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Model simplification by asymptotic order of magnitude reasoning

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Abstract

One of the hardest problems in reasoning about a physical system is finding an approximate model that is mathematically tractable and yet captures the essence of the problem. This paper describes an implemented program AOM which automates a powerful simplification method. AOM is based on two domain-independent ideas: self-consistent approximations and asymptotic order of magnitude reasoning. The basic operation of AOM consists of five steps: (1) assign order of magnitude estimates to terms in the equations, (2) find maximal terms of each equation, i.e., terms that are not dominated by any other terms in the same equation, (3) consider all possible n-term dominant balance assumptions, (4) propagate the effects of the balance assumptions, and (5) remove partial models based on inconsistent balance assumptions. AOM also exploits constraints among equations and submodels. We demonstrate its power by showing how the program simplifies difficult fluid models described by coupled nonlinear partial differential equations with several parameters. We believe the derivation given by AOM is more systematic and easily understandable than those given in published papers.

1. Introduction

Model simplification—the derivation of simpler equations from more general ones—is a recurrent problem in many areas of science and engineering. Few physical theories can be matched directly with experiments. To an experimenter wondering why a certain value is got for a measurement, say, why water boils at about 100°C under ordinary conditions, a complicated system of equations describing 10^{23} molecules, interacting with one another with a complicated force law, governed by the fundamental laws of quantum statistical mechanics would not be exactly helpful. The real problem is to reduce these equations, so complex and general, with so many variables linked together to a form responsive to analysis and interpretation. Approximate models are not just

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useful; they are essential. Without it, we cannot make predictions. With it, we can guide experiments and assess numerical results.

Professional scientists simplify problems by all sorts of methods. Among them are dimensional and order of magnitude estimates, exploitation of small parameters, and symmetry considerations. These methods and others besides, which belong to the "bag of tricks" of a skilled practitioner, are seldom articulated in a systematic, constructive manner that beginners can apply directly in scientific works. To simplify a model the scientist must exercise judgement in choices of what idealizations or approximations to make. Making such judgement often requires an understanding of the gross features of the solution, knowledge of the relative importance of terms in the model, and consideration of limiting cases. The purpose of this paper is to demonstrate how this kind of knowledge can be embodied in a computer program to tackle the difficult problem of model approximation in fluid dynamics.

The implemented program, called AOM, is based on two domain-independent ideas: self-consistent approximations, and asymptotic order of magnitude reasoning. Given a system of fluid equations, such as the Navier–Stokes equations, and a few order of magnitude estimates for the variables in the equations, AOM finds all the simplified equations consistent with the problem specification. The spirit of the analysis is heuristic and exploratory. We will not be able to prove that the simplified equations have strict validity. Rather these equations are the only ones which could possibly be valid.

Fluid dynamics is a good domain for testing simplification ideas for three reasons. First, the domain is extremely important. Knowledge of fluid dynamics is critical for the solutions of many scientific and technological problems—from life in moving fluids, to drag on ship hulls, to heat transfer in reentering spacecrafts, to motion of air masses, and to evolution of galaxies. Second, the fluid equations are in general systems of coupled nonlinear partial differential equations, which present enormous analytical and numerical difficulties. So methods of simplification are likely to have large payoff. Third, the domain is appropriate because a list of identifiable, significant approximations—important for the development of that subject—has accumulated over the years.

Despite the appearance of the program as an application of AI techniques to a specialized domain, we want to stress our more general concerns for this line of research:

- To study the nature of scientific reasoning as practiced in normal science. We would like to codify some of the skills that professionals have in formulating problems, making approximations, explaining data, and testing theories.
- To solve real problems in an area of significance to modern science.
- To provide scientists with an intelligent workbench consisting of a library of powerful heuristic and qualitative methods.

Our work is thus rooted in the tradition of focusing on the problem solving behavior of articulate professionals in well-structured domains and formalizing their methods so that a computer can exhibit similar behavior on similar problems. Works with a similar intent range from the early expert systems (such as Slagle's SAINT, Moses' SIN, and Dendral), to the engineering problem solving project and dynamicist workbench in MIT [1,5,33], and to the recent researches in qualitative reasoning [9].

This work is closely related to the researches concerning automatic model generation and order of magnitude reasoning. Raiman introduces *order of magnitude scales* to ex-

tend the power of qualitative algebra [20]. Weld explores related ideas in a technique called *exaggeration* in the context of comparative analysis [27]. Mavrovouniotis and Stephanopoulos combine numerical and symbolic order of magnitude relations in analyzing chemical processes [14]. Raiman and Williams explore the use of simplifying assumptions in finding dominant equilibrium behaviors of an acid–base system [28].

Our project differs from these works in two major aspects. First, whereas all the previous works deal with either qualitative models or models specified by algebraic or ordinary differential equations (ODEs), we believe AOM is one of the first programs to handle systems of nonlinear partial differential equations (PDEs). Second, we base our programs on a theory of asymptotic order of magnitude of *functions*, which, besides being closer to what applied mathematicians or fluid dynamicists use,[1] applies to algebraic equations and ODEs as well as to PDEs.

The use of asymptotic order of magnitude in qualitative reasoning originally appeared in [32]. The program has since been extended to deal with more substantial, non-textbook, examples; in particular it can reproduce a well-known approximate model for laminar flow, the triple-deck model, a composite model having interacting subparts. This paper demonstrates the additional capability of AOM, and greatly expands the explanation of new examples and the underlying algorithm.

The paper is organized as follows. We first introduce some terminology necessary to understand the fluid domain. We then explain in Section 3 the model simplification task in the context of a fluid problem. Despite the relatively complicated equations in the paper, the essential ideas can be understood by a careful reading of the motivating examples in Sections 4 and 5 where we describe asymptotic order of magnitude, and the method of dominant balance to find self-consistent approximations. The simplification algorithm is described in Section 6. We show the program's performance in Sections 7 and 8. In Section 9, we present a detailed evaluation of the strengths and weaknesses of the program. Section 10 puts AOM in the context of previous works on order of magnitude reasoning. We finally conclude with a summary of the technical advances this paper presents.

2. Characteristics of the problem domain

2.1. Some terminology

Fluids obey Newton's laws of motion. The basic equations are expressions for conservation of momentum and conservation of mass. The three momentum equations in Fig. 1 are just examples of Newton's Second Law ($F = ma$). In fluid mechanics, it is customary to have the acceleration or the inertia terms written on the left-hand side of the equation, and the remaining force terms on the right.

Since the motion of a fluid particle can change with both time and space, the inertia consists of two parts: the *local acceleration* (i.e., rate of change of velocity with respect

[1] The asymptotic theory is also commonly used in the analysis of algorithms.

Conservation of momentum:

$$\overbrace{\underbrace{\frac{\partial u}{\partial t}}_{\text{local}} + \overbrace{u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}}^{\text{convective}}}^{\text{inertia forces}} = \overbrace{\underbrace{-\frac{\partial p}{\partial x}}_{\text{pressure}} + \overbrace{\frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)}^{\text{viscous}}}^{\text{applied forces}}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{Re}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

Conservation of mass:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Fig. 1. Meaning of terms in the 3D incompressible Navier–Stokes equations. When the velocity is independent of time, and the w-component of the velocity is zero, the general equations reduce to the 2D steady incompressible flow.

to time), and the *convective acceleration* (i.e., product of velocity and the velocity gradient).

A *steady flow* is one in which the local acceleration is zero. The applied forces on the fluid can be divided into two types: (1) *surface forces*, caused by molecular attractions, include pressure and friction forces due to viscosity, and (2) *body forces* resulting from external force fields like gravity or magnetic field. It is often convenient to define the pressure term to include gravity (i.e., $p+\rho gy$, where ρ is density of fluid, g gravitational constant, and y is the vertical coordinate).

The mass conservation law is expressed in a differential form: the divergence of the fluid velocity is zero. This equation just means that in each region, no matter how small, the net flow-in must be equal to the net flow-out. In other words, whatever is flowing is not permitted to accumulate. Such a flow is called *incompressible*. Incompressibility is an idealization, but it is a good approximation for most flows under ordinary temperature.

The momentum equations express a balance of opposing forces on the fluid: the inertia forces keep the fluid moving steadily against the effects of pressure gradient and viscous forces. The ratio, called the *Reynolds number*:

$$Re = \frac{uL}{\nu},$$

where u is a characteristic velocity, L a length scale, and ν the kinematic viscosity, can be thought of as a comparative measure of the inertia and the viscous forces; it is an indication of the relative importance of viscosity—actually the unimportance since high Reynolds numbers are associated with slightly viscous flow. If we gradually increase the Reynolds number, e.g., by increasing the flow speed, then at some critical value Re^*, the viscosity will no longer dissipate energy quick enough to stabilize the flow. A

transition from laminar flow to turbulence occurs. It is remarkable that the whole world of laminar and turbulent flow is described by four lines of mathematical symbols—well, not exactly the whole world because there are flows that are compressible.

2.2. *Ontology*

Description of fluid motion involves a variety of quantities: (1) the fundamental quantities: time, space, and mass, (2) the usual dynamical quantities from particle mechanics such as velocity, acceleration, force, pressure, and momentum, (3) quantities that are less familiar but can be easily derived from the more basic ones: velocity gradient and pressure gradient, convective acceleration, viscous shearing forces, and turbulent stress, (4) dimensionless parameters such as the Reynolds number, and (5) scale parameters, such as δ, which determine the length, time, or velocity scale of interest.

3. The task

We are interested in the task of model simplification. It occurs in the early heuristic phase in the mathematical modeling of a scientific problem. Model simplification takes three inputs: (1) a detailed model, (2) a description of the parameters, dependent variables, and independent variables of the model, and (3) essential physical effects. Its output is one or more simplified models with constraints on the parameters. These constraints might delimit the region of validity of the approximation. They might also contain important information about the character of the solution.

Detailed fluid models are usually available from standard textbooks and so are the physical meanings of parameters and variables. If the detailed model is composite, the input description specifies detailed equations for each of its component models. The description of variables is problem-dependent; it often includes their boundary values and estimated maximum order of magnitude. The order of magnitude estimates can be obtained from experimental observations concerning the phenomenon or from solutions of simple flows. Knowledge of essential physical effects can also be obtained from experiments. For instance, neglecting all the viscous terms in the equation will predict non-physical results, the so-called d'Alembert paradox to be discussed shortly.

In general, a simplified model is valid only under a range of parameter values—within a certain range of Reynolds numbers, inside a narrow region around a body, for example.

As our first motivating example, we will derive Prandtl's boundary layer approximation for high Reynolds number flows (but not too high so that the flow remains laminar), which is probably the single most important approximation made in the history of fluid mechanics. For ease of exposition, we consider the case of two-dimensional, steady, incompressible flow over a flat plate (Fig. 2). The same technique will work for three-dimensional, unsteady flow over arbitrary bodies.

The detailed model is the 2D steady incompressible Navier–Stokes equations (Fig. 3). Eqs. (1) and (2) are the momentum equations, while (3) is the equation of continuity (or conservation of mass). The model is a system of three coupled PDEs containing

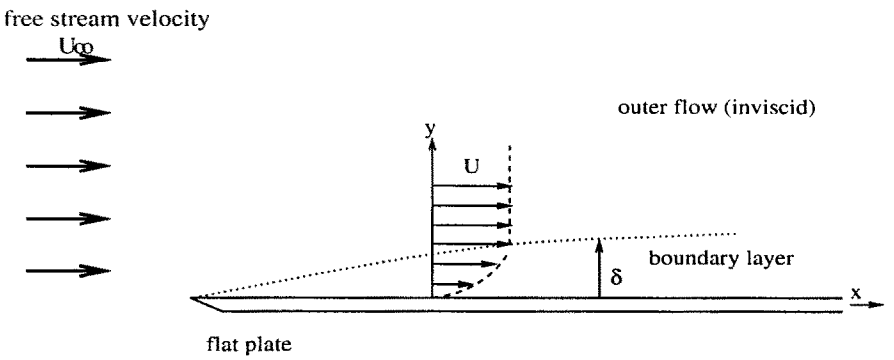

Fig. 2. Boundary layer over a semi-infinite flat plate. The velocity gradient near the surface is large because of the no-slip condition. As one moves away from the surface, along the y-direction, the local velocity increases steadily as it approaches the free stream velocity. The width of the boundary layer is denoted by δ; it grows downstream.

three unknowns u, v, and p. The objective is to simplify the model in the limit of large Reynolds numbers.

