Primitive flag-transitive generalized hexagons and octagons

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Abstract

Suppose that an automorphism group G acts flag-transitively on a finite generalized hexagon or octagon $\mathcal{S}$, and suppose that the action on both the point and line set is primitive. We show that G is an almost simple group of Lie type, that is, the socle of G is a simple Chevalley group.
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Keywords: Generalized hexagons; Generalized octagons; Generalized polygons; Primitive permutation groups;
Flag-transitivity; O'Nan–Scott Theorem; Almost simple groups; Groups of Lie type

1. Introduction

The classification of all finite flag-transitive generalized polygons is a long-standing important open problem in finite geometry. Generalized polygons, introduced by Tits in [18], are among the most notable and prominent examples of discrete geometries, they have a lot of applications and are the building bricks of the Tits buildings. The determination of all finite flag-transitive examples would have a great impact on many problems, not in the least because of a significant weakening of the hypotheses of many results. It is generally considered as an "NC-hard" problem, where NC stands for "No Classification (of finite simple groups allowed)." By a result of Feit

and Higman [5], we must only consider generalized triangles (which are the projective planes), generalized quadrangles, generalized hexagons and generalized octagons. In each case there are nontrivial examples of finite flag transitive geometries, and it is believed that we know all of them. The most far reaching results are known for the class of projective planes, where the only counterexamples would have a sharply transitive group on the flags, and the number of points must be a prime number, see [9]. For generalized quadrangles, besides the well-known classical (and dual classical, in the terminology of [14]) cases there are exactly two other examples, both arising from transitive hyperovals in Desarguesian projective planes, namely of respective order 4 and 16 (and the hyperovals are the regular one and the Lunelli–Sce hyperoval, respectively). Both quadrangles have an affine representation, that is, their point set can be identified with the point set of a 3-dimensional affine space, and the line set is the union of some parallel classes of lines of that space (more precisely, those parallel classes of lines that define the corresponding hyperoval in the plane at infinity). For generalized hexagons and octagons, only the classical (Moufang) examples are known to exist, and the conjecture is that they are the only flag-transitive ones (and some even conjecture that they are the only finite ones!). An affine construction similar to the one above for quadrangles can never lead to a generalized hexagon or octagon, and this observation easily leads to the nonexistence of generalized hexagons and octagons admitting a primitive point-transitive group and whose O'Nan–Scott type is HA (see below for precise definitions).

This observation is the starting point of the present paper. Since the classification of finite flag-transitive generalized polygons is NC-hard, we have to break the problem down to a point where we must start a case-by-case study involving the different classes of finite simple groups. One celebrated method is the use of the famous O'Nan–Scott Theorem. This theorem distinguishes several classes of primitive permutation groups, one being the class HA above. Another class is the class AS, the Almost Simple case, and this class contains all known examples of finite flag-transitive generalized hexagons and octagons. Ideally, one would like to get rid of all O'Nan–Scott classes except for the class AS. The rest of the proof would then consist of going through the list of finite simple groups and try to prove that the existing examples are the only possibilities. In the present paper, we achieve this goal. We even do a little better and prove that we can restrict to Chevalley groups, that is, we rule out the almost simple groups with alternating socle, the sporadic groups being eliminated already in [2]. The treatment of the different classes of Chevalley groups is a nontrivial but—so it appears—a feasible job, and shall be pursued elsewhere. Note that the classical hexagons and octagons have a flag-transitive automorphism group of almost simple type with socle the simple Chevalley groups of type G_2, $^3\mathsf{D}_4$ and $^2\mathsf{F}_4$. Their construction is with the natural BN-pair. The automorphism group of these polygons is primitive on both the point-set and the line-set, and it is also flag-transitive.

We note, that our assumptions include primitive actions on both the point and the line set of the generalized hexagon or octagon. In some case, this can be weakened, and we have stated our intermediate and partial results each time under the weakest hypotheses. This could be important for future use when trying to reduce the general case to the primitive one handled in large in this paper.

A similar treatment for the finite generalized quadrangles seem out of reach for the moment. Therefore, we restrict ourselves to the cases of hexagons and octagons for the rest of the paper.

2. Setting

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a finite point-line geometry, where $\mathcal{P}$ is a point set, $\mathcal{L}$ is a line set, and $\mathtt{I}$ is a binary symmetric incidence relation. The *incidence graph* of $\mathcal{S}$ is the graph with vertex set

$\mathcal{P} \cup \mathcal{L}$, where the adjacency relation is given by the incidence relation $\mathtt{I}$. The *diameter* of $\mathcal{S}$ is by definition the diameter of the incidence graph of $\mathcal{S}$, and the *gonality* of $\mathcal{S}$ is by definition half of the girth of the incidence graph of $\mathcal{S}$ (which is a bipartite graph and therefore has even girth). For $n \geqslant 2$, the geometry $\mathcal{S}$ is a *weak generalized n-gon*, if both the diameter and the gonality of $\mathcal{S}$ are equal to n. If every point is incident with at least three lines, and every line carries at least three points, then we say that $\mathcal{S}$ is *thick*, and we call it a *generalized n-gon*, or *generalized polygon*. In this case, there are positive integers $s, t \geqslant 2$ such that every line is incident with $s + 1$ points, and every point is incident with $t + 1$ lines. We call (s, t) the *order* of $\mathcal{S}$. If $n = 2$, then $\mathcal{S}$ is a trivial geometry where every point is incident with every line. If $n = 3$, then $\mathcal{S}$ is a projective plane.

A generalized 6-gon (or *hexagon*) $\mathcal{S}$ with order (s, t) has $(1 + s)(1 + st + s^2 t^2)$ points and $(1 + t)(1 + st + s^2 t^2)$ lines. The number of flags, that is the number of incident point-line pairs, of $\mathcal{S}$ is equal to $(1 + s)(1 + t)(1 + st + s^2 t^2)$. Also, it is well known that st is a perfect square (see [5,19]). A generalized 8-gon (or *octagon*) $\mathcal{S}$ with order (s, t) has $(1 + s)(1 + st)(1 + s^2 t^2)$ points and $(1 + t)(1 + st)(1 + s^2 t^2)$ lines. The number of flags of $\mathcal{S}$ is equal to $(1 + s)(1 + t)(1 + st)(1 + s^2 t^2)$. Also, it is well known that $2st$ is a perfect square (see [5,19]). Hence one of s, t is even and consequently, either the number of points or the number of lines of $\mathcal{S}$ is odd.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized hexagon or octagon. A *collineation* or *automorphism* of G is a permutation of the point set $\mathcal{P}$, together with a permutation of the line set $\mathcal{L}$, preserving incidence. The group of automorphisms is denoted by $\mathsf{Aut}\,\mathcal{S}$ and is referred to as the *automorphism group* of $\mathcal{S}$. If G is a group of automorphisms of $\mathcal{S}$, then G can be viewed as a permutation group on $\mathcal{P}$ and also as a permutation group on $\mathcal{L}$. The main theorem of this paper is the following.

Theorem 2.1. *Suppose that G is a group of automorphisms of a generalized hexagon or octagon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$. If G is primitive on both $\mathcal{P}$ and $\mathcal{L}$ and G is flag-transitive then G must be an almost simple group of Lie type.*

3. Some preliminary results

The next result will be useful to rule out the existence of generalized polygons with a certain number of points. Suppose that n is a natural number and suppose that $n = 3^\alpha p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ where the p_i are pairwise distinct primes all different from 3, $\alpha \geqslant 0$ and $\alpha_i \geqslant 1$ for all i. Then we define the following quantities:

$$a(n) = 3^{\max\{0, \alpha - 1\}} \prod_{p_i \not\equiv 1 \bmod 3} p_i^{\alpha_i};$$

$$b(n) = \prod_{p_i \not\equiv 1 \bmod 4} p_i^{\alpha_i}.$$

We obtain the following result about the number of points of a generalized hexagon or octagon.

Lemma 3.1. *Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized hexagon or octagon.*

(i) *If $\mathcal{S}$ is a generalized hexagon, then $a(|\mathcal{P}|)^3 \leqslant |\mathcal{P}|$.*
(ii) *If $\mathcal{S}$ is a generalized octagon, then $b(|\mathcal{P}|)^2 \leqslant |\mathcal{P}|$.*

Proof. (i) Suppose that $\mathcal{S}$ is a generalized hexagon with order (s, t). Then $|\mathcal{P}| = (1 + s)(1 + st + s^2 t^2)$. As mentioned in the previous section, st is a square, and it was proved in the last paragraph of [2, p. 90] that if p is a prime such that $p | 1 + st + s^2 t^2$, then $p \equiv 1 \pmod 3$; in

addition, $1+st+s^2t^2$ is not divisible by 9. Thus $a(|\mathcal{P}|)$ must divide $1+s$ and $|\mathcal{P}|/a(|\mathcal{P}|)$ must be divisible by $1+st+s^2t^2$. On the other hand, since $t \geqslant 2$, we obtain that $(1+s)^2 \leqslant (1+st+s^2t^2)$, which implies that $a(|\mathcal{P}|)^2 \leqslant |\mathcal{P}|/a(|\mathcal{P}|)$, and so part (i) is valid.

(ii) Suppose that $\mathcal{S}$ is a generalized octagon with order (s,t). Then $|\mathcal{P}| = (1+s)(1+st)(1+s^2t^2)$. As mentioned above, $2st$ is a square, and it was proved in [2, p. 99] that, if p is a prime such that $p \mid 1+s^2t^2$, then $p \equiv 1 \pmod 4$. Thus $b(|\mathcal{P}|)$ must divide $(1+s)(1+st)$ and $|\mathcal{P}|/b(|\mathcal{P}|)$ must be divisible by $(1+s^2t^2)$. On the other hand, since $s, t \geqslant 2$, it follows that $(1+s)(1+st) \leqslant (1+s^2t^2)$, and so $b(|\mathcal{P}|) \leqslant |\mathcal{P}|/b(|\mathcal{P}|)$, and statement (ii) holds.   □

We will use the following notation: if x is a point collinear with the point y, that is, x and y are incident with a common line, then we write $x \sim y$. Dually, the notation $L \sim M$ for lines L, M means that L and M are concurrent; that is, they share a common point. If x and z are non-collinear points collinear to a common point y, then, assuming that the gonality is at least 5, the point y is unique with this property and we write $y = x \bowtie z$.

If G is a permutation group acting on a set Ω then the image of $\omega \in \Omega$ under $g \in G$ is denoted by ωg, while the stabilizer in G of ω is denoted by G_ω. The group G is said to be *semiregular* if $G_\omega = 1$ for all $\omega \in \Omega$, and it is said to be *regular* if it is transitive and semiregular.

