \mathcal{E}_0 \). Thus

\[
\sup_{\varphi \in X} \left[ \int_{\Omega} \frac{d\lambda'}{dm} \cdot \varphi \, dm - J(\varphi) \right] = \int_{\Omega} k(\cdot, \frac{d\lambda'}{dm}) \, dm
\]

with \( k(x, \cdot) = [j^*(x, \cdot) \nabla \delta^*(\cdot | I(x))]^** \) and \( \Gamma = \text{ess sup} \cup_{u \in X} \{u(\cdot)\} \). Again \( \Gamma(x) = \text{cl}\{\varphi(x) | \varphi \in \mathcal{H}\} \) and one can end the proof as in Theorem 4.

(b) Suppose that the supremum is on the whole space \( \mathcal{E}_c \) (or \( \mathcal{E}_0 \)) and that \( \psi \) is continuous. Proceeding as in (a), but with \( \mathcal{H} = \mathcal{E}_c \) or \( \mathcal{E}_0 \), the difficulty is to check that, denoting \( \Gamma = \text{ess sup} \cup_{u \in X} \{u(\cdot)\} = \text{cl}\{\varphi(x) | \varphi \in \mathcal{E}_c \text{ or } \mathcal{E}_0 \text{ and } \int \psi f^*(\cdot, \varphi) \, d\mu < \infty\} \), one has \( \psi(x) \delta^*(z | \Gamma(x)) = \psi(x) h(x, z) \). We may suppose \( \psi(x) > 0 \). There exists a compact neighborhood \( K \) of \( x \) such that \( \inf_K \psi = \delta > 0 \). The remainder is routine.

4. SOME PROPERTIES OF \( h \) AND \( g \)

Throughout this section the duality pair is either \((\mathcal{M}, \mathcal{E}_c)\) or \((\mathcal{M}^b, \mathcal{E}_0)\). Hypotheses (H1) and (H2) are assumed, so

\[
h(x, z) = \sup \{z : \varphi(x) | \varphi \in \mathcal{E}_c \cap \text{dom} \, I_{f^*}\} = \sup \{z : \varphi(x) | \varphi \in \mathcal{E}_0 \cap \text{dom} \, I_{f^*}\}
\]

(see the proof of Theorem 4').

We will sometimes use in place of (H1) the stronger

\[(H1)' \exists \lambda_0 \in ]0, \infty[, \exists a \in L^1 \text{ such that a.e., } \forall z, f(x, z) \geq \lambda_0 |z| - a(x).\]

(Remark that \((H1)' \Rightarrow (H1) \) with \( \varphi_0 = 0 \).)

Recall that the recession or asymptotic function \( f_\infty(x, \cdot) \) of the convex l.s.c. proper function \( f(x, \cdot) \) satisfies

\[
\forall z_0 \in \text{dom} \, f(x, \cdot), \quad f_\infty(x, z) = \lim_{r \to \infty} \frac{f(x, z_0 + rz)}{r}
\]

and \( f_\infty(x, z) = \delta^*(z | \text{dom} \, f^*(x, \cdot)) \) (Rockafellar [34, Theorem 8.5, p. 66, and Theorem 13.3, p. 116]).
Proposition 6. Let

\[ E(x) = \left\{ z \in \mathbb{R}^d \mid \exists V \text{ open, } V \ni x, \exists \varphi \text{ continuous on } V \text{ such that} \right\} \]

\[ \varphi(x) = z \text{ and } \int_V f^*(\cdot, \varphi) \, d\mu < \infty \]

\[ E_1(x) = \left\{ z \in \mathbb{R}^d \mid \exists V \text{ open, } V \ni x \text{ such that} \right\} \]

\[ \int_V f^*(\cdot, z) \, d\mu < \infty \}

Then

1. \( \forall (x, z), h(x, z) = \delta^*(z \mid E(x)) \),

2. if \( x \in \Omega \setminus \text{supp } \mu, E(x) = E_1(x) = \mathbb{R}^d \) and \( h(x, \cdot) = \delta(\cdot \mid \{0\}) \),

3. under (H1)\(^{-}\), \( \forall x, E_1(x) \subset E(x) \subset E_1(x) \).

