Non-adaptability measures in the pseudo-questionnaires context

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Received 22 January 2007; received in revised form 20 August 2007; accepted 12 September 2007
Available online 21 September 2007

Abstract

Measuring the degree of non-adaptability of a partition to a criterion, represented by a reference partition, is an essential step in pseudo-questionnaires theory. In this work we characterize axiomatically the measure of non-adaptability in a general context. We base it on pre-orders on the set of all the possible experiences (complete or incomplete partitions). The construction of this measure is crucial for practical applications. It can be done in a natural way starting from the atoms of the partitions and constructing the non-adaptability measure by a process of successive aggregations made by suitable aggregation operators.

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Keywords: Pseudo-questionnaire; Partition; T-norm and t-conorm; Non-additive measure; Adaptability; Compositive non-adaptability measure

1. Introduction

Picard [14] introduced in 1965 the questionnaire theory in order to formalize the classification processes (such as census) according to the answers given to some questions. In 1970, Terrenoire [18,19] proposed an extension of this formalization, the pseudo-questionnaires, which allows to deal with the inquiring processes, that is, to assign an element to a subset of a given classification. These studies were later continued by some
other authors (see, e.g., [5,7]). The main difference between questionnaires and pseudo-questionnaires is that questionnaires always consider direct questions (for instance “Are you European?”; if we are interested in classifying European and no European people) and pseudo-questionnaires consider indirect questions (for instance “Do you have headache?”; if we are interested in determining the disease of the patient).

Both, questionnaires and pseudo-questionnaires, have been defined in terms of probabilities of questions and answers. However, in many situations no probability distributions are available and nevertheless classifying processes have to be built up. Thus, some generalizations of questionnaire and pseudo-questionnaire theory have been proposed by Bertoluzza [3] and Bertoluzza and Salas [4]. In these generalizations the answers, which are propositions, are represented, via the Stone Theorem, by subsets of a suitable reference space $\Omega$. In this framework the questions are described by the expected answers, that is, by a collection of subsets which is a complete or incomplete partition.

In Picard’s formalization, the questions determine the ultimate classification, whereas in the Terrenoire approach the desired final partition (the reference partition) is given “a priori”; questions are used to reach the goal, that is, to assign an individual to a class of the a priori classification.

A classical example of application of the pseudo-questionnaires is given in the medical environment. In this framework, the goal is “pick out individual diseases”. Thus, the final partition is the class of the sets $A_i$ each of which represents the proposition “the disease is $d_i$”, $i = 1, 2, \ldots, n$ ($A_i$ are supposed to be pairwise disjoint). Questions can be clinical tests, radiographs and so on; in general, they do not reach an unquestionable classification, but only give a sequence of partitions which approximate more and more the right individuation of diseases. The sequence of questions is complete if the obtained partition at certain point is a refinement of the reference partition. Many times, a weaker purpose can be obtained: individuate a particular disease. In this case, the process concludes if we can obtain a path composed of questions (tests) and answers (results) such that the logical conjunction of the answers is represented by a collection of disjoint sets, each of them is contained in a element of the partition.

There exist many problems of this kind in economy, social sciences, physic and so on. They are characterized by the impossibility of making direct questions such as “Are you in the class $A_i$?”. In these cases, it is only possible try to guess the result by means of indirect questions.

Since it is very difficult to obtain the complete individuation process, it is natural to ask questions until a partition is obtained, which approximates the reference partition, and therefore a measure of “fitting” is needed. Montes et al. [11–13] proposed a partial answer in the case where the obtained partition and the reference partition have the same number of elements. However, in the general case (in particular in medical tests), the number of elements of both partitions are not equal and nevertheless a measure of “fitting” is necessary. The necessity of comparing any two partitions motivated this study. In this paper we present the results obtained in this line of research. Our goal is to provide a proper definition of measure of non-adaptability, as a tool to compare a partition obtained by posing questions with a given one. Apart from the general study, some specific kind of measures of non-adaptability are studied, the composite ones, in order to use them in practical situations.

Our paper is organised as follows. Section 2 features a brief review of known concepts of partitions, non-additive measures and some specific aggregation operators: $t$-norms and $t$-conorms. In Section 3 we introduce the concept of non-adaptability measure, we relate it with the adaptability order and we find its main and natural properties. This study is concluded in Section 4 with the characterization of left, right-atom and totally compositive non-adaptability measures. In these cases the non-adaptability measure can be built by combining the measure of non-adaptability of each atom, and such combination is done by means of $t$-norms and $t$-conorms. A summarizing conclusion is provided.

2. Basic concepts

2.1. Partitions

Since we would like to measure the degree of non-adaptability of a partition, obtained by a questionnaire, to an objective (partition of reference), we should, first of all, establish the space in which we are working. Let us consider a measurable space $(\Omega, \mathcal{F})$, where $\Omega$ is the set of the elementary events and $\mathcal{F}$ is the algebra of subsets of $\Omega$ formed by the observable events. In this space, $\mathcal{E}$ denotes the collection of possible experiences
(finite partitions completed or not\(^1\)) on \((\Omega, \mathcal{A})\). For the sake of simplicity, let us suppose that \(\{A\} \in \mathcal{A}\) for all \(A \in \mathcal{A}\). Besides that, let us consider on \(\mathcal{A}\) a partial order relation \(\sqsubseteq\) and two operations (\(\land\) and \(\lor\)), such that, for any \(\Pi_1\) and \(\Pi_2\) in \(\mathcal{A}\),

- \(\Pi_1\) is said to be a subpartition of \(\Pi_2\), denoted by \(\Pi_1 \sqsubseteq \Pi_2\), if and only if any element of \(\Pi_1\) is a subset of an element of \(\Pi_2\), that is,
  \[
  \Pi_1 \sqsubseteq \Pi_2 \iff \forall A_i \in \Pi_1, \exists B_j \in \Pi_2 | A_i \subseteq B_j.
  \]
  Note that it is not required \(\Pi_1\) and \(\Pi_2\) to have the same support, although it is evident that \(\text{Supp}(\Pi_1) \subseteq \text{Supp}(\Pi_2)\).

In the particular case, \(\text{Supp}(\Pi_1) = \text{Supp}(\Pi_2)\) a subpartition is a refinement. Thus, the concept of subpartition generalizes the classical concept of refinement.

- The product of \(\Pi_1\) and \(\Pi_2\), denoted by \(\Pi_1 \land \Pi_2\), is the partition formed by the intersections of all the elements of \(\Pi_1\) with all the elements of \(\Pi_2\), that is,
  \[
  \Pi_1 \land \Pi_2 = \{A_i \cap B_j | A_i \in \Pi_1, B_j \in \Pi_2\}.
  \]
  Evidently, the support of \(\Pi_1 \land \Pi_2\) is \(\text{Supp}(\Pi_1) \cap \text{Supp}(\Pi_2)\).

- If the supports \(\text{Supp}(\Pi_1)\) and \(\text{Supp}(\Pi_2)\) are disjoint, the union (or disjunction) of \(\Pi_1\) and \(\Pi_2\) will be denoted by \(\Pi_1 \lor \Pi_2\) and it is defined as the partition formed by all the elements of \(\Pi_1\) and all the elements of \(\Pi_2\), that is,
  \[
  \Pi_1 \lor \Pi_2 = \{A_i | A_i \in \Pi_1\lor A_i \in \Pi_2\}.
  \]
  Evidently, the support of \(\Pi_1 \lor \Pi_2\) is \(\text{Supp}(\Pi_1) \cup \text{Supp}(\Pi_2)\).

2.2. T-norms and t-conorms

In practical cases, an explicit form of the non-adaptability measures would be very useful. This explicit form can be obtained when we work with compositive measures. We will see in Section 4 the appropriate composition of the atoms will be made by means of t-norms and t-conorms (for a complete overview, see [9]), which are particular types of aggregation operators. Thus, we will recall briefly these concepts.

Triangular norms (t-norms for short) were originally introduced by Menger [10] as operators for fusion of distribution functions, needed when generalizing the triangle inequality from classical metric spaces to probabilistic metric spaces. Nowadays, the axioms of t-norms are a little more restrictive. They were given by Schweizer and Sklar [16] and they require, besides the conditions imposed by Menger, associativity and a boundary condition (1 is the neutral element).

Mathematically, a t-norm is an increasing, commutative and associative binary operation on \([0, 1]\) with neutral element 1. The three most important t-norms are the minimum operator \(T_{\text{M}}(x, y) = \min(x, y)\), the algebraic product \(T_p(x, y) = xy\) and the Łukasiewicz t-norm \(T_L(x, y) = \max(x + y - 1, 0)\). The associativity allows us to extend each t-norm \(T\) in a unique way to an n-ary operation in the usual way by induction, defining for each n-tuple \((x_1, x_2, \ldots, x_n) \in [0, 1]^n\)

\[
T(x_1, x_2, \ldots, x_n) = T(\underbrace{T_{\ldots}(x_1, x_2)}_{n-1}, x_n), \quad n > 2.
\]

On the other hand, a t-conorm is an increasing, commutative and associative binary operation on \([0, 1]\) with neutral element 0. T-norms and t-conorms come in dual pairs: to any t-norm \(T\) there corresponds a t-conorm \(S\) through the relationship \(S(x, y) = 1 - T(1 - x, 1 - y)\). For the above three t-norms this yields the maximum operator \(S_{\text{M}}(x, y) = \max(x, y)\), the probabilistic sum \(S_p(x, y) = x + y - xy\) and the Łukasiewicz t-conorm (bounded sum) \(S_L(x, y) = \min(x + y, 1)\).

\(^1\) A partition \(\Pi\) is said to be complete if, and only if, \(\text{Supp}(\Pi) = \Omega\), where \(\text{Supp}(\Pi)\) denotes the support of \(\Pi\), that is, the union of any element of \(\Pi\).
2.3. Non-additive measures

The concept of measure is one of the most important concepts in mathematics. Classically, measures are supposed to be additive, that is, if \( A_1, A_2, \ldots, A_n \) is a collection of disjoint elements of \( \mathcal{A} \), then

\[
m\left( \bigcup_{i=1}^{n} A_i \right) = \sum_{i=1}^{n} m(A_i).
\]

This property can be very appropriate in some situations, but can also be somewhat inadequate in many reasoning environments of the real world. For example, if the efficiency of a set of workers is being measured, the efficiency of the people doing teamwork is not the addition of the efficiency of each individual working on their own. In 1974, Sugeno introduced the concept of fuzzy measure [17], which only requires monotonicity related to the inclusion of sets instead of additivity.