Lemma 3.2. *If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized hexagon or octagon with order (s,t), then the following is true.*

(i) *If $\gcd(s,t) \neq 1$ and g is an automorphism of $\mathcal{S}$, then either g has a fixed point or there is a point $x \in \mathcal{P}$ such that $x \sim xg$.*

(ii) *If $\gcd(s,t) \neq 1$ and g is an automorphism of $\mathcal{S}$ with order 2, then g has either a fixed point or a fixed line. In particular, if G is an automorphism group of $\mathcal{S}$ with even order, then G cannot be semiregular on both $\mathcal{P}$ and $\mathcal{L}$.*

(iii) *Let x be a point and let y_1 and y_2 be two points collinear with x such that y_1 is not collinear with y_2. Suppose there are automorphisms g_1, g_2 mapping x to y_1, y_2, respectively. If g_1 and g_2 commute, then $y_1 g_2 = y_2 g_1 = x$.*

(iv) *If G is an automorphism group of $\mathcal{S}$ which is transitive on $\mathcal{P}$, then $\mathbb{C}_{\mathsf{Aut}\,\mathcal{S}}(G)$ is intransitive on $\mathcal{P}$.*

(v) *If G is an automorphism group of $\mathcal{S}$ acting faithfully and flag transitively, then $|G| \leqslant |G_x|^{12}$ for all $x \in \mathcal{P}$.*

Proof. Claim (i) is shown in [16]. To show (ii), let g be an automorphism with order 2 and assume that g has no fixed point. Then, by (i), there is a point $x \in \mathcal{P}$, such that $x \sim xg$. Suppose that L is the line that is incident with x and xg. Then the image Lg of L is incident with xg and $xg^2 = x$, and so $Lg = L$. Thus L must be a fixed line of g. If G is an automorphism group with even order then G contains an automorphism with order 2. If G is semiregular on $\mathcal{P}$ then g has no fixed point in $\mathcal{P}$. Thus, by the argument above, g must have a fixed line, and so G cannot be semiregular on $\mathcal{L}$. Thus (ii) is proved. In claim (iii), as $x \sim y_1$, the point $y_2 = xg_2$ is collinear with $y_1 g_2 = xg_1 g_2$. Similarly, $y_1 = xg_1$ is collinear with $y_2 g_1 = xg_2 g_1 = xg_1 g_2$. Hence if $x \neq xg_1 g_2$, then the gonality of $\mathcal{S}$ would be at most 4, which is a contradiction. Let us now show (iv). Set $C = \mathbb{C}_{\mathsf{Aut}\,\mathcal{S}}(G)$ and assume that C is transitive on $\mathcal{P}$. Let x and y be vertices of $\mathcal{S}$ such that $x \sim y$. Then there is some $g \in G$ such that $xg = y$. On the other hand, as $\mathcal{S}$ is thick and its gonality is at least 6, we can choose distinct vertices y_1 and y_2 such that $x \sim y_1$, $x \sim y_2$, $y \nsim y_1$, and $y \nsim y_2$. By assumption, C is transitive, and so there are $c_1, c_2 \in C$ such that

$xc_1 = y_1$ and $xc_2 = y_2$. Then we obtain that $y_1g = y_2g = x$, which is a contradiction, and so (iv) is valid.

Finally, we verify (v). Suppose first that $\mathcal{S}$ is a generalized hexagon with order (s, t), let $x \in \mathcal{P}$ and let G_x denote the stabilizer in G of x. Since G is flag-transitive, G_x must be transitive on the $t + 1$ lines that are incident with x and, in particular, $|G_x| \geqslant t + 1$. Therefore, using the Orbit–Stabilizer Theorem and the inequality $s \leqslant t^3$ (see [7] and [19, Theorem 1.7.2(ii)]),

$$\frac{|G|}{|G_x|} = |\mathcal{P}| = (1+s)\left(1 + st + s^2t^2\right) \leqslant \left(1 + t^3\right)\left(1 + t^4 + t^8\right) \leqslant (1+t)^{11} \leqslant |G_x|^{11},$$

and the statement for hexagons follows. If $\mathcal{S}$ is a generalized octagon with order (s, t), then, using the inequality $s \leqslant t^2$ (see [8] and [19, 1.7.2(iii)]), we obtain similarly that

$$\frac{|G|}{|G_x|} = |\mathcal{P}| = (1+s)(1+st)\left(1 + s^2t^2\right) \leqslant \left(1+t^2\right)\left(1+t^3\right)\left(1+t^6\right) \leqslant (1+t)^{11} \leqslant |G_x|^{11},$$

and the statement for octagons also follows. □

We note that a generalized hexagon or octagon is a self-dual concept, and so the dual of a true statement is also true. For instance, taking the dual of statement (iv), we obtain the following fact: *if G is a line-transitive automorphism group of $\mathcal{S}$, then $\mathbb{C}_{\mathsf{Aut}\,\mathcal{S}}(G)$ is intransitive on the lines.* In this paper we do not state the dual of each of the results, but we often use the dual statements in our arguments.

We will also need the following group theoretic lemma. Recall that a group G is said to be *almost simple* if it has a unique minimal normal subgroup T which is non-abelian and simple. In this case, T is the *socle* of G and the group G can be considered as a subgroup of the automorphism group of T containing all inner automorphisms.

Lemma 3.3.

(a) *Let S be an almost simple group with socle T and let H be a maximal subgroup of S such that $T \nleqslant H$. Then $\mathbb{N}_T(H \cap T) = H \cap T$.*

(b) *Suppose that $T_1, \ldots, T_k$ are pairwise isomorphic finite simple groups and, for $i = 2, \ldots, k$, the map $\alpha_i : T_1 \to T_i$ is an isomorphism. Then the subgroup*

$$D = \left\{ \left(t, \alpha_2(t), \ldots, \alpha_k(t)\right) \,\middle|\, t \in T_1 \right\}$$

is self-normalizing in $T_1 \times \cdots \times T_k$.

Proof. (a) If S and T are as in the lemma, then $H \cap T \lhd H$. Hence $H \leqslant \mathbb{N}_S(H \cap T)$. Note that S can be considered as a primitive group acting on the right coset space $[S : H]$ with point-stabilizer H. Since the socle of such a primitive group is non-regular, we obtain that $H \cap T \neq 1$. Hence $H \cap T$ is a proper, nontrivial subgroup of T, which cannot be normal in S. Thus, since H is a maximal subgroup of S, we obtain that $\mathbb{N}_S(H \cap T) = H$. Hence $\mathbb{N}_T(H \cap T) = \mathbb{N}_S(H \cap T) \cap T = H \cap T$.

(b) Let $G = T_1 \times \cdots \times T_k$ and let $(t_1, \ldots, t_k) \in \mathbb{N}_G(D)$. Then, for all $t \in T_1$,

$$\left(t, \alpha_2(t), \ldots, \alpha_k(t)\right)^{(t_1, \ldots, t_k)} = \left(t^{t_1}, \alpha_2(t)^{t_2}, \ldots, \alpha_k(t)^{t_k}\right) \in D.$$

Thus, for all $i \in \{2, \ldots, k\}$, we obtain that $\alpha_i(t^{t_1}) = \alpha_i(t)^{t_i}$. Therefore $t_1 \alpha_i^{-1}(t_i)^{-1} \in \mathbb{C}_{T_i}(t)$. As this is true for all $t \in T_1$, we obtain that $t_1 \alpha_i^{-1}(t_i)^{-1} \in Z(T_1)$. As T_1 is a non-abelian, finite, simple group, this yields that $\alpha_i(t_1) = t_i$. Hence $(t_1, \ldots, t_k) \in D$, and so $\mathbb{N}_G(D) = D$. □

4. Hexagons and octagons with primitive automorphism group

The structure of a finite primitive permutation group is described by the O'Nan–Scott Theorem (see [3, Sections 4.4–4.5] or [4, Section 4.8]). In the mathematics literature, one can find several versions of this theorem, and in this paper we use the version that can, for instance, be found in [1, Section 3]. Thus we distinguish between 8 classes of finite primitive groups, namely HA, HS, HC, SD, CD, PA, AS, TW. A description of these classes can be found below.

Recall that in a finite group G, the *socle* of G is the product of the minimal normal subgroups in G and it is denoted by $\operatorname{Soc} G$. In fact, $\operatorname{Soc} G$ is the direct product of the minimal normal subgroups of G. As a minimal normal subgroup of G is a direct product of pairwise isomorphic finite simple groups, the socle of G is also the direct product of finite simple groups.

Suppose that $G_1, \ldots, G_k$ are groups, set $G = G_1 \times \cdots \times G_k$, and, for $i \in \{1, \ldots, k\}$, let φ_i denote the natural projection map $\varphi_i : G \to G_i$. A subgroup H of G is said to be *subdirect with respect to the given direct decomposition of G* if $\varphi_i(H) = G_i$ for $i = 1, \ldots, k$. If the G_i are non-abelian finite simple groups then the G_i are precisely the minimal normal subgroups of G. In this case, a subgroup H is said to be *subdirect* if it is subdirect with respect to the decomposition of G into the direct product of its minimal normal subgroups. If G is a finite group then the holomorph $\operatorname{Hol} G$ is defined as the semidirect product $G \rtimes \operatorname{Aut} G$.

The O'Nan–Scott type of a finite primitive permutation group G can be recognized from the structure and the permutation action of $\operatorname{Soc} G$. Let $G \leqslant \operatorname{Sym} \Omega$ be a finite primitive permutation group, let M be a minimal normal subgroup of G, and let $\omega \in \Omega$. Note that M must be transitive on Ω. Further, M is a characteristically simple group, and so it is isomorphic to the direct product of pairwise isomorphic finite simple groups. The main characteristics of G and M in each primitive type are as follows.

HA M is abelian and regular, $\mathbb{C}_G(M) = M$ and $G \leqslant \operatorname{Hol} M$.
HS M is non-abelian, simple, and regular; $\operatorname{Soc} G = M \times \mathbb{C}_G(M) \cong M \times M$ and $G \leqslant \operatorname{Hol} M$.
HC M is non-abelian, non-simple, and regular; $\operatorname{Soc} G = M \times \mathbb{C}_G(M) \cong M \times M$ and $G \leqslant \operatorname{Hol} M$.
SD M is non-abelian and non-simple; M_ω is a simple subdirect subgroup of M and $\mathbb{C}_G(M) = 1$.
CD M is non-abelian and non-simple; M_ω is a non-simple subdirect subgroup of M and $\mathbb{C}_G(M) = 1$.
PA M is non-abelian and non-simple; M_ω is a not a subdirect subgroup of M and $M_\omega \neq 1$; $\mathbb{C}_G(M) = 1$.
AS M is non-abelian and simple; $\mathbb{C}_G(M) = 1$, and so G is an almost simple group.
TW M is non-abelian and non-simple; $M_\omega = 1$; $\mathbb{C}_G(M) = 1$.

We pay special attention to the groups of type AS. In this class, the group G has a unique minimal normal subgroup which is non-abelian and simple. Therefore G is isomorphic to a subgroup of $\operatorname{Aut} T$ which contains all inner automorphisms. Such an abstract group is referred to as *almost simple*. The next result shows that under certain conditions a primitive automorphism group of a generalized hexagon or octagon must be an almost simple group.

Theorem 4.1. *If G is a point-primitive, line-primitive and flag-transitive group of automorphisms of a generalized hexagon or octagon, then the type of G must be* AS *on both the points and the lines. In particular, G, as an abstract group, must be almost simple.*

Theorem 4.1 is a consequence of the following lemma.

Lemma 4.2. *If G is a group of automorphisms of a generalized hexagon or octagon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ then the following holds.*

(i) *If G is primitive on $\mathcal{P}$ then the type of G on $\mathcal{P}$ is not HA, HS, HC. Dually, if G is primitive on $\mathcal{L}$ then the type of G on $\mathcal{L}$ is not HA, HS, or HC.*

(ii) *If G is flag-transitive and it is primitive on $\mathcal{P}$ then the type of G on $\mathcal{P}$ is not PA or SD. Dually, if G is flag-transitive and it is primitive on $\mathcal{L}$ then the type of G on $\mathcal{L}$ is not PA or SD.*

(iii) *If G is flag-transitive and it is primitive on both $\mathcal{P}$ and $\mathcal{L}$, then the O'Nan–Scott type of G on $\mathcal{P}$ and on $\mathcal{L}$ is not SD or TW.*

Proof. Let $\mathcal{S}$ and G be as assumed in the theorem. Suppose further that G is primitive on $\mathcal{P}$ and let M be a fixed minimal normal subgroup of G. In this case, $M = T_1 \times \cdots \times T_k$ where the T_i are finite simple groups; let T denote the common isomorphism type of the T_i.

(i) As M is transitive on $\mathcal{P}$, Lemma 3.2(iv) implies that $\mathbb{C}_G(M)$ must be intransitive. Since $\mathbb{C}_G(M)$ is a normal subgroup of G, we obtain that $\mathbb{C}_G(M) = 1$. Hence the O'Nan–Scott type of G on $\mathcal{P}$ is not HA, HS, HC. The dual argument proves the dual statement.

(ii) Assume now that G is flag-transitive and it is primitive on $\mathcal{P}$. We claim that the O'Nan–Scott type of G on $\mathcal{P}$ is not PA or CD. Assume by contradiction that this O'Nan–Scott type is PA or CD. In this case $\mathcal{P}$ can be identified with the Cartesian product Γ^ℓ in such a way that G can be viewed as a subgroup of the wreath product $H \operatorname{wr} \mathsf{S}_\ell$ where H is a primitive subgroup of $\operatorname{Sym} \Gamma$ and the projection of G into S_ℓ is transitive. Set $N = \operatorname{Soc} H$ and let $\gamma \in \Gamma$. We must have that N^ℓ, considered as a subgroup of $H \operatorname{wr} \mathsf{S}_\ell$, is a subgroup of G, and, in fact, $N^\ell = \operatorname{Soc} G = M$. Further, we have the following two possibilities:

PA If the type of G is PA then the type of H is AS and we have that $N \cong T$, $\ell = k$ and N_γ is a proper subgroup of N.