It might be thought that since Re is large ($\gg 1$), the coefficient multiplying the viscous terms in the momentum equations is small and therefore we can neglect these terms in first approximation. However, a model that neglects viscosity will predict zero drag on a solid body in steady flow—theoretical predictions diverge from reality. The observation that viscosity—no matter how small it is—can't be neglected in drag calculation is known as the d'Alembert paradox. The resolution of this paradox depends on Prandtl's idea of boundary layer, an idea that brought much credibility to the whole theoretical fluid dynamics.

Prandtl's idea is that at high Reynolds numbers viscosity remains important near the body surface even if it could be disregarded everywhere else. As long as the "no-slip" condition holds, i.e., that fluids do not slip with respect to solids, there will be a thin layer around the body where rapid changes of velocity produce notable effects, despite the small coefficient Re^{-1}. The layer in question is called *boundary layer*. Although boundary layers are typically thin compared to the body dimension, their effects on drag and transport properties can be enormous.

To get a feel of the type of reasoning involved in the derivation of the boundary layer approximation, we will quote a passage, slightly edited for our purpose, from a standard fluid dynamics textbook [30]:

> To start with we assume that δ^*, the width of the boundary layer, is small compared with L, the length of the flat plate if Re is large. That means $\delta = \delta^*/L \ll 1$, and the range of the boundary layer y is δ. Since u and x are all of order of unity, Eq. (3) states that v is of order δ. Now the convective terms in Eq. (1) are all of O(1). A glance at the viscous terms in Eq. (1) reveals that $\partial^2 u/\partial x^2 \ll \partial^2 u/\partial y^2$ so that the first can be neglected and the viscous terms can be replaced by $Re^{-1}\partial^2 u/\partial y^2$. Since in the boundary layer the viscous terms are of the same order of magnitude as the inertial terms, $Re^{-1}\partial^2 u/\partial y^2 = \mathrm{O}(1)$; this shows that:

Model name: prandtl-boundary-layer
Independent variables:
x lower-bound = 0 upper-bound = 1
physical features = space, streamwise
y ...
Dependent variables:
u depends-on = x, y
lower-bound = 0 upper-bound = 1
physical features = velocity, streamwise
v ...
Parameters:
Re type = large-parameter
physical features = dimensionless-number
δ type = small-parameter
physical features = dimensionless-number, length, transverse

Essential terms: viscous, inertia
Equations:
Streamwise-momentum:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1}$$

Transverse-momentum:

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{2}$$

Continuity:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3}$$

Order of Magnitude estimates:

$u = \mathrm{O}(1)$

$x = \mathrm{O}(1)$

$y = \mathrm{O}(\delta)$

Fig. 3. Problem specification for flow over semi-infinite plate. It consists of three parts: (1) declarations of variables and parameters, (2) detailed equations of motion (2D steady, incompressible Navier–Stokes Equations: u and v are the horizontal and normal components of the velocity, p is the pressure, and Re is the Reynolds number), and (3) a few order of magnitude estimates. A parameter is large if its magnitude is $\gg 1$, and small if $\ll 1$. The essential terms specification rules out simplified models that do not contain any viscous or inertia terms.

$$Re = \mathrm{O}\left(\frac{1}{\delta^2}\right). \tag{4}$$

To see how p varies, we turn to Eq. (2). Again the term $\partial^2 v/\partial x^2$ can be neglected since it is added to a much larger term $\partial^2 v/\partial y^2$. Then all the terms involving v are of $\mathrm{O}(\delta)$. Hence the pressure variation with respect to y in the boundary layer is of $\mathrm{O}(\delta^2)$, and can be neglected. Thus we take the pressure outside the boundary layer to be the pressure inside. But outside the boundary layer, the pressure distribution $p(x)$ is a function of x only. So we can replace the partial derivative of the pressure term by the total derivative. Thus the flow in the boundary layer is governed by:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}, \tag{5}$$

to which must be added the equation of continuity (3).

Much can be learned from this explanation. First, we notice that the simplified model consists of only two equations (5) and (3), and two unknowns u and v; the momentum equation (2) is discarded. The pressure p becomes a known boundary term to be given by the solution to the outer flow, the far field approximation, where viscosity can be totally ignored. Second, the explanation refers to physical meanings of the terms in the equations; we have *inertia terms, convective terms, viscous terms*, and *pressure terms*. Third, the reasoning makes heavy use of order of magnitude estimates to justify the elimination of small terms. Fourth, given a few basic order of magnitude estimates (such as those of δ, u, and x), estimates for more complicated quantities involving partial derivatives are automatically inferred. In particular, it derives the important conclusion that the dependency of the pressure on y, i.e., the variation across the thin boundary layer, can be neglected at this level of approximation. Finally, by balancing the inertia terms and the viscous terms, it obtains a semi-quantitative relationship, Eq. (4), between the boundary layer thickness δ and the Reynolds number Re. Since Re is proportional to x, the distance downstream from the leading edge, and the dimensional thickness δ^* is defined to be $\delta^* = \delta \times x$, we can predict that the boundary layer will increase parabolically downstream—first rapidly and then gradually. [2]

4. Asymptotic order of magnitude of functions

Exploiting the smallness of parameters or variables in a mathematical formulation to construct approximate models is a powerful idea that finds applications in almost every branch of physics. In projectiles, we ignore air resistance. In planetary motion, we ignore all but the forces exerted by the sun. In water waves, we ignore viscosity and compressibility. There is not always a guarantee that approximations like these will give accurate predictions. But the simplicity of the models allows us to make qualitative

[2] The details: $Re = \mathrm{O}(1/\delta^2)$ implies $\delta \propto 1/\sqrt{Re} \propto 1/\sqrt{x}$. But $\delta^* = \delta \times x \propto x/\sqrt{x} = \sqrt{x}$.

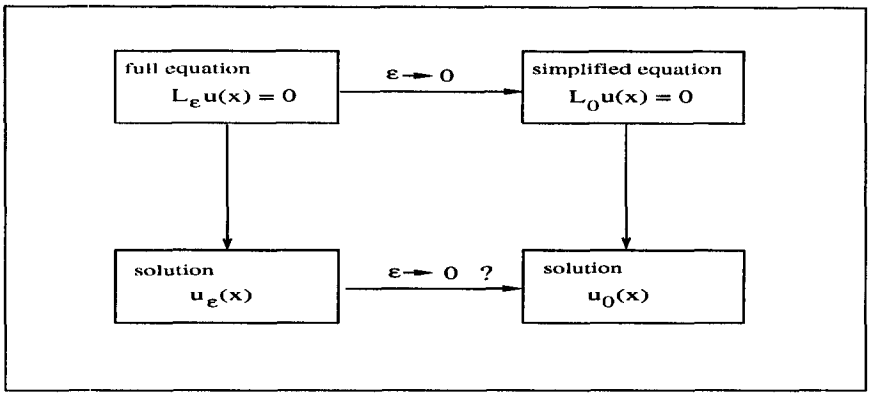

Fig. 4. The simplified equation $L_0 u(x) = 0$ is obtained by letting the parameter $\varepsilon \to 0$ in the full equation $L_\varepsilon u(x) = 0$. The solutions to the full and simplified equations are denoted by $u_\varepsilon(x)$ and $u_0(x)$ respectively. The validity of the approximation depends on whether $u_\varepsilon(x) \to u_0(x)$ as $\varepsilon \to 0$.

or semi-quantitative predictions that can be checked with measurements. If the match is not satisfactory, we probably throw away too much in the original formulation, and should reexamine some of the neglected terms.

Let's be more explicit about the approximation problem. Consider an equation $L_\varepsilon u(x) = 0$, where L_ε is an operator (could be algebraic, differential, or integral) depending on a small parameter ε and acting on the function $u(x)$ (see Fig. 4). An approximate equation $L_0 u(x) = 0$ is obtained by taking the limit $\varepsilon \to 0$. Suppose $u_\varepsilon(x)$ is the solution to the complete equation, and $u_0(x)$ the solution to the approximate equation. Do the solutions exist? Are they unique? If so, how do they compare?

Except for the happy rare cases, existence and uniqueness theorems for PDEs are almost non-existent. So the discussion of error estimates is largely academic. Rather we would like to introduce a language to describe the sense in which an approximate solution is "close" to the true one, if they both exist. One obvious candidate for closeness measure is the *approximate equality*, namely,

$$u_\varepsilon(x) \approx u_0(x) \iff |u_\varepsilon(x) - u_0(x)| \ll 1 \quad \text{as } \varepsilon \to 0.$$

In other words, the *absolute* difference between the approximate solution and the true one is close to zero. While sometimes useful, approximate equality is less informative than it might seem. Suppose we want to approximate the function $\sin x$ near $x = 0$. Since $\sin x \approx 0$ for small x, the constant function 0 might be a good approximation. But so are the functions $\sqrt{x} \approx x \approx x^2 \approx 0$. The "shape" of $\sin x$, however, looks more like that of x than like those of $\sqrt{x}$ or x^2. Specifically the function $\sin x$ approaches 0 at the *same rate* as the function x.

Approximate equality is also not a good measure when we approximate a function near its singular points, namely, the places at which the function blows up. For an example, consider the two functions $e^x + x$ and e^x as $x \to \infty$. Although the difference between them is infinite, they look remarkly the same for large x.

For these two reasons, the notion of "asymptotic equality", denoted by the symbol $\sim$, which measures the *relative* difference between an approximation and the true solution, is widely used in approximation theory. We will consider the *asymptotic behavior* of a function $f(\varepsilon)$ as ε approaches some critical value ε_0. Without loss of generality, we can assume $\varepsilon_0 = 0$, since translation $(\varepsilon - \varepsilon_0)$ and inversion $(1/\varepsilon)$ can be used to handle any nonzero finite and infinite limiting values.

There are several ways to describe the asymptotic behavior of a function with varying degrees of precision. For instance, we could describe the limiting value $f(\varepsilon)$ as $\varepsilon \to 0$ qualitatively, i.e., whether it is bounded, vanishing, or infinite. Or, we could describe the limiting value *quantitatively* by giving a numerical value for the bound. But we find it most useful to describe the *shape of the function qualitatively* as a limit is approached. The description uses the order symbols O ("big oh"), o ("little oh"), and $\sim$ ("asymptotically equal") to express the relative magnitudes of two functions.

Definition 1. $f(\varepsilon) = O(g(\varepsilon))$, $\varepsilon \to 0$ if $\lim_{\varepsilon \to 0} |f(\varepsilon)/g(\varepsilon)| = K$ where K is a finite number.

Definition 2. $f(\varepsilon) = o(g(\varepsilon))$, $\varepsilon \to 0$ if $\lim_{\varepsilon \to 0} f(\varepsilon)/g(\varepsilon) = 0$.

Definition 3. $f(\varepsilon) \sim g(\varepsilon)$, $\varepsilon \to 0$ if $\lim_{\varepsilon \to 0} |f(\varepsilon)/g(\varepsilon)| = 1$.

The notation $f \ll g$, which reads "f is negligible compared to g", is synonymous with $f = o(g)$. The notation $f = O(g)$ implies the ratio of f/g is bounded, whereas $f \sim g$ means the *relative error* between f and g goes to 0.

Typically, we will use a convenient set of simple functions inside an order symbol; they are called the *gauge functions* because they are used to describe the shape of an arbitrary function in the neighborhood of a critical point. Common gauge functions include the powers and inverse powers of ε. For example, $\sin(\varepsilon) = O(\varepsilon)$ as $\varepsilon \to 0$. For more complicated problems, logarithms and exponentials of powers of ε may also be used. But we shall not consider this larger class of gauge functions in this paper. Throughout the paper, we will assume the unknown gauges all have the form ε^n where n is any rational number and its value will be determined as part of the simplification analysis.

The asymptotic order of magnitude must be distinguished from the numerical order of magnitude. If $f = 10^6 g$, then f and g differ by 6 numerical orders of magnitude, but they are still of the same asymptotic order.

Below we list some useful rules of operation on order symbols [10]:

(1) $O(fg) = O(f)O(g)$.
(2) $O(f + g) = \max(O(f), O(g))$.
(3) $O(f) + o(f) = O(f)$.
(4) $o(fg) = O(f)o(g) = o(f)o(g)$.
(5) If $f = O(g)$, then $\int_0^\varepsilon f(t)\,dt = O(\int_0^\varepsilon |g(t)|\,dt)$ as $\varepsilon \to 0$.