Formally, a fuzzy measure is a function \( m : \mathcal{A} \to [0,1] \) such that \( m(\emptyset) = 0 \), \( m(\Omega) = 1 \) and it satisfies the properties of monotonicity (\( \forall A, B \in \mathcal{A} \), if \( A \subseteq B \), then \( m(A) \leq m(B) \)) and continuity (\( \forall n \in \mathbb{N}, A_n \in \mathcal{A} \) and \( (A_n)_{n \in \mathbb{N}} \) monotonic \( (A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots \) or \( A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n \supseteq \cdots \)) implies that \( \lim_{n \to \infty} m(A_n) = m(\lim_{n \to \infty} A_n) \). As examples of fuzzy measures, we could consider several measures on finite algebras, such as probability, credibility and plausibility measures (see [1]).

For any \( \ell \)-conorm \( S \), the concept of \( S \)-decomposition is proposed by Weber in [22] as a generalization of the additivity. This concept allow us to obtain some important kinds of fuzzy measures. More precisely, a measure \( m \) is said to be \( S \)-decomposable if \( m(A \cup B) = S(m(A), m(B)) \), for all \( A, B \in \mathcal{A} \) such that \( A \cap B = \emptyset \).

An overview on non-additive measures is given in [6]. For a more advanced presentation we could see [8,21]. For a more axiomatic foundation see [15,20].

3. Non-adaptability measure

In the set of partitions \( \mathcal{E} \), we have considered suitable, mainly in the questionnaires framework, the following definition of order of adaptability. The idea behind this concept is that for any partition \( \Pi \) in \( \mathcal{E} \), we are interested in classifying the partitions by means of a pre-order, so that the “smallest” partitions are the best adapted to \( \Pi \) and the “greatest” ones the worst adapted to \( \Pi \).

Definition 1. Let \( (\Omega, \mathcal{A}) \) a measurable space and let \( \mathcal{E} \) be the collection of finite partitions on \( (\Omega, \mathcal{A}) \). A function \( f : \mathcal{E} \to \{ \text{Total pre-orders of } \mathcal{E} \} \) is said to be an order of adaptability if, and only if, it satisfies the following conditions:

1. If \( \Pi_1 \subseteq \Pi_2 \) then \( \Pi_1 \leq_{H} \Pi_2 \) for all \( \Pi_1, \Pi_2 \in \mathcal{E} \), where \( f(\Pi) = \leq_{H} \).
2. For all \( \Pi \in \mathcal{E} \), \( \Pi \) is the minimum element of \( (\mathcal{E}/\sim_{H}, \leq_{H}) \), being \( \sim_{H} \) the equivalence relation defined by \( \leq_{H}(\Pi_1 \sim_{H} \Pi_2) \iff \Pi_1 \leq_{H} \Pi_2 \) and \( \Pi_2 \leq_{H} \Pi_1 \).
3. For all \( \Pi \in \mathcal{E} \), there exists a lattice isomorphism \( \varphi_{\Pi} : (\mathcal{E}/\sim_{H}, \leq_{H}) \to (A, \leq) \) associated to \( f \) such that if \( \Pi \subseteq \Pi' \) then \( \varphi_{\Pi} \circ p \geq \varphi_{\Pi'} \circ p \), being \( A \subseteq \mathbb{R} \), \( p : \mathcal{E} \to \mathcal{E}/\sim_{H} \) the canonical projection and \( \leq \) the usual order of \( \mathbb{R} \).

In the previous definition, we could consider \( \Pi \) as the reference partition and then the relation \( \leq_{H} \) orders the different partitions (obtained by different pseudo-questionnaires or in different step of a pseudo-questionnaire) by their adaptability to \( \Pi \). In other words, \( \Pi_1 \leq_{H} \Pi_2 \) means that \( \Pi_1 \) is more adapted to \( \Pi \) than \( \Pi_2 \), that is, \( \Pi_1 \) is a better classification.

Below we present an example of this concept, in which we can start to look at the idea of non-adaptability measure.

Example 2. Let \( (\Omega, \mathcal{A}) \) a measurable space, let \( \mathcal{E} \) be the collection of finite partitions on \( (\Omega, \mathcal{A}) \) and let \( m \) be a fuzzy measure. For any two partitions \( \Pi_1 = \{ A_i \}_{i=1}^{n} \) and \( \Pi_2 = \{ C_j \}_{j=1}^{r} \) in \( \mathcal{E} \), we define the map \( A : \mathcal{E} \times \mathcal{E} \to [0,1] \) as follows
\[ \Delta(P_1, P) = \sum_{\sigma \in \mathcal{A}p_l(n,r)} \left( m \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma(i)}) \right) \right), \]

where \( Ap(n,r) \) denotes the set of applications from \( \{1, \ldots, n\} \) to \( \{1, \ldots, r\} \).

Now, let us consider the map \( f: \mathcal{E} \to \{\text{Total pre-orders of } \mathcal{E}\} \) defined by \( f(P) = \leq_P \) for any \( P \in \mathcal{E} \), where \( \leq_P \) denotes the following pre-order

\[ P_1 \leq_P P_2 \iff \Delta(P_1, P_2) \leq \Delta(P_2, P_1). \]

It is easy to see that \( f \) is well defined because \( f(P) \) is defined in terms of the usual total order of \( \mathbb{R} \).

In order to prove that this map \( f \) is an order of adaptability, we only have to prove \( f \) fulfils the three conditions imposed in Definition 1.

1. Let \( P_1 = \{A_i\}_{i=1}^n, P_2 = \{B_k\}_{k=1}^s, P = \{C_j\}_{j=1}^r \) be three partitions in \( \mathcal{E} \) such that \( P_1 \sqsubseteq P_2 \). Then,

\[ \Delta(P_2, P) = \sum_{\sigma \in \mathcal{A}p_l(n,r)} \left( m \left( \bigcup_{B_k \in P_2} (B_k - C_{\sigma(k)}) \right) \right) = m \left( \bigcup_{B_k \in P_2} (B_k - C_{\sigma_2(k)}) \right) \]

being \( \sigma_2 \) the application where the minimum is reached.

Moreover, as \( P_1 \sqsubseteq P_2 \) and \( m \) is a fuzzy measure, we have that

\[ m \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma(i)}) \right) \]

\[ = m \left( \bigcup_{B_k \in P_2} \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma_2(k)}) \right) \right) = m \left( \bigcup_{B_k \in P_2} \left( \bigcup_{A_i \in P_1} A_i \right) - C_{\sigma_2(k)} \right) \]

\[ \leq m \left( \bigcup_{B_k \in P_2} (B_k - C_{\sigma_2(k)}) \right) = \Delta(P_2, P), \]

where \( \sigma \) is the element of \( Ap(n,r) \) defined by \( \sigma(i) = \sigma_2(k) \) if \( A_i \sqsubseteq B_k \).

Finally,

\[ \Delta(P_1, P) = \sum_{\sigma \in \mathcal{A}p_l(n,r)} \left( m \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma(i)}) \right) \right) \leq m \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma_1(i)}) \right) \]

and, therefore, \( \Delta(P_1, P) \leq \Delta(P_2, P) \), that is, \( P_1 \leq_P P_2 \).

2. Let \( P = \{C_j\}_{j=1}^r \) be any partition in \( \mathcal{E} \). As \( m \) is a fuzzy measure, \( m(A) \geq 0 \) for all \( A \in \mathcal{E} \) and then,

\[ 0 \leq \Delta(P, P) = \sum_{\sigma \in \mathcal{A}p_l(n,r)} \left( m \left( \bigcup_{C_j \in P} (C_j - C_{\sigma(j)}) \right) \right) \leq m \left( \bigcup_{C_j \in P} (C_j - C_j) \right) = m(\emptyset) = 0. \]

Therefore, \( \Delta(P, P) = 0 \), and we conclude that \( P \leq_P P \).

3. Let \( \varphi_P : (\mathcal{E}/\sim_P, \leq_P) \to (\mathbb{R}, \leq) \) be the application defined by \( \varphi_P([P_1]) = \Delta(P_1, P) \), being \( P \) a representative of the equivalence class \([P_1]\). It is easy to prove that \( \varphi_P : (\mathcal{E}/\sim_P, \leq_P) \to (A, \leq) \) is a well defined lattice isomorphism, being \( A = \text{Im}(\varphi_P) \).

In order to prove the last inequality, let \( P_1 = \{A_i\}_{i=1}^n, P_2 = \{C_j\}_{j=1}^r \) and \( P' = \{D_k\}_{k=1}^s \) be three partitions in \( \mathcal{E} \) such that \( P \sqsubseteq P' \). On one hand,

\[ \Delta(P_1, P) = \sum_{\sigma \in \mathcal{A}p_l(n,r)} \left( m \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma(i)}) \right) \right) = m \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma_1(i)}) \right) \]

being \( \sigma_1 \) the application where the minimum is reached.

On the other hand, as \( P \sqsubseteq P' \), we know that each \( C_{\sigma(i)} \in P \), there exists an element \( D_k \) such that \( C_{\sigma(i)} \sqsubseteq D_j \). Thus, \( (A_i - C_{\sigma(i)}) \supseteq (A_i - D_j) \) and \( \bigcup_{A_i \in P_1} (A_i - C_{\sigma(i)}) \supseteq \bigcup_{A_i \in P_1} (A_i - D_j) \). By the axioms of fuzzy measure,

\[ \min \left( \bigcup_{A_i \in P_1} (A_i - C_{\sigma(i)}) \right) \geq \min \left( \bigcup_{A_i \in P_1} (A_i - D_j) \right) = \min \left( \bigcup_{A_i \in P_1} (A_i - D_{\sigma(i)}) \right) \]

being \( \sigma(i) = j \), an element of \( Ap(n,s) \).
As
\[
m\left(\bigcup_{A_i \in \Pi_1} (A_i - D_{\sigma(i)})\right) \geq \min_{\sigma \in \text{Ap}(\Pi,\alpha)} \left\{ m\left(\bigcup_{A_i \in \Pi_1} (A_i - D_{\sigma(i)})\right) \right\} = \Delta(\Pi_1, \Pi'),
\]
we conclude that $\varphi_{\Pi_1}(p(\Pi_1)) \geq \varphi_{\Pi'}(p(\Pi_1))$ for all $\Pi_1 \in \mathcal{E}$.