CD If the type of G is CD, then the type of H is SD, $N \cong T^s$ where $s \geqslant 2$ and $s = k/\ell$. In this case, N_γ is a diagonal subgroup in N which is isomorphic to T.

Since H is primitive on Γ, the normal subgroup N must be transitive on Γ. If $\gamma \in \Gamma$, then H_γ is a maximal subgroup of H. Thus Lemma 3.3 implies that $\mathbb{N}_N(N_\gamma) = N_\gamma$ (part (a) of the lemma applies in the PA case, and part (b) applies in the CD case). Suppose that $\gamma, \delta \in \Gamma$ such that $N_\gamma = N_\delta$. Then there is $n \in N$ such that $\gamma n = \delta$, and $(N_\gamma)^n = N_\delta = N_\gamma$. Hence n normalizes N_γ, and so $n \in N_\gamma$, and we obtain that $\gamma = \delta$. Therefore different points of Γ must have different stabilizers in N.

Let α be an arbitrary element of Γ and consider the point $x \in \mathcal{P}$ represented by the ℓ-tuple $(\alpha, \alpha, \ldots, \alpha)$. We claim first that there exists a point $y \sim x$ such that every entry of the ℓ-tuple representing y is equal to α, except for one entry. Indeed, let y be any point collinear with x. Then, if the claim were not true, we may assume without loss of generality that y is represented by $(\beta_1, \beta_2, \ldots)$, where $\beta_1 \neq \alpha \neq \beta_2$. By the argument in the previous paragraph, the stabilizers in N of α and β_1 are distinct, and so there exists $g \in N_\alpha$ such that $\beta_1' := \beta_1 g \neq \beta_1$. Put $\bar{g} = (g, 1_N, 1_N, \ldots, 1_N)$ (ℓ factors) and $y' = y\bar{g}$. Let $h \in N_{\beta_2}$ be such that $\alpha' := \alpha h \neq \alpha$ (such an h exists by the argument in the previous paragraph). Put $\bar{h} = (1_N, h, 1_N, 1_N, \ldots, 1_N)$ (ℓ factors), and put $x' = x\bar{h}$. Then $x' \neq x$, and both y and y' are fixed under $\bar{h}$. Since $y \sim x \sim y'$, we deduce $y \sim x' \sim y'$. This implies (because the gonality of $\mathcal{S}$ is at least 5) that x, x', y, y' are incident

with a common line. But all entries, except the second, of x' are equal to α. Thus our claim is proved.

So we may pick $y \sim x$ with $y = (\beta, \alpha, \alpha, \ldots, \alpha)$ (ℓ entries) and $\beta \neq \alpha$. By the flag-transitivity, there exists $g \in G_x$ mapping y to a point not collinear with y. There are two possibilities:

(a) We can choose g such that the first entry of yg is equal to α.
(b) For every such g, the first entry of yg differs from α.

In case (a), as $x = (\alpha, \ldots, \alpha)$ and $g \in G_x$, we may suppose without loss of generality that $y' := yg = (\alpha, \beta', \alpha, \ldots, \alpha)$. Choose $h, h' \in N$ such that $\alpha h = \beta$, and $\alpha h' = \beta'$. Put $\bar{h} = (h, 1_N, \ldots, 1_N)$ and $\bar{h}' = (1_N, h', 1_N, \ldots, 1_N)$. Then $\bar{h}$ and $\bar{h}'$ commute and Lemma 3.2(iii) implies that $x = x\bar{h}\bar{h}'$. Hence $\alpha = \beta = \beta'$, a contradiction.

In case (b), we consider an arbitrary such g and put $z = yg$. Also, consider an arbitrary $g' \in G_x$ not preserving the first component of $\Gamma \times \Gamma \times \cdots \times \Gamma$. By assumption, yg' is incident with the line through x and y, and we put $z' = yg'$. If we now let y and y' in the previous paragraph play the role of z and z', respectively, of the present paragraph, then we obtain a contradiction again.

Thus we conclude that the type of G on $\mathcal{P}$ is not PA or CD and the dual statement can be verified using the dual argument.

(iii) Suppose that $\mathcal{S}$ is a generalized hexagon or octagon and G is a group of automorphisms such that G is flag-transitive and G is primitive on $\mathcal{P}$ and $\mathcal{L}$ of type either SD or TW. First we claim that $\mathcal{S}$ must be a generalized hexagon and $\gcd(s, t) = 1$. If $\mathcal{S}$ is a generalized octagon with order (s, t), then either $|\mathcal{P}|$ or $|\mathcal{L}|$ must be odd. However, the degree of a primitive group with type SD or TW is a power of the size of a non-abelian finite simple group. Therefore $\mathcal{S}$ must be a hexagon as claimed. Assume now by contradiction that $\gcd(s, t) \neq 1$ and consider the subgroup T_1 of the socle M. Since G is either SD or TW on $\mathcal{P}$ and also on $\mathcal{L}$ we have that T_1 is semiregular on both $\mathcal{P}$ and on $\mathcal{L}$. However, as T_1 is a non-abelian finite simple group, T_1 has even order, and this is a contradiction, by Lemma 3.2(ii).

So we may suppose for the remainder of this proof that $\mathcal{S}$ is a generalized hexagon with parameters (s, t) such that $\gcd(s, t) = 1$. Note that the number of lines is $(t + 1)(1 + st + s^2 t^2)$, and the number of points is $(s + 1)(1 + st + s^2 t^2)$. If G has the same O'Nan–Scott type on the set of points and the set of lines, then $|\mathcal{P}| = |\mathcal{L}|$, which implies $s = t$. Since $\gcd(s, t) = 1$, this is impossible, and we may assume without loss of generality that the type of G is SD on $\mathcal{P}$ and it is TW on $\mathcal{L}$. Hence $|\mathcal{P}| = (s + 1)(1 + st + s^2 t^2) = |T|^{k-1}$ and $|\mathcal{L}| = (t + 1)(1 + st + s^2 t^2) = |T|^k$. Thus $|T| = (t + 1)/(s + 1)$ and so $t = s|T| + |T| - 1$.

We digress in this paragraph to show that the order of the non-abelian finite simple group T is divisible by 4. It seems to be well known that this assertion follows immediately from the Feit–Thompson Theorem which states that $|T|$ is even. The following simple argument was showed to us by Michael Giudici in private communication. Recall that the right-regular representation ϱ of T is a homomorphism from T to $\mathsf{Sym}\, T$ that maps $t \in T$ to the permutation $\varrho(t) \in \mathsf{Sym}\, T$ where $\varrho(t)$ is defined by the equation $x\varrho(t) = xt$ for all $x \in T$. It is easy to see that $\varrho(T)$ is a regular subgroup of $\mathsf{Sym}\, T$; that is $\varrho(T)$ is transitive, and, for all $t \in T \setminus \{1\}$, $\varrho(t)$ has no fixed-points. Now $\varrho(T) \cong T$ and $\varrho(T) \cap \mathsf{Alt}\, T$ is a normal subgroup of $\varrho(T)$ with index at most 2. Thus $\varrho(T) \leqslant \mathsf{Alt}\, T$, and so every element of $\varrho(T)$ is an even permutation on T. By the Feit–Thompson Theorem referred to above, we have that T contains an involution g. Since $\varrho(g)$ is also an involution, it must be the product of disjoint transpositions. As $\varrho(g)$ is an even permutation, the number of transpositions in $\varrho(g)$ must be even. Further, as $\varrho(g)$ has no fixed-points, every

element of T must be involved in precisely one of these transpositions. This implies that $4 \mid |T|$, as claimed.

We now continue with the main thrust of the proof. In order to derive a contradiction, we show that the equations for s, t and $|T|$ above imply that $4 \nmid |T|$. Indeed, note that st is a square, and so, as $\gcd(s, t) = 1$, we have that t must be a square. If 4 divides $|T|$ then $t = s|T| + |T| - 1 \equiv 3 \pmod 4$. However, 3 is not a square modulo 4, which gives the desired contradiction. Hence, in this case, G cannot be primitive with type SD or TW. $\quad \square$

The reader may wonder whether it is possible for an abstract group G to have two faithful primitive permutation actions, one with type TW and one with type SD. Gross and Kovács in [6] show that if G is a twisted wreath product of A_5 and A_6 where the twisting subgroup in A_6 is isomorphic to A_5, then G is isomorphic to the straight wreath product $A_5 \text{ wr } A_6$. Hence in this case G can be a primitive permutation group of type TW and also of type SD.

Now we can prove Theorem 4.1.

Proof of Theorem 4.1. Suppose that G is a point-primitive, line-primitive and flag-transitive group of automorphisms of a generalized hexagon or octagon. Using parts (i)–(iii) of Lemma 4.2, we obtain that the type of G on both the points and lines must be AS. In particular G, as an abstract group, must be almost simple. $\quad \square$

5. Hexagons and octagons with an almost simple automorphism group

In this section we prove the following theorem.

Theorem 5.1. *If $\mathcal{S}$ is a generalized hexagon or octagon and G is a flag-transitive and point-primitive automorphism group of $\mathcal{S}$, then G is not isomorphic to an alternating or symmetric group with degree at least 5.*

Our strategy to prove Theorem 5.1 is to show that a maximal subgroup of an alternating or symmetric group cannot be a point-stabilizer. To carry out this strategy, we need some arithmetic results about the maximal subgroups of A_n and S_n.

Lemma 5.2. *If $n \in \mathbb{N}$ and $n \geqslant 107$ then*

$$n^{12 + 12 \lfloor \log_2 n \rfloor} \leqslant n!/2. \tag{1}$$

Proof. Checking the numbers between 107 and 208, we can see that (1) holds for all $n \in \{107, \ldots, 208\}$. So suppose without loss of generality in the remaining of this proof that n is at least 209. The Stirling Formula gives, for each $n \geqslant 1$, that there is $\vartheta_n \in [0, 1]$ such that $n! = (n/e)^n \sqrt{2\pi n} e^{\vartheta_n/(12n)}$ (see [10, Theorem 2, Chapter XII]), which gives that $(n/e)^n \leqslant n!/2$. We claim that $n^{n/2} \leqslant (n/e)^n$ for $n \geqslant 8$. Easy calculation shows that the inequality holds for $n = 8$. We assume that it holds for some n and prove by induction that it holds for $n + 1$. Let us compute that

$$\left(\frac{(n+1)^{(n+1)/2}}{n^{n/2}} \right)^2 = \frac{(n+1)^{n+1}}{n^n} = (n+1) \left(\frac{n+1}{n} \right)^n$$

and

$$\left(\frac{((n+1)/e)^{n+1}}{(n/e)^n}\right)^2 = e^{-2}\left(\frac{(n+1)^{n+1}}{n^n}\right)^2 = e^{-2}(n+1)^2\left(\frac{n+1}{n}\right)^{2n}.$$

This shows that

$$\frac{(n+1)^{(n+1)/2}}{n^{n/2}} \leqslant \frac{((n+1)/e)^{n+1}}{(n/e)^n},$$

and the assumption that $n^{n/2} \leqslant (n/e)^n$ gives the claimed inequality for $n+1$. Therefore it suffices to show that $n^{12+12\lfloor \log_2 n \rfloor} \leqslant n^{n/2}$, and, in turn, we only have to show that $12 + 12\log_2 n \leqslant n/2$ for $n \geqslant 209$. Again, easy computation shows that the inequality holds for $n = 209$. Since $x \mapsto 12 + 12\log_2 x$ is a concave function and $x \mapsto x/2$ is a linear function, the inequality must hold for all $n \geqslant 209$. $\quad\square$

Lemma 5.3. *Suppose that G is an alternating or symmetric group with degree n ($n \geqslant 5$) and H is a primitive and maximal subgroup of G such that $|H|^{12} \geqslant |G|$. Then G and H must be as one of the groups in Table* 1 *of Appendix* A.