Order relations cannot in general be differentiated. That is, if $f = O(g)$, then it is not generally true that $f' = O(g')$. However, using the definition of the total differential of a differentiable function $f(x, y)$,

$$df = \underbrace{\frac{\partial f}{\partial x}\,dx}_{df\text{-}x} + \underbrace{\frac{\partial f}{\partial y}\,dy}_{df\text{-}y},$$

where df-x and df-y are the partial differentials, we can derive some useful rules involving partial derivatives:

(1) $O(\partial f/\partial x)O(dx) = O(df\text{-}x)$.

(2) $O(\partial f/\partial y)O(dy) = O(df\text{-}y)$.

(3) $O(df) = \max(O(df\text{-}x), O(df\text{-}y))$.

These rules are used to approximate the amount the function f changes, df (which is not infinitesimal in general), when the independent variables x and y change.

One final remark about the formal properties of the gauge functions. This property will be important to those who want to formalize asymptotic order of magnitude reasoning. Let's consider the powers of ε, and compare their orderings with those of real numbers in the interval $(0, \infty)$. Evidently we have:

$$\frac{1}{a} < \frac{1}{b} \quad \text{iff} \quad \varepsilon^a \ll \varepsilon^b.$$

However, the reals have ordering properties that are not shared by the powers of ε. Two notable ones are the least upper bound, and the greatest lower bound properties. For instance, the decreasing sequence of reals $\{1/n\}$, where $n = 1, 2, 3, \ldots$, converges to 0, whereas the decreasing sequence $\{\varepsilon^n\}$ does *not* converge to the zero function. Between any ε^n and the zero function, there are infinitely many transcendentally small functions. One example would be $0 \ll e^{-1/\varepsilon} \ll \varepsilon^n$ for all n (as $\varepsilon \to 0$).[3]

5. Theory of simplification

5.1. Simplification by balancing terms

A powerful idea to simplify an equation is to identify the small terms in the equation, drop these terms, solve the simplified equation, and check for consistency [2,13]. But this does not always work. Consider the following simple polynomial:

$$3\varepsilon^2 x^3 + x^2 - \varepsilon x - 4 = 0$$

in the limit $\varepsilon \to 0$. We might naively drop the cubic and the linear terms because their coefficients are small. But if we do that, we only get two roots $x = \pm 2$, losing the third root. Thus, the process of simplification can lead to a loss of important information. We have a singular perturbation problem when the reduced equation changes order. Many interesting problems in fluid mechanics have this singular character (see [26]).

What went wrong? The problem is that terms that appear small are not really small. The missing root depends inversely on ε in such a way that the cubic term is not

[3] Note that this is an asymptotic relation, namely, it holds as $\varepsilon \to 0$. For any *finite* ε, we can indeed find a value of n for which the statement is false. But the statement is valid in the limit of small ε.

negligible even its coefficient becomes small. To fix this problem, we introduce three concepts: an *undetermined gauge*, a *significant gauge*, and a *maximal set*. To begin, we will assume $x = O(\varepsilon^n)$ where n is still undetermined—hence the name undetermined gauge. The order of each term is then:

$$\underbrace{3\varepsilon^2 x^3}_{O(\varepsilon^{3n+2})} + \underbrace{x^2}_{O(\varepsilon^{2n})} - \underbrace{\varepsilon x}_{O(\varepsilon^{n+1})} - \underbrace{4}_{O(1)} = 0.$$

Since n is still unknown, we could not determine which terms are large and which are small. To proceed, we could assume the equation is dominated by two terms, the remaining being negligible. We *balance* the two dominant terms, i.e., assert that their magnitudes are asymptotically equal, thereby obtaining an equation for the unknown n. With n determined, we could check whether the neglected terms are in fact negligible. If they are, we have a self-consistent approximation.

A little technicality before we state the general method of dominant balance. Since we must allow the situation where two or more terms may have the same asymptotic order, we group terms into equivalence classes by the relation $\sim$. A maximal set is any such class that is not smaller than any other classes. As an example, the cubic polynomial above has four maximal sets each containing one term. The heuristic can then be stated as follows:

Method of Dominant Balance. If the equation has n maximal sets, balance m ($\leqslant n$) of them; these m maximal sets are called *dominant*. Assume the remaining $(n-m)$ sets are negligible. Self-consistent choices of dominant maximal sets correspond to significant simplified equations.

Our heuristic is a straightforward generalization of the more common 2-term dominant balance procedure [2]. The m-term balance is needed to handle equations for which balancing two maximal sets does not give any self-consistent approximations. For instance, the ODE

$$\frac{dy}{dx} - \frac{y}{x} = \frac{\cos x}{x^2}$$

requires a 3-term balance because all the pairwise balances are inconsistent. In this simple case, the 3-term balance does not produce any simplification. However, in more complicated equations, like the fluid models to be discussed later, the 3-term balance can produce significant simplifications. Values of m larger than 3 have not found to be that useful because first the resulting simplification will not be much different from the equation we start with, and second it rapidly increases the number of possible approximate models to be tested.

To understand how the heuristic works, let's consider the cubic equation. Here the number of maximal sets n is 4. All 1-set balances are inconsistent because they are singleton sets. Six possibilities are given by 2-set balances. For instance, one possibility is that the first two terms are dominant, i.e., $\varepsilon^2 x^3 \sim x^2 \gg \varepsilon x, 4$. Equating the two undetermined gauges, we get $3n + 2 = 2n$ and this implies $n = -2$. The remaining terms are $O(\varepsilon^{-1})$ and $O(1)$, which is consistent with the assumption that the first two

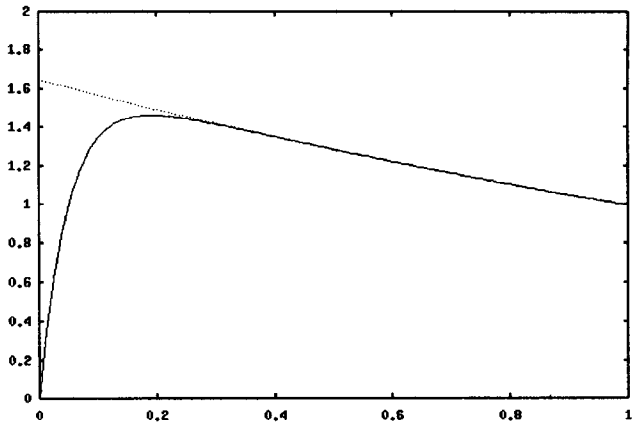

Fig. 5. The solution of the differential equation $\varepsilon(d^2y/dx^2) + 2(dy/dx) + y = 0$ for $\varepsilon = 0.1$ The "inner solution" is a rapidly increasing function inside a region of $O(\varepsilon)$ around the origin. The "outer solution", indicated by the dotted line, is a slowly decaying function. (Adapted from [13, Fig. 9.1, p. 187].)

terms are dominant. So this possibility is included. On the other hand, if we assume $\varepsilon^2x^3 \sim \varepsilon x \gg x^2$,4, we get $n = -1/2$. But then $x^2 = O(\varepsilon^{-1}) \gg O(\varepsilon^{1/2})$, violating the assumption that it should be much smaller than the first term. This possibility must be excluded. A similar analysis shows that only one more possibility, when the second and fourth terms are dominant, i.e., $n = 0$, is self-consistent. So the heuristic concludes that we should consider *two* simplified polynomials:

$$3\varepsilon^2x^3 + x^2 = 0 \Rightarrow x \sim \frac{1}{3\varepsilon^2} \quad \text{and} \quad x^2 - 4 = 0 \Rightarrow x \sim \pm 2.$$

The values of ε^n for which we get self-consistent dominant maximal sets are called *significant gauges*. Notice that each significant gauge generates an equation that captures a qualitatively significant asymptotic behavior.

The balancing procedure can also be used to simplify differential equations. Consider the equation:

$$\varepsilon\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant \varepsilon \ll 1, \tag{6}$$

with boundary conditions $y(0) = 0$ and $y(1) = 1$. The equation can be solved exactly (see Fig. 5) [13]. We see that the solution curve is composed of two parts: a slowly varying part in the region away from $x = 1$ and a rapidly changing part in the region near $x = 0$. Pretending we don't know the exact solution, we will see how the balancing procedure recovers the two significant approximations.

As in the cubic equation, just dropping the apparently small term $\varepsilon(d^2y/dx^2)$ does not quite work because the resulting first-order equation cannot possibly satisfy the two boundary conditions. The problem here is the derivative term (d^2y/dx^2) is not $O(1)$ when x approaches 0. So we rescale x by an undetermined gauge $\delta(\varepsilon)$:

$$X = \frac{x}{\delta(\varepsilon)}.$$

The exact form of the unknown gauge $\delta(\varepsilon)$ will come out of the balancing procedure.[4]

Substitution of this change of variable into the original equation yields:

$$\frac{\varepsilon}{\delta^2}\frac{d^2y}{dX^2} + \frac{2}{\delta}\frac{dy}{dX} + y = 0.$$

The orders of magnitude of the three terms are ε/δ^2, $1/\delta$, and 1. Of the three possible 2-term balances, two are self-consistent, namely, $1/\delta \sim 1$ and $1/\delta \sim \varepsilon/\delta^2$. The first consistent balance implies $\delta \sim 1$, giving

$$2\frac{dy}{dx} + y = 0, \tag{7}$$

while the second implies $\delta \sim \varepsilon$, giving

$$\frac{d^2y}{dX^2} + 2\frac{dy}{dX} = 0. \tag{8}$$

Since the approximation is derived under the assumption that $x = O(1)$, Eq. (7) should satisfy the boundary condition on the right $y(1) = 1$. In the applied mathematics literature, this equation is called the *outer approximation*. Eq. (8) is presumed to be valid when $x = O(\varepsilon)$ and so it satisfies the left boundary condition $y(0) = 0$; this is the so-called *inner approximation*. Ascribing the second boundary condition for Eq. (8) needs more work; we will return to this problem in a later section.

To illustrate the use of maximal sets, we consider the boundary layer equations. Scaling the independent variable y by an unknown gauge $\delta \ll 1$ and applying the order of magnitude rules, we get the following estimates for the terms in the streamwise equation (see Section 6):

$$u\frac{\partial u}{\partial x} = O(1); \qquad v\frac{\partial u}{\partial y} = O(1); \qquad \frac{\partial p}{\partial x} = ?;$$
$$\frac{1}{Re}\frac{\partial^2 u}{\partial x^2} = O\left(\frac{1}{Re}\right); \qquad \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} = O\left(\frac{1}{Re}\frac{1}{\delta^2}\right);$$

where the symbol "?" indicates that the estimate for a quantity is unknown. Since $\delta \ll 1$, we have $Re^{-1} \ll Re^{-1}\delta^{-2}$. So the term $Re^{-1}(\partial^2 u/\partial x^2)$ is not maximal. The remaining four terms are maximal and grouped into three sets:

$$A = \left\{u\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y}\right\}, \qquad B = \left\{\frac{\partial p}{\partial x}\right\}, \qquad C = \left\{\frac{1}{Re}\frac{\partial^2 u}{\partial y^2}\right\}.$$

Similarly, the estimates for the transverse equation are:

[4] It should be noted that δ is *postulated* to be the unknown gauge that makes the derivatives O(1). The balancing procedure then reveals that for that to happen, $\delta = O(\varepsilon)$, i.e., $n = 1$ (see text). The substitution step for this class of gauge functions does not require any user input.

$$u\frac{\partial v}{\partial x} = O(\delta); \qquad v\frac{\partial v}{\partial y} = O(\delta); \qquad \frac{\partial p}{\partial y} = ?;$$
$$\frac{1}{Re}\frac{\partial^2 v}{\partial x^2} = O\left(\frac{\delta}{Re}\right); \qquad \frac{1}{Re}\frac{\partial^2 v}{\partial y^2} = O\left(\frac{1}{Re}\frac{1}{\delta}\right);$$

and the maximal sets are:

$$D = \left\{u\frac{\partial v}{\partial x}, v\frac{\partial v}{\partial y}\right\}, \qquad E = \left\{\frac{\partial p}{\partial y}\right\}, \qquad F = \left\{\frac{1}{Re}\frac{\partial^2 v}{\partial y^2}\right\}.$$

5.2. *Constraints from multiple equations*

For a model containing multiple equations, one can often exploit constraints among terms in different equations to reduce the number of possible dominant balances. For instance, a direct application of m-set balance (for $m \leqslant 3$) to the streamwise equation generates five possible balances:

(1) A,
(2) A, B,
(3) A, C,
(4) B, C,
(5) A, B, C.