In particular, let us consider $\Omega = \{1, 2, 3, 4, 5\}$, $\mathcal{A}$ is the power set of $\Omega$, that is, $\mathcal{A} = \mathcal{P}(\Omega)$ and the fuzzy measure $m$ is defined as $m(A) = \frac{\text{card}(A)}{\text{card}(\Omega)}$ for any $A \in \mathcal{A}$, where $\text{card}$ denotes the cardinality of the set. In this context, if $\Pi_1 = \{\{1, 2\}, \{3\} \}, \Pi_2 = \{\{3\}, \{4, 5\}\}$ and $\Pi = \{\{1, 3\}, \{2, 4, 5\}\}$, then
\[
\Delta(\Pi_1, \Pi) = T_M \left( m\left(\{\{1, 2\}, \{3\}\} \right) \right) = \min\{m(\{\{1, 2\}, \{3\}\})\} = \min\{1, 2\} = 1/5
\]
and
\[
\Delta(\Pi_2, \Pi) = T_M \left( m\left(\{\{3\}, \{4, 5\}\} \right) \right) = \min\{m(\{\{3\}, \{4, 5\}\})\} = \min\{1/5, 2/5, 1/5, 5/5\} = 1/5
\]
where $C_i$ denotes the elements of $\Pi_i$. Thus, $\Delta(\Pi_2, \Pi) \leq \Delta(\Pi_1, \Pi)$ and therefore, $\Pi_2 \leq_{\Pi} \Pi_1$ and $\Pi_1 \not\leq_{\Pi} \Pi_2$.

With the order of adaptability we have obtained a pre-order in $\mathcal{E}$ associated to any partition, which allow us to classify the partition in a proper way. However, this concept is not easy to be used in practical situations and therefore, we are going to present an equivalent definition, whose use is simpler. Thus, we can write Definition 1 in terms of the divergence between two partitions.

**Definition 3.** Let $(\Omega, \mathcal{A})$ be a measurable space and let $\mathcal{E}$ be the collection of finite partitions on $(\Omega, \mathcal{A})$. A map $\Delta : \mathcal{E} \times \mathcal{E} \to \mathbb{R}^+$ is a non-adaptability measure if, and only if,

1. $\Delta(\Pi, \Pi) = 0 \ \forall \Pi \in \mathcal{E}$.
2. $\Delta(\Pi_1, \Pi) \leq \Delta(\Pi_2, \Pi)$ and $\Delta(\Pi, \Pi_1) \geq \Delta(\Pi, \Pi_2)$, for all $\Pi_1, \Pi_2, \Pi \in \mathcal{E}$ such that $\Pi_1 \supseteq \Pi_2$.

These two properties arise in a natural way and they can be easily interpreted. The first axiom is understood as follows: if the goal is reached, that is, if we have obtained the final partition, then the adaptability is maximum ($\Delta(\Pi, \Pi) = 0$). For the second one, if we do more questions, then the adaptability obtained is, at least, as good as if we stop at this moment ($\Delta(\Pi_1, \Pi) \leq \Delta(\Pi_2, \Pi)$); moreover, if the final partition has more elements, then it is more difficult to reach to it ($\Delta(\Pi, \Pi_1) \geq \Delta(\Pi, \Pi_2)$).

Definitions 1 and 3 are equivalent as it follows from the next theorem.

**Theorem 4.** Let us consider the set $\mathcal{M}$ formed by all the non-adaptability measures and the set $\mathcal{C}$ formed by all the adaptability orders such that the lattice isomorphism $\varphi_{\Pi}$ fulfils that $\varphi_{\Pi}(\{\Pi\}) = 0$ for all $\Pi \in \mathcal{E}$. There is a one-to-one correspondence between $\mathcal{M}$ and $\mathcal{C}$.

**Proof.** Let $\mathcal{E}$ be an order of adaptability, we can define the map $\Psi : \mathcal{C} \to \mathcal{M}$ as
\[
\Psi(f)(\Pi_1, \Pi) = \varphi_{\Pi}(\{\Pi_1\}).
\]
Since

1. $\Psi(f)(\Pi, \Pi) = \varphi_{\Pi}(\{\Pi\}) = 0$.
2. For any $\Pi_1, \Pi_2, \Pi$ in $\mathcal{E}$ such that $\Pi_1 \subseteq \Pi_2$, we have that $\Pi_1 \leq_{\Pi} \Pi_2$ and $\Psi(f)(\Pi_1, \Pi) = \varphi_{\Pi}(\{\Pi_1\}) \leq \varphi_{\Pi}(\{\Pi_2\}) = \Psi(f)(\Pi_2, \Pi)$. In other hand, $\varphi_{\Pi_1} \circ p \geq \varphi_{\Pi_2} \circ p$ and $\Psi(f)(\Pi, \Pi_1) = (\varphi_{\Pi_1} \circ p)(\Pi) \geq (\varphi_{\Pi_2} \circ p)(\Pi) = \Psi(f)(\Pi, \Pi_2)$,

then, $\Psi(f)$ is a non-adaptability measure.
Conversely, let us define the inverse of $\Psi$, which will be denoted by $\Phi$. Thus, if $\Delta$ is a non-adaptability measure, $\Phi(\Delta)$ is defined, for any $\Pi \in \mathcal{E}$, as $\Phi(\Delta)(\Pi) = \leq_H$, where $\leq_H$ is given by

$$\Pi_1 \leq_H \Pi_2 \iff \Delta(\Pi_1, \Pi) \leq \Delta(\Pi_2, \Pi).$$

It is easy to check that $\Phi(\Delta)(\Pi)$ is a pre-order and

1. If $\Pi_1 \subseteq \Pi_2$ then $\Delta(\Pi_1, \Pi) \leq \Delta(\Pi_2, \Pi)$ and $\Pi_1 \leq_H \Pi_2$.
2. For all $\Pi, \Pi_1 \in \mathcal{E}$, we have that $\Delta(\Pi, \Pi_1) = 0$ and $\Delta(\Pi_1, \Pi) \geq 0$, therefore $\Pi \leq_H \Pi_1$ and $\Pi$ is the minimum element of $\left( \mathcal{E} \setminus \Pi, \leq_H \right)$.
3. We can define an application $\varphi_H : (\mathcal{E} \setminus \Pi, \leq_H) \to (\mathbb{R}, \leq)$ by $\varphi_H(\Pi) = \Delta(\Pi, \Pi_1)$. Let $\Pi_1, \Pi, \Pi'$ be four partitions in $\mathcal{E}$ such that $\Pi \subseteq \Pi'$ then $\varphi_H \circ p(\Pi_1) = \Delta(\Pi_1, \Pi) \geq \Delta(\Pi_1, \Pi') = \varphi_H \circ p(\Pi_1)$. As this inequality is fulfilled for any $\Pi_1 \in \mathcal{E}$, then $\varphi_H \circ p \geq \varphi_H \circ p$.

Therefore, $\Phi(\Delta)$ is an order of adaptability with $\varphi_H(\Pi) = \Delta(\Pi, \Pi) = 0 \ \forall \Pi \in \mathcal{E}$.

Moreover, it is trivial to prove that $\Phi \circ \Psi(f) = f \ \forall f \in \mathcal{C}$ and $\Psi \circ \Phi(\Delta) = \Delta(\Delta(\Delta, \Pi))$ and therefore, the proof is concluded. $\Box$.

Once we have established that the definitions are equivalent, we can consider some examples of non-adaptability measures. By means of Example 2 and Theorem 4 we obtain immediately a first example.

**Example 5.** Let $(\Omega, \mathcal{A})$ be a measurable space, let $\mathcal{E}$ be the collection of finite partitions on $(\Omega, \mathcal{A})$ and let $m$ be a fuzzy measure. Let us consider the map $\Delta : \mathcal{E} \times \mathcal{E} \to [0, 1]$ defined as

$$\Delta(\Pi_1, \Pi) = \left| \frac{\sum_{\sigma \in \Pi_1} m\left( \bigcup_{A_i \in \Pi_1} \left( A_i - B_{\sigma(i)} \right) \right) - \sum_{\sigma \in \Pi} m\left( \bigcup_{A_i \in \Pi} \left( A_i - B_{\sigma(i)} \right) \right)}{\sum_{\sigma \in \Pi_1} m\left( \bigcup_{A_i \in \Pi_1} \left( A_i - B_{\sigma(i)} \right) \right) - \sum_{\sigma \in \Pi} m\left( \bigcup_{A_i \in \Pi} \left( A_i - B_{\sigma(i)} \right) \right)} \right|,$$

where $\Pi_1 = \{A_1\}_{i=1}^n$ and $\Pi = \{B_k\}_{k=1}^s$ are partitions in $\mathcal{E}$. It is trivial to prove that $\Delta$ is a non-adaptability measure because it is the image of the order of adaptability of Example 2 by the one-to-one correspondence of Theorem 4.

As a simple example of this concept, we could consider again the particular case of $\Omega = \{1, 2, 3, 4, 5\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$, which was developed at the end of Example 2. In that case we obtain that $\Delta(\Pi_2, \Pi) = 0$, that is, $\Pi_2$ is totally adapted to $\Pi$ (the classification is perfect). However,

$$\Delta(\Pi, \Pi_2) = \sum_{\sigma \in \Pi_2} m\left( \left( \{1, 3\} - B_{\sigma(1)} \right) \cup \left( \{2, 4, 5\} - B_{\sigma(2)} \right) \right) = \min\{m(\{1, 2, 4, 5\}), m(\{1, 2\}), m(\Omega), m(\{1, 2, 3\})\} = \min\{4/5, 2/5, 1, 3/5\} = 2/5,$$

where $B_i$ denotes the elements of $\Pi_2$. This is an example which shows that the measure of non-adaptability is not symmetric. This is logical, since the degree of adaptability of any partition $\Pi$ to another partition $\Pi'$ has not to be the same that the degree of adaptability of $\Pi'$ to $\Pi$. In fact, the degree of non-adaptability of any partition to $\Omega$ is zero, but the degree of non-adaptability of $\{\Omega\}$ to any partition is maximum (no questions are considered in the pseudo-questionnaire).

**Example 6.** If we consider an additive measure $m$ in Example 5, the above measure can be written as follows:

$$\Delta(\Pi_1, \Pi) = \sum_{A_i \in \Pi_1} \sum_{B_k \in \Pi} m(A_i - B_k)$$

for any $\Pi_1$ and $\Pi$ in $\mathcal{E}$.