Proof. Suppose that H is a primitive and maximal subgroup of G. Using the classification of maximal subgroups of the alternating and symmetric groups [11] and Maróti's theorem [13, Theorem 1.1], we have that one of the following must hold:

(1) $n = k^\ell$ for some $k \geqslant 5$ and $\ell \geqslant 2$ and H is permutationally isomorphic to $(\mathsf{S}_k \text{ wr } \mathsf{S}_\ell) \cap G$ in product action;
(2) G is isomorphic to M_n for $n \in \{11, 12, 23, 24\}$ in its 4-transitive action;
(3) $|G| < n^{1+\lfloor \log_2 n \rfloor}$.

Suppose that case (1) is valid and let H be permutationally isomorphic to the group $(\mathsf{S}_k \text{ wr } \mathsf{S}_\ell) \cap G$ in product action for some $k \geqslant 5$ and $\ell \geqslant 2$. Then we obtain that

$$|H|^{12} \leqslant (k!)^{12\ell} \cdot (\ell!)^{12}.$$

We claim that $(k!)^{12\ell} \cdot (\ell!)^{12} < (k^\ell)!/2$ except for finitely many pairs (k, ℓ). First note that all primes p dividing $(k!)^{12\ell} \cdot (\ell!)^{12}$ will also divide $(k^\ell)!/2$. For an integer x, let $|x|_p$ denote the largest non-negative integer α such that $p^\alpha | x$. It suffices to show that there are only finitely many pairs (k, ℓ) such that $|(k!)^{12\ell} \cdot (\ell!)^{12}|_p \geqslant |(k^\ell)!/2|_p$, where p is an arbitrary prime which is not greater than $\max\{k, \ell\}$. It is routine to check that if x is an integer then

$$|x!|_p = \sum_{u=1}^{\infty} \left\lfloor \frac{x}{p^u} \right\rfloor \leqslant \sum_{u=1}^{\infty} \frac{x}{p^u} = \frac{x}{p} \sum_{u=0}^{\infty} \frac{1}{p^u} = \frac{x}{p} \cdot \frac{p}{p-1} = \frac{x}{p-1}. \qquad (2)$$

Thus

$$\left|(k!)^{12\ell} \cdot (\ell!)^{12}\right|_p \leqslant 12\ell \frac{k}{p-1} + 12\frac{\ell}{p-1} = \frac{12\ell k + 12\ell}{p-1} \leqslant \frac{24\ell k + 24\ell}{p}.$$

Clearly, $k^\ell \geqslant 8$. Further, as $k \geqslant 5$, $\ell \geqslant 2$, and $p \leqslant \max\{k, \ell\}$, we obtain that $p^2 \leqslant k^\ell$. Hence we obtain from the first equality in (2) that

$$\left|(k^\ell)!/2\right|_p \geqslant \frac{k^\ell}{p}.$$

Routine computation shows that the set of pairs (k, ℓ) for which $k \geqslant 5$ and $\ell \geqslant 2$ and $24\ell k + 24\ell \geqslant k^\ell$ is $\{5, \ldots, 48\} \times \{2\} \cup \{5, \ldots, 8\} \times \{3\}$. Then checking finitely many possibilities it is easy to compute that $(k!)^{12\ell} \cdot (\ell!)^{12} \geqslant (k^\ell)!/2$ if and only if $(k, l) \in \{5, \ldots, 10\} \times \{2\}$. In particular, the degree of H is at most 100.

(2) Easy computation shows that $|\mathsf{M}_n|^{12} \geqslant n!/2$ for $n \in \{11, 12, 23, 24\}$.

(3) Lemma 5.2 shows that if $n \geqslant 107$ then $n^{12 + 12\lfloor \log_2 n \rfloor} \leqslant n!/2$. Hence if $n \geqslant 107$ and H is a maximal subgroup of A_n or S_n which is as in part (3) of the theorem, then $|H|^{12} < n!/2$. Thus, in this case, the degree of H must be at most 106.

Summarizing the argument above: if H is a primitive maximal subgroup of G such that $|H|^{12} \geqslant |G|$ then the degree of H is at most 106. It remains to prove that H must be one of the groups in Table 1 in Appendix A. Various classifications of primitive groups of small degree can be found in the literature; for convenience we use the classification by Roney-Dougal [15], as it can be accessed through the computational algebra system GAP [17]. In what follows we explain how we obtained Table 1 in Appendix A using the GAP system. First, for a fixed $n \in \{5, \ldots, 106\}$, let P_n denote the list of primitive groups with degree n. For $H \in P_n$ we check whether or not $H \leqslant \mathsf{A}_n$. Then we check whether $|H|^{12} \geqslant |G|$ where G is either A_n (if $H \leqslant \mathsf{A}_n$) or S_n (otherwise). If H satisfies this condition then we keep it in P_n, otherwise we erase it from P_n. The next step is to eliminate those groups which are clearly not maximal subgroups in A_n or S_n. If $H_1, H_2 \in P_n$ such that $H_1, H_2 \leqslant \mathsf{A}_n$ and $H_1 < H_2$ then H_1 is erased from P_n. Similarly, if $H_1, H_2 \nleqslant \mathsf{A}_n$ such that $H_1 < H_2$, then H_1 is thrown away. We do this calculation for all $n \in \{5, \ldots, 106\}$ and the subgroups H that we obtain are in Appendix A. $\quad\square$

Let us note that Lemma 5.3 is not an "if and only if" statement. Indeed, Table 1 in Appendix A may be redundant in the sense, that a subgroup in Table 1 may not be maximal in A_n or S_n.

Let us now prove Theorem 5.1.

Proof of Theorem 5.1. Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized hexagon or octagon and G is a point-primitive, flag-transitive automorphism group of $\mathcal{S}$ such that G is isomorphic to A_n or S_n with some $n \geqslant 5$. By Buekenhout and Van Maldeghem [2], we may assume that $n \geqslant 14$. Let $x \in \mathcal{P}$. Then G_x, as a subgroup of S_n, is either intransitive, or it is transitive and imprimitive, or it is primitive. We consider these three cases below.

G_x is intransitive. Here, G_x is the stabilizer in G of a partition of the underlying set into two blocks, one with size k and one with size ℓ, where $k + \ell = n$, $k \neq \ell$. Let us also allow here the case when $k = \ell$, though in this case G_x may not be intransitive. Assume without loss of generality that $k \leqslant \ell$. Then G_x contains a subgroup isomorphic to $(\mathsf{A}_k \times \mathsf{A}_\ell) \rtimes C_2$. Hence the points of $\mathcal{S}$ can be labelled with the subsets of $\{1, 2, \ldots, n\}$ of size k. We may label x as $\{1, 2, \ldots, k\}$. Let $k_1 < k$ be maximal with the property that there is a point y of $\mathcal{S}$ collinear with x and the label of y intersects $\{1, 2, \ldots, k\}$ in k_1 elements. Without loss of generality, we may assume that $y \sim x$ has label $\{1, 2, \ldots, k_1, k+1, \ldots, 2k-k_1\}$. First suppose that $k_1 = k - 1$. Note that, since the permutation rank of G is at least 4, we may assume $k \geqslant 3$. By transitivity of G_x on $\{1, 2, \ldots, k\}$, and by transitivity of the pointwise stabilizer of $\{1, 2, \ldots, k\}$ on the complement $\{k+1, k+2, \ldots, n\}$, every point with a label sharing exactly $k-1$ elements with $\{1, 2, \ldots, k\}$ is adjacent with x. An arbitrary element g of G_x now maps y onto a point y' with label, without loss of generality, either $\{1, 2, \ldots, k-1, k+2\}$ or $\{2, 3, \ldots, k, k+1\}$ or $\{2, 3, \ldots, k, k+2\}$. In the first two cases y' is collinear with y. Since, by flag-transitivity, we can choose g such that it does not preserve the line xy, and hence does not map the point y onto a collinear point, we

may assume that the point y' with label $\{2, 3, \ldots, k, k+2\}$ is not collinear with y, and hence has distance 4 to y (in the incidence graph). But now the automorphism $(1\ k+1)(k\ k+2)$ (if $k' = k+1$) fixes both y and y', but not $x = y \bowtie y''$. Hence $k_1 < k-1$.

Now the automorphism $(k-1\ k)(k+1\ 2k-k_1+1)$ belongs to G_x and maps y to a point z whose label shares $k-1$ elements with y. Hence z cannot be collinear with y (otherwise, mapping y to x, the image of z produces a point with a label contradicting the maximality of k_1 which is less than $k-1$). On the other hand, z is collinear with x. If $k_1 > 0$, then the automorphism $(1\ k+2)(k-1\ k)$ belongs to G, preserves y and z, but not $x = y \bowtie z$. Now suppose that $k_1 = 0$. If $2k+1 < n$, then the automorphism $(1\ 2k+2)(2\ 3)$ fixes y and z, but not $x = y \bowtie z$, a contradiction. If $2k+1 = n$, then, by the maximality of k_1, and the transitivity of A_k, we see that there are precisely $k+1$ points collinear with x on which G_x acts 2-transitively. This easily implies that either $s = 1$ or $t = 0$, either way a contradiction.

G_x is imprimitive. Here G_x is the stabilizer of a partition of the underlying set into ℓ blocks each with size k. Let x be a point of $\mathcal{S}$, which we may assume without loss of generality to correspond to the partition of $\{1, 2, \ldots, n\}$ into ℓ subsets of size k given by $\{ik+1, ik+2, \ldots, ik+k\}$, $0 \leqslant i < \ell$. We may assume that $\ell > 2$, the case $\ell = 2$ being completely similar to the intransitive case, as noticed above. (If $\ell = 2$ then, as the number of point is greater that 4, we may also assume that $k \geqslant 3$.)

We first claim that there is some point $y \sim x$ such that y corresponds to a partition sharing at least one partition class with x (we will identify the points with their corresponding partition). Let y be any point collinear with x and suppose that y has no partition class in common with x. If $k = 2$, then $\ell > 6$ and so the automorphism $(1\ 2)(3\ 4)$ destroys at most 4 classes of y, while it fixes x. Hence the image z of y has at least three classes $\{i_1, i_2\}, \{i_3, i_4\}, \{i_5, i_6\}$ in common with y, and therefore we may assume that $y \nsim z$. The group generated by $(i_1\ i_2)(i_3\ i_4)$, $(i_1\ i_3)(i_2\ i_4)$ and $(i_1\ i_5)(i_2\ i_6)$ fixes both y and z but cannot fix x, a contradiction. Suppose now $k > 2$. Then the automorphism $(1\ 2\ 3)$ destroys at most 3 classes of y and maps y to a point z sharing at least $\ell - 3$ classes with y. This is at least one if $\ell > 3$. If $\ell = 3$, then $k > 3$ and hence some class of y shares at least two elements with some class of x. Without loss of generality, we may assume that $1, 2$ are in some class of y and hence the automorphism $(1\ 2\ 3)$ destroys at most two classes of y, resulting in the fact that z shares at least one class with y again. Let this common class be given by $\{i_1, i_2, i_3, \ldots\}$, where we may suppose without loss of generality that i_1, i_2, i_3 do not belong to a common class of x. The automorphism $(i_1\ i_2\ i_3)$ fixes both y and z, but not $x = y \bowtie z$, a contradiction. Our claim is proved.

Now let ℓ_1 be maximal with respect to the property that there exist two collinear points sharing ℓ_1 classes. By the foregoing, $\ell_1 > 0$, and we may assume that the class $\{1, 2, \ldots, k\}$ belongs to the point $y \sim x$. Suppose that $\ell_1 < \ell - 2$. In particular, it follows from our assumptions that $\ell \geqslant 4$. It also follows from our assumptions that there is a transposition $(j_1\ j_2)$ fixing x and not fixing y. Hence the automorphism $(1\ 2)(j_1\ j_2)$ preserves x and maps y to a point z sharing $\ell - 2$ classes with y. By the maximality of ℓ_1, we see that $y \nsim z$. Also, y and z contain a common class which is not a class of x. So there exist elements j_3, j_4 contained in a common class of both y and z, but belonging to different classes of x. The automorphism $(1\ 2)(j_3\ j_4)$ fixes y and z, but not $x = y \bowtie z$, a contradiction. We have shown that $\ell_1 = \ell - 2$, and this now holds for all $\ell \geqslant 3$.

