Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 228 (2011) 241–265
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Higher order spt-functions

F.G. Garvan [1]

Department of Mathematics, University of Florida, Gainesville, FL 32611-8105, United States

Received 5 November 2010; accepted 17 May 2011

Available online 28 May 2011

Communicated by George E. Andrews

Dedicated to the memory of A.O.L. (Oliver) Atkin

Abstract

Andrews' spt-function can be written as the difference between the second symmetrized crank and rank moment functions. Using the machinery of Bailey pairs a combinatorial interpretation is given for the difference between higher order symmetrized crank and rank moment functions. This implies an inequality between crank and rank moments that was only known previously for sufficiently large n and fixed order. This combinatorial interpretation is in terms of a weighted sum of partitions. A number of congruences for higher order spt-functions are derived.

MSC: primary 11P82, 11P83; secondary 05A17, 11F33, 33D15

Keywords: Bailey's Lemma; Bailey pair; Andrews' spt-function; Partitions; Rank moments; Crank moments; Quasimodular forms; Congruences

1. Introduction

Andrews [3] defined the function $\mathrm{spt}(n)$ as the number of smallest parts in the partitions of n. He related this function to the second rank moment. He also proved some surprising congruences mod 5, 7 and 13. Namely, he showed that

E-mail address: fgarvan@ufl.edu.

[1] The author was supported in part by NSA Grant H98230-09-1-0051. Most of the research for this paper was done while the author was visiting Mike Hirschhorn at the University of New South Wales in June 2010. The first version of this paper was submitted to the arXiv on August 6, 2010.

doi:10.1016/j.aim.2011.05.013

$$\mathrm{spt}(n) = np(n) - \frac{1}{2}N_2(n), \tag{1.1}$$

where $N_2(n)$ is the second rank moment function and $p(n)$ is the number of partitions of n, and he proved that

$$\mathrm{spt}(5n+4) \equiv 0 \pmod 5,$$
$$\mathrm{spt}(7n+5) \equiv 0 \pmod 7,$$
$$\mathrm{spt}(13n+6) \equiv 0 \pmod{13}.$$

As noted in [16], (1.1) can be rewritten as

$$\mathrm{spt}(n) = \frac{1}{2}\big(M_2(n) - N_2(n)\big),$$

where $M_2(n)$ is the second crank moment function. Rank and crank moments were introduced by A.O.L. Atkin and the author [6]. Bringmann [8] studied analytic, asymptotic and congruence properties of the generating function for the second rank moment as a quasi-weak Maass form. Further congruence properties of Andrews' spt-function were found by the author [16], Folsom and Ono [14] and Ono [21]. In [16] it was conjectured that

$$M_{2k}(n) > N_{2k}(n), \tag{1.2}$$

for all $k \geqslant 1$ and $n \geqslant 1$. Here $M_{2k}(n)$ and $N_{2k}(n)$ are the $2k$-th crank and $2k$-th rank moment functions. For each fixed k, the inequality was proved for sufficiently large n by Bringmann, Mahlburg and Rhoades [12], who determined the asymptotic behavior for the difference $M_{2k}(n) - N_{2k}(n)$ (see Section 7). The first few cases of the conjecture were previously proved by Bringmann and Mahlburg [9]. In this paper we prove the inequality unconditionally for all n and k by finding a combinatorial interpretation for the difference between symmetrized crank and rank moments. Analytic and arithmetic properties of higher order rank moments were studied by Bringmann, Lovejoy and Osburn [11] and by Bringmann, the author and Mahlburg [10].