In particular, if $\Pi$ is a complete partition of $\Omega$, then $\Delta$ can be obtained as

$$\Delta(\Pi_1, \Pi) = \sum_{A_i \in \Pi_1} \sum_{k \neq J(A_i, \Pi)} m(A_i \cap B_k),$$

where $J(A_i, \Pi)$ is an index such that $m(A_i \cap B_{J(A_i, \Pi)}) = \sum_{B_k \in \Pi} m(A_i \cap B_k)$.
Since the fuzzy measure $m$ considered at the end of Example 2 is additive, then we could calculate the measure of non-adaptability as follows
\[
\Delta(\Pi_1, \Pi_2) = \sum_{A_i \in \Pi_1} \sum_{B_k \in \Pi} T_M m(A_i - B_k)
\]
\[
= \min\{m(\{1, 2\} - \{1, 3\}), m(\{1, 2\} - \{2, 4, 5\})\} + \min\{m(\{3\} - \{1, 3\}), m(\{3\} - \{2, 4, 5\})\}
\]
\[
= \min\{1/5, 1/5\} + \min\{0, 1/5\} = 1/5.
\]

By means of Definition 3, we can prove some properties of a non-adaptability measure. These properties emphasize the intuitive idea of non-adaptability and they will be very important when we consider some specific non-adaptability measures.

**Proposition 7.** Let $(\Omega, \mathcal{A})$ be a measurable space, let $\mathcal{E}$ be the collection of finite partitions on $(\Omega, \mathcal{A})$ and let $\Delta$ be a non-adaptability measure on $\mathcal{E}$. The following properties are satisfied:

1. $\Delta(\Pi_1, \Pi_2) = 0 \forall \Pi_1, \Pi_2 \in \mathcal{E}$ such that $\Pi_1 \subseteq \Pi_2$.
2. $\Delta(\Pi_1, \Pi_2) \leq \Delta(\{\Omega\}, \emptyset) \forall \Pi_1, \Pi_2 \in \mathcal{E}$.
3. $\Delta(\Pi_1 \vee \Pi_2, \Pi) \leq \min\{\Delta(\Pi_1, \Pi), \Delta(\Pi_2, \Pi)\} \forall \Pi_1, \Pi_2, \Pi \in \mathcal{E}$.
4. $\Delta(\Pi, \Pi_1 \vee \Pi_2) \geq \max\{\Delta(\Pi, \Pi_1), \Delta(\Pi, \Pi_2)\} \forall \Pi_1, \Pi_2, \Pi \in \mathcal{E}$.
5. $\Delta(\Pi_1 \wedge \Pi_2, \Pi) = 0, i = 1, 2 \forall \Pi_1, \Pi_2 \in \mathcal{E}$.
6. $\min\{\Delta(\Pi_1, \Pi), \Delta(\Pi_2, \Pi)\} \leq \Delta(\Pi_1 \wedge \Pi_2, \Pi) \forall \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ such that $\text{Supp}(\Pi_1) \cap \text{Supp}(\Pi_2) = \emptyset$.
7. $\Delta(\Pi_1, \Pi_1) = \Delta(\Pi, \Pi_1) \geq \Delta(\Pi, \Pi_2) \forall \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ such that $\text{Supp}(\Pi_1) \cap \text{Supp}(\Pi_2) = \emptyset$.
8. $\Delta(\Pi_i, \Pi_1 \vee \Pi_2) = 0, i = 1, 2 \forall \Pi_1, \Pi_2 \in \mathcal{E}$ such that $\text{Supp}(\Pi_i) \cap \text{Supp}(\Pi_2) = \emptyset$.
9. $\Delta(\Pi_1, \Pi) = \Delta(\Pi_2, \Pi)$ and $\Delta(\Pi, \Pi_1) = \Delta(\Pi, \Pi_2) \forall \Pi_1, \Pi_2, \Pi \in \mathcal{E}$ such that $\Pi_1 = \Pi_2 \vee \{\emptyset\}$.

**Proof.** Let us consider $\Pi_1$, $\Pi_2$ and $\Pi$ are elements in $\mathcal{E}$.

1. If $\Pi_1 \subseteq \Pi_2$ then $0 \leq \Delta(\Pi_1, \Pi_2) \leq \Delta(\Pi_2, \Pi_2) = 0$.
2. For any $\Pi \in \mathcal{E}$, $\emptyset \subseteq \Pi \subseteq \{\Omega\}$. Thus, $\Delta(\Pi_1, \Pi_2) \leq \Delta(\{\Omega\}, \Pi_2) \leq \Delta(\{\Omega\}, \emptyset)$.
3. Since $\Pi_1 \cap \Pi_2 \subseteq \Pi_i, i = 1, 2$, the proof of this property is immediate from Axiom 2 in Definition 3.
4. By using again Axiom 2 in Definition 3.
5. In this case, we consider Property (1).
6–8 Since $\Pi_i \subseteq \Pi_1 \vee \Pi_2$, $i = 1, 2$, it is immediate, from Definition 3 the proof of these properties.
9. From Property (6), $\Delta(\Pi_1, \Pi) \geq \Delta(\Pi_2, \Pi)$. On the other hand, since $\Pi_1 \subseteq \Pi_2$, from Axiom 2 in Definition 3, $\Delta(\Pi_1, \Pi) \leq \Delta(\Pi_2, \Pi)$. The other equality is totally analogous, by considering Property (7) instead of (6). □

**Remark 8.** We could see these properties as natural requirements in the context of pseudo-questionnaires as follows:

1. If we have achieved an appropriate classification for all the individuals, the adaptability is perfect, that is, the non-adaptability is null.
2. Since any partition is a subpartition of $\{\Omega\}$, for a given partition, $\{\Omega\}$ is the partition worst adapted to it. Moreover, the degree of non-adaptability increases for “smaller” partitions, in the sense proposed by means of the partial order called subpartition. Thus, the maximum non-adaptability possible is from $\{\Omega\}$ to $\{\emptyset\}$. This property will allow us to normalize the measures of non-adaptability and to work in the interval $[0, 1]$ instead of $\mathbb{R}^+$, with the advantages we will comment at the end of this section.
3. If we combine two pseudo-questionnaires over the same population, the classification will be, at least, as good as if we consider any of them separately.
4. The product is a partition with more elements than any of them separately. If we have more eventualities, the classification is always more difficult.
5. If we have achieved, at least, to classify any individual in its group, the adaptability is total.
(6) Similar interpretation as (3).
(7) Similar interpretation as (4).
(8) Similar interpretation as (5).
(9) In the context of Information Theory, it makes sense to consider partitions with empty sets as elements of it (see [7]), which is against the classical definition of partition, usually considered in Algebra. However, this inclusion should not produce any modification, since it is a simple mathematical agreement, without a real interpretation. The no influence of this element is clear in this property.

By the second property of the non-adaptability measures, although they assume values in \( \mathbb{R}^+ \), we could normalize them. Thus, for any \( A \in \mathcal{A} \) we could define the function \( A' : \mathcal{A} \times \mathcal{A} \rightarrow [0,1] \) by

\[
A'(\Pi_1, \Pi_2) = \frac{A(\Pi_1, \Pi_2)}{A(\Omega, \emptyset)}
\]

provided that \( A(\Omega, \emptyset) < +\infty \) and prove that \( A' \) is again a non-adaptability measure. Thus, from now on, we will consider a non-adaptability measure as a function on \([0,1]\), that is, we deal with \( A \) normalized. This will be an advantage in order to use \( t \)-norms and \( t \)-conorms to obtain non-adaptability measures, since these operations are usually defined on the unit square \([0,1]^2\).

4. Composite non-adaptability measures

At the moment of using this measure in practical problems, an explicit form is needed. Since the axiomatic definition allows a great number of choices, it seems to be useful to require some other properties, each of them characterizing a particular class of such measures. Compositivity will allow us to achieve this purpose.

We have considered some different ways to decompose non-adaptability and we have found examples for some of them, in which the composition does not make sense, since we arrived at a contradiction. After these considerations, we have established the following classes of non-adaptability measures.

4.1. Left-compositive non-adaptability

**Definition 9.** A non-adaptability measure \( A \) is said to be **left compositive** if there exists a map \( \psi : [0,1]^2 \rightarrow [0,1] \) such that

\[
A(\Pi_1 \cup \Pi_2, \Pi) = \psi(A(\Pi_1, \Pi), A(\Pi_2, \Pi))
\]

for all \( \Pi_1, \Pi_2, \Pi \in \mathcal{A} \) such that \( \text{Supp}(\Pi_1) \cap \text{Supp}(\Pi_2) = \emptyset \).

**Example 10.** The non-adaptability measure proposed in Example 5 is not left-compositive in general. Thus, for instance, let us consider \( \Omega = \{x_1, x_2, \ldots, x_6\} \), \( \mathcal{A} = \mathcal{P}(\Omega) \), the non-additive fuzzy measure defined by

\[
m(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ 1/3, & \text{if } A = \{x_i\} \quad \forall i = 1, \ldots, 6, \\ 3/4, & \text{if } A = \{x_1, x_4\}, \\ 1, & \text{otherwise}, \end{cases}
\]

and the partitions \( \Pi_1 = \{x_1, x_2\}, \Pi_2 = \{x_3, x_4\}, \Pi_3 = \{x_5, x_6\}, \Pi = \{x_2, x_3\} \) and \( \Pi' = \{x_4, x_5\} \).

If \( A \) is left-compositive, then there exists a map \( \psi \) such that \( A(\Pi_1 \cup \Pi_2, \Pi) = \psi(A(\Pi_1, \Pi), A(\Pi_2, \Pi)) \) and \( A(\Pi_2 \cup \Pi_3, \Pi') = \psi(A(\Pi_2, \Pi'), A(\Pi_3, \Pi')) \). For the fixed partitions, \( A(\Pi_1, \Pi) = A(\Pi_2, \Pi) = A(\Pi_3, \Pi') = A(\Pi_3, \Pi') = 1/3, \quad \text{but} \quad A(\Pi_1 \cup \Pi_2, \Pi) = 3/4 \quad \text{and} \quad A(\Pi_2 \cup \Pi_3, \Pi') = 1, \) which implies \( \psi(1/3, 1/3) = 3/4 \) and \( \psi(1/3, 1/3) = 1 \). This is not possible and, therefore, there does not exist such a map \( \psi \), that is, \( A \) is not left-compositive.