Now let k_1 be the maximal number of elements in the intersection of two distinct classes of two collinear elements sharing $\ell - 2$ classes. Note that $k_1 \geqslant k/2 > 0$. First we show that $k_1 < k - 1$. So we assume by way of contradiction that $k_1 = k - 1$. By transitivity of G_x, every point with a partition sharing $\ell - 2$ classes with x and for which the distinct classes share $k - 1$

elements, is collinear with x. By flag-transitivity and thickness, at least two such points y', y'' are not collinear with y. If the different classes of y' (compared with the classes of x) are the same as those of y, then, for $k > 2$, the same arguments as in the intransitive case lead to a contradiction. For $k = 2$, y'' does not have this property (since there are only three points with $\ell - 2$ given partition classes), and we switch the roles of y' and y'' in this case. So y' differs from y in three or four classes. We distinguish between two possibilities.

(1) *y and y' differ in exactly three partition classes.* We may assume that y contains the classes $\{1, 2, \ldots, k - 1, k + 1\}$ and $\{k, k + 2, k + 3, \ldots, 2k\}$ (and the other classes coincide with classes of x). Without loss of generality, there are two possibilities for y'. Either y' contains the classes $\{1, 2, \ldots, k - 1, 2k + 1\}$ and $\{k, 2k + 2, 2k + 3, \ldots, 3k\}$ (and the other classes coincide with classes of x), or y' contains the classes $\{1, 2, \ldots, k - 2, k, 2k + 1\}$ and $\{k - 1, 2k + 2, 2k + 3, \ldots, 3k\}$ (and the other classes coincide with classes of x). In the first case the automorphism $g = (k + 1\ 2k + 1\ k)$ maps y onto y', and y' onto a point collinear with x. Since $y'g$ is not collinear with $yg = y'$, we see that g must preserve $y \bowtie y' = yg \bowtie y'g = x$. But it clearly does not, a contradiction. In the second case the automorphism $(k - 1\ 2k + 1)(k\ k + 1)$ interchanges y with y', but does not fix $x = y \bowtie y'$, a contradiction.

(2) Hence *y and y' differ in four partition classes.* We take y as in (1), and we can assume that y' contains the classes $\{2k + 1, 2k + 2, \ldots, 3k - 1, 3k + 1\}$ and $\{3k, 3k + 2, 3k + 3, \ldots, 4k\}$. Now the automorphism $(k\ k + 1)(3k\ 3k + 1)$ interchanges y with y' without fixing $x = y \bowtie y'$, a contradiction.

Hence we have shown $k_1 < k - 1$. But now the rest of the proof is similar to the last paragraph of the intransitive case, where the subcase $k_1 = 0$ cannot occur. We conclude that G_x is primitive on $\{1, 2, \ldots, n\}$.

G_x is primitive. By Lemma 3.2(v), $|G| \leqslant |G_x|^{12}$, and so Lemma 5.3 implies that G and G_x must be in Table 1 of Appendix A. Set $u = |\mathcal{P}| = |G : G_x|$ and let $a(u)$ and $b(u)$ be the quantities defined before Lemma 3.1. Then Lemma 3.1 implies that if $\mathcal{S}$ is a hexagon then $a(u)^3 \leqslant u$ and, if $\mathcal{S}$ is an octagon, then $b(u)^2 \leqslant u$. For each pair (G, G_x) in Appendix A, one can compute using, for instance, the GAP computational algebra system, the quantities u, $a(u)$, and $b(u)$. The computation shows that $a(u)^3 > u$ and $b(u)^2 > u$ holds in each of the cases. The computation of $a(u)$ and $b(u)$ are presented in Appendices B and C. Therefore none of the groups in Appendix A can occur, and so we exclude this case as well.

Thus G cannot be an alternating or symmetric group. □

Now we can prove our main theorem.

Proof of Theorem 2.1. Suppose that $\mathcal{S}$ and G are as in the theorem. Then Theorem 4.1 implies that G must be an almost simple group. Let T denote the unique minimal normal subgroup of G. Note that T is a non-abelian simple group. By [2], T cannot be a sporadic simple group, and by Theorem 5.1, T cannot be an alternating group. Thus T must be a simple group of Lie type and G must be an almost simple group of Lie type. □

6. Directions of future work

Now that Theorem 2.1 is proved, the next step in the full classification of generalized hexagons and octagons satisfying the conditions of Theorem 2.1 is to treat the class of almost simple groups of Lie type. It is not our intention to be as detailed as possible regarding these groups, as we think the only worthwhile job now is to complete the classification in full. We noted in the proof

Lemma 4.2 that in a generalized octagon either the number of points or the number of lines is odd. Therefore it is meaningful to investigate which almost simple groups of Lie type with odd degree can occur in Theorem 2.1. Another possible task is to use Lemma 3.2 to characterize the case when the parameters are not co-prime. We conclude this paper by presenting a couple of examples to illustrate that Lemmas 3.1 and 3.2 can be used, to some extent, in this direction. However, our examples also show that a complete treatment of these groups is beyond the scope of this paper and will probably require new ideas.

Let us assume that G is an almost simple group of Lie type with socle T and that G is a group of automorphisms of a generalized hexagon or octagon $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ acting primitively both on the point set and on the line set, and transitively on the set of flags. Suppose, in addition, that the number $|\mathcal{P}|$ of points is odd and let x be a point. The possibilities for T and the point stabilizer T_x can be found in [9,12]. One possibility, for instance, is that $q = 3^{2m+1}$ with some $m \geqslant 1$, $T \cong {}^2\mathsf{G}_2(q)$ and $|T_x| = q(q^2 - 1)$. We claim that it follows from our results that this case cannot occur. Note that $|\mathcal{P}| = q^2(q^2 - q + 1)$. If $\mathcal{S}$ is a hexagon, then Lemma 3.1 implies that $a(q^2(q^2 - q + 1))^3 \leqslant q^2(q^2 - q + 1)$ (the function a is defined before Lemma 3.1). However, $a(q^2(q^2 - q + 1))^3 \geqslant 3^{12m+3}$ which would imply that $3^{12m+3} \leqslant 3^{8m+4}$ which does not hold for $m \geqslant 1$. Thus such a hexagon does not exist, and similar argument shows that neither does such an octagon.

Another case is that $T \cong \mathsf{F}_4(q)$, $|T_x| = q^{16}(q^2 - 1)(q^4 - 1)(q^6 - 1)(q^8 - 1)$, and so $|\mathcal{P}| = q^8(q^8 + q^4 + 1)$. Computer calculation shows that among the prime-powers that are smaller then 10^4, there are 626 values of q such that $a(|\mathcal{P}|)^3 \leqslant |\mathcal{P}|$, and there are 625 such values with $b(|\mathcal{P}|)^2 \leqslant |\mathcal{P}|$. Therefore Lemma 3.1 cannot directly be used to exclude this case.

We conclude this paper with an example that shows how Lemma 3.2 may be applied. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ be as above and let us assume that the parameters s and t of $\mathcal{S}$ are not co-prime. By Lemma 3.2(ii), an involution in G either fixes a point or fixes a line. Now if G is isomorphic to A_n or S_n with some $n \geqslant 5$, then, by possibly taking the dual polygon, we may assume that a double transposition (in the natural representation of G) is contained in a point stabilizer G_x. Therefore, as a subgroup of S_n, G_x has minimal degree at most 4 (see [4, p. 76] for the definition of the minimal degree). Now if G_x is primitive then [4, Example 3.3.1] shows that $n \leqslant 8$, and hence G is ruled out by [2]. This argument shows that under the additional condition that $\gcd(s, t) \neq 1$, the proof of Theorem 5.1 can be significantly simplified.

Acknowledgments

Appendix A

The following Table 1 contains the primitive maximal subgroups H of A_n and S_n ($n \geqslant 5$) such that $|H|^{12} \geqslant n!$ and $|H|^{12} \geqslant n!/2$ if $H \leqslant A_n$. Note that Table 1 may contain non-maximal subgroups; see the remarks at the end of the proof of Lemma 5.3. Table 1 was automatically generated from a GAP output, and so the notation follows the GAP system.

Table 1

`C(5)` $\leqslant A_5$	`D(2*5)` $\leqslant A_5$	`AGL(1, 5)` $\leqslant S_5$	`PSL(2,5)` $\leqslant A_6$
`PGL(2,5)` $\leqslant S_6$	`7:3` $\leqslant A_7$	`L(3, 2)` $\leqslant A_7$	`AGL(1, 7)` $\leqslant S_7$
`ASL(3, 2)` $\leqslant A_8$	`PSL(2, 7)` $\leqslant A_8$	`PGL(2, 7)` $\leqslant S_8$	`3^2:(2'A(4))` $\leqslant A_9$
`PGammaL(2, 8)` $\leqslant A_9$	`AGL(2, 3)` $\leqslant S_9$	`A(5)` $\leqslant A_{10}$	`M(10)` $\leqslant A_{10}$
`S(5)` $\leqslant S_{10}$	`PGammaL(2, 9)` $\leqslant S_{10}$	`11:5` $\leqslant A_{11}$	`M(11)` $\leqslant A_{11}$
`AGL(1, 11)` $\leqslant S_{11}$	`M(11)` $\leqslant A_{12}$	`M(12)` $\leqslant A_{12}$	`PSL(2, 11)` $\leqslant A_{12}$
`PGL(2, 11)` $\leqslant S_{12}$	`13:6` $\leqslant A_{13}$	`L(3, 3)` $\leqslant A_{13}$	`AGL(1, 13)` $\leqslant S_{13}$
`PSL(2,13)` $\leqslant A_{14}$	`PGL(2,13)` $\leqslant S_{14}$	`PSL(4, 2)` $\leqslant A_{15}$	`2^4.PSL(4, 2)` $\leqslant A_{16}$
`17:8` $\leqslant A_{17}$	`L(2, 2^4):4 = PGammaL(2, 2^4)` $\leqslant A_{17}$	`AGL(1, 17)` $\leqslant S_{17}$	`PSL(2,17)` $\leqslant A_{18}$
`PGL(2,17)` $\leqslant S_{18}$	`19:9` $\leqslant A_{19}$	`AGL(1, 19)` $\leqslant S_{19}$	`PSL(2,19)` $\leqslant A_{20}$
`PGL(2,19)` $\leqslant S_{20}$	`A(7)` $\leqslant A_{21}$	`PGL(3, 4)` $\leqslant A_{21}$	`PGL(2, 7)` $\leqslant S_{21}$
`S(7)` $\leqslant S_{21}$	`PGammaL(3, 4)` $\leqslant S_{21}$	`M(22)` $\leqslant A_{22}$	`M(22):2` $\leqslant S_{22}$
`23:11` $\leqslant A_{23}$	`M(23)` $\leqslant A_{23}$	`AGL(1, 23)` $\leqslant S_{23}$	`M(24)` $\leqslant A_{24}$
`PSL(2, 23)` $\leqslant A_{24}$	`PGL(2, 23)` $\leqslant S_{24}$	`ASL(2, 5):2` $\leqslant A_{25}$	`(A(5) x A(5)):2^2` $\leqslant A_{25}$
`AGL(2, 5)` $\leqslant S_{25}$	`(S(5) x S(5)):2` $\leqslant S_{25}$	`PSigmaL(2, 25)` $\leqslant A_{26}$	`PGammaL(2, 25)` $\leqslant S_{26}$
`ASL(3, 3)` $\leqslant A_{27}$	`PSp(4, 3):2` $\leqslant A_{27}$	`AGL(3, 3)` $\leqslant S_{27}$	`PGammaL(2, 8)` $\leqslant A_{28}$
`PGammaU(3, 3)` $\leqslant A_{28}$	`PSp(6, 2)` $\leqslant A_{28}$	`S(8)` $\leqslant A_{28}$	`PSL(2, 27):3` $\leqslant A_{28}$
`PGammaL(2, 27)` $\leqslant S_{28}$	`29:14` $\leqslant A_{29}$	`AGL(1, 29)` $\leqslant S_{29}$	`PSL(2,29)` $\leqslant A_{30}$
`PGL(2,29)` $\leqslant S_{30}$	`L(3, 5)` $\leqslant A_{31}$	`L(5, 2)` $\leqslant A_{31}$	`AGL(1, 31)` $\leqslant S_{31}$
`ASL(5, 2)` $\leqslant A_{32}$	`PSL(2, 31)` $\leqslant A_{32}$	`PGL(2, 31)` $\leqslant S_{32}$	`PGammaL(2, 32)` $\leqslant A_{33}$
`S(8)` $\leqslant A_{35}$	`PGammaU(3, 3)` $\leqslant A_{36}$	`PSp(4, 3):2` $\leqslant A_{36}$	`PSp(6, 2)` $\leqslant A_{36}$
`A(9)` $\leqslant A_{36}$	`S(9)` $\leqslant S_{36}$	`(S(6) x S(6)):2` $\leqslant S_{36}$	`(S(5) x S(5)):2` $\leqslant S_{36}$
`PSL(2,37)` $\leqslant A_{38}$	`PGL(2,37)` $\leqslant S_{38}$	`PSp(4, 3):2` $\leqslant A_{40}$	`PSp(4, 3)` $\leqslant A_{40}$
`PSL(4, 3)` $\leqslant A_{40}$	`PSp(4, 3):2` $\leqslant S_{40}$	`PGL(4, 3)` $\leqslant S_{40}$	`PSL(2,41)` $\leqslant A_{42}$
`PGL(2,41)` $\leqslant S_{42}$	`PSL(2,43)` $\leqslant A_{44}$	`PGL(2,43)` $\leqslant S_{44}$	`S(10)` $\leqslant A_{45}$
`PSp(4, 3):2` $\leqslant S_{45}$	`(A(7) x A(7)):4` $\leqslant A_{49}$	`(S(7) x S(7)):2` $\leqslant S_{49}$	`PSU(3, 5):2` $\leqslant A_{50}$
`Alt(11)` $\leqslant A_{55}$	`Sym(11)` $\leqslant S_{55}$	`PSL(3, 7).3` $\leqslant A_{57}$	`PSL(6, 2)` $\leqslant A_{63}$
`AGL(6, 2)` $\leqslant A_{64}$	`Sym(8) wreath Sym(2)` $\leqslant A_{64}$	`Sym(12)` $\leqslant A_{66}$	`Sym(13)` $\leqslant S_{78}$
`Alt(9)^2.2^2` $\leqslant A_{81}$	`Sym(9) wreath Sym(2)` $\leqslant S_{81}$	`Sym(10) wreath Sym(2)` $\leqslant S_{100}$	