Andrews [2] defined the k-th symmetrized rank function by

$$\eta_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m,n),$$

where $N(m,n)$ is the number of partitions of n with rank m. Andrews gave a new interpretation of the symmetrized rank function in terms of Durfee symbols. As a natural analog to the symmetrized rank function we define the k-th symmetrized crank function by

$$\mu_k(n) = \sum_{m=-n}^{n} \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m,n),$$

where $M(m,n)$ is number of partitions of n with crank m, for $n \neq 1$. For $n = 1$ we define

$$M(-1,1) = 1, \qquad M(0,1) = -1, \qquad M(1,1) = 1, \quad \text{and otherwise} \quad M(m,1) = 0.$$

One of our main results is the following identity

$$\sum_{n=1}^{\infty}\bigl(\mu_{2k}(n)-\eta_{2k}(n)\bigr)q^n = \sum_{n_k\geqslant n_{k-1}\geqslant\cdots\geqslant n_1\geqslant 1}\frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2(q^{n_1+1};q)_\infty}. \tag{1.3}$$

When $k=1$ this result reduces to (1.1). In Eq. (1.3) and throughout this paper we use the standard q-notation [17]. We compare Eq. (1.3) with the identity

$$\sum_{n=1}^{\infty}\mu_{2k}(n)q^n = \frac{1}{(q)_\infty}\sum_{n_k\geqslant n_{k-1}\geqslant\cdots\geqslant n_1\geqslant 1}\frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2}, \tag{1.4}$$

which is proved in Section 3. Some remarks about this identity are also given in Section 7.

In Section 2 we show that many of Andrews' results [2] for symmetrized rank moments can be extended to symmetrized crank moments. In Section 3 we prove a general result for Bailey pairs from which our main identity (1.3) follows. In Section 4, we use an analog of Stirling numbers of the second kind to show how ordinary moments can be expressed in terms of symmetrized moments and how our main identity implies the inequality (1.2). For each $k\geqslant 1$, we are able to define a higher order spt-function $\mathrm{spt}_k(n)$ so that

$$\mathrm{spt}_k(n)=\mu_{2k}(n)-\eta_{2k}(n),$$

for all $k\geqslant 1$ and $n\geqslant 1$. In Section 5 we give the combinatorial definition of $\mathrm{spt}_k(n)$ in terms of a weighted sum over the partitions of n. We note that when $k=1$, $\mathrm{spt}_k(n)$ coincides with Andrews' spt-function.

In Section 6 we prove a number of congruences for the higher order spt-functions. In Section 7 we make some concluding remarks and close the paper with a table of $\mathrm{spt}_k(n)$ for small n and k.

2. Symmetrized crank moments

In this section we collect some results for symmetrized crank moments. Many of Andrews' results and proofs for symmetrized rank moments have analogs for symmetrized crank moments; thus we omit some details.

$$\begin{aligned}
C(z,q) &:= \sum_{n=1}^{\infty}\sum_{m=-n}^{n}M(m,n)z^m q^n\\
&= \frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2}(1+q^n)}{(1-zq^n)(1-z^{-1}q^n)}\right)\\
&\quad\bigl(\text{by [15, Eq. (7.15), p. 70]}\bigr)\\
&= \frac{1}{(q)_\infty}\left(1+\sum_{n=1}^{\infty}(-1)^n q^{n(n+1)/2}\left(\frac{1-z}{1-zq^n}+\frac{1-z^{-1}}{1-z^{-1}q^n}\right)\right)
\end{aligned}$$

$$= \frac{(1-z)}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-zq^n},$$

and

$$C^{(j)}(z,q) = \left(\frac{\partial}{\partial z}\right)^j C(z,q) = \frac{-j!}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(n-1)/2+jn}(1-q^n)}{(1-zq^n)^{j+1}},$$

for $j \geqslant 0$.

By [2, Theorem 1] we know that $\eta_k(n) = 0$ if k is odd. In a similar fashion we find that $\mu_k(n) = 0$ if k is odd.

We will need

Theorem 2.1. *(See Andrews [2].)*

$$\sum_{n=1}^{\infty} \eta_{2k}(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}}$$
$$= \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2+kn}}{(1-q^n)^{2k}}. \tag{2.1}$$

This theorem has a crank analog.