Similarly, five balances are possible for the transverse equation. The total number of balances is $5 \times 5 = 25$ for the momentum equations. But many of these balances are inconsistent. For instance, if we assume the set A is the dominant set in the streamwise equation, then the set F can't be dominant in the transverse equation because $A \gg C$ implies $D \gg F$. Or, if the sets A and C balance, then it would imply both D and F have the same order of magnitude and hence they belong to the same maximal set.

Since the self-consistency checks are the most expensive part of the procedure, it is useful to rule out certain balancing possibilities quickly. A useful criterion uses the concept of significant set. Of the three maximal sets in the streamwise equation, two of them—A and C—have additional properties: (1) they contain terms that are essential (Fig. 3), (2) no other maximal sets in the equation contain the same type of essential terms, and (3) they dominate the same type of essential terms in all other equations, namely, $A \gg D$ and $C \gg F$. Maximal sets satisfying these three properties are called significant sets. The terms in A and C are called significant terms. None of the maximal sets in the transverse equation are significant.

Significant Set Filter. Of the possible consistent m-set balances, remove those that do not contain the significant sets.

Applying the filter to the streamwise equation, we remove three out of the five balances, leaving only two balances $\{A, C\}$ and $\{A, B, C\}$ to consider. Either of the balances would imply that the maximal sets D and F have the same order of magnitude. Thus, only two of the five possible balances in the transverse equation—$\{D, F\}$ and $\{D, E, F\}$—are consistent.

The rationale for the significant set filter comes from the observation that a simplified model not containing a significant set cannot contain all essential terms. Consider the significant set A. It contains the only inertia terms in the streamwise momentum equation. If A is neglected, then by order of magnitude argument the set D, which contains the only inertia terms in the transverse equation, should also be neglected. Thus, no inertia terms will be kept in the simplification, violating the input requirement that some inertia terms must be present. The significant set filter is quite useful in constraining the generation of candidate dominant balances. The filter is applicable whenever we have an a priori knowledge that certain types of terms must be present in the simplified model.

5.3. *Constraints from matching submodels*

The approximation for the ODE (Eq. (6)) is an example of a composite model with two nearly independent subparts: the inner and the outer approximation. In Section 5.1, we are able to determine the inner and outer equations, and all but one boundary conditions by the balancing procedure. The missing boundary condition for the inner approximation can only be found by matching the detailed solutions of the two approximations. The conclusion that the interaction between the submodels supplies additional constraints to complete the specification of the submodels is general. We will first show how this is done in the ODE by explicitly solving the individual approximations. Then, we will illustrate how additional constraints can be generated even when we *cannot* explicitly solve the submodels.

The idea behind solution matching is the hope that the regions of validity of the inner and outer approximations overlap [13]. In this intermediate region, both solutions to the approximate equations should give reasonable approximations to the true solution. Solving the outer approximation gives:

$$y_{\text{out}} = e^{(1-x)/2},$$

and the inner approximation gives:

$$y_{\text{in}} = K(1 - e^{-2X}),$$

where K is the coefficient to be determined. Denote the intermediate region by the variable $s = O(x/\varepsilon^{\alpha})$, where $0 < \alpha < 1$. Rewriting the solutions y_{out} and y_{in} in terms of the intermediate variable s and taking the limit $\varepsilon \to 0$ yields:

$$y_{\text{out}} = e^{(1-\varepsilon^{\alpha}s)/2} \xrightarrow{\varepsilon \to 0} e^{1/2},$$
$$y_{\text{in}} = K(1 - e^{-2(s/(\varepsilon^{1-\alpha}))}) \xrightarrow{\varepsilon \to 0} K.$$

By assumption, the two solutions agree in the intermediate region, which implies $K = e^{1/2}$.[5]

[5] This matching procedure will not always work, but because it works for a sufficiently large class of problems, it is among the most popular methods in the singular perturbation literature. Rigorous justification of the matching procedure is still a subject of research. For a useful review, consult [12].

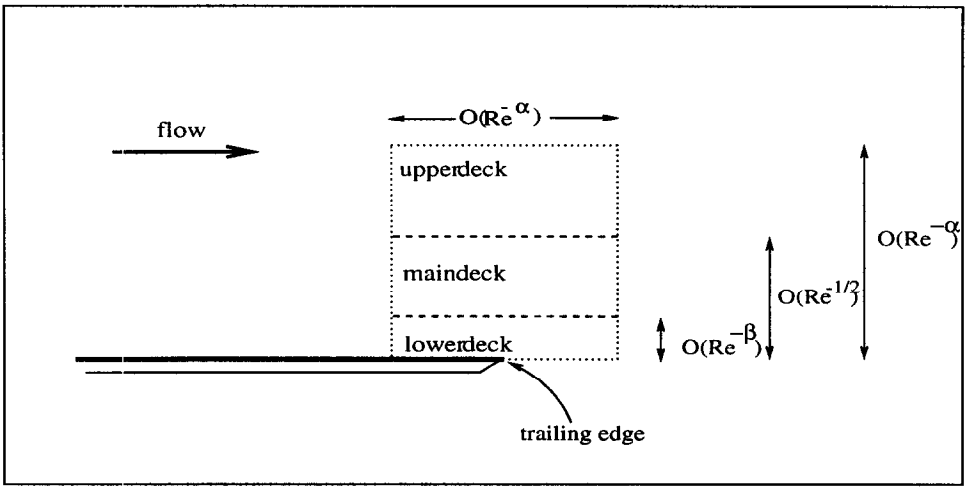

Fig. 6. Schematic picture of the triple deck structure around the trailing edge of a finite plate. The three decks are marked by dotted lines. The horizontal dimension of the decks is $O(Re^{-\alpha})$. The vertical dimensions of the decks are as follows: $O(Re^{-\beta})$ for the lower deck, $O(Re^{-1/2}$ for the main deck, and $O(Re^{-\alpha})$ for the upper deck. The qualitative three deck structure is given to the program. The task of the program is to (1) deduce the values for unknown parameters such as α and β, and (2) simplify the flow equations in each deck.

Prandtl's approximation is another example of a composite model with nearly independent subparts, i.e., the role of the free stream non-viscous flow is limited to setting the external pressure for the boundary layer, and to a first approximation the free stream flow is not affected by what happens inside the thin boundary layer. In more complicated fluid problems, we find that the submodels often interact non-weakly. To determine the structure of each submodel and its region of validity, we have to consider not only the balancing of effects in the region, but also the matching conditions at the boundaries of adjacent regions.

A good example of a composite model with non-weakly interacting subparts is Stewartson and Messiter's triple deck model which successfully models flows near the trailing edge of a flat plate [24]. The model is considered a major landmark in theory of laminar flow because the same technique can be applied immediately to many other problems, including flow near a corner, a hump, a tilted trailing edge, and even supersonic flow.

Roughly, the triple deck model has three subparts (Fig. 6): a "lower deck" reacts to the solid boundary just like Prandtl's thin boundary layer, an "upper deck" is controlled by the free stream condition, and a "main deck" transmits the displacement effect of the lower deck to the upper deck.

The problem specification (Fig. 7) describes the qualitative triple deck structure.

Let us go over the encoding and explain the new features. This problem has eight parameters. *Re* is the usual Reynolds number. To make the expressions easier to read, we use the convention that hat ^ denotes the upper deck quantities, tilde ~ the main deck quantities, and bar ¯ the lower deck quantities. For example, $\hat{\pi}$, $\tilde{\pi}$, and $\bar{\pi}$ represent

Model name: flat-plate-with-trailing-edge
Dependent variables:
P0 depends-on: x
lower-bound = 0 upper-bound = 1
physical features = pressure, external
UB ...
Parameters:
Re type = large-parameter
physical features = dimensionless-number
$\hat{\pi}$ type = small-parameter
physical features = dimensionless-number, pressure
$\tilde{\pi}$ type = small-parameter
physical features = dimensionless-number, pressure
$\bar{\pi}$ type = small-parameter
physical features = dimensionless-number, pressure
$\hat{\sigma}$ type = small-parameter
physical features = dimensionless-number, velocity
$\tilde{\sigma}$ type = small-parameter
physical features = dimensionless-number, velocity
α type = scalar
β type = scalar
Relations:
$\alpha < \frac{1}{2}$, $\beta > \frac{1}{2}$
Regions:
layout: :vertical (lower-deck, main-deck, upper-deck)
lower-deck
inherit: prandtl-boundary-layer
dependent variables: $u_1, \ldots$
expansion: $u = Re^{1/2-\beta}u_1$
estimates: $x = O(Re^{-\alpha})$, $y = O(Re^{-\beta})$
...
main-deck
inherit: 2d-incompressible-steady-flow
dependent variables: $u_1, \ldots$
expansion: $u = UB + u_1$
estimates: $x = O(Re^{-\alpha})$, $y = O(Re^{-1/2})$
...
upper-deck
inherit: 2d-incompressible-steady-flow
dependent variables: $u_1, \ldots$
expansion: $u = U_\infty + u_1$
estimates: $x = O(Re^{-\alpha})$, $y = O(Re^{-\alpha})$
...

Fig. 7. The problem specification for flow over a finite plate. It is a composite model consisting of three subregions. Subregions can be specified by the type of flow they inherit. Local declarations of parameters and variables are permitted. Some variable declarations and all boundary conditions are omitted. The π's are orders of magnitude of the pressures; the σ's are the velocities.

the symbolic orders of magnitude for the pressure in the upper, main, and lower deck respectively. The two parameters, $\tilde{\sigma}$and $\hat{\sigma}$, are the unknown magnitudes of the streamwise velocity in the main and upper decks. Finally, α and β are unknowns characterizing the dimensionless size of the upper and lower decks respectively. AOM's task is two-fold: (1) find constraints governing these parameters, and (2) solve the constraints to determine the dependency of the seven unknowns on *Re*.

Two new features appear in the encoding. First, the `Regions` slot allows the flow model to be made up of several submodels. In this case, the three submodels are named lower-deck, main-deck, and upper-deck. Declaration conventions within a submodel are the same as in the parent model. Variables declared inside a submodel are considered local to that model. The `layout` descriptor specifies that the subregions are stacked vertically. Currently only two types of layout—vertical and horizontal—are supported. Second, the `inherit` slot allows the momentum and continuity equations to be inherited from a previously defined flow model. For instance, the lower deck submodel inherits the Prandtl boundary layer equations (`model-4` in Fig. 10). We could have started with more general equations in which case AOM would repeat the analysis for a non-composite model to rederive the boundary layer equations. The main and upper decks inherit the steady 2D incompressible Navier–Stokes equations.

Two types of hints are given in the encoding to AOM to solve this problem. First, the constraints on α and β encode the qualitative information that the lower deck is inside the Prandtl boundary layer (which is $O(Re^{-1/2})$), while the upper deck is outside. Second, the relations defining u_1 in the lower and main deck encode the assumption that the flows inside the decks are small perturbations from the Blasius profile, $UB(y)$. (The profile is shown in Fig. 2.) The size of the perturbations will come out of the analysis.

The triple deck model is fairly complicated: its specification requires the values of seven unknown parameters $(\alpha, \beta, \bar{\pi}, \tilde{\pi}, \hat{\pi}, \hat{\sigma}, \tilde{\sigma})$ in terms of the Reynolds number *Re*. Seven equations relating these parameters are therefore needed for a well-posed problem. We will explain how four of these seven equations come from interactions among the submodels.

Applying the balancing procedure to the main deck yields four possible approximate models, none of which can be solved explicitly (see Section 7). However, one of the models, MD-1, is simple enough so that two important constraints can be deduced from the form of the equations.

$$\text{MD-1:} \quad UB\frac{\partial u_1}{\partial x} + v\frac{\partial UB}{\partial y} = 0,$$

$$\frac{\partial p}{\partial y} = 0,$$

$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

First, $\partial p/\partial y = 0$ implies that the main deck transmits the pressure from the upper to lower deck without modifying it, which generates two constraints relating the pressures in the three decks:

$$\bar{\pi} = \tilde{\pi}, \qquad \hat{\pi} = \tilde{\pi}.$$

In other words, the pressures in all three decks have the same magnitude. Note that this important piece of information is not deducible from dominant term balancing alone: a simplified equation, the second equation of MD-1, has to be solved.