Appendix B

The following[5] Table 2 contains the values of u and $a(u)$ where $u = |G : H|$, G and H are as in Appendix A, and $\deg G = \deg H \geqslant 14$. One can read off that in each of the cases $a(u)^3 > u$, which implies that G cannot be an automorphism group of a flag-transitive generalized hexagon with point stabilizer H. See the proof of Theorem 5.1.

Table 2

PSL(2,13) $\leqslant$ A$_{14}$	PGL(2,13) $\leqslant$ S$_{14}$	PSL(4, 2) $\leqslant$ A$_{15}$	2^4.PSL(4, 2) $\leqslant$ A$_{16}$
$a(u)^3 \approx 6.86 \cdot 10^{18}$	$a(u)^3 \approx 6.86 \cdot 10^{18}$	$a(u)^3 \approx 1.67 \cdot 10^{15}$	$a(u)^3 \approx 1.67 \cdot 10^{15}$
$u \approx 3.99 \cdot 10^7$	$u \approx 3.99 \cdot 10^7$	$u \approx 3.24 \cdot 10^7$	$u \approx 3.24 \cdot 10^7$
17:8 $\leqslant$ A$_{17}$	L(2, 2^4):4 = PGammaL(2, 2^4) $\leqslant$ A$_{17}$	AGL(1, 17) $\leqslant$ S$_{17}$	PSL(2,17) $\leqslant$ A$_{18}$
$a(u)^3 \approx 3.20 \cdot 10^{26}$	$a(u)^3 \approx 1.85 \cdot 10^{20}$	$a(u)^3 \approx 3.20 \cdot 10^{26}$	$a(u)^3 \approx 3.20 \cdot 10^{26}$
$u \approx 1.30 \cdot 10^{12}$	$u \approx 1.08 \cdot 10^{10}$	$u \approx 1.30 \cdot 10^{12}$	$u \approx 1.30 \cdot 10^{12}$
PGL(2,17) $\leqslant$ S$_{18}$	19:9 $\leqslant$ A$_{19}$	AGL(1, 19) $\leqslant$ S$_{19}$	PSL(2,19) $\leqslant$ A$_{20}$
$a(u)^3 \approx 3.20 \cdot 10^{26}$	$a(u)^3 \approx 6.44 \cdot 10^{33}$	$a(u)^3 \approx 6.44 \cdot 10^{33}$	$a(u)^3 \approx 6.44 \cdot 10^{33}$
$u \approx 1.30 \cdot 10^{12}$	$u \approx 3.55 \cdot 10^{14}$	$u \approx 3.55 \cdot 10^{14}$	$u \approx 3.55 \cdot 10^{14}$
PGL(2,19) $\leqslant$ S$_{20}$	A(7) $\leqslant$ A$_{21}$	PGL(3, 4) $\leqslant$ A$_{21}$	PGL(2, 7) $\leqslant$ S$_{21}$
$a(u)^3 \approx 6.44 \cdot 10^{33}$	$a(u)^3 \approx 2.17 \cdot 10^{34}$	$a(u)^3 \approx 1.57 \cdot 10^{30}$	$a(u)^3 \approx 7.34 \cdot 10^{37}$
$u \approx 3.55 \cdot 10^{14}$	$u \approx 1.01 \cdot 10^{16}$	$u \approx 4.22 \cdot 10^{14}$	$u \approx 1.52 \cdot 10^{17}$
S(7) $\leqslant$ S$_{21}$	PGammaL(3, 4) $\leqslant$ S$_{21}$	M(22) $\leqslant$ A$_{22}$	M(22):2 $\leqslant$ S$_{22}$
$a(u)^3 \approx 2.17 \cdot 10^{34}$	$a(u)^3 \approx 1.57 \cdot 10^{30}$	$a(u)^3 \approx 4.25 \cdot 10^{31}$	$a(u)^3 \approx 4.25 \cdot 10^{31}$
$u \approx 1.01 \cdot 10^{16}$	$u \approx 4.22 \cdot 10^{14}$	$u \approx 1.26 \cdot 10^{15}$	$u \approx 1.26 \cdot 10^{15}$
23:11 $\leqslant$ A$_{23}$	M(23) $\leqslant$ A$_{23}$	AGL(1, 23) $\leqslant$ S$_{23}$	M(24) $\leqslant$ A$_{24}$
$a(u)^3 \approx 8.12 \cdot 10^{42}$	$a(u)^3 \approx 4.25 \cdot 10^{31}$	$a(u)^3 \approx 8.12 \cdot 10^{42}$	$a(u)^3 \approx 4.25 \cdot 10^{31}$
$u \approx 5.10 \cdot 10^{19}$	$u \approx 1.26 \cdot 10^{15}$	$u \approx 5.10 \cdot 10^{19}$	$u \approx 1.26 \cdot 10^{15}$
PSL(2, 23) $\leqslant$ A$_{24}$	PGL(2, 23) $\leqslant$ S$_{24}$	ASL(2, 5):2 $\leqslant$ A$_{25}$	(A(5) x A(5)):2^2 $\leqslant$ A$_{25}$
$a(u)^3 \approx 8.12 \cdot 10^{42}$	$a(u)^3 \approx 8.12 \cdot 10^{42}$	$a(u)^3 \approx 1.31 \cdot 10^{47}$	$a(u)^3 \approx 9.51 \cdot 10^{45}$
$u \approx 5.10 \cdot 10^{19}$	$u \approx 5.10 \cdot 10^{19}$	$u \approx 1.29 \cdot 10^{21}$	$u \approx 5.38 \cdot 10^{20}$
AGL(2, 5) $\leqslant$ S$_{25}$	(S(5) x S(5)):2 $\leqslant$ S$_{25}$	PSigmaL(2, 25) $\leqslant$ A$_{26}$	PGammaL(2, 25) $\leqslant$ S$_{26}$
$a(u)^3 \approx 1.31 \cdot 10^{47}$	$a(u)^3 \approx 9.51 \cdot 10^{45}$	$a(u)^3 \approx 1.31 \cdot 10^{50}$	$a(u)^3 \approx 1.31 \cdot 10^{50}$
$u \approx 1.29 \cdot 10^{21}$	$u \approx 5.38 \cdot 10^{20}$	$u \approx 1.29 \cdot 10^{22}$	$u \approx 1.29 \cdot 10^{22}$
ASL(3, 3) $\leqslant$ A$_{27}$	PSp(4, 3):2 $\leqslant$ A$_{27}$	AGL(3, 3) $\leqslant$ S$_{27}$	PGammaL(2, 8) $\leqslant$ A$_{28}$
$a(u)^3 \approx 2.81 \cdot 10^{51}$	$a(u)^3 \approx 3.21 \cdot 10^{49}$	$a(u)^3 \approx 2.81 \cdot 10^{51}$	$a(u)^3 \approx 2.84 \cdot 10^{58}$
$u \approx 3.59 \cdot 10^{22}$	$u \approx 1.05 \cdot 10^{23}$	$u \approx 3.59 \cdot 10^{22}$	$u \approx 1.00 \cdot 10^{26}$
PGammaU(3, 3) $\leqslant$ A$_{28}$	PSp(6, 2) $\leqslant$ A$_{28}$	S(8) $\leqslant$ A$_{28}$	PSL(2, 27):3 $\leqslant$ A$_{28}$
$a(u)^3 \approx 5.54 \cdot 10^{55}$	$a(u)^3 \approx 3.21 \cdot 10^{49}$	$a(u)^3 \approx 1.49 \cdot 10^{54}$	$a(u)^3 \approx 8.41 \cdot 10^{57}$
$u \approx 1.26 \cdot 10^{25}$	$u \approx 1.05 \cdot 10^{23}$	$u \approx 3.78 \cdot 10^{24}$	$u \approx 5.17 \cdot 10^{24}$
PGammaL(2, 27) $\leqslant$ S$_{28}$	29:14 $\leqslant$ A$_{29}$	AGL(1, 29) $\leqslant$ S$_{29}$	PSL(2,29) $\leqslant$ A$_{30}$
$a(u)^3 \approx 8.41 \cdot 10^{57}$	$a(u)^3 \approx 3.57 \cdot 10^{64}$	$a(u)^3 \approx 3.57 \cdot 10^{64}$	$a(u)^3 \approx 3.57 \cdot 10^{64}$
$u \approx 5.17 \cdot 10^{24}$	$u \approx 1.08 \cdot 10^{28}$	$u \approx 1.08 \cdot 10^{28}$	$u \approx 1.08 \cdot 10^{28}$

Table 2 (continued)