However, the measure proposed in Example 6 is left-composite, where \( \psi \) is the sum. This measure could also be obtained by means of the bounded sum \( t \)-conorm as follows

\[
A(\Pi_1, \Pi) = \sum_{A_i \in \Pi_1} T_M m(A_i - B_k).
\]
From this starting point, we could define a class of left-compositive non-adaptability measures, which are defined by

$$\Delta_s(P_1, P_2) = \sum_{A \in P_1, B \in P_2} \mu(A - B),$$

where $S$ is a $t$-conorm and $m$ is a $S$-decomposable measure. By the properties of $t$-conorm and the minimum $t$-norm, it is easy to prove that $\Delta_s$ fulfils the two axioms in Definition 3, that is, $\Delta_s$ is left-composite with respect to $S$.

It seems natural that $\psi$ has to be a “universal” function in the sense proposed by Benvenuti [2] to the uncertainty and information measures. In our context, universality implies that if we have defined a non-adaptability measure to the partitions. Since $A$ has to be a non-adaptability measure over the atoms, some compatibility conditions are needed.

**Definition 11.** For any triple $(\Omega, \mathcal{A}, \mathcal{E})$ and any map $\psi : [0, 1]^2 \to [0, 1]$, we say that a family of numbers $\{a_{A, P}\}_{A \in \mathcal{A}, P \in \mathcal{E}}$ in the interval $[0, 1]$ is left-compatible with $\psi$ if it fulfils the following conditions:

1. $a_{A, P} = 0$, for any $A \in \mathcal{A}$ and $P \in \mathcal{E}$ such that $\{A\} \subseteq P$;
2. $a\psi(x_{A_{i-1}, \ldots, A_{i-1}}, a_{A_{i}, P}) \leq a_{B, P}$ for any partitions $\{A_i\}_{i=1}^n \subseteq \mathcal{E}$ and any element $B \in \mathcal{A}$ such that $\{A_i\}_{i=1}^n \subseteq \{B\}$, where $\psi_1 = \text{Id}$, and $\psi_n$ is given by $\psi_n(x_1, x_2, \ldots, x_n) = \psi(\psi_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n)$, for any $n \geq 2$;
3. $a_{A, P} \geq a_{A', P}$, for any $A \in \mathcal{A}$ and $P, P' \in \mathcal{E}$ such that $P \subseteq P'$.

Universality also means that for any element $x$ in $[0, 1]$, it is possible to obtain two partitions whose non-adaptability measure coincides with $x$. Thus, with all these ideas, we can precisate the concept of left-universal map with respect to the non-adaptability measures.

**Definition 12.** The map $\psi : [0, 1]^2 \to [0, 1]$ is said to be left-universal w.r.t. the non-adaptability measures if it fulfils the following conditions:

1. For every triple $(\Omega, \mathcal{A}, \mathcal{E})$ and for every family of numbers $\{a_{A, P}\}_{A \in \mathcal{A}, P \in \mathcal{E}}$ left-compatible with $\psi$, the map $\Delta : \mathcal{E} \times \mathcal{E} \to [0, 1]$ defined by

$$\Delta(\{A_i\}_{i=1}^n, P) = \psi_n(\{a_{A_{i-1}, \ldots, A_{i-1}}, a_{A_{i}, P}\}, \{A_i\}_{i=1}^n, P)$$

is a non-adaptability measure.
2. For every $x \in [0, 1]$, there exist a triple $(\Omega, \mathcal{A}, \mathcal{E})$ and a family of numbers $\{a_{A, P}\}_{A \in \mathcal{A}, P \in \mathcal{E}}$ left-compatible with $\psi$ such that $x \in \{a_{A, P}\}_{A \in \mathcal{A}, P \in \mathcal{E}}$.

In any case, an important consequence of this definition is that it allows us to characterize the left-universal functions by means of $t$-conorms.

**Theorem 13.** Given a function $\psi : [0, 1]^2 \to [0, 1]$, the following assessments are equivalent:

1. $\psi$ is left-universal w.r.t. the non-adaptability measures;
2. $\psi$ is a $t$-conorm.

**Proof.** If $\psi$ is left-universal w.r.t. the non-adaptability measures, from Condition (2) in Definition 12, we have that for any $x \in [0, 1]$, there exists a triple $(\Omega, \mathcal{A}, \mathcal{E})$, an element $A \in \mathcal{A}$ and an element $P \in \mathcal{E}$ such that $x = a_{A, P}$, where $\{a_{A, P}\}_{A \in \mathcal{A}, P \in \mathcal{E}}$ is a family of numbers left-compatible with $\psi$. Then, by applying Condition (1) of left-universality and Property 9 in Proposition 7, we have that

$$x = a_{A, P} = \Delta(\{A\}, P) = \Delta(\{A\} \cup \{\emptyset\}, P) = \psi(\Delta(\{A\}, P), \Delta(\{\emptyset\}, P)) = \psi(x, 0),$$

$$x = a_{A, P} = \Delta(\{A\}, P) = \Delta(\{\emptyset\} \cup \{A\}, P) = \psi(\Delta(\{\emptyset\}, P), \Delta(\{A\}, P)) = \psi(0, x),$$

that is, the left-universality of $\psi$ forces 0 to be its neutral element ($\psi(x, 0) = \psi(0, x) = x \ \forall x \in [0, 1]$).
Now, let us consider \( \Omega = \{a, b, c\} \), \( \mathcal{A} = \mathcal{P}(\Omega) \) and \( \mathcal{E} \) the set of all the finite partitions formed by elements of \( \mathcal{A} \). For any \((x, y, z, t) \in [0, 1]^4\) such that \( x \leq y \), we could define the family of numbers \( \mathcal{A}_{A,n} \) as follows:

\[
\mathcal{A}_{A,}\{a, c\}, \{b\}, \{c\}, \{0\} = x, \mathcal{A}_{A,}\{b, c\}, \{a\}, \{0\} = t, \mathcal{A}_{A,}\{a, b\}, \{c\}, \{0\} = \mathcal{A}_{A,}\{a, c\}, \{b\}, \{0\} = 1 \quad \text{and} \quad \mathcal{A}_{A,}\{a, b, c\}, \{0\} = 0 \quad \text{in any other case, where } f: [0, 1] \rightarrow [0, 1] \text{ is defined by } f(t) = 1 \forall t \in (0, 1) \text{ and } f(0) = y.
\]

It is easy to prove that this family of numbers fulfills the left-compatibility conditions for the left-universal map \( \psi \), and therefore \( \Delta(\{a_1, a_2, \ldots, a_n\}) = \psi_n(\mathcal{A}_{A,1}, \mathcal{A}_{A,2}, \ldots, \mathcal{A}_{A,n}) \forall \{a_i\}_{i=1}^n, \Pi \in \mathcal{E} \) is a non-adaptability measure.

Let us consider the case \( t = 0 \), as \( \{\{a\}, \{b\}\} \subseteq \{\{a, c\}, \{b\}\} \) from Axiom 2 in Definition 3, we obtain that \( \Delta(\{\{a\}, \{b\}\}, \{0\}) \leq \Delta(\{\{a, c\}, \{b\}\}, \{0\}) \), that is, \( \psi(x, z) \leq \psi(y, z) \) and, therefore, \( \psi \) is increasing.

Moreover, since the order of the elements in a partition is non-relevant, it is immediate that \( \Delta(\{\{a\}, \{b\}\}, \{0\}) = \psi(x, z) = \psi(z, x) = \Delta(\{\{b\}, \{a\}\}, \{0\}) \). As this equality is fulfilled for any \( x, z \in [0, 1] \), it is proven that \( \psi \) is commutative.

Let us now consider \( t \) is any element in \([0, 1]\), then \( \Delta(\{\{a\}, \{b\}, \{c\}\}, \{0\}) = \psi_3(\mathcal{A}_{\{a\}, \{0\}}, \mathcal{A}_{\{b\}, \{0\}}, \mathcal{A}_{\{c\}, \{0\}}) = \psi(\psi(\mathcal{A}_{\{a\}, \{0\}}, \mathcal{A}_{\{b\}, \{0\}}, \mathcal{A}_{\{c\}, \{0\}}) = \psi(\psi(x, z), t) \) and, analogously, \( \Delta(\{\{b\}, \{c\}, \{a\}\}, \{0\}) = \psi(\psi(z, x), t) \). By the proven commutativity of \( \psi \), \( \psi(A, (\{a\}, \{c\}, \{0\}) = \psi(x, \psi(\psi(z, x), t)) \) and, applying again the non-relevancy of the order of the elements in a partition, we obtain that \( \psi(\psi(x, z), t) = \psi(x, \psi(z, t)) \). Since \( x, z \) and \( t \) are any three elements in \([0, 1]\), it is proven that \( \psi \) is associative.

Thus, we have proven that any left-universal function is increasing, commutative, associative, with neutral element 0, that is, it is a \( t \)-conorm.

Conversely, let \( S \) be a \( t \)-conorm. We consider any triple \( (\Omega, \mathcal{A}, \mathcal{E}) \) and any family of numbers \( \{\mathcal{A}_{A,}\}_{A \in \mathcal{A}, \Pi \in \mathcal{E}} \) satisfying the left-compatibility conditions with respect to \( S \) and let us define the map \( \Delta: \mathcal{E} \times \mathcal{E} \rightarrow [0, 1] \) as

\[
\Delta(\Pi_1, \Pi_2) = \max_{A \in \Pi_1} \mathcal{A}_{A,\Pi_2} \quad \forall \Pi_1, \Pi_2 \in \mathcal{E}.
\]

Then,

(1) For any \( \Pi = \{A_i\}_{i=1}^n \in \mathcal{E} \), by the left-compatibility condition (1), \( \mathcal{A}_{A_i,\Pi} = 0 \) \( \forall i \in \{1, 2, \ldots, n\} \) and therefore \( \Delta(\Pi, \Pi) = S_{A_i,\Pi} = 0 \).

(2) Let us consider \( \Pi_1, \Pi_2 \) in \( \mathcal{E} \) such that \( \Pi_1 \subseteq \Pi_2 \) and any other partition \( \Pi \) in \( \mathcal{E} \). Since \( \Pi_1 \) is a subpartition of \( \Pi_2 \), \( S \) is a \( t \)-conorm and \( \{\mathcal{A}_{A,}\}_{A \in \mathcal{A}, \Pi \in \mathcal{E}} \) is a left-compatible family with respect to \( S 

\Delta(\Pi_1, \Pi_2) = \max_{A \in \Pi_1} \mathcal{A}_{A,\Pi_2} = \max_{A \in \Pi_1} \mathcal{A}_{A,\Pi_2} \leq \max_{A \in \Pi_2} \mathcal{A}_{A,\Pi_2} = \Delta(\Pi_2, \Pi_2).
\]

On the other hand, by the left-compatibility conditions, we have that \( \mathcal{A}_{C,\Pi_1} \geq \mathcal{A}_{C,\Pi_2}, \forall C \in \Pi \). Thus, since \( S \) is increasing,

\[
\Delta(\Pi, \Pi_1) = \max_{C \in \Pi_2} \mathcal{A}_{C,\Pi_2} \geq \max_{C \in \Pi_2} \mathcal{A}_{C,\Pi_1} = \Delta(\Pi, \Pi_2).
\]

Since \( \Delta \) fulfills the requirements imposed in Definition 3, then it is a non-adaptability measure.