Theorem 2.2.

$$\sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}}$$
$$= \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2+kn}}{(1-q^n)^{2k}}. \tag{2.2}$$

Proof. As in the proof of [2, Theorem 2] we have

$$\sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(2k)!} \left(\left(\frac{\partial}{\partial z}\right)^{2k} z^{k-1} C(z,q) \right) \Bigg|_{z=1}$$
$$= \frac{1}{(2k)!} \sum_{j=0}^{k-1} \binom{2k}{j} (k-1)\cdots(k-j) C^{(2k-j)}(1,q)$$
$$= \frac{1}{(q)_\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2+(2k-j)n}(1-q^n)}{(1-q^n)^{2k-j+1}}$$

$$= \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2+2kn}}{(1-q^n)^{2k}} \left(1 + \frac{q^{-n}}{(1-q^n)^{-1}}\right)^{k-1}$$

$$= \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2+kn}}{(1-q^n)^{2k}}. \qquad \square$$

3. Rank moments, crank moments and Bailey chains

In [22], Alexander Patkowski used a limiting form of Bailey's Lemma to obtain a partition identity analogous to (1.1), which relates an spt-like function to the second rank moment. We consider a similar limiting form that iterates Bailey's Lemma and obtain a general theorem for Bailey pairs (see Theorem 3.3 below). Then we show how our main identity (1.3) for the difference between symmetrized crank and rank moments follows from using well-known Bailey pairs. In this section we use the standard notation found in [17].

Definition 3.1. A pair of sequences $(\alpha_n(a,q), \beta_n(a,q))$ is called a Bailey pair with parameters (a,q) if

$$\beta_n(a,q) = \sum_{r=0}^{n} \frac{\alpha_r(a,q)}{(q;q)_{n-r}(aq;q)_{n+r}}$$

for all $n \geqslant 0$.

Theorem 3.2 *(Bailey's Lemma). Suppose $(\alpha_n(a,q), \beta_n(a,q))$ is a Bailey pair with parameters (a,q). Then $(\alpha'_n(a,q), \beta'_n(a,q))$ is another Bailey pair with parameters (a,q), where*

$$\alpha'_n(a,q) = \frac{(\rho_1,\rho_2;q)_n}{(aq/\rho_1, aq/\rho_2;q)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n(a,q)$$

and

$$\beta'_n(a,q) = \sum_{k=0}^{n} \frac{(\rho_1,\rho_2;q)_k (aq/\rho_1\rho_2;q)_{n-k}}{(aq/\rho_1, aq/\rho_2;q)_n (q;q)_{n-k}} \left(\frac{aq}{\rho_1\rho_2}\right)^k \beta_k(a,q).$$

For more information on Bailey's Lemma and its applications see [1, Chapter 3]. We will need the following limit which is an easy exercise:

$$\lim_{\rho_2\to 1} \lim_{\rho_1\to 1} \frac{1}{(1-\rho_1)(1-\rho_2)} \left(1 - \frac{(q)_k (q/\rho_1\rho_2)_k}{(q/\rho_1)_k (q/\rho_2)_k}\right) = \sum_{j=1}^{k} \frac{q^j}{(1-q^j)^2}. \tag{3.1}$$

Theorem 3.3. *Suppose $(\alpha_n, \beta_n) = (\alpha_n(1,q), \beta_n(1,q))$ is a Bailey pair with $a = 1$, and $\alpha_0 = \beta_0 = 1$. Then*

$$\sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{(q)_{n_1}^2 q^{n_1+n_2+\cdots+n_k} \beta_{n_1}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2}$$
$$= \sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{kr}\alpha_r}{(1-q^r)^{2k}}.$$

Proof. From Bailey's Lemma we have

$$\sum_{j=1}^{n} \frac{(\rho_1)_j(\rho_2)_j(q/\rho_1\rho_2)_{n-j}(q/\rho_1\rho_2)^j \beta_j}{(q)_{n-j}}$$
$$= \frac{(q/\rho_1)_n(q/\rho_2)_n}{(q)_n^2}\left(1 - \frac{(q)_n(q/\rho_1\rho_2)_n}{(q/\rho_1)_n(q/\rho_2)_n}\right)$$
$$+ (q/\rho_1)_n(q/\rho_2)_n \sum_{r=1}^{n} \frac{(\rho_1)_r(\rho_2)_r}{(q)_{n-r}(q)_{n+r}(q/\rho_1)_r(q/\rho_2)_r}\left(\frac{q}{\rho_1\rho_2}\right)^r \alpha_r.$$