Second, the streamwise momentum equation and the continuity equation can be combined to give:

$$\frac{\partial}{\partial y}\left(\frac{v}{UB}\right) = 0.$$

where the quantity v/UB, the ratio of the vertical velocity to the horizontal velocity, is called the *streamline slope*. So we conclude that the main deck also transmits without modification the streamline slope, which gives rise to two additional constraints relating the streamline slopes in the three decks:

$$\tilde{\sigma} Re^{\alpha-1/2} = Re^{2\alpha-4\beta+3/2}, \qquad \tilde{\sigma} Re^{\alpha-1/2} = \hat{\sigma}.$$

The remaining three constraints are consequences of balancing assumptions in the lower and the upper deck; they will be explained in Section 7.

5.4. Domain-specific knowledge in simplification

Two types of domain-specific knowledge play an important role in the simplification process. The first type, given as part of the input specification, concerns order of magnitude and bound estimates on quantities. For instance, in the specification for the triple deck problem, the horizontal extent of the lower deck is given by $x = O(Re^{-\alpha})$ where $\alpha < 1/2$. This order of magnitude estimate follows from a general requirement of the boundary layer that the horizontal extent must be much greater than the boundary layer thickness, which scales like $O(Re^{-1/2})$. We will derive the latter scaling relationship in Section 6. Another example is the expansion of the velocity u in the lower deck specification. The leading order of magnitude estimate, $Re^{1/2-\beta}$, is based on the Blasius profile solution to the Prandtl boundary layer problem [30].

The second type of domain-specific knowledge consists of formulas relating velocities and pressure in simple flows. For instance, the program knows that in a potential flow the pressure is proportional to the streamline slope [16]. Declarative knowledge like this can be used to generate additional constraints when the program already establishes the character of a flow.

6. Simplification algorithm

Given a detailed model, the simplification algorithm finds all self-consistent simplified models by the maximal set balancing procedure. A simplified model is *self-consistent* if (1) the terms neglected are consistent with the dominant balance assumptions, and (2) it contains essential terms specified in the input. The algorithm terminates when every equation has only one maximal set, and all the constraints among unknown parameters have been computed and resolved. The algorithm consists of two procedures: `simplify-composite-model` and `simplify-atomic-model`.

The principal steps of the procedure `simplify-composite-model` are:

(1) Preprocess the detailed model specification.
(2) Call `simplify-atomic-model` for each submodel.
(3) Generate all possible composite models from the simplified submodels.
(4) Compute consequences of each composite model.
(5) Propagate all the transmission constraints among submodels.
(6) Collect and solve the constraint set for all unknown parameters.

The procedure `simplify-atomic-model` operates on an atomic model, i.e. one that has no submodels. Its purpose is to simplify all the equations in the model. It has six steps:

(1) Assign order of magnitude estimates to all the terms of the equations.
(2) Find the maximal sets.
(3) If every equation has one maximal set, then return the model and stop.
(4) Otherwise, pick one unsimplified equation and consider all m-set balances.
(5) Propagate the effects of the balance assumptions to eliminate inconsistent balances. Each consistent balance generates a new partially simplified model.
(6) Call `simplify-atomic-atom` recursively on all the partially simplified models.

The `simplify-atomic-model` algorithm will terminate because during each call of simplification, the number of maximal sets is reduced by at least one. So each recursive call will return either a simplified model or `nil` if the partially simplified model is not self-consistent.

We will first use the Prandtl problem as a running example to explain the steps of the `simplify-atomic-model` procedure. In the next section, we use the triple deck problem to explain the `simplify-composite-model` procedure.

6.1. *Preprocessing the input specification*

Given the input specification of a detailed model, the preprocessor creates (1) internal representations for quantities and equations, and (2) a constraint network connecting the quantities. The model specification consists of a name, a list of quantity descriptions, the momentum and continuity equations in infix form, relations defining external pressure and free stream velocities, and a list of estimated orders of magnitude.

Quantities

Quantities are represented by CLOS objects. They are divided into three types: (1) independent variables (x and y for space), (2) dependent variables (e.g., P and P_0 for pressure, and u and v for velocity), and (3) parameters (e.g., Re for Reynolds number, δ for length scale). Each quantity has slots for its upper bound, lower bound, boundary values, physical features, and relations which other quantities. A dependent variable contains additional information about its dependency on the independent variables. For example, the dependent variable u depends on both x and y.

The input for Prandtl's problem specifies nine quantities—x, y, u, v, U_∞, P_0, P, Re, and δ. The preprocessor generates a total of 60 quantities. The reason is that for each dependent variable, quantities corresponding to its total differential, partial differentials, and derivatives are also automatically generated. For instance, the dependent variable u

generates five additional quantities: du, du-x, du-y, $\partial u/\partial x$, and $\partial u/\partial y$. Quantities are also generated for each term in the equations and relations. An example would be the dependent variable *d2Udx2/RE* corresponding to the viscous term $Re^{-1}(\partial^2 u/\partial x^2)$ in the streamwise equation.

Input quantities have associated physical features such as space, velocity, and pressure. These features are used to determine the physical meaning of derived quantities by simple rewrite rules. For instance, a velocity quantity differentiated by a space quantity gives a velocity-gradient quantity. The physical meaning of a term in the equation is determined in a similar fashion. For example, a term that is the product of a velocity quantity and a velocity gradient represents the convective inertia term.

A constraint language

Equations are represented as constraints on quantities so that unknown quantities can be deduced locally from the known ones. Our constraint language has seven primitives:

(1) The constant constraint, `(constant q v)`, asserts that $O(q) = v$.

(2) The maximum constraint, `(maximum q1 q2 q3)`, asserts that

$$O(q3) = \max(O(q1), O(q2)).$$

Example: `(maximum du-x du-y du)`.

(3) The multiplier constraint, `(multiplier q1 q2 q3)`, asserts that

$$O(q1) \times O(q2) = O(q3).$$

Example: `(multiplier u dudx ududx)`.

(4) The exponential constraint, `(exponential q1 q2 q3)`, asserts that

$$O(q1^{q2}) = O(q3).$$

Example: `(exponential x 2 x2)`.

(5) The equality constraint, `(== q1 q2)`, asserts that

$$O(q1) = O(q2).$$

Example: the continuity equation (3) is represented by `(== dudx dvdy)`.

(6) The variation constraint, `(variation f x df-x)`, captures the inference that when the partial differential of a function $f(x, y)$ with respect to x is much less than the value of f at its outer boundary, then f is asymptotically equal to its boundary value. Symbolically, $df\text{-}x = o(f_0) \Rightarrow O(f) = O(f_0)$, where f_0 is the value of f at its outer boundary in the x-direction.

(7) The total-variation constraint, `(total-variation f df)`, asserts that

$$O(df) = O(\text{upperbound}(f) - \text{lowerbound}(f)).$$

Most of these constraints are self-explanatory. In the variation constraint, it should be noted that the differentials df or $df\text{-}x$ do not have to be infinitesimals. They represent the amount the function f changes when the independent variables change. For instance, let

$$f(x, y) = x^2 + y^3.$$

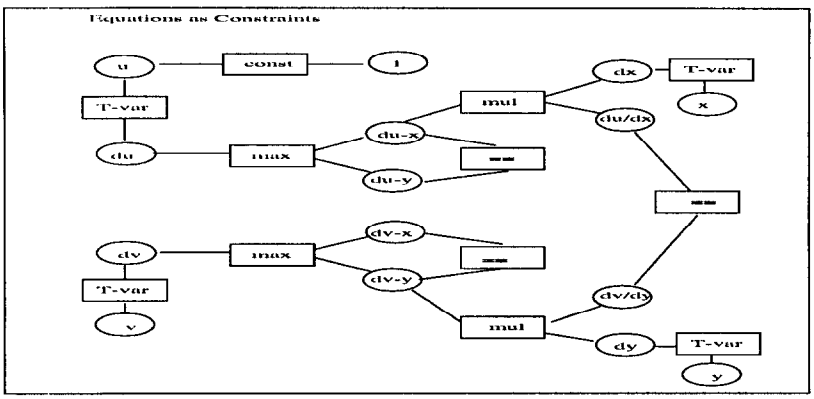

Fig. 8. Equations are represented as a network of constraints for forward inferences.

The change of f at a point $(x, y) = (1, 2)$ can be approximated by:

$$df = 2dx + 12dy.$$

If $\triangle x = 0.1$ and $\triangle y = 0.2, \triangle f = 2.6 > 1$. Here we use the notation $\triangle$ to differentiate $\triangle f$, the result of the action of the mapping df (see below) from the mapping df itself.

The same approximation relation holds for order of magnitude estimates. Mathematicians prefer to think of df as a (constant) differential 1-form, which exists to be integrated. Formally, it is a mapping from the set of finite intervals to the set of real numbers. The definition turns the usual way of looking at integrals upside down. Instead of viewing integration as something we do to a function, we regard it as the action of the differential on a finite interval. This blends well with order of magnitude operations because they are well-behaved under integration.

Even when df happens to be an infinitesimal, in AOM it is not true that it is less than any nonzero quantity—unlike most other order of magnitude systems. The reason is that asymptotic order of magnitude provides an ordering of infinitesimals (e.g., $\varepsilon, \varepsilon^2, \ldots$). So if $f_0 = O(\varepsilon^2)$ and $df\text{-}x = O(\varepsilon)$, then $df\text{-}x \neq o(f_0)$.

6.2. *Assignment of order of magnitude*

The constraint language allows simple inferences about quantities to be made. Consider the network generated by the continuity equation (Fig. 8).

Given that $u = O(1)$, $x = O(1)$, $y = O(\delta)$, the value $v = O(\delta)$ is automatically deduced.[6]

The estimates for quantities that result from forward inferences are given in Table 1.

[6] The deduction steps are as follows: (1) $u = O(1)$ and lowerbound(u) = 0 $\Rightarrow$ $du = O(1)$, (2) $du = O(1)$ and $O(du\text{-}x) = O(du\text{-}y) \Rightarrow du\text{-}x = O(1)$ and $du\text{-}y = O(1)$, (3) $x = O(1)$ and lowerbound(x) = 0 $\Rightarrow$ $dx = O(1)$, (4) $y = O(\delta)$ and lowerbound(y) = 0 $\Rightarrow$ $dy = O(\delta)$, (5) $du\text{-}x = O(1)$ and $dx = O(1) \Rightarrow du/dx = O(1)$, (6) $du/dx = O(1)$ and $O(du/dx) = O(dv/dy) \Rightarrow dv/dy = O(1)$, (7) $dv/dy = O(1)$ and $dy = O(\delta) \Rightarrow dv\text{-}y = O(\delta)$, (8) $dv\text{-}y = O(\delta)$ and $O(dv\text{-}x) = O(dv\text{-}y) \Rightarrow dv\text{-}x = O(\delta)$, (9) $dv\text{-}x = O(\delta)$ and $dv\text{-}y = O(\delta) \Rightarrow dv = O(\delta)$, (10) $dv = O(\delta)$ and lowerbound(v) = 0 $\Rightarrow$ $v = O(\delta)$.

Table 1
Estimates for quantities

Streamwise equation	Transverse equation	Continuity equation
$u\partial u/\partial x = \mathrm{O}(1)$	$u\partial v/\partial x = \mathrm{O}(\delta)$	$\partial u/\partial x = \mathrm{O}(1)$
$v\partial u/\partial y = \mathrm{O}(1)$	$v\partial v/\partial y = \mathrm{O}(\delta)$	$\partial v/\partial y = \mathrm{O}(1)$
$\partial p/\partial x = ?$	$\partial p/\partial y = ?$	
$Re^{-1}\partial^2 u/\partial x^2 = \mathrm{O}(Re^{-1})$	$Re^{-1}\partial^2 v/\partial x^2 = \mathrm{O}(Re^{-1}\delta)$	
$Re^{-1}\partial^2 u/\partial y^2 = \mathrm{O}(Re^{-1}1/\delta^2)$	$Re^{-1}\partial^2 v/\partial y^2 = \mathrm{O}(Re^{-1}1/\delta)$	

6.3. *Finding the maximal sets*

Finding the maximal sets in an equation depends on resolving inequalities between order of magnitude expressions. Three mechanisms are used to infer inequalities: (1) algebraic rewrite rules, (2) bounding algorithms, and (3) graph search.