Example. Without (H1)\(^{-}\), (3) may be false. Let \( \Omega = \mathbb{R} \), \( \mu \) the Lebesgue measure, \( d = 2 \),

\[ D_x = \{ \lambda (\cos x, \sin x) \mid \lambda \in \mathbb{R} \}, \quad f(x, \cdot) = \delta(\cdot \mid D_x). \]

Then \( f^*(x, \cdot) = \delta(\cdot \mid D_x^+) \) and \( E_1(0) = \{(0,0)\}, E(0) = \{0\} \times \mathbb{R} \).

Proof. (1) This is proved in Valadier [42, Proposition 7, p. 22] and is known since Olech [28].

(2) If \( x \notin \text{supp } \mu, V = \Omega \setminus \text{supp } \mu \) is an open neighborhood of \( x \) and \( \int_V f^*(x, z) \mu(dx) = 0 \) for any \( z \). So \( E(x) = E_1(x) = \mathbb{R}^d \) and \( h(x, \cdot) = \delta(\cdot \mid \{0\}) \).

(3) The inclusion \( E_1(x) \subset E(x) \) is obvious. Let \( z \in E(x) \). Let \( V \) and \( \varphi \) corresponding to \( z \). We may, changing \( V \) in a smaller neighborhood, suppose \( \varphi \) bounded. For any \( \varepsilon > 0 \), let \( V_\varepsilon = \{ y \in V \mid |\varphi(y) - z| < \varepsilon \} \) and \( W_\varepsilon \) a compact neighborhood of \( x \) contained in \( V_\varepsilon \). There exists \( \theta_\varepsilon : V \to [0, 1] \) continuous such that \( \theta_\varepsilon(x) = 1 \) on \( W_\varepsilon \) and \( \text{supp } \theta_\varepsilon \subset V_\varepsilon \). Define

\[ \varphi_\varepsilon = \theta_\varepsilon z + (1 - \theta_\varepsilon) \varphi. \]

Then \( \varphi_\varepsilon(x) = z \) and \( \sup_{y \in V} |\varphi_\varepsilon(y) - \varphi(y)| \leq \varepsilon \).

By (H1)\(^{-}\) the functional \( I \) on \( L^\infty(V, \mu) \), defined by \( I(v) = \int_V f^*(\cdot, v) \, d\mu \), is bounded on a (norm) neighborhood of \( 0 \), so it is continuous on \( \text{int}(\text{dom } I) \), which contains \( [0, \varphi[ \). Hence if \( r \in [0, 1[ \)

\[ \lim_{\varepsilon \to 0} \int_V f^*(\cdot, r\varphi_\varepsilon) \, d\mu = \int_V f^*(\cdot, r\varphi) \, d\mu. \]

So for \( \varepsilon \) sufficiently small, \( f^*(\cdot, r\varphi_\varepsilon) \in L^1 \), hence \( \int_{\text{int}(W_\varepsilon)} f^*(\cdot, rz) \, d\mu < \infty \) and \( rz \in E_1(x) \). Finally, \( z \in E_1(x) \).
PROPOSITION 7. (1) One has \( \mu \text{-a.e.} \)
\[
g(x, \cdot) \leq f(x, \cdot) \quad h(x, \cdot) = g_\infty(x, \cdot) \leq f_\infty(x, \cdot),
\]

(2) \( I_f \) is \( \sigma(L^1, C_0) \) (resp. \( \sigma(L^1_{\text{loc}}, C_0) \)) l.s.c. iff \( \mu \text{-a.e.} \) \( h(x, \cdot) = f_\infty(x, \cdot) \) (equivalently \( h(x, \cdot) \geq f_\infty(x, \cdot) \)).

(3) If \( \Omega' \) is an open subset of \( \Omega \) and if \( x \mapsto \text{epi } f^*(x, \cdot) \) is l.s.c. on \( \Omega' \), then \( \forall x \in \Omega', \ f_\infty(x, \cdot) \leq h(x, \cdot) \). As a consequence if \( \mu(\Omega \setminus \Omega') = 0 \), \( I_f \) is l.s.c.