We divide both sides by $(1-\rho_1)(1-\rho_2)$, let $\rho_1 \to 1$, $\rho_2 \to 1$, and use (3.1) to obtain

$$\sum_{j=1}^{n} (q)_{j-1}^2 q^j \beta_j = \sum_{j=1}^{n} \frac{q^j}{(1-q^j)^2} + (q)_n^2 \sum_{r=1}^{n} \frac{q^r \alpha_r}{(q)_{n-r}(q)_{n+r}(1-q^r)^2}.$$

Letting $n \to \infty$ we have

$$\sum_{j=1}^{\infty} (q)_{j-1}^2 q^j \beta_j = \sum_{j=1}^{\infty} \frac{q^j}{(1-q^j)^2} + \sum_{r=1}^{\infty} \frac{q^r \alpha_r}{(1-q^r)^2},$$

which is the case $k = 1$ of the theorem.

Now we suppose that the theorem is true for $k = K - 1$, so that

$$\sum_{n_K \geqslant n_{K-1} \geqslant \cdots \geqslant n_2 \geqslant 1} \frac{(q)_{n_2}^2 q^{n_2+\cdots+n_K} \beta_{n_2}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2 \cdots (1-q^{n_2})^2}$$
$$= \sum_{n_K \geqslant n_{K-1} \geqslant \cdots \geqslant n_2 \geqslant 1} \frac{q^{n_2+\cdots+n_K}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2 \cdots (1-q^{n_2})^2} + \sum_{r=1}^{\infty} \frac{q^{(K-1)r}\alpha_r}{(1-q^r)^{2K-2}}.$$

We now replace (α_n, β_n) by the Bailey pair (α_n', β_n') in Bailey's Lemma to obtain

$$\sum_{\substack{n_K \geqslant n_{K-1} \geqslant \cdots \geqslant n_2 \geqslant 1 \\ n_2 \geqslant n_1 \geqslant 0}} \frac{(q)_{n_2}^2 q^{n_2+\cdots+n_K} (\rho_1)_{n_1}(\rho_2)_{n_1}(q/\rho_1\rho_2)_{n_2-n_1}(q/\rho_1\rho_2)^{n_1}\beta_{n_1}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2 \cdots (1-q^{n_2})^2 (q)_{n_2-n_1}(q/\rho_1)_{n_2}(q/\rho_2)_{n_2}}$$
$$= \sum_{n_K \geqslant n_{K-1} \geqslant \cdots \geqslant n_2 \geqslant 1} \frac{q^{n_2+\cdots+n_K}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2 \cdots (1-q^{n_2})^2}$$

$$+\sum_{r=1}^{\infty}\frac{q^{(K-1)r}(\rho_1)_r(\rho_2)_r(q/\rho_1\rho_2)^r\alpha_r}{(1-q^r)^{2K-2}(q/\rho_1)_r(q/\rho_2)_r},$$

and

$$\sum_{n_K\geqslant n_{K-1}\geqslant\cdots\geqslant n_2\geqslant n_1\geqslant 1}\frac{(q)_{n_2}^2q^{n_2+\cdots+n_K}(\rho_1)_{n_1}(\rho_2)_{n_1}(q/\rho_1\rho_2)_{n_2-n_1}(q/\rho_1\rho_2)^{n_1}\beta_{n_1}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2\cdots(1-q^{n_2})^2(q)_{n_2-n_1}(q/\rho_1)_{n_2}(q/\rho_2)_{n_2}}$$
$$=\sum_{n_K\geqslant n_{K-1}\geqslant