Algebraic rewrite rules

Many simple inequalities can be determined by rewrite rules. A predicate `lessp` provides the interface between the rules and expressions. The predicate, which is implemented on top of an algebraic simplifier (see Section 7), handles inequality between extended real numbers, infinitesimals and hyperreals (such as 0+, 1−),[7] and algebraic expressions involving quantities. Some examples of rules are:

$$
\begin{aligned}
&x < vx, && \text{if } x > 0 \text{ and } v > 1,\\
&xu < xv, && \text{if } x > 0 \text{ and } u < v,\\
&x^u < x^v, && \text{if } x > 1 \text{ and } u < v,\\
&x^u < x^v, && \text{if } 0 < x < 1 \text{ and } u > v.
\end{aligned}
$$

As an illustration, let's see how the inequality $\sigma Re^{\alpha-1/2} < \sigma Re^{2\alpha-1/2}$, where $Re \gg 1$, $0 < \alpha < \frac{1}{2}$ and $0 < \sigma \ll 1$, is decided:

$$
\begin{aligned}
\sigma Re^{\alpha-1/2} < \sigma Re^{2\alpha-1/2} &\xrightarrow{R2} Re^{\alpha-1/2} < Re^{2\alpha-1/2}\\
&\xrightarrow{R3} \alpha - \tfrac{1}{2} < 2\alpha - \tfrac{1}{2}\\
&\xrightarrow{*} \alpha < 2\alpha\\
&\xrightarrow{R1} \texttt{true}
\end{aligned}
$$

where the third step, indicated by an asterisk, is the action of an ordinary simplification rule.

Bounding algorithms

Computing the upper and lower bound of an expression allows a wider class of inequalities to be decided. We implement part of the inequality bounder described in

[7] See the description of hyperreal numbers in [27]. In particular 0+ corresponds to (HALO 0 1).

[21]: the `ub-lb` and the `sup-inf` algorithm. The basic idea of the algorithms has been explored by many researchers [4,6,22].

The `ub-lb` algorithm computes numeric bounds for an expression. The bound of an expression is computed recursively from the bounds of its parts. For example, the upper bound of an expression $x + y$ is given by: $ub(x + y) = ub(x) + ub(y)$. The only nontrivial part of the algorithm is to handle an exponential expression x^y. We add two special cases which frequently arise in comparing order of magnitude expressions.

Case 1. When x is a large parameter (such as $Re \gg 1$),

$$ub(x^y) = \begin{cases} [ub(x)]^{ub(y)}, & \text{if } ub(y) > 0, \\ [lb(x)]^{ub(y)} & \text{otherwise.} \end{cases}$$

Case 2. When x is a small parameter (such as $0 < \delta \ll 1$),

$$ub(x^y) = \begin{cases} [ub(x)]^{lb(y)} & \text{if } lb(y) > 0, \\ [lb(x)]^{lb(y)} & \text{otherwise.} \end{cases}$$

As an illustration, consider the inequality $Re^{2\alpha-1} < 1$, where $0 < \alpha < \frac{1}{2}$. The algorithm computes:

$ub(Re^{2\alpha-1})$
  $lb(Re) = 1+$
  $ub(2\alpha - 1)$
    $ub(\alpha) = \frac{1}{2}^-$
    $= 0-$
  $= (1+)^{0-}$
  $= 1-$
  $< 1.$

The hyperreals are manipulated by rules like:
(1) $p \times 1+ = p+$,
(2) $n \times 1+ = n-$,
(3) $p \times 0+ = 0+$,
(4) $n \times 0+ = 0-$,
(5) $p+ \times q+ = pq+$,
where p and q are finite positive real numbers, and n is a finite negative real number.

Graph search

Algebraic techniques to resolve inequalities become increasingly complicated when the quantities are related by symbolic constraints. We initially employed the `sup-inf` algorithm, but its exponential time complexity quickly made it a bottleneck. Although inferentially weaker than the `sup-inf` method, Simmons' idea of searching a quantity lattice works much faster [23].

The basic idea is to represent explicitly the order relationships between quantities in a directed graph whose nodes are quantities and edges are labeled order relations, and

Fig. 9. Graph search to determine order relations.

to use a breadth-first search to find paths between quantities. We generalize Simmons' representation to include symbolic factors in the order relations. Let's look at an example (Fig. 9(a)). We have four quantities: A, B, C, and D. Assume δ is a small parameter. The following relations are also known: (1) $O(A) = O(B)$, (2) $O(B) = \delta O(D)$, and (3) $O(A) = \delta O(C)$. To show that $O(C) = O(D)$, we find the shortest path between them, collecting the symbolic factor of each edge of the path. The symbolic factors are divided into two groups: the $\ll$-factors, and the $\gg$-factors depending on whether the edge is labeled $\ll$ or $\gg$. In the example, the $\ll$-factors consists of one factor δ, while the $\gg$-factors consists of one factor $1/\delta$.

The inference procedure can also handle partial information. For instance, in the graph shown in Fig. 9(b), it will correctly conclude that $E \gg H$ even if it is not told what the symbolic factor of edge $F \gg G$ is.

The time complexity of the graph search is $O(E)$, where E is the number of edges in the graph [8]. To avoid a dense graph, the initial lattice only includes order relations that can be easily computed by the algebraic rewrite rules. For instance, terms having the same physical features (e.g. the inertia terms in the momentum equations) usually have similar order of magnitude estimates; they can be easily compared. New order relations are introduced as maximal sets are balanced.

6.4. Balancing maximal sets

The maximal sets for the boundary layer problem with their order of magnitude estimates, first described in Section 5.1, are repeated here:

$$A = \left\{ u\frac{\partial u}{\partial x}, v\frac{\partial u}{\partial y} \right\}, \qquad B = \left\{ \frac{\partial p}{\partial x} \right\}, \qquad C = \left\{ \frac{1}{Re}\frac{\partial^2 u}{\partial y^2} \right\},$$

$$1, \qquad ?, \qquad \frac{1}{Re}\frac{1}{\delta^2},$$

$$D = \left\{ u\frac{\partial v}{\partial x}, v\frac{\partial v}{\partial y} \right\}, \qquad E = \left\{ \frac{\partial p}{\partial y} \right\}, \qquad F = \left\{ \frac{1}{Re}\frac{\partial^2 v}{\partial y^2} \right\}$$

$$\delta, \qquad ?, \qquad \frac{1}{Re}\frac{1}{\delta}.$$

The sets A and C are significant in the streamwise equation because $A \gg D$ and $C \gg F$. Use of the significant set filter gives two dominant balances: $\{A, C\}$ and $\{A, B, C\}$. Balancing A and C yields the constraint $1 \sim Re^{-1}\delta^{-2}$, which says that the boundary layer thickness $\delta = O(1/\sqrt{Re})$. Note that this important qualitative relationship is derived without solving any PDEs.

Constraints derived from balancing assumptions are propagated through the quantity lattice. With the new constraint on δ, the graph searcher is able to infer that the maximal sets D and F must have the same order of magnitude. The maximal set E is a singleton set and so $O(E)$ is at most δ. Therefore, at $O(1)$ approximation, none of the terms in the transverse equation contribute. In particular, we have $\partial p/\partial y = 0$. Details of the two possible simplified models are shown in Fig. 10.

7. An example of composite model: the triple deck

In this section we will explain the steps of the procedure `simplify-composite-model`. The triple deck problem will be used as the running example. In particular we show how the values of the seven parameters in the problem—$\alpha, \beta, \bar{\pi}, \tilde{\pi}, \hat{\pi}, \hat{\sigma}, \tilde{\sigma}$—are determined. The first two steps of the procedure, preprocessing input and simplifying the submodels, have already been described. We will start at step (3).

7.1. Generating composite models

The results of calling `simplify-atomic-model` on each of the submodels of the triple deck are displayed in Fig. 11. The upper deck equations are reduced to the potential flow equations (UD-1). The lower deck is the usual Prandtl boundary layer equations (LD-1 and LD-2). Balancing the inertia terms, pressure term, and the viscous term of the streamwise boundary layer equation in LD-2 produces two parameter constraints:

$$\bar{\pi} = Re^{\beta - 1/2 - \alpha},$$

$$\alpha = 3\beta - \tfrac{3}{2}.$$

The main deck has four possible simplifications: MD-1, MD-2, MD-3, and MD-4, and the lower deck has two. Therefore eight composite models are possible.

7.2. Computing consequences

The purpose of this step is to extract further consequences from the simplified equations of each submodel. Two kinds of deductions are made. First, domain-specific rules are run on the submodels. For instance, a rule relating pressure and streamline slope asserts that $\hat{\pi} = O(v/u) = \hat{\sigma}$, after the program determines that the upper deck is a potential flow. Second, simplified equations are checked to see if they can be further resolved to a simpler form. For example, the equations for MD-1 can be resolved to give:

> (simplify-atomic-model *model*)
Making ⟨MODEL-2: PRANDTL-BOUNDARY-LAYER-1⟩ from
⟨MODEL-1: PRANDTL-BOUNDARY-LAYER⟩ ...
Assigning the leading order terms of TRANSVERSE-MOMENTUM-EQUATION to be: DPDY
Significant terms in STREAMWISE-MOMENTUM-EQUATION are:
UDUDX VDUDY D2UDY2/RE
STREAMWISE-MOMENTUM-EQUATION: 2 candidate dominant sets:
(D2UDY2/RE VDUDY UDUDX)
(D2UDY2/RE DPDX VDUDY UDUDX)

Making ⟨MODEL-3: PRANDTL-BOUNDARY-LAYER-1⟩ from
⟨MODEL-2: PRANDTL-BOUNDARY-LAYER-1⟩ ...
Balancing 2 terms:
D2UDY2/RE (VISCOUS STRESS TRANSVERSE) and
VDUDY (CONVECTIVE-ACCELERATION INERTIA TRANSVERSE)
in STREAMWISE-MOMENTUM-EQUATION with 1 parameter assumption: $Re = 1/\delta^2$
Assigning the leading order terms of STREAMWISE-MOMENTUM-EQUATION to be:
D2UDY2/RE and VDUDY and UDUDX
⟨MODEL-3: PRANDTL-BOUNDARY-LAYER-1⟩ is self-consistent.

Making ⟨MODEL-4: PRANDTL-BOUNDARY-LAYER-2⟩ from
⟨MODEL-2: PRANDTL-BOUNDARY-LAYER-1⟩ ...
Balancing 3 terms:
D2UDY2/RE (VISCOUS STRESS TRANSVERSE) and
DPDX (PRESSURE-GRADIENT) and
VDUDY (CONVECTIVE-ACCELERATION INERTIA TRANSVERSE)
in STREAMWISE-MOMENTUM-EQUATION with 1 parameter assumption: $Re = 1/\delta^2$
Assigning the leading order terms of STREAMWISE-MOMENTUM-EQUATION to be:
D2UDY2/RE and DPDX and VDUDY and UDUDX
⟨MODEL-4: PRANDTL-BOUNDARY-LAYER-2⟩ is self-consistent.

Two simplified models are found.

Model-3:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Model-4:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

The drag coefficient is $O(1/\sqrt{Re})$.