EXAMPLE. Let \( \Omega = \mathbb{R} \), \( \mu \) the Lebesgue measure, \( d = 1 \), \( K \) a compact subset of \( \mathbb{R} \) with \( \text{int}(K) = \emptyset \) and \( \mu(K) > 0 \), and
\[
f(x, z) = \begin{cases} |z| & \text{if } x \in K \\ 0 & \text{otherwise,} \end{cases}
\]
Then \( I_f^*(\varphi) = \delta(\varphi \mid \{0\}) \), so \( I_f = 0 \neq I_f^* \).

Proof. Parts (1) and (2) have been proved in Valadier [41, 42]. For a somewhat more direct proof see Bouchitte [5, 7].

(3) Let \( x_0 \in \Omega' \). If \( z_0 \in \text{dom } f^*(x_0, \cdot) \), by the Michael theorem [26] there exists a continuous selector \((\varphi, \psi)\) of \( x \mapsto \text{epi } f^*(x, \cdot) \) such that \((\varphi(x_0), \psi(x_0)) = (z_0, f^*(x_0, z_0))\). Let \( K \) be a compact neighborhood of \( x \) contained in \( \Omega' \). Then
\[
\int_{\text{int } K} f^*(\cdot, \varphi) \, d\mu \leq \int_K \psi \, d\mu < \infty.
\]
Hence \( z_0 \in E(x_0) \). Therefore \( f_\infty(x_0, \cdot) \leq h(x_0, \cdot) \). The last assertion follows from (2).

THEOREM 8. (1) Under one of the hypotheses

\( \text{H3) } \forall z, f^*(\cdot, z) \) is u.c.s. on \( \Omega \),

\( \text{H4) } f \) is l.s.c. on \( \Omega \times \mathbb{R}^d \) and \( f(\cdot, 0) \) is locally bounded,

one has \( \forall x \in \Omega, f_\infty(x, \cdot) \leq h(x, \cdot) \) (hence \( I_f \) is l.s.c.).

(2) Under (H3) or (H4) and moreover

\( \text{H5) } \forall z, f_\infty(\cdot, z) \) is u.c.s.,

one has
\[
h(x, z) = \begin{cases} f_\infty(x, z) & \text{if } x \in \text{supp } \mu \\ \delta(z \mid \{0\}) & \text{if } x \in \Omega \setminus \text{supp } \mu. \end{cases}
\]
Remarks and Comments. (1) For $I_f$ being $\sigma(L^1, \mathcal{C}_0)$ l.s.c. it is sufficient to have (H3) or (H4) on an open set $\Omega'$ such that $\mu(\Omega \setminus \Omega') = 0$, for example (as said in [24]) if

$$f(x, z) = \begin{cases} f_1(z) & \text{if } x \in \Omega_1 \\ f_2(z) & \text{if } x \in \Omega_2, \end{cases}$$

where $\Omega_1$ and $\Omega_2$ are disjoint open subsets such that

$$\mu(\Omega \setminus (\Omega_1 \cup \Omega_2)) = 0.$$

(2) If $f(x, \cdot)$ does not depend on $x$, (H3) and (H5) are obviously satisfied. If moreover $\text{supp } \mu = \Sigma$, the formula of Theorem 5 becomes, $\forall \psi$ Borel bounded positive function,

$$\sup \left\{ \int \psi \varphi \cdot d\lambda - \int \psi f^*(\cdot, \varphi) \, d\mu \mid \varphi \in \mathcal{C}_0 \cap \text{dom } I_{\psi} \right\}$$

$$= \int \psi f \left( \frac{d\lambda}{d\mu} \right) \, d\mu + \int \psi f^\infty \left( \frac{d\lambda}{d|\lambda|} \right) \, d|\lambda|.$$

This is the starting formula (for $\psi$ continuous) of Temam [37] and Demengel and Temam [19].