Moreover, for any \( x \in [0, 1] \) we could consider the triple \( (\Omega, \mathcal{A}, \mathcal{E}) \) where \( \Omega = \{1\} \), \( \mathcal{A} = \mathcal{P}(\Omega) \) and \( \mathcal{E} = \{\{0\}, \{\{1\}\}\} \) and the family of numbers \( \{\mathcal{A}_{A,}\}_{A \in \mathcal{A}, \Pi \in \mathcal{E}} \) defined by

\[
\mathcal{A}_{A,\Pi} = \begin{cases} x, & \text{if } A = \{1\} \text{ and } \Pi = \{0\}, \\ 0, & \text{otherwise.} \end{cases}
\]

It is immediate to prove that this family is left-compatible with \( S \) and \( x \) is included in it.

Thus, we have proven that \( S \) is left-universal w.r.t. the non-adaptability measures. As the proof is done for any \( t \)-conorm \( S \), this implication is proven. \( \square \)

As we have proven that the family of left-universal functions coincides with the family of \( t \)-conoms, we have that any non-adaptability measure \( \Delta \) which is left-composite with respect to a left-universal \( \psi \) satisfies that

\[
\Delta(\Pi_1, \Pi_2) = \max_{A \in \Pi_1} \Delta(\{A_i\}, \Pi_2) \quad \forall \Pi_1, \Pi_2 \in \mathcal{E}
\]

for some \( t \)-conorm \( S \).
An example of this kind of left-universal composition was already obtained in Example 10, when we replaced the bounded sum $t$-conorm for any $t$-conorm.

### 4.2. Right atom-compositive non-adaptability

In this section, we are going to obtain similar results for the other kind of appropriate composition, for the second component of the non-adaptability measures.

**Definition 14.** A non-adaptability measure is said to be right atom-compositive if there exists a map $\psi : [0, 1]^2 \to [0, 1]$ such that

$$\Delta(\{A\}, P_1 \cup P_2) = \psi(\Delta(\{A\}, P_1), \Delta(\{A\}, P_2))$$

for all $P_1, P_2 \in \mathcal{E}$ such that $\text{Supp}(P_1) \cap \text{Supp}(P_2) = \emptyset$ and for all $A \in \mathcal{A}$.

**Example 15.** Since $\Delta(\{A\}, P)$ assumes the value $T_M m(A - B_j)$ for both non-adaptability measures proposed in Examples 5 and 6, it is immediate that their normalized versions are right atom-compositive for the map $\psi = T_M$.

Now, we are going to present an example of non-right atom-compositive non-adaptability measure. For that, we are going to consider a measure which measures the adaptability to the support of the partition of reference.

Thus, let $(\Omega, \mathcal{A})$ be a measurable space, let $\mathcal{E}$ be the collection of finite partitions on $(\Omega, \mathcal{A})$ and let $m$ be a $S$-decomposable fuzzy measure. Let us consider the map $D : E \to [0, 1]$ defined as

$$D(P_1, P) = \sum_{A_i \in P_1} m(A_i - \bigcup_{B_j \in P} B_j) \quad \forall P_1, P \in \mathcal{E}.$$ 

Let $P$ be any partition in $\mathcal{E}$. Since,

1. The non-adaptability from $P$ to itself is

$$\Delta(P, P) = \sum_{A_i \in P} m(A_i - \bigcup_{A_j \in P} A_j) = \sum_{A_i \in P} 0 = 0.$$

2. For any partitions $P_1, P_2$ and $P$ in $\mathcal{E}$ such that $P_1 \subseteq P_2$ we have that

$$\Delta(P_1, P_2) = \sum_{A_i \in P_1} m(A_i - \bigcup_{C_j \in P_2} C_j) = \sum_{B_k \in P_2} m\left(\bigcup_{A_i \subseteq B_k} A_i - \bigcup_{C_j \in P_2} C_j\right)$$

$$\leq \sum_{B_k \in P_2} m\left(B_k - \bigcup_{C_j \in P_2} C_j\right) = \Delta(P_2, P),$$

and

$$\Delta(P_1, P) = \sum_{C_j \in P} m\left(C_j - \bigcup_{A_i \in P_1} A_i\right) \geq \sum_{C_j \in P} m\left(C_j - \bigcup_{B_k \in P_2} B_k\right) = \Delta(P, P_2)$$

then $\Delta$ fulfills the axioms imposed in Definition 3, that is, $\Delta$ is a non-adaptability measure.

It is trivial that it is left-compositive for the map $S$. Let us now consider $\Omega = [0, 1]$, $\mathcal{A} = \mathcal{P}([0, 1])$, the set function $m : \mathcal{P}([0, 1]) \to [0, 1]$ defined by

$$m(A) = \begin{cases} 0, & \text{if } A = \emptyset, \\ \sup A, & \text{if } A \text{ is finite or countably infinite}, \\ 1, & \text{otherwise}, \end{cases}$$

which is a $S_M$-decomposable fuzzy measure (see [9]) and the following elements of $\mathcal{A}$: $A = \{0, 1/3, 2/3\}$, $B_1 = \{2/3\}$, $B_2 = \{0, 2/3\}$, $C_1 = \{0\}$, $C_2 = \{1/3\}$.

If $A$ is right atom-compositive, then there exists a map $\psi$ such that

$$\Delta(\{A\}, \{B_1\} \cup \{C_1\}) = \psi(\Delta(\{A\}, \{B_1\}), \Delta(\{A\}, \{C_1\})), $$

$$\Delta(\{A\}, \{B_2\} \cup \{C_2\}) = \psi(\Delta(\{A\}, \{B_2\}), \Delta(\{A\}, \{C_2\})).$$

However, we have that $\Delta(\{A\}, \{B_1\}) = \Delta(\{A\}, \{B_2\}) = 1/3$ and $\Delta(\{A\}, \{C_1\}) = \Delta(\{A\}, \{C_2\}) = 2/3$, but $\Delta(\{A\}, \{B_1\} \cup \{C_1\}) = 1/3 \neq 0 = \Delta(\{A\}, \{B_2\} \cup \{C_2\})$. Thus, we have arrived to a contradiction and therefore, $A$ is not right atom-compositive.

If we consider again the appropriate requirements for universality in this context, in the sense introduced by Benvenuti [2] for uncertainty and information measures, we obtain the following definitions.

**Definition 16.** For any triple $(\Omega, \mathcal{A}, \mathcal{E})$ and any map $\psi : [0, 1]^2 \rightarrow [0, 1]$, we say that a family of numbers $\{x_{A,B}\}_{A,B \in \mathcal{A}}$ in the interval $[0, 1]$ is right-compatible with $\psi$ if it fulfills the following conditions:

1. $x_{A,B} = 0$, for any $A, B \in \mathcal{A}$ such that $A \subseteq B$;
2. $x_{A,B} \leq x_{A',B}$, for any $A', B \in \mathcal{A}$ such that $A \subseteq A'$;
3. $\psi_n(x_{A_1,B_1}, \ldots, x_{A_k,B_k}) \geq x_{A,B}$, for any partition $\{B_j\}_{j=1}^n \in \mathcal{E}$ and for any $A, B \in \mathcal{A}$ such that $\{B_j\}_{j=1}^n \subseteq \{B\}$, where $\psi_1 = Id$, and $\psi_n$ is given by $\psi_n(x_1, x_2, \ldots, x_n) = \psi(\psi_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n)$, for any $n \geq 2$.

**Definition 17.** The map $\psi : [0, 1]^2 \rightarrow [0, 1]$ is said to be right-universal w.r.t. the non-adaptability measures if it fulfills the following conditions:

1. For every triple $(\Omega, \mathcal{A}, \mathcal{E})$ and for every family of numbers $\{x_{A,B}\}_{A,B \in \mathcal{A}}$ right-compatible with $\psi$, the map $\Delta : \mathcal{E} \times \mathcal{E} \rightarrow [0, 1]$ defined by

$$\Delta(\{A\}, \{B\}) = \psi_n(x_{A_1,B_1}, x_{A_2,B_2}, \ldots, x_{A_k,B_k}) \quad \forall A \in \mathcal{A}, \forall \{B_j\}_{j=1}^n \in \mathcal{E}$$

is the restriction to $\mathcal{E} \times \mathcal{E}$ of a non-adaptability measure on $\mathcal{E}$, where $\mathcal{E}' = \{\{A\} | A \in \mathcal{A}\}$.

2. For every $x \in [0, 1]$, there exists a triple $(\Omega, \mathcal{A}, \mathcal{E})$ and a family of numbers $\{x_{A,B}\}_{A,B \in \mathcal{A}}$ right-compatible with $\psi$ such that $x, 1 \in \{x_{A,B}\}_{B \in \mathcal{A}}$.

With an analogous reasoning to the previous subsection, we obtain that $\psi$ is a $t$-norm.

**Theorem 18.** Given a function $\psi : [0, 1]^2 \rightarrow [0, 1]$, the following assessments are equivalent:

1. $\psi$ is right-universal w.r.t. the non-adaptability measures;
2. $\psi$ is a $t$-norm.

**Proof.** If $\psi$ is right-universal w.r.t. the non-adaptability measures, from Condition (2) in Definition 17, we have that for any $x \in [0, 1]$, there exists a triple $(\Omega, \mathcal{A}, \mathcal{E})$ and two elements $A, B \in \mathcal{A}$ such that $x = x_{A,B}$, $1 = x_{A,B}$, where $\{x_{A,B}\}_{A,B \in \mathcal{A}}$ is a family of numbers right-compatible with $\psi$. Since $\{\emptyset\} \subseteq \{B\}$, by Condition (3) in Definition 16, we have that $1 = x_{A,B} \leq x_{A,\emptyset} \leq 1$, that is, $x_{A,\emptyset} = 1$.