PGL(2,29) $\leqslant$ S$_{30}$	L(3, 5) $\leqslant$ A$_{31}$	L(5, 2) $\leqslant$ A$_{31}$	AGL(1, 31) $\leqslant$ S$_{31}$
$a(u)^3 \approx 3.57 \cdot 10^{64}$	$a(u)^3 \approx 1.09 \cdot 10^{62}$	$a(u)^3 \approx 1.92 \cdot 10^{60}$	$a(u)^3 \approx 5.58 \cdot 10^{70}$
$u \approx 1.08 \cdot 10^{28}$	$u \approx 1.10 \cdot 10^{28}$	$u \approx 4.11 \cdot 10^{26}$	$u \approx 8.84 \cdot 10^{30}$
ASL(5, 2) $\leqslant$ A$_{32}$	PSL(2, 31) $\leqslant$ A$_{32}$	PGL(2, 31) $\leqslant$ S$_{32}$	PGammaL(2, 32) $\leqslant$ A$_{33}$
$a(u)^3 \approx 1.92 \cdot 10^{60}$	$a(u)^3 \approx 5.58 \cdot 10^{70}$	$a(u)^3 \approx 5.58 \cdot 10^{70}$	$a(u)^3 \approx 1.50 \cdot 10^{72}$
$u \approx 4.11 \cdot 10^{26}$	$u \approx 8.84 \cdot 10^{30}$	$u \approx 8.84 \cdot 10^{30}$	$u \approx 2.65 \cdot 10^{31}$
S(8) $\leqslant$ A$_{35}$	PGammaU(3, 3) $\leqslant$ A$_{36}$	PSp(4, 3):2 $\leqslant$ A$_{36}$	PSp(6, 2) $\leqslant$ A$_{36}$
$a(u)^3 \approx 5.70 \cdot 10^{78}$	$a(u)^3 \approx 9.86 \cdot 10^{84}$	$a(u)^3 \approx 3.65 \cdot 10^{80}$	$a(u)^3 \approx 5.70 \cdot 10^{78}$
$u \approx 1.28 \cdot 10^{35}$	$u \approx 1.53 \cdot 10^{37}$	$u \approx 3.58 \cdot 10^{36}$	$u \approx 1.28 \cdot 10^{35}$
A(9) $\leqslant$ A$_{36}$	S(9) $\leqslant$ S$_{36}$	(S(6) x S(6)):2 $\leqslant$ S$_{36}$	(S(5) x S(5)):2 $\leqslant$ S$_{36}$
$a(u)^3 \approx 2.92 \cdot 10^{81}$	$a(u)^3 \approx 2.92 \cdot 10^{81}$	$a(u)^3 \approx 3.65 \cdot 10^{77}$	$a(u)^3 \approx 1.70 \cdot 10^{82}$
$u \approx 1.02 \cdot 10^{36}$	$u \approx 1.02 \cdot 10^{36}$	$u \approx 3.58 \cdot 10^{35}$	$u \approx 1.29 \cdot 10^{37}$
PSL(2,37) $\leqslant$ A$_{38}$	PGL(2,37) $\leqslant$ S$_{38}$	PSp(4, 3):2 $\leqslant$ A$_{40}$	PSp(4, 3) $\leqslant$ A$_{40}$
$a(u)^3 \approx 8.72 \cdot 10^{90}$	$a(u)^3 \approx 8.72 \cdot 10^{90}$	$a(u)^3 \approx 5.05 \cdot 10^{87}$	$a(u)^3 \approx 4.04 \cdot 10^{88}$
$u \approx 1.03 \cdot 10^{40}$	$u \approx 1.03 \cdot 10^{40}$	$u \approx 7.86 \cdot 10^{42}$	$u \approx 1.57 \cdot 10^{43}$
PSL(4, 3) $\leqslant$ A$_{40}$	PSp(4, 3):2 $\leqslant$ S$_{40}$	PGL(4, 3) $\leqslant$ S$_{40}$	PSL(2,41) $\leqslant$ A$_{42}$
$a(u)^3 \approx 6.92 \cdot 10^{84}$	$a(u)^3 \approx 4.04 \cdot 10^{88}$	$a(u)^3 \approx 6.92 \cdot 10^{84}$	$a(u)^3 \approx 8.79 \cdot 10^{97}$
$u \approx 6.72 \cdot 10^{40}$	$u \approx 1.57 \cdot 10^{43}$	$u \approx 6.72 \cdot 10^{40}$	$u \approx 2.03 \cdot 10^{46}$
PGL(2,41) $\leqslant$ S$_{42}$	PSL(2,43) $\leqslant$ A$_{44}$	PGL(2,43) $\leqslant$ S$_{44}$	S(10) $\leqslant$ A$_{45}$
$a(u)^3 \approx 8.79 \cdot 10^{97}$	$a(u)^3 \approx 3.87 \cdot 10^{107}$	$a(u)^3 \approx 3.87 \cdot 10^{107}$	$a(u)^3 \approx 5.83 \cdot 10^{101}$
$u \approx 2.03 \cdot 10^{46}$	$u \approx 3.34 \cdot 10^{49}$	$u \approx 3.34 \cdot 10^{49}$	$u \approx 1.64 \cdot 10^{49}$
PSp(4, 3):2 $\leqslant$ S$_{45}$	(A(7) x A(7)):4 $\leqslant$ A$_{49}$	(S(7) x S(7)):2 $\leqslant$ S$_{49}$	PSU(3, 5):2 $\leqslant$ A$_{50}$
$a(u)^3 \approx 4.66 \cdot 10^{105}$	$a(u)^3 \approx 6.52 \cdot 10^{116}$	$a(u)^3 \approx 6.52 \cdot 10^{116}$	$a(u)^3 \approx 2.43 \cdot 10^{125}$
$u \approx 2.30 \cdot 10^{51}$	$u \approx 1.19 \cdot 10^{55}$	$u \approx 1.19 \cdot 10^{55}$	$u \approx 6.03 \cdot 10^{58}$
Alt(11) $\leqslant$ A$_{55}$	Sym(11) $\leqslant$ S$_{55}$	PSL(3, 7).3 $\leqslant$ A$_{57}$	PSL(6, 2) $\leqslant$ A$_{63}$
$a(u)^3 \approx 1.62 \cdot 10^{142}$	$a(u)^3 \approx 1.62 \cdot 10^{142}$	$a(u)^3 \approx 8.06 \cdot 10^{156}$	$a(u)^3 \approx 1.12 \cdot 10^{164}$
$u \approx 3.18 \cdot 10^{65}$	$u \approx 3.18 \cdot 10^{65}$	$u \approx 3.59 \cdot 10^{69}$	$u \approx 4.91 \cdot 10^{76}$
AGL(6, 2) $\leqslant$ A$_{64}$	Sym(8) wreath Sym(2) $\leqslant$ A$_{64}$	Sym(12) $\leqslant$ A$_{66}$	Sym(13) $\leqslant$ S$_{78}$
$a(u)^3 \approx 1.12 \cdot 10^{164}$	$a(u)^3 \approx 2.35 \cdot 10^{167}$	$a(u)^3 \approx 7.71 \cdot 10^{174}$	$a(u)^3 \approx 5.28 \cdot 10^{214}$
$u \approx 4.91 \cdot 10^{76}$	$u \approx 1.95 \cdot 10^{79}$	$u \approx 5.68 \cdot 10^{83}$	$u \approx 1.81 \cdot 10^{105}$
Alt(9)^2.2^2 $\leqslant$ A$_{81}$	Sym(9) wreath Sym(2) $\leqslant$ S$_{81}$	Sym(10) wreath Sym(2) $\leqslant$ S$_{100}$	
$a(u)^3 \approx 2.97 \cdot 10^{220}$	$a(u)^3 \approx 2.97 \cdot 10^{220}$	$a(u)^3 \approx 2.74 \cdot 10^{293}$	
$u \approx 2.20 \cdot 10^{109}$	$u \approx 2.20 \cdot 10^{109}$	$u \approx 3.54 \cdot 10^{144}$	

Appendix C

The following Table 3 contains the values of u and $b(u)$, where $u = |G : H|$, G and H are as in Appendix A, and $\deg G = \deg H \geqslant 14$. One can read off that in each of the cases $b(u)^2 > u$, which implies that G cannot be an automorphism group of a flag-transitive generalized octagon with point stabilizer H. See the proof of Theorem 5.1.

Table 3

$\texttt{PSL(2,13)} \leqslant \mathsf{A}_{14}$	$\texttt{PGL(2,13)} \leqslant \mathsf{S}_{14}$	$\texttt{PSL(4, 2)} \leqslant \mathsf{A}_{15}$	$\texttt{2\^{}4.PSL(4, 2)} \leqslant \mathsf{A}_{16}$
$b(u)^2 \approx 3.61 \cdot 10^{12}$	$b(u)^2 \approx 3.61 \cdot 10^{12}$	$b(u)^2 \approx 1.41 \cdot 10^{10}$	$b(u)^2 \approx 1.41 \cdot 10^{10}$
$u \approx 3.99 \cdot 10^7$	$u \approx 3.99 \cdot 10^7$	$u \approx 3.24 \cdot 10^7$	$u \approx 3.24 \cdot 10^7$
$\texttt{17:8} \leqslant \mathsf{A}_{17}$	$\texttt{L(2, 2\^{}4):4 = PGammaL(2, 2\^{}4)} \leqslant \mathsf{A}_{17}$	$\texttt{AGL(1, 17)} \leqslant \mathsf{S}_{17}$	$\texttt{PSL(2,17)} \leqslant \mathsf{A}_{18}$
$b(u)^2 \approx 4.68 \cdot 10^{17}$	$b(u)^2 \approx 3.25 \cdot 10^{13}$	$b(u)^2 \approx 4.68 \cdot 10^{17}$	$b(u)^2 \approx 4.68 \cdot 10^{17}$
$u \approx 1.30 \cdot 10^{12}$	$u \approx 1.08 \cdot 10^{10}$	$u \approx 1.30 \cdot 10^{12}$	$u \approx 1.30 \cdot 10^{12}$
$\texttt{PGL(2,17)} \leqslant \mathsf{S}_{18}$	$\texttt{19:9} \leqslant \mathsf{A}_{19}$	$\texttt{AGL(1, 19)} \leqslant \mathsf{S}_{19}$	$\texttt{PSL(2,19)} \leqslant \mathsf{A}_{20}$
$b(u)^2 \approx 4.68 \cdot 10^{17}$	$b(u)^2 \approx 3.46 \cdot 10^{22}$	$b(u)^2 \approx 3.46 \cdot 10^{22}$	$b(u)^2 \approx 3.46 \cdot 10^{22}$
$u \approx 1.30 \cdot 10^{12}$	$u \approx 3.55 \cdot 10^{14}$	$u \approx 3.55 \cdot 10^{14}$	$u \approx 3.55 \cdot 10^{14}$
$\texttt{PGL(2,19)} \leqslant \mathsf{S}_{20}$	$\texttt{A(7)} \leqslant \mathsf{A}_{21}$	$\texttt{PGL(3, 4)} \leqslant \mathsf{A}_{21}$	$\texttt{PGL(2, 7)} \leqslant \mathsf{S}_{21}$
$b(u)^2 \approx 3.46 \cdot 10^{22}$	$b(u)^2 \approx 7.79 \cdot 10^{22}$	$b(u)^2 \approx 1.35 \cdot 10^{20}$	$b(u)^2 \approx 1.75 \cdot 10^{25}$
$u \approx 3.55 \cdot 10^{14}$	$u \approx 1.01 \cdot 10^{16}$	$u \approx 4.22 \cdot 10^{14}$	$u \approx 1.52 \cdot 10^{17}$
$\texttt{S(7)} \leqslant \mathsf{S}_{21}$	$\texttt{PGammaL(3, 4)} \leqslant \mathsf{S}_{21}$	$\texttt{M(22)} \leqslant \mathsf{A}_{22}$	$\texttt{M(22):2} \leqslant \mathsf{S}_{22}$
$b(u)^2 \approx 7.79 \cdot 10^{22}$	$b(u)^2 \approx 1.35 \cdot 10^{20}$	$b(u)^2 \approx 1.21 \cdot 10^{21}$	$b(u)^2 \approx 1.21 \cdot 10^{21}$
$u \approx 1.01 \cdot 10^{16}$	$u \approx 4.22 \cdot 10^{14}$	$u \approx 1.26 \cdot 10^{15}$	$u \approx 1.26 \cdot 10^{15}$
$\texttt{23:11} \leqslant \mathsf{A}_{23}$	$\texttt{M(23)} \leqslant \mathsf{A}_{23}$	$\texttt{AGL(1, 23)} \leqslant \mathsf{S}_{23}$	$\texttt{M(24)} \leqslant \mathsf{A}_{24}$
$b(u)^2 \approx 4.04 \cdot 10^{28}$	$b(u)^2 \approx 1.21 \cdot 10^{21}$	$b(u)^2 \approx 4.04 \cdot 10^{28}$	$b(u)^2 \approx 1.21 \cdot 10^{21}$
$u \approx 5.10 \cdot 10^{19}$	$u \approx 1.26 \cdot 10^{15}$	$u \approx 5.10 \cdot 10^{19}$	$u \approx 1.26 \cdot 10^{15}$
$\texttt{PSL(2, 23)} \leqslant \mathsf{A}_{24}$	$\texttt{PGL(2, 23)} \leqslant \mathsf{S}_{24}$	$\texttt{ASL(2, 5):2} \leqslant \mathsf{A}_{25}$	$\texttt{(A(5) x A(5)):2\^{}2} \leqslant \mathsf{A}_{25}$
$b(u)^2 \approx 4.04 \cdot 10^{28}$	$b(u)^2 \approx 4.04 \cdot 10^{28}$	$b(u)^2 \approx 2.58 \cdot 10^{31}$	$b(u)^2 \approx 4.49 \cdot 10^{30}$
$u \approx 5.10 \cdot 10^{19}$	$u \approx 5.10 \cdot 10^{19}$	$u \approx 1.29 \cdot 10^{21}$	$u \approx 5.38 \cdot 10^{20}$
$\texttt{AGL(2, 5)} \leqslant \mathsf{S}_{25}$	$\texttt{(S(5) x S(5)):2} \leqslant \mathsf{S}_{25}$	$\texttt{PSigmaL(2, 25)} \leqslant \mathsf{A}_{26}$	$\texttt{PGammaL(2, 25)} \leqslant \mathsf{S}_{26}$
$b(u)^2 \approx 2.58 \cdot 10^{31}$	$b(u)^2 \approx 4.49 \cdot 10^{30}$	$b(u)^2 \approx 2.58 \cdot 10^{33}$	$b(u)^2 \approx 2.58 \cdot 10^{33}$
$u \approx 1.29 \cdot 10^{21}$	$u \approx 5.38 \cdot 10^{20}$	$u \approx 1.29 \cdot 10^{22}$	$u \approx 1.29 \cdot 10^{22}$
$\texttt{ASL(3, 3)} \leqslant \mathsf{A}_{27}$	$\texttt{PSp(4, 3):2} \leqslant \mathsf{A}_{27}$	$\texttt{AGL(3, 3)} \leqslant \mathsf{S}_{27}$	$\texttt{PGammaL(2, 8)} \leqslant \mathsf{A}_{28}$
$b(u)^2 \approx 1.99 \cdot 10^{34}$	$b(u)^2 \approx 1.01 \cdot 10^{33}$	$b(u)^2 \approx 1.99 \cdot 10^{34}$	$b(u)^2 \approx 9.31 \cdot 10^{38}$
$u \approx 3.59 \cdot 10^{22}$	$u \approx 1.05 \cdot 10^{23}$	$u \approx 3.59 \cdot 10^{22}$	$u \approx 1.00 \cdot 10^{26}$
$\texttt{PGammaU(3, 3)} \leqslant \mathsf{A}_{28}$	$\texttt{PSp(6, 2)} \leqslant \mathsf{A}_{28}$	$\texttt{S(8)} \leqslant \mathsf{A}_{28}$	$\texttt{PSL(2, 27):3} \leqslant \mathsf{A}_{28}$
$b(u)^2 \approx 1.45 \cdot 10^{37}$	$b(u)^2 \approx 1.01 \cdot 10^{33}$	$b(u)^2 \approx 1.30 \cdot 10^{36}$	$b(u)^2 \approx 4.13 \cdot 10^{38}$
$u \approx 1.26 \cdot 10^{25}$	$u \approx 1.05 \cdot 10^{23}$	$u \approx 3.78 \cdot 10^{24}$	$u \approx 5.17 \cdot 10^{24}$
$\texttt{PGammaL(2, 27)} \leqslant \mathsf{S}_{28}$	$\texttt{29:14} \leqslant \mathsf{A}_{29}$	$\texttt{AGL(1, 29)} \leqslant \mathsf{S}_{29}$	$\texttt{PSL(2,29)} \leqslant \mathsf{A}_{30}$
$b(u)^2 \approx 4.13 \cdot 10^{38}$	$b(u)^2 \approx 1.08 \cdot 10^{43}$	$b(u)^2 \approx 1.08 \cdot 10^{43}$	$b(u)^2 \approx 1.08 \cdot 10^{43}$
$u \approx 5.17 \cdot 10^{24}$	$u \approx 1.08 \cdot 10^{28}$	$u \approx 1.08 \cdot 10^{28}$	$u \approx 1.08 \cdot 10^{28}$