(3) In case $f$ is l.s.c. on whole the space $\Omega \times \mathbb{R}^d$, hypothesis (H4), Giaquinta, Modica, and Soucek [21], and Dal Maso [16] obtain, thanks to a result of Reschetenjak [30] about sublinear functions of measures, that the functional

$$G \left[ \begin{array}{c} \lambda \\ [\mathcal{M}^b]^d \rightarrow \end{array} \right] \rightarrow -\infty, \infty \right]$$

is $\sigma(\mathcal{M}^b, \mathcal{C}_0)$ l.s.c. As a consequence $I_f$ is $\sigma(L^1, \mathcal{C}_0)$ l.s.c., hence $g(x, \cdot) = f(x, \cdot)$ $\mu$-a.e. But it can happen that $G \neq F_1$. Indeed consider the following example suggested in [16, 4.4, p. 414].

Example. Let $\Omega = \mathbb{R}$, $\mu = dx$, $d = 1$.

$$f(x, z) = \begin{cases} |z| & \text{if } |z| \leq \frac{1}{2} \\ 2|z| - |x|^{-1/2} & \text{if } |z| \geq \frac{1}{2} \end{cases}$$

Then $f$ is continuous on $\Omega \times \mathbb{R}$, (H4) is satisfied, but (H5) does not hold.

One can check that $\forall x$, $\text{sgn}(x) = 2|z|$ and $f(0, z) = |z|$. Thus $G(\delta_0) = 1$ and $F_1(\delta_0) = 2$. 
(4) In [1, 16] (where the more difficult problem of a functional depending on the gradient is studied), a sufficient condition ensuring \( F = G \) is set. This condition implies that \( f \) is continuous in \( x \) and has linear growth in \( z \); more precisely,

\[
\forall \varepsilon > 0, \exists \delta > 0, |x_1 - x_2| < \delta \Rightarrow \forall z, |f(x_1, z) - f(x_2, x)| \leq \varepsilon (1 + |z|).
\]

This hypothesis is far more stringent that the one of (2) of Theorem 8. Indeed (H3) or (H4) supplemented with (H5) does not imply the continuity of \( f(\cdot, z) \) but only the continuity of \( f_{\infty}(\cdot, z) \) (remark that \( f \) being l.s.c., \( f_{\infty}(\cdot, z) \) is l.s.c. too).

**Proof of Theorem 8.** (1) By Proposition 7 it is sufficient to prove that the multifunction \( Q : x \mapsto \text{epi} f^*(x, \cdot) \) is l.s.c.

(a) Under (H3). Let \( U \) be an open subset of \( \mathbb{R}^d \times \mathbb{R} \). Then

\[
\{ x \in \Omega | Q(x) \cap U \neq \emptyset \} = \{ x | \exists (z, r) \in U \text{ such that } f^*(x, z) \leq r \} = \bigcup_{(z, r) \in U} \{ x | f^*(x, z) < r \}
\]

(the change from \( \leq \) to \( < \) is easy) which is open.

(b) Under (H4). Recall that, for \((z, t) \in \mathbb{R}^d \times \mathbb{R}, \)

\[
\tilde{f}(x, z, t) = \delta^*((z, t) | Q(x)) = \begin{cases} -tf(x, z/t) & \text{if } t < 0 \\ f_{\infty}(x, z) & \text{if } t = 0 \\ +\infty & \text{if } t > 0. \end{cases}
\]

From Lemma A2 it is sufficient to prove that \( \tilde{f} \) is l.s.c. This is a consequence of Dal Maso [16].

(2) Under (H5)

\[
V = \{ x \in \Omega | \exists z \in \mathbb{R}^d \text{ such that } f_{\infty}(x, z) < h(x, z) \} = \bigcup_z \{ x | f_{\infty}(x, z) < h(x, z) \}
\]

is open (\( h \) defined in Theorem 4 is l.s.c.). From Proposition 7(1) a.e. \( f_{\infty}(x, \cdot) \geq h(x, \cdot) \), so \( V \) is negligible, hence \( V \cap \text{supp } \mu = \emptyset \).

If \( x \in \text{supp } \mu, x \notin V \) and then using (1), \( f_{\infty}(x, \cdot) = h(x, \cdot) \). If \( x \notin \text{supp } \mu, \) the result follows from Proposition 6(2).