Fig. 10. The program finds two self-consistent simplified models. It deduces that $Re = 1/\delta^2$ in both of these models. It also computes the drag coefficient based on the magnitude of the velocity gradient $\partial u/\partial y$.

$$\frac{\partial}{\partial y}\left(\frac{v}{UB}\right) = 0.$$

Equations of the form:

$$\frac{\partial}{\partial y}Q = 0$$

Upper deck

UD-1:

$$\frac{\partial u_1}{\partial x} = -\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y}$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Main deck

MD-1:

$$UB\frac{\partial u_1}{\partial x} + v\frac{\partial UB}{\partial y} = 0$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial y} = 0$$

MD-2:

$$UB\frac{\partial u_1}{\partial x} + v\frac{\partial UB}{\partial y} = -\frac{\partial p}{\partial x}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial y} = 0$$

MD-3:

$$UB\frac{\partial u_1}{\partial x} + v\frac{\partial UB}{\partial y} = \frac{1}{Re}\frac{\partial^2 UB}{\partial y^2}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial y} = 0$$

MD-4:

$$UB\frac{\partial u_1}{\partial x} + v\frac{\partial UB}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 UB}{\partial y^2}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u_1}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Lower deck

LD-1:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

LD-2:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \frac{1}{Re}\frac{\partial^2 u}{\partial y^2}$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Fig. 11. Results after calling `simplify-atomic-model` on each of the submodels of the triple deck problem. The simplified model for the upper deck is a potential flow. The main deck has four consistent models. The lower deck reduces to the Prandtl boundary layer equations. Eight composite models are possible. Only the combination {UD-1, MD-1, LD-2 } produces enough matching constraints for solving the seven unknowns.

are called *transmission constraints*, and Q is a transmission quantity. Physically, a transmission constraint asserts that the quantity Q does not vary in the y-direction across two regions. It implies a matching constraint on the quantity Q between two adjacent regions.

7.3. Propagation of transmission constraints

For each transmission constraint, the program asserts that the quantities corresponding to Q in two adjacent regions have the same order of magnitude. For example, the transmission constraint $\partial p/\partial y$ in MD-1 leads to two assertions:

$$\tilde{\pi} = \tilde{\pi},$$
$$\hat{\pi} = \tilde{\pi}.$$

Similarly, the transmission constraint $(\partial/\partial y)(v/UB) = 0$ leads to two assertions:

$$\tilde{\sigma} Re^{\alpha - 1/2} = Re^{2\alpha - 4\beta + 3/2},$$
$$\tilde{\sigma} Re^{\alpha - 1/2} = \hat{\sigma}.$$

7.4. Solving the constraint set

The final step is to solve all the parameter constraints that have been asserted. Seven such constraints are asserted for the triple deck problem. They are collected here for convenience:

$$\tilde{\sigma} Re^{\alpha - 1/2} = Re^{2\alpha - 4\beta + 3/2}, \qquad \tilde{\sigma} Re^{\alpha - 1/2} = \hat{\sigma},$$
$$\tilde{\pi} = \bar{\pi}, \qquad \tilde{\pi} = \hat{\pi}, \qquad \hat{\pi} = \hat{\sigma}, \qquad \bar{\pi} = Re^{\beta - 1/2 - \alpha},$$
$$\alpha = 3\beta - \tfrac{3}{2}.$$

This set of nonlinear equations is easily solvable by hand. But like many other simple-looking equations, they are beyond the capability of a commercial computer algebra system like Mathematica. For example, the Mathematica 2.2 `Solve` procedure will not be able to solve for x in terms of a in a simple equation (see Fig. 12):

$$\log(x + 1) + \log(x - 1) = a.$$

Our solution is to build an algebraic simplifier on top of Mathematica to complement its equation solving capability. Our algebraic simplifier borrows an important idea from previous AI works: Bundy and Wellham's meta-rules for equation solving [7]. The meta-rule approach organizes algebraic rewrite rules into packets. Each packet performs a particular function on an expression. For instance, one has an `isolate` packet whose job is to isolate an unknown from an equation containing a single occurrence of the unknown. A `collect` and an `attract` packet reduce the number of occurrences of unknowns to one so that the `isolate` packet can apply. Our simplifier has three rule sets: (1) meta-rules for equation and inequality solving, (2) algebraic rewrite rules for standard simplification, and (3) rules for implementing the `ub-lb` and `sup-inf` algorithms. A

To solve: $\log(x+1)+\log(x-1)=a$
Apply attract rule: $\{\log ?x+\log ?y \rightarrow \log(?x*?y)\}$
to get: $\log(x+1)*(x-1)=a$
Apply collect rule: $\{(?x+?y)(?x-?y) \rightarrow ?x^2-?y^2\}$
to get: $\log(x^2-1)=a$
Apply isolate rule: $\{\log ?x=?y \rightarrow ?x=e^{?y}\}$
to get: $x^2-1=e^a$
Apply isolate rule: $\{?x-?y=?w \rightarrow ?x=?y+?w\}$
to get: $x^2=1+e^a$
Apply isolate rule: $\{?x^{?n}=?y \rightarrow ?x=?y^{1/?n}\}$
to get: $x=(1+e^a)^{1/2}$.

Fig. 12. Repeatedly apply meta-rules to the equation to solve for x.

$$\alpha=\tfrac{3}{8}$$
$$\beta=\tfrac{5}{8}$$
$$\hat{\pi}=\tilde{\pi}=\bar{\pi}=Re^{1/4}$$
$$\hat{\sigma}=Re^{1/4}$$
$$\tilde{\sigma}=Re^{1/8}$$

Fig. 13. Complete solution for the seven parameters in the triple deck problem.

significant portion of the rules handle exponentiation, maximum/minimum, derivatives, and inequalities.

The solutions for the parameters are given in Fig. 13.

Important physical information can be deduced from these parameters. The α and β values establish the extent of the lower deck: the horizontal extent is $O(Re^{-3/8})$ and vertical $O(Re^{-5/8})$. The horizontal velocity u in the lower deck is $O(Re^{(1/2-\beta)}) = O(Re^{-1/8})$ from which we calculate the velocity gradient $\partial u/\partial y = O(Re^{1/2})$. Using the velocity gradient and the definition of drag coefficient C_D [30], and integrating over the horizontal extent of the lower deck, we arrive at the following correction to the drag due to the finite edge effect:

$$C_D = O(Re^{-7/8}).$$

The conclusion that the finite edge effect contributes $O(Re^{-7/8})$ is highly nontrivial. Researchers had for years assumed that the next order correction to the drag on the flat plate is $O(Re^{-1})$ [26]. Notice that we arrive at the conclusion without solving the triple deck equations!

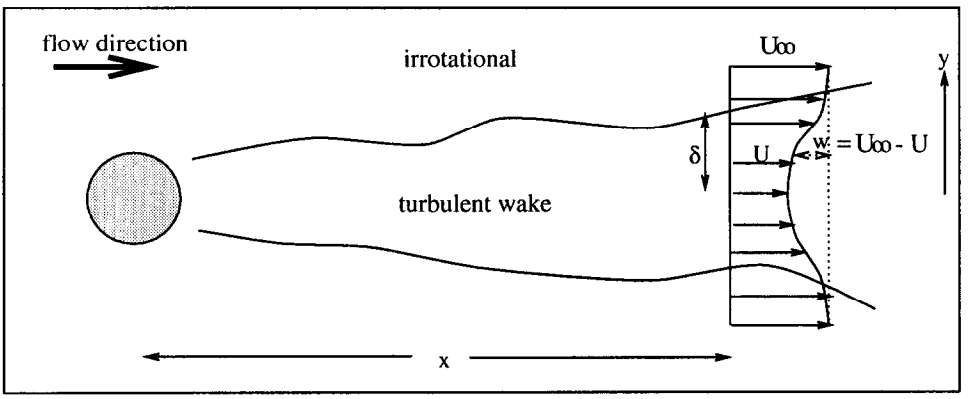

Fig. 14. Free turbulent flow over solid body. The width of the wake grows as the square root of the distance downstream. The velocity profile for the mean flow shows a deficit near the centerline. The velocity defect w is defined as the difference between the free stream velocity and the mean flow velocity. The velocity profile recovers the free stream velocity u_∞ in a distance of $O(\delta)$.

8. A third example: turbulent wake

Turbulent flows occur frequently in nature and engineering. However our understanding of turbulence is rather primitive. Few analytical results are available. In this section, we will illustrate the generality of our model simplification algorithm with a problem in free turbulent flow, a flow that is free of wall boundaries.

The problem concerns the shape of the wake and the behavior of the velocity profile in the downstream of a plane flow over a solid body (Fig. 14). The flow is studied at a place far away from the body so that (1) the details of the body won't matter, and (2) the averaged velocity profile reaches a steady state.

Turbulent flows are chaotic and unpredictable. Almost all theoretical studies use some type of statistical averaging. The hope is that averaged properties are easier to calculate and would represent the macroscopic properties important for engineering purposes. A particular type of averaging—the ensemble averaging—is commonly used. The idea is to average the results obtained from a series of identical experiments. For instance, in each experiment one measures the velocity and pressure at specific locations. Then the experiment is repeated many times with identical initial and boundary conditions. An average of these experiments is the ensemble average. The difference between the ensemble average and an individual experiment value is the fluctuating part. So we can write the velocity and pressure variables as follows:

$$u = U + u',$$
$$p = P + p',$$

where U and P are the mean velocity and pressure, and the prime quantities are the fluctuating parts. Substituting the mean and fluctuating variables into the Navier–

Model name: turbulent-free-wake
Independent variables:
x lower-bound = 0 upper-bound = 1
physical features = space, streamwise
y ...
Dependent variables:
U depends-on = x, y
lower-bound = 0 upper-bound = 1
physical features = velocity, streamwise, mean-flow
u' ...
Parameters:
Re type = large-parameter
physical features = dimensionless-number
δ type = small-parameter
physical features = dimensionless-number, length, transverse
u_0 type = small-parameter
physical features = dimensionless-number, velocity, turbulent
w_0 type = small-parameter
physical features = dimensionless-number, velocity, streamwise
Essential terms: turbulent, inertia
Equations:
Streamwise-momentum:

$$\frac{\partial U}{\partial x} + v\frac{\partial U}{\partial y} = -\frac{\partial P}{\partial x} + \frac{1}{Re}\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\right) + \frac{\partial}{\partial x}\left(-\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(-\overline{u'v'}\right) \tag{9}$$

Transverse-momentum:

$$U\frac{\partial V}{\partial x} + V\frac{\partial V}{\partial y} = -\frac{\partial P}{\partial y} + \frac{1}{Re}\left(\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2}\right) + \frac{\partial}{\partial x}\left(-\overline{u'v'}\right) + \frac{\partial}{\partial y}\left(-\overline{v'^2}\right) \tag{10}$$

Continuity:

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{11}$$

Relations: $U = U_\infty - w$
Order of Magnitude estimates:

$$U_\infty = \mathrm{O}(1), \quad U = \mathrm{O}(1), \quad x = \mathrm{O}(1), \quad y = \mathrm{O}(\delta),$$

$$u' = \mathrm{O}(u_0), \quad v' = \mathrm{O}(u_0), \quad w = \mathrm{O}(w_0)$$

Fig. 15. Problem specification for flow in the asymptotic downstream of a turbulent wake. The detailed model gives the 2D Reynolds averaged equations for the mean flow U and V. The equations are similar to the 2D Navier–Stokes equations except for the extra terms (like $(\partial/\partial y)(-\overline{u'v'})$) involving the fluctuating quantities u' and v', the Reynolds stress terms. These are called stress terms—even though they have nothing to do with molecular stresses—because mathematically they behave as if the total stress on the flow were composed of the viscous stress plus an additional component due to the mean product of the fluctuating velocities. The Reynolds stress terms are never negligible in a turbulent flow, but their analytical forms are in general unknown. The overbar stands for ensemble averaging. The problem has four dimensionless parameters: Re, δ, u_0, and w_0. Important analytical relationships can be deduced by order of magnitude reasoning without explicitly modeling the details of the Reynolds stress.

Two simplified models are found.

Model-1:

$$U\frac{\partial U}{\partial x} + \frac{\partial}{\partial y}\left(\overline{u'v'}\right) = 0$$
$$\frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left(\overline{v'^2}\right) = 0$$
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Parameter constraints:

$$w_0 = \frac{u_0^2}{\delta} \quad \text{and} \quad Re \gg \frac{1}{\delta^2}$$

Model-2:

$$U\frac{\partial U}{\partial x} - \frac{1}{Re}\frac{\partial^2 U}{\partial y^2} = 0$$
$$\frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left(-\overline{v'^2}\right) = 0$$
$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0$$

Parameter constraints:

$$w_0 \gg \frac{u_0^2}{\delta} \quad \text{and} \quad Re = \frac{1}{\delta^2}$$

Fig. 16. The program finds two self-consistent simplified models for the turbulent free wake problem. Notice it also deduces constraints among the dimensionless parameters. With an additional hypothesis of self-preserving flow, i.e., the various turbulent quantities are self-similar at various downstream locations, Model-1 can be solved analytically to produce formula relating the velocity defect and the width of the wake as functions of the downstream distance.