Table 3 (continued)

PGL(2,29) $\leqslant$ S$_{30}$	L(3, 5) $\leqslant$ A$_{31}$	L(5, 2) $\leqslant$ A$_{31}$	AGL(1, 31) $\leqslant$ S$_{31}$
$b(u)^2 \approx 1.08 \cdot 10^{43}$	$b(u)^2 \approx 2.28 \cdot 10^{41}$	$b(u)^2 \approx 1.54 \cdot 10^{40}$	$b(u)^2 \approx 1.46 \cdot 10^{47}$
$u \approx 1.08 \cdot 10^{28}$	$u \approx 1.10 \cdot 10^{28}$	$u \approx 4.11 \cdot 10^{26}$	$u \approx 8.84 \cdot 10^{30}$
ASL(5, 2) $\leqslant$ A$_{32}$	PSL(2, 31) $\leqslant$ A$_{32}$	PGL(2, 31) $\leqslant$ S$_{32}$	PGammaL(2, 32) $\leqslant$ A$_{33}$
$b(u)^2 \approx 1.54 \cdot 10^{40}$	$b(u)^2 \approx 1.46 \cdot 10^{47}$	$b(u)^2 \approx 1.46 \cdot 10^{47}$	$b(u)^2 \approx 1.31 \cdot 10^{48}$
$u \approx 4.11 \cdot 10^{26}$	$u \approx 8.84 \cdot 10^{30}$	$u \approx 8.84 \cdot 10^{30}$	$u \approx 2.65 \cdot 10^{31}$
S(8) $\leqslant$ A$_{35}$	PGammaU(3, 3) $\leqslant$ A$_{36}$	PSp(4, 3):2 $\leqslant$ A$_{36}$	PSp(6, 2) $\leqslant$ A$_{36}$
$b(u)^2 \approx 3.19 \cdot 10^{52}$	$b(u)^2 \approx 4.59 \cdot 10^{56}$	$b(u)^2 \approx 5.11 \cdot 10^{53}$	$b(u)^2 \approx 3.19 \cdot 10^{52}$
$u \approx 1.28 \cdot 10^{35}$	$u \approx 1.53 \cdot 10^{37}$	$u \approx 3.58 \cdot 10^{36}$	$u \approx 1.28 \cdot 10^{35}$
A(9) $\leqslant$ A$_{36}$	S(9) $\leqslant$ S$_{36}$	(S(6) x S(6)):2 $\leqslant$ S$_{36}$	(S(5) x S(5)):2 $\leqslant$ S$_{36}$
$b(u)^2 \approx 2.04 \cdot 10^{54}$	$b(u)^2 \approx 2.04 \cdot 10^{54}$	$b(u)^2 \approx 5.11 \cdot 10^{51}$	$b(u)^2 \approx 6.62 \cdot 10^{54}$
$u \approx 1.02 \cdot 10^{36}$	$u \approx 1.02 \cdot 10^{36}$	$u \approx 3.58 \cdot 10^{35}$	$u \approx 1.29 \cdot 10^{37}$
PSL(2,37) $\leqslant$ A$_{38}$	PGL(2,37) $\leqslant$ S$_{38}$	PSp(4, 3):2 $\leqslant$ A$_{40}$	PSp(4, 3) $\leqslant$ A$_{40}$
$b(u)^2 \approx 4.23 \cdot 10^{60}$	$b(u)^2 \approx 4.23 \cdot 10^{60}$	$b(u)^2 \approx 2.94 \cdot 10^{58}$	$b(u)^2 \approx 1.17 \cdot 10^{59}$
$u \approx 1.03 \cdot 10^{40}$	$u \approx 1.03 \cdot 10^{40}$	$u \approx 7.86 \cdot 10^{42}$	$u \approx 1.57 \cdot 10^{43}$
PSL(4, 3) $\leqslant$ A$_{40}$	PSp(4, 3):2 $\leqslant$ S$_{40}$	PGL(4, 3) $\leqslant$ S$_{40}$	PSL(2,41) $\leqslant$ A$_{42}$
$b(u)^2 \approx 3.63 \cdot 10^{56}$	$b(u)^2 \approx 1.17 \cdot 10^{59}$	$b(u)^2 \approx 3.63 \cdot 10^{56}$	$b(u)^2 \approx 1.97 \cdot 10^{65}$
$u \approx 6.72 \cdot 10^{40}$	$u \approx 1.57 \cdot 10^{43}$	$u \approx 6.72 \cdot 10^{40}$	$u \approx 2.03 \cdot 10^{46}$
PGL(2,41) $\leqslant$ S$_{42}$	PSL(2,43) $\leqslant$ A$_{44}$	PGL(2,43) $\leqslant$ S$_{44}$	S(10) $\leqslant$ A$_{45}$
$b(u)^2 \approx 1.97 \cdot 10^{65}$	$b(u)^2 \approx 5.31 \cdot 10^{71}$	$b(u)^2 \approx 5.31 \cdot 10^{71}$	$b(u)^2 \approx 6.98 \cdot 10^{67}$
$u \approx 2.03 \cdot 10^{46}$	$u \approx 3.34 \cdot 10^{49}$	$u \approx 3.34 \cdot 10^{49}$	$u \approx 1.64 \cdot 10^{49}$
PSp(4, 3):2 $\leqslant$ S$_{45}$	(A(7) x A(7)):4 $\leqslant$ A$_{49}$	(S(7) x S(7)):2 $\leqslant$ S$_{49}$	PSU(3, 5):2 $\leqslant$ A$_{50}$
$b(u)^2 \approx 2.79 \cdot 10^{70}$	$b(u)^2 \approx 7.52 \cdot 10^{77}$	$b(u)^2 \approx 7.52 \cdot 10^{77}$	$b(u)^2 \approx 3.89 \cdot 10^{83}$
$u \approx 2.30 \cdot 10^{51}$	$u \approx 1.19 \cdot 10^{55}$	$u \approx 1.19 \cdot 10^{55}$	$u \approx 6.03 \cdot 10^{58}$
Alt(11) $\leqslant$ A$_{55}$	Sym(11) $\leqslant$ S$_{55}$	PSL(3, 7).3 $\leqslant$ A$_{57}$	PSL(6, 2) $\leqslant$ A$_{63}$
$b(u)^2 \approx 6.40 \cdot 10^{94}$	$b(u)^2 \approx 6.40 \cdot 10^{94}$	$b(u)^2 \approx 4.02 \cdot 10^{104}$	$b(u)^2 \approx 2.32 \cdot 10^{109}$
$u \approx 3.18 \cdot 10^{65}$	$u \approx 3.18 \cdot 10^{65}$	$u \approx 3.59 \cdot 10^{69}$	$u \approx 4.91 \cdot 10^{76}$
AGL(6, 2) $\leqslant$ A$_{64}$	Sym(8) wreath Sym(2) $\leqslant$ A$_{64}$	Sym(12) $\leqslant$ A$_{66}$	Sym(13) $\leqslant$ S$_{78}$
$b(u)^2 \approx 2.32 \cdot 10^{109}$	$b(u)^2 \approx 3.81 \cdot 10^{111}$	$b(u)^2 \approx 3.90 \cdot 10^{116}$	$b(u)^2 \approx 1.40 \cdot 10^{143}$
$u \approx 4.91 \cdot 10^{76}$	$u \approx 1.95 \cdot 10^{79}$	$u \approx 5.68 \cdot 10^{83}$	$u \approx 1.81 \cdot 10^{105}$
Alt(9)^2.2^2 $\leqslant$ A$_{81}$	Sym(9) wreath Sym(2) $\leqslant$ S$_{81}$	Sym(10) wreath Sym(2) $\leqslant$ S$_{100}$	
$b(u)^2 \approx 9.59 \cdot 10^{146}$	$b(u)^2 \approx 9.59 \cdot 10^{146}$	$b(u)^2 \approx 4.22 \cdot 10^{195}$	
$u \approx 2.20 \cdot 10^{109}$	$u \approx 2.20 \cdot 10^{109}$	$u \approx 3.54 \cdot 10^{144}$	

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