5. EXAMPLES

The proofs of the results stated in Examples 1 to 4 are left to the reader. For details see Bouchitté [5, 7].
EXAMPLE 1. Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $\Sigma$ be an $N-1$-dimensional hypersurface contained in $\Omega$. Let $\mu$ denote the measure $dx + H^{N-1}(\Sigma \cap \cdot)$, where $H^{N-1}$ is the $N-1$-dimensional Hausdorff measure (thus $H^{N-1}(\Sigma \cap \cdot)$ is the area measure of $\Sigma$). We suppose that $\Sigma$ is regular, that is, $\mu$ is finite on compact sets and $\Omega \setminus \Sigma$ is dense in $\Omega$.

Let

$$f(x, z) = \begin{cases} |z| & \text{if } x \in \Omega \setminus \Sigma \\ \frac{1}{2}|z|^2 & \text{if } x \in \Sigma, \end{cases}$$

Let

$$\beta(z) = \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| \leq 1 \\ |z| - \frac{1}{2} & \text{if } |z| > 1 \end{cases}$$

Remark that $\beta = \frac{1}{2} |\cdot|^2 \nabla |\cdot|$. Then, if $\lambda_a + \lambda_s$ is the $\mu$-decomposition of $\lambda$,

$$F(\lambda) = \int_{\Omega \setminus \Sigma} d |\lambda_a| + \int_{\Sigma} \beta \left( \frac{d\lambda_a}{d\mu} (x) \right) dH^{N-1}(x) + |\lambda_s| (\Omega).$$

EXAMPLE 2. Let $\Omega$ be an open subset of $\mathbb{R}^N$, $\mu$ the Lebesgue measure, $a: \Omega \to [0, \infty]$ a locally integrable function, and $f(x, z) = a(x) |z|$. Then, if

$$\tilde{a}(x) = \lim_{\delta \to 0^+} [\mu(B(x, \delta))]^{-1} \int_{B(x, \delta)} a(y) dy$$

and $\tilde{a}$ is the l.s.c. hull of $\tilde{a}$,

$$\tilde{F}(\lambda) = \int_{\Omega} \tilde{a} d |\lambda|.$$

Remark. As soon as $f^*(x, \cdot)$ is an indicator, $F(\lambda) = \delta^*(\lambda | \Phi)$, where $\Phi$ is the set of $G$-selectors of a l.s.c. multifunction $\Gamma$. For the existence of $\Gamma$ see Valadier [44]. In Examples 2 and 4 below, it is possible to “calculate” $\Gamma$.

EXAMPLE 3. Let $\Omega$ be an open subset of $\mathbb{R}^N$, $\mu$ the Lebesgue measure, $a: \Omega \to [0, \infty]$ a measurable function, and $f(x, z) = \frac{1}{2} a(x) |z|^2$. Then, if $\Omega'$ is the greatest open subset on which $1/a$ is locally integrable (with the convention $1/0 = +\infty$), one has

$$F(\lambda) = \begin{cases} \frac{1}{2} \int_{\Omega'} a(x) \left( \frac{d\lambda_a}{dx} \right)^2 dx & \text{if } |\lambda_s| (\Omega') = 0 \\ +\infty & \text{if } |\lambda_s| (\Omega') > 0. \end{cases}$$
**Example 4.** Let $\Omega$ be an open subset of $\mathbb{R}^N$, $\mu$ the Lebesgue measure, and $A: \Omega \to \mathbb{R}^d$ a measurable function such that $|A(x)| = 1$ a.e. Let $f(x, z) = [A(x) \cdot z]^*$. If $\tilde{A}$ is defined as in Example 2 but coordinate-wise, that is,

$$\forall i \in \{1, \ldots, d\}, \quad \tilde{A}_i(x) = \lim_{\delta \to 0^+} \left[ \mu(B(x, \delta)) \right]^{-1} \int_{B(x, \delta)} A_i(y) \, dy,$$

and if $\Omega'$ is the greatest open subset on which $\tilde{A}$ is continuous, then $F(\lambda) = \int_{\Omega} \left[(\partial \lambda_i/\partial |\lambda|)(x) \cdot \tilde{A}(x)\right]^+ |\lambda| \, (dx)$.

**Remarks.**

(1) On $\Omega'$, $|\tilde{A}(x)| = 1$ because $\tilde{A}(x) = A(x)$ a.e.