From Condition (1) in Definition 17, $x = x_{A,\emptyset} = \Delta(\{\Omega\}, \{A\})$. Moreover, $\Delta(\{\Omega\}, \{A\}) = \Delta(\{\Omega\}, \{A\} \cup \{\emptyset\}) = \Delta(\{\Omega\}, \{\emptyset\} \cup \{A\})$ and therefore

$$\Delta(\{\Omega\}, \{A\} \cup \{\emptyset\}) = \psi(\Delta(\{\Omega\}, \{A\}), \Delta(\{\Omega\}, \{\emptyset\})) = \psi(x, 1), $$

$$\Delta(\{\Omega\}, \{\emptyset\} \cup \{A\}) = \psi(\Delta(\{\Omega\}, \{\emptyset\}), \Delta(\{\Omega\}, \{A\})) = \psi(1, x),$$

that is, the right-universality of $\psi$ forces 1 to be its neutral element ($\psi(1, x) = \psi(1, x) = x \forall x \in [0, 1]$).

Now, let us consider $\Omega = \{a, b, c\}$, $\mathcal{A} = \mathcal{P}(\Omega)$ and $\mathcal{E}$ the set of all the finite partitions formed by elements of $\mathcal{A}$. For any $(x, y, z, t)$ in $[0, 1]^4$ such that $x \leq y$, we could define the family of numbers $x_{A,B}$ as follows: $x_{A,\emptyset} = 1,$
\[ x_{\Omega,\{a\}} = y, x_{\Omega,\{b\}} = z, x_{\Omega,\{c\}} = t, x_{\Omega,\{a,c\}} = g(t) \text{ and } x_{A,B} = 0 \text{ in any other case, where } g : [0,1] \rightarrow [0,1] \text{ is defined by } g(t) = 0 \text{ \forall } t \in [0,1] \text{ and } g(1) = x. \]

It is easy to prove that this family of numbers fulfils the right-compatibility conditions for the right-universal map \( \psi \), and therefore \( A(\{A\}, \{B_j\}_{j=1}^n) = \psi_{\delta}(x_{A,B_1}, x_{A,B_2}, \ldots, x_{A,B_n}) \forall A \in \mathcal{A} \forall \{B_j\}_{j=1}^n \in \delta' \) is the restriction to \( \delta' \times \delta' \) of a non-adaptability measure on \( \delta \times \delta \), where \( \delta' = \{A\} \in \mathcal{A} \).

Let us consider the case \( t = 1 \), as \( \{a\}, \{b\} \subseteq \{a,c\}, \{b\} \), from Axiom 2 in Definition 3, we obtain that \( A(\{\Omega\}, \{\{a\}, \{b\}\}) \geq A(\{\Omega\}, \{\{a,c\}, \{b\}\}) \), that is, \( \psi(y,z) \geq \psi(x,z) \) and, therefore, \( \psi \) is increasing.

Besides that, since the order of the elements in a partition is not relevant, we have that \( A(\{\Omega\}, \{\{a\}, \{b\}\}) = \psi(y,z) = \psi(z,y) = A(\{\Omega\}, \{\{b\}, \{a\}\}) \). As this equality is fulfilled for any \( y, z \in [0,1] \), it is proven that \( \psi \) is commutative.

Now, let us consider \( t \) is any element in \([0,1] , \) then \( A(\{\Omega\}, \{\{a\}, \{b\}, \{c\}\}) = \psi_3(x_{\Omega,\{a\}}, x_{\Omega,\{b\}}, x_{\Omega,\{c\}}) = \psi(\psi(x_{\Omega,\{a\}}, x_{\Omega,\{b\}}), x_{\Omega,\{c\}}) = \psi(\psi(y,z), t) \). Analogously, \( A(\{\Omega\}, \{\{b\}, \{c\}, \{a\}\}) = \psi(\psi(y,z), t) = \psi(\psi(z,y), t) \), by the commutativity of \( \psi \). Applying again the non-relevancy of the order of the elements in a partition, we obtain that \( \psi(\psi(y,z), t) = \psi(y, \psi(z,t)) \). Since \( y, z \) and \( t \) are any three elements in \([0,1] , \) it is proven that \( \psi \) is associative.

Thus, we have proven that any right-universal function is increasing, commutative, associative, with neutral element 1, that is, it is a \( t \)-norm.

Conversely, let \( T \) be a \( t \)-norm. Let us consider any triple \((\Omega, \mathcal{A}, \delta)\) and any family of numbers \( \{x_{A,B}\}_{A,B \in \mathcal{A}} \) satisfying the right-compatibility conditions with respect to \( T \) and let us define the map \( A : \delta' \times \delta' \rightarrow [0,1] \) as

\[ A(\{A\}, \{B_j\}_{j=1}^n) = T_n(x_{A,B_1}, x_{A,B_2}, \ldots, x_{A,B_n}) \forall A \in \mathcal{A} \forall \{B_j\}_{j=1}^n \in \delta', \]

and the map \( \overline{A} : \delta' \times \delta' \rightarrow [0,1] \) defined from \( A \) as follows:

\[ \overline{A}(\Pi_1, \Pi_2) = S_{M_1} \Delta(\{A_i\}, \Pi_1) = S_{M_1} \Delta(\{A_i\}, \Pi_1) \forall \Pi_1, \Pi_2 \in \delta'. \]

Then,

1. For any \( \Pi = \{A_i\}_{i=1}^n \in \delta' \), by the right-compatibility condition (1), \( x_{A_i,A_i} = 0 \forall i \in \{1,2,\ldots,n\} \). Thus, \( A(\{A_i\}, \Pi) = T(T_{A_i \in \Pi_2, j \neq i} x_{A_i,A_i,0}) = 0 \forall A_i \in \Pi \) and therefore \( A(\Pi, \Pi) = S_{M_1}(0,0,\ldots,0) = 0 \).
2. Let us consider \( \Pi_1 = \{A_i\}_{i=1}^n \), \( \Pi_2 = \{B_k\}_{k=1}^m \) in \( \delta \) such that \( \Pi_1 \subseteq \Pi_2 \) and any other partition \( \Pi \in \delta \).

Since \( \Pi_1 \) is a subpartition of \( \Pi_2 \), we could rename the elements of \( \Pi_1 \) as \( \{A_1,1,\ldots,A_{1,n}, A_{2,1},\ldots,A_{2,n},\ldots,A_{s,1},\ldots,A_{s,n}\} \), where \( \{A_{k,i}\}_{i=1}^n \) are the elements of \( \Pi_1 \) included in the element \( B_k \) of \( \Pi_2 \), for any \( k = 1,2,\ldots,s \). Thus, by the properties of a \( t \)-norm and the maximum \( t \)-conorm, we have that

\[ \overline{A}(\Pi_1, \Pi_2) = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta'). \]

On the other hand, from the right-compatibility conditions we have that \( T_n(x_{A_i,A_{i,1}}, x_{A_{i,1},A_{i,2}}, \ldots, x_{A_{i,1},A_{i,n}}) \geq x_{C_{j},B_k} \forall C_j \in \Pi, \) for any \( B_k \in \Pi_2 \). Thus, since \( T \) is increasing,

\[ \overline{A}(\Pi_1, \Pi_2) = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta') = S_{M_1} \Delta(\{A_i\}, \delta'). \]

Thus, \( \overline{A} \) fulfils the requirements imposed in Definition 3 to be a non-adaptability measure.

Besides that, it is clear that the restriction of \( \overline{A} \) to \( \delta' \times \delta' \) is \( A \) and therefore, the first condition in Definition 17 is fulfilled by \( T \).

Moreover, for any \( x \in [0,1] \) we could consider the triple \((\Omega, \mathcal{A}, \delta)\) where \( \Omega = \{1,2\} \subseteq \mathbb{R} \), \( \mathcal{A} = \mathcal{P}(\Omega) = \{\emptyset, \{1\}, \{2\}, \Omega\} \) and the family of numbers \( \{x_{A,B}\}_{A,B \in \mathcal{A}} \) defined by

\[ x_{A,B} = \begin{cases} 1, & \text{if } A = \Omega \text{ and } B = \emptyset, \\ x, & \text{if } A = \Omega \text{ and } B = \{1\}, \\ 0, & \text{otherwise}. \end{cases} \]
It is immediate to prove that this family is right-compatible with any \( t \)-norm \( T \), \( x = x_{\Omega,0} \) and \( 1 = x_{\Omega,1} \). Thus, the \( t \)-norm \( T \) also fulfills the second condition in Definition 17.

Thus, we have proven that any \( t \)-norm \( T \) is right-universal w.r.t. the non-adaptability measures.

From this theorem we could conclude that for any non-adaptability measure \( \Delta \) right atom-compositive with respect to a right-universal function \( \psi \), the non-adaptability between an atom and a partition can be obtained from the atoms by means of a \( t \)-norm as follows:

\[
\Delta(\{A\}, \Pi) = T_{B_{\in \Pi}} \Delta(\{A\}, \{B_{i}\}) \quad \forall A \in \mathcal{A} \quad \forall \Pi \in \mathcal{E}.
\]

4.3. Totally compositive non-adaptability

Starting from the two previous subsections it seems natural to propose the following definition.

**Definition 19.** A non-adaptability measure is said to be **totally compositive** if it is left compositive and right atom-compositive.

**Example 20.** Since the measure defined in Example 5 is right atom-compositive, but not left-compositive and the measure defined in Example 15 is left-compositive, but not right atom-compositive, the only example of totally compositive non-adaptability measure presented in this paper is the general measure showed in Example 10 (which has as a particular case the measure proposed in Example 6).

We conclude this paper with a result where we obtain the expression of the general non-adaptability measure from the non-adaptability of the atoms, for the different kind of studied compositions.