Stokes equations and performing the averaging, we will arrive at the Reynolds averaged equations for the mean flow (Fig. 15).

Using 2-set balance, the program finds two simplified models (Fig. 16). If the program uses a 3-set balance, it will find five simplified models.

The turbulent wake problem is interesting for two reasons. First, a host of other free turbulence problems—jet and shear flow for example—can be simplified in an identical way [25]. Second, the simplified models are simple enough to be solved explicitly if the flow is assumed to be self-similar. The solution gives an explicit dependence of the velocity defect and the width of the wake on the downstream distance. Such analytical results are extremely rare in turbulence. One also expects the results to be reliable because the simplification does not depend on the details of the Reynolds stress terms.

9. Evaluation

In conjunction with a program that automates formal perturbation expansion (an early version is described in [31]), AOM is used by the author to simplify many fluid models, including flows in turbulent wake, turbulent jet, chaotic wave motion in a tank, shallow water waves, and deep water waves. We plan to extend the program to handle flows over non-simple boundaries, such as shallow water waves with a sloping bottom or a slowly varying channel width. Modeling such flows is known to involve soliton equations with time-varying coefficients [15]. Many aspects of these flows are still subject of current research in the fluid dynamics literature.

What has been automated?

The method of dominant balance is one of the most general and powerful simplification methods. However, the technique has never been automated. Its application to partial differential equations with several parameters is not entirely straightforward because of the multitude of possible simplifications, parameter assumptions, and consistency checks. The explanations of choice of scalings found in typical published papers are usually ambiguous and not well-motivated. We do not usually understand them on first reading. The original papers on the triple decks are good examples of this problem.[8] By explicitly enumerating the balancing possibilities with the associated boundary matching and parameter assumptions, we obtain an explanation of the triple deck scalings which we believe is much clearer than the published accounts.

In general, simplification of a flow model by the dominant balance method involves seven steps:

(1) Break up the region of interest into qualitatively distinct subregions.
(2) Estimate the orders of magnitude for a subset of quantities.
(3) Determine the form of gauge functions.
(4) Enumerate balancing possibilities.
(5) Deduce consequences of balancing assumptions on parameters.
(6) Match boundaries and solve simplified equations to get additional constraints on parameters.
(7) Solve the resulting set of constraints.

AOM automates the last five steps. It depends on hints about the qualitative structure of the simplified models. For instance, AOM is told that there should be three decks in modeling the trailing edge of a flat plate. AOM would be much more useful if it can nondeterministically try out some simple models, compare their predictions with experimental data, and use the discrepancy between prediction and data to guide the construction of more elaborate models. For instance, in calculating the viscous drag on the flat plate, it might first propose the simple potential flow model. Noticing the potential flow model gives zero drag, it conjectures the importance of viscosity and looks for regions where a viscous term becomes important. Near the trailing edge of the plate, it finds the horizontal length scale is no longer large compared to the $O(Re^{-1/2})$ boundary layer, and therefore a new inner viscous layer may come into existence.

Step 3, the substitution of powers of ε, is automatic as explained in the paper. Several classical fluid problems require a more general class of gauge functions (such as $\varepsilon \log \varepsilon$); they frequently arise when a perturbation expansion is carried to higher-order. Not much systematic theory about these gauge function is known.

[8] The original author, K. Stewartson, came up with the triple deck in 1968 and published a paper in which the scalings of the regions are *wrong*. It took another year before the correct scalings as given in this paper were derived. Although the balancing idea is straightforward, applying the problem to coupled equations with several parameters and boundary conditions can be quite hairy even for experts. Scaling choices—no matter how non-obvious—are typically *not* explained in published papers. Disagreements among experts over the choice of scaling are quite common. The present author had personal experiences with this type of discussions when I was a coauthor of a fluid mechanics paper.

When does the simplification heuristic fail?

The simplification algorithm described in this paper has two major limitations. The first one can be handled by appropriate extensions to the current program. The second one requires a completely new method of attack.

The first limitation has to do with the class of gauge functions. The program currently recognizes only powers and negative powers of ε. It is well known that more general gauge functions, such as logarithms and exponentials of ε,[9] are required for many problems: the slender body approximation, and higher order matching in Stokes flow, just to name a few. Including more general gauges would complicate the solution matching procedure [26], a topic we plan to pursue.

The second limitation is more serious. The simplification algorithm assumes that a few significant length scales characterizes the physics of a problem. Consider a glass of water at room temperature. There are only a few length scales: the random motion of water molecules, the surface waves, and perhaps viscous effects in the thin boundary layer. But as the temperature and pressure of the water are raised to critical values (647 K and 217 atm), the density of the water and steam is equal. Density fluctuations that used to dominate at the molecular scale become increasingly macroscopic. Therefore at the critical region, effects of many wavelength scales have to be incorporated into a mathematical model [29]. A similar phenomenon occurs during the energy cascade via eddies of many length scales in a turbulent flow. Problems like these fall outside the framework of model simplification described in this paper.

How good are the approximate models?

There is no simple answer to this question. First, like many simplification methods, the accuracy of the dominant balance depends on a continuity property between the equation and the solution space, namely, small changes in the equation correspond to small changes in its solution. So one can expect a poor approximation when equations do not have this property. For instance, in finding the intersection of two nearly parallel lines, small changes in the coefficients of the linear equations can lead to a large change in the solution. In numerical analysis, such set of equations is called ill-conditioned.

Second, existence and uniqueness theorems for PDEs are rare. One often does not know whether the approximate model has a solution at all or whether the solution if exists will be unique. The strongest claim one can make seems to be this: An approximate model that is *not* self-consistent is certainly a poor approximation. This claim is easy to justify.[10] Let x and x' be the true and approximate solutions to an equation, and let $T(x)$ be a term of the equation to be neglected. Assuming T varies continuously with x, we can state the continuity property as follows:

$$T(x) \text{ small } \Rightarrow\ x \approx x' \ \Rightarrow\ T(x') \text{ small.}$$

Hence, if $T(x)$ is large, then $T(x')$ is large and therefore $x \not\approx x$.

[9] These are the so-called Hardy's L-functions [11].

[10] A similar argument is made in [13].

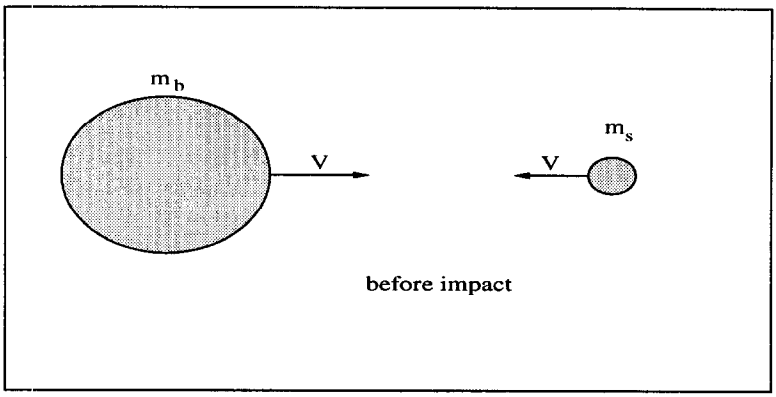

Fig. 17. Raiman's colliding ball examples. Two balls with masses m_b and m_s are approaching each other with an equal and opposite velocity V. Assume that the collision is elastic and that $m_b \gg m_s$. What is the velocity of the small ball after impact?

In practice, an approximate model is validated by subjecting its predictions to experimental and numerical checks. In fact, there still exists no theorem which speaks to the validity and accuracy of Prandtl's boundary layer approximation, but ninety years of experimental results leave little doubt of its validity and its value.

10. Related work

Previous work in equation simplification involves the various styles of order of magnitude reasoning.

Raiman's *Estimates* [20] is based on the notion of order of magnitude scales. The intuition is that an order of magnitude description is a coarse description of a quantity. The coarse value $V(q)$ of a quantity q is defined as a set of values which contains the value of q. Two quantities p and q are said to be equal in order of magnitude if the sets $V(p)$ and $V(q)$ overlap. Two primitive scales, *small* and *rough*, are provided; they can be used to build up finer order of magnitude scales.

Order of magnitude equality, under Raiman's definition, is *not* transitive. As a consequence, one cannot substitute equals for equals, which complicates the inference machinery. Part of the justification for the intransitivity comes from considering the colliding ball example (Fig. 17). We will show that the problem can be easily solved by the use of gauge function without sacrificing the substitution of equals for equals.

Let m_b and m_s be the masses of the big and small ball respectively. Let v and u with the corresponding subscripts to denote their velocities before and after impact. The initial conditions are $m_b \gg m_s$, $v_b = V$, and $v_s = -V$. The question is: What are u_b and u_s? Rewriting Raiman's equations (30) and (31) in terms of a small parameter $\varepsilon = m_s/m_b \ll 1$ yields:

$$u_b - V = \varepsilon(-V - u_s), \qquad u_b + V = u_s - V.$$

The maximal terms in the first equation are u_b and V on the left-hand side. Balancing them gives $u_b = V$. Substituting $u_b = V$ into the second equation immediately gives the desired answer: $u_s = 3V$.

Mavrovouniotis and Stephanopoulos' O[M] program [14] improved on Raiman's earlier FOG formalism [19]. O[M] provides seven primitive relations, such as `much smaller than` and `moderately smaller than`. These relations are based on an interpretation of the positive real line as an ordered set of disjoint intervals delimited by five distinguished points, ε, $1/(1+\varepsilon)$, 1, $1+\varepsilon$, and ε^{-1}. This interpretation is reminiscent of the ordering of gauge functions in the asymptotic theory of AOM. But there are three major differences between the two schemes. First, AOM uses a much richer set of gauge functions, namely, all powers of ε, which are necessary for handling complicated singular equations. Second, AOM's order of magnitude scales are hierarchical. Terms of $O(1)$ are first compared, then those of $O(\varepsilon^n)$, followed by those of $O(\varepsilon^{2n})$, and so on. Third, AOM's gauges are *functions* whereas O[M]'s ε is a problem-dependent scalar. Thus, O[M] formalizes the concept of numerical order of magnitude rather than that of asymptotic order of magnitude.

Murthy [17] introduces a four-level quantity space, incorporating some of Raiman's infinitesimal relations, and logarithms. The use of logarithmic scales in approximation is explored by Bennett [3] and, more extensively, in Nayak's Napier [18]. These authors define the order of magnitude of a quantity $O(q)$ to be $\log_b |q|$. The choice of the base b is problem-dependent. Although the logarithmic scales are more refined than the scales in O[M], they are designed with a similar purpose, namely, to handle numerical order of magnitude.

In summary, most of the previous works explored order of magnitude reasoning based on either numerical order or some form of infinitesimal. These programs handle algebraic equations and qualitative differential equations. AOM complements them by a theory of asymptotic order of functions, which provides a uniform foundation for simplifying algebraic equations, ODEs, and PDEs.

11. Conclusion

Building approximate models is not just useful: they are essential for a clear understanding of the relevant physics in a difficult problem. This paper describes a simplification method based on three ideas: asymptotic order of functions, the dominant balance procedure, and constraint formulation due to parameter assumptions and boundary matching. The paper introduces a number of technical advances:

(1) It introduces asymptotic order of magnitude reasoning to extend the range of problems solvable by qualitative reasoning techniques.
(2) It articulates and explains the dominant balance method so that even a beginning graduate student can use the technique to simplify partial differential equations with several parameters.
(3) It produces more systematic explanations of the scalings used in classical fluid models like the triple deck.

(4) It provides the scalings necessary for the automatic application of formal perturbation expansion procedures.

Professional scientists have the ability to simplify problems. They routinely use many powerful heuristic and qualitative methods to simplify a problem and get at its essence before any lengthy calculation is attempted. Dimensional and order of magnitude estimates, exploitation of small parameters, consideration of limiting cases, perturbation expansions, use of analytical properties of physical quantities, consequences of symmetry, diagrammatic representations, and renormalizability of theory—these methods are not mere tricks to solve specific problems; rather they embody general ideas for strategic reformulation of problems and intelligent guess of the form of the solutions. Isolating these ideas and articulating them explicitly would go a long way towards understanding some of the core skills that make up a theoretical scientist.

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