(2) The existence of $\Omega'$ and $\tilde{A}$ can be proved without the $\sim$ operation. Indeed $\Omega'$ is the greatest open subset on which $A$ is a.e. equal to a (unique) continuous function. The existence of $\Omega'$ follows from the Lindelöf property. One can treat also $f(x, z) = |A(x) \cdot z|$: in this case it is necessary to topologize the unit sphere identifying opposite points.

**Example 5** (which describes the usual case in plasticity theory). Let $\Omega$ be an open bounded subset of $\mathbb{R}^N$ and $E$ the space of symmetric tensors of order 2 ($\dim E = N(N+1)/2$). Recall that $E$ has a Euclidean structure for which the orthogonal of $\mathbb{R}I$ (the one-dimensional subspace of diagonal tensors) is the space $E^D$ of tensors whose traces vanish.

Let $B$ be a closed convex-valued l.s.c. multifunction such that $\forall x, 0 \in B(x)$. We suppose moreover that $\forall \varphi \in C_0$, $\varphi(x) \in B(x)$ a.e. if $\varphi \in B(x)$ everywhere (remark that this avoids $\Omega = ]-1, 1[ \cup \{0\}$ if $x \neq 0$, $B(0) = \{0\}$). There exist many l.s.c. discontinuous multifunctions which satisfy this hypothesis. In practice $B(x) = B^D(x) + \mathbb{R}I$, where $B^D(x)$ is a convex compact subset of $E^D$ containing 0.

Let $\gamma: \mathbb{R} \to \mathbb{R}$ be continuous and $\psi$ be a convex normal integrand on $\Omega \times E$ such that $0 \leq \psi(x, \cdot) \leq \gamma(\cdot)$. The useful integrand in plasticity is

$$f(x, \cdot) = [\psi(x, \cdot) + \delta(\cdot | B(x))].$$

Let $h(x, \cdot) = \delta^*(\cdot | B(x))$. Then

$$\forall \lambda \in \mathcal{M}^b(\Omega; E), \quad F_1(\lambda) = \int_{\Omega} f \left( x, \frac{d\lambda}{dx} \right) \, dx + \int_{\Omega} h(x, d\lambda),$$

**Remark.** When $B(x) = B^D(x) + \mathbb{R}I$,

$$h(x, z) = \begin{cases} \delta^*(z | B^D(x)) & \text{if } z \in E^D \\ +\infty & \text{otherwise.} \end{cases}$$
Hence \( \text{dom } h(x, \cdot) = E^D \) and, if \( u \in BD(\Omega) \) and \( Du = \frac{1}{2}(u_u + u_y) \) satisfies \( F_1(Du) < \infty \), the singular part of the measure \( \text{div } u = \text{tr}(Du) \) vanishes.

**Proof.** Since \( f^*(x, \cdot) = \psi(x, \cdot) + \delta(\cdot \mid B(x)) \), one has for \( \varphi \in \mathcal{C}_0 \)

\[
J_f(\varphi) < \infty \iff \varphi(x) \in B(x) \text{ a.e.} \iff \forall x, \varphi(x) \in B(x).
\]

Thanks to the Michael theorem [26], for any \( z \in B(x) \), there exists \( \varphi \in \mathcal{C}_0 \) with \( \varphi(x) = z \) and \( \forall y, \varphi(y) \in B(y) \). Thus \( h(x, \cdot) = \delta^*(\cdot \mid B(x)) \) and, since \( g^* = f^* + h^* \), \( \forall x, g(x, \cdot) = f(x, \cdot) \).

**APPENDIX 1**

**Lemma A1.** Let \( g: \mathbb{R}^d \to [-\infty, \infty] \) be convex l.s.c., \( D \subset \text{dom } g \). Suppose \( \delta \) convex. Then the l.s.c. hull

\[
g + \delta(\cdot \mid D)
\]

of \( g + \delta(\cdot \mid D) \) is equal to \( g + \delta(\cdot \mid \bar{D}) \).

In particular \( \inf_D g = \inf_{\bar{D}} g \).