**Theorem 21.** Let \((\Omega, \mathcal{A}, \mathcal{E})\) be the triple formed by the sample space \( \Omega \), the algebra \( \mathcal{A} \) and the family of partitions \( \mathcal{E} \). For any map \( \Delta : \mathcal{E} \times \mathcal{E} \rightarrow [0,1] \) we have that

1. \( \Delta \) is a non-adaptability measure left-compositive with respect to a left-universal function if, and only if, there exist a \( t \)-conorm \( S \) and a family of numbers \( \{x_{A,\Pi}\}_{A \in \mathcal{A}, \Pi \in \mathcal{E}} \) in \([0,1]\) left-compatible with \( S \) such that
   \[
   \Delta(\Pi_1, \Pi) = S_{A \in \Pi_1} x_{A,\Pi} \quad \forall \Pi_1, \Pi \in \mathcal{E}.
   \]

2. A non-adaptability measure \( \Delta \) is right atom-compositive with respect to a right-universal function if, and only if, there exist a \( t \)-norm \( T \) and a family of numbers \( \{x_{A,B}\}_{A,B \in \mathcal{A}} \) in \([0,1]\) right-compatible with \( T \) such that
   \[
   \Delta(\{A\}, \Pi) = T_{B \in \Pi} x_{A,B} \quad \forall A \in \mathcal{A} \quad \forall \Pi \in \mathcal{E}.
   \]

3. \( \Delta \) is a non-adaptability measure totally compositive with respect to two universal functions (left and right) if, and only if, there exist a \( t \)-conorm \( S \), a \( t \)-norm \( T \) and a family of numbers \( \{x_{A,B}\}_{A,B \in \mathcal{A}} \) in \([0,1]\) right-compatible with \( T \) such that
   (a) The family of numbers \( \{\beta_{A,\Pi}\}_{A \in \mathcal{A}, \Pi \in \mathcal{E}} \) in \([0,1]\), where \( \beta_{A,\Pi} = T_{B \in \Pi} x_{A,B} \) \( \forall A \in \mathcal{A} \quad \forall \Pi \in \mathcal{E} \) satisfies the Condition (2) in Definition 11 with respect to \( S \).
   (b) For any two partitions \( \Pi_1, \Pi \) in \( \mathcal{E} \) we have that
      \[
      \Delta(\Pi_1, \Pi) = S_{A \in \Pi_1} T_{B \in \Pi_2} x_{A,B}.
      \]

**Proof**

1. If \( \Delta \) is a non-adaptability measure left-compositive with respect to a left-universal function, then, by Theorem 13, it is left-compositive with respect to a \( t \)-conorm \( S \). Thus, by Definition 9, for any \( \Pi_1 \) and \( \Pi \) in \( \mathcal{E} \), we have that
   \[
   \Delta(\Pi_1, \Pi) = S_{A \in \Pi_1} \Delta(\{A_i\}, \Pi).
   \]
If we defined \( \alpha_{A,B} = \Delta(A, B) \), \( \forall A \in \mathcal{A} \), \( \forall B \in \mathcal{B} \), it is an immediate consequence of \( \Delta \) being a non-adaptability measure, that the family of numbers \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) is left-compatible with \( S \).

Conversely, for a family of numbers \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) left-compatible with a \( t \)-conorm \( S \), since \( S \) is left-universal w.r.t. the non-adaptability measures (Theorem 13), by Definition 12, the map \( \Delta \) defined as

\[
\Delta(P_1, P_2) = S_{A \in P_1} \alpha_{A,B} \quad \forall P_1, P_2 \in \mathcal{P}
\]

is a non-adaptability measure. It is trivial that it is left-compositive with respect to \( S \).

(2) If \( \Delta \) is a non-adaptability measure right atom-compositive with respect to a right-universal function, then, by Theorem 18, it is right atom-compositive with respect to a \( t \)-norm \( T \). Thus, by Definition 14, for any \( A \) in \( \mathcal{A} \) and any \( P \) in \( \mathcal{P} \), we have that

\[
\Delta(A, P) = T_{B \in P} \Delta(A, \{B\}).
\]

If we defined \( \alpha_{A,B} = \Delta(A, B) \), \( \forall A, B \in \mathcal{A} \), it is an immediate consequence of being \( \Delta \) a non-adaptability measure, that the family of numbers \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) is right-compatible with \( T \).

Conversely, for a non-adaptability measure \( \Delta \), if there exist a \( t \)-norm \( T \) and a family of numbers \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) in \([0,1]\) right-compatible with \( T \) such that

\[
\Delta(A, P) = T_{B \in P} \alpha_{A,B} \quad \forall A \in \mathcal{A} \quad \forall P \in \mathcal{P},
\]

it is trivial that it is right atom-compositive with respect to \( T \). Since, from Theorem 18, \( T \) is a right-universal function, the proof of this item is concluded.

(3) If \( \Delta \) is a non-adaptability measure totally compositive with respect to a left-universal and a right-universal functions, then, by Theorems 13 and 18, it is totally compositive with respect to a \( t \)-conorm \( S \) and a \( t \)-norm \( T \), respectively. Thus, by Definitions 9 and 14, for any \( P_1 \) and \( P_2 \) in \( \mathcal{P} \), we have that

\[
\Delta(P_1, P_2) = S_{A \in P_1} T_{B \in P_2} \Delta(A, \{B\}).
\]

If we defined \( \beta_{A,B} = \Delta(A, B) \), \( \forall A, B \in \mathcal{A} \), it is an immediate consequence of being \( \Delta \) a non-adaptability measure, that the family of numbers \( \{\beta_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) is right-compatible with \( T \).

Now, if we consider the family of numbers \( \{\beta_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) defined as \( \beta_{A,B} = T_{B \in P} \alpha_{A,B} \forall A \in \mathcal{A} \forall P \in \mathcal{P} \), then, for any \( P_1, P_2 \in \mathcal{P} \) and any \( B \in \mathcal{B} \) with \( P_1 \subseteq \{B\} \), we have that

\[
S_{A \in P_1} \beta_{A,B} = S_{A \in P_1} T_{B \in P_2} \alpha_{A,B} = \Delta(P_1, P_2) \leq \Delta(P_1, \{B\}) = T_{B \in P_1} \Delta(P_1, \{B\}) = T_{B \in P_1} \beta_{A,B} = \beta_{A,B},
\]

that is, the family \( \{\beta_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) satisfies Condition (2) in Definition 11 with respect to \( S \).

Thus, the family of numbers \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) is a right-compatible with \( T \) family, fulfilling the two conditions imposed at the statement. Conversely, let \( \Delta \) be the map defined by

\[
\Delta(P_1, P_2) = S_{A \in P_1} T_{B \in P_2} \alpha_{A,B} \quad \forall P_1, P_2 \in \mathcal{P}
\]

for a family of numbers \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) right-compatible with \( T \) and such that the family \( \{\beta_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \), where \( \beta_{A,B} = T_{B \in P} \alpha_{A,B} \), fulfills Condition (2) in Definition 11 with respect to \( S \). Then,

- For any \( A \in \mathcal{A} \) and any \( P \in \mathcal{P} \), if \( \{A\} \subseteq P \), we have that

\[
\beta_{A,B} = T_{B \in P} \alpha_{A,B} = T_{B \in P} \alpha_{A,B} = 0,
\]

since \( T \) is a \( t \)-norm and \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) is a family right-compatible with \( T \).

- For any \( A \in \mathcal{A} \) and for any \( P, P' \in \mathcal{P} \) such that \( P \subseteq P' \), we have that

\[
\beta_{A,B} = T_{B \in P} \alpha_{A,B} = T_{C_k \in P'} \alpha_{A,B} = T_{C_k \in P'} \alpha_{A,B} = \beta_{A,B'},
\]

since \( \{B \in P | B \subseteq C_k\} \subseteq \{C_k\} \forall C_k \in P' \) and the family \( \{\alpha_{A,B}\}_{A \in \mathcal{A}, B \in \mathcal{B}} \) holds Condition (2) in Definition 16.
Thus, we have proven that the family of numbers \( \{ \beta_{\mathcal{A}, \mathcal{B}} \}_{\mathcal{A}, \mathcal{B} \in \mathcal{H}} \) fulfills, Conditions (1) and (3) in Definition 11. By hypothesis it also fulfills Condition (2) in Definition 11 and, therefore, \( \Delta \) can be obtained as

\[
\Delta(\Pi_1, \Pi) = \sum_{\mathcal{A} \in \Pi_1} \beta_{\mathcal{A}, \mathcal{B}} \quad \forall \Pi_1, \Pi \in \mathcal{H}
\]

for a family of numbers \( \{ \beta_{\mathcal{A}, \mathcal{B}} \}_{\mathcal{A}, \mathcal{B} \in \mathcal{H}} \) left-compatible with \( S \).

From we have proven in (1), \( \Delta \) is a non-adaptability measure, left-compositive w.r.t. the left-universal function \( S \). Moreover, from the definition of \( \Delta \), we have that \( \Delta \) is a non-adaptability measure fulfilling that

\[
\Delta(\{ A \}, \Pi) = \sum_{B \in \Pi} \mathcal{A}_{A,B} \quad \forall A \in \mathcal{A} \quad \forall \Pi \in \mathcal{H}
\]

for the \( t \)-norm \( T \) and the family of numbers \( \{ \mathcal{A}_{A,B} \}_{A,B \in \mathcal{A}} \) right-compatible with \( T \) and therefore, as we have proven in (2), it is right-atom compositive w.r.t. the right-universal function \( T \).

Thus, we have proven that \( \Delta \) is totally compositive w.r.t. the left-universal function \( S \) and the right-universal function \( T \). \( \square \)

5. Concluding remarks

In this work we have introduced a natural measure to quantify the degree of “non-adaptability” from the classification obtained by a pseudo-questionnaire to the objective (partition of reference). This measure could allow us to obtain a degree of reliability of a pseudo-questionnaire before its generalized use. Although this was our initial goal, the obtained measures could be considered in any field where a classification is required.

We have presented an axiomatic definition of the concept of non-adaptability measure, starting from some natural requirements. However, in practical situations, an explicit expression is very useful. Thus, we have mainly focused our attention on the non-adaptability measures which present some kind of compositive property. The left-composite non-adaptability measures are useful to aggregate the “non-adaptability” degree of two samples without common elements (for instance, two disjoint families of patients) to a specific objective (a collection of diseases). The right atom-composite non-adaptability measures are considered when we are interested in aggregating the comparison of one element (for instance, a patient) with an objective (a collection of diseases) and its comparison with another objective (a different collection of diseases). When we consider totally compositive non-adaptability measures, both aggregations can be considered. In these cases, we have proven that the appropriate way of composing the non-adaptability from the atoms is by means of \( t \)-conorm (left-composite) and \( t \)-norm (right atom-composite).

In the future, we will focus our attention on the branching property. As this property characterizes all the uncertainty measures involved in questionnaires and pseudo-questionnaires, it seems to be reasonable to impose this condition also to the non-adaptability measure, as we will use this function in a process, the pseudo-questionnaire, which, at the end, splits subsets in smaller ones. The idea is to characterize the class of measures of non-adaptability such that the difference between the measure of non-adaptability of a partition \( \Pi_1 \) to another partition \( \Pi_2 \) and the measure of non-adaptability of the subpartition \( \Pi_1' \) to \( \Pi_2 \) only depends on the modified sets, where \( \Pi_1' \) is obtained by breaking a element of \( \Pi_1 \) in two sets.

References