**Proof.** Obviously \( g + \delta(\cdot \mid \bar{D}) \leqslant g + \delta(\cdot \mid D) \). Without loss of generality we may suppose that the affine subspace generated by \( D \) is \( \mathbb{R}^d \) itself. So \( \text{int}(\text{co } D) \neq \emptyset \). Let \( x_0 \in \text{int}(\text{co } D) \), one has \( x_0 \in \text{int}(\text{dom } g) \cap \bar{D} \).

(a) As \( g \) is continuous at \( x_0 \),

\[
g + \delta(\cdot \mid D)(x_0) = \lim_{x \to x_0} g(x) = g(x_0)
\]

\[
= [g + \delta(\cdot \mid \bar{D})](x_0).
\]

(b) Let \( x_1 \in \bar{D}, x_1 \neq x_0 \), and prove \( g + \delta(\cdot \mid D)(x_1) \leqslant g(x_1) \). Let \( x_2 = \lambda x_1 + (1 - \lambda)x_0 \). When \( \lambda \) runs through \([0, 1[\), \( x_2 \) belongs to \( \text{int}(\text{dom } g) \cap \bar{D} \), hence, by (a),

\[
g + \delta(\cdot \mid D)(x_2) = g(x_2).
\]

On a one-dimensional interval like \([x_0, x_1]\), a convex function is u.s.c., so when it is l.s.c. it is continuous. Hence

\[
g + \delta(\cdot \mid D)(x_1) = \lim_{\lambda \to 1^-} g + \delta(\cdot \mid D)(x_2)
\]

\[
= \lim_{\lambda \to 1^-} g(x_2) = g(x_1).
\]

The last formula is easy.
Lemma A2. Let $Q$ be a multifunction on a topological space $\Omega$ to the convex subsets of $\mathbb{R}^d$. Then $Q$ is l.s.c. on $\Omega$ iff $(x, z') \mapsto \delta^*(z' | Q(x))$ is l.s.c. on $\Omega \times \mathbb{R}^d$.

Proof. Let $\varphi(x, z') = \delta^*(z' | Q(x))$.

(1) Suppose $Q$ is l.s.c. Let $(x_0, z'_0) \in \Omega \times \mathbb{R}^d$ and $r \in \mathbb{R}$, $r < \varphi(x_0, z'_0)$. The set $W = \{(z, z') \in (\mathbb{R}^d)^2 | z \cdot z' > r\}$ is open. There exists $z_0 \in Q(x_0)$ such that $(z_0, z'_0) \in W$. There exists $U$ an open neighborhood of $z_0$ and $U'$ an open neighborhood of $z'_0$ such that $U \times U'$ is contained in $W$. As $Q$ is l.s.c. and $z_0 \in Q(x_0) \cap U$, there exists a neighborhood $V$ of $x_0$ such that $\forall x \in V$, $Q(x) \cap U \neq \emptyset$. Hence

$$\varphi(x, z') = \delta^*(z' | Q(x)) > r.$$ 

Thus $\varphi$ is l.s.c. at $(x_0, z'_0)$.

(2) Suppose $\varphi$ is l.s.c. and $Q$ is not l.s.c. at $x_0$. Let $U$ be an open subset of $\mathbb{R}^d$ such that $Q(x_0) \cap U \neq \emptyset$. We may suppose $U$ convex and $0 \in Q(x_0) \cap U$. Thus $\varphi(x_0, \cdot) \geq 0$. There exists a generalized sequence $(y_a)$ such that $y_a \to x_0$ and $Q(y_a) \cap U = \emptyset$. By the Hahn–Banach theorem $\exists z'_a$ and $r \in \mathbb{R}$ such that

$$\varphi(y_a, z'_a) \leq r \leq \inf_{z \in U} z \cdot z'_a.$$ 

We may suppose $r = -1$. Thus $z'_a \in \{z' | \forall z \in U, z \cdot z' \geq -1\}$, which is an equicontinuous set (here a bounded subset of $\mathbb{R}^d$). Let $z'$ be a cluster point of the generalized sequence $(z'_a)$. By the lower semi-continuity of $\varphi$, $\varphi(x_0, z') \leq -1$, which is a contradiction.

Remark. This improves in one direction II.21 of Castaing and Valadier [15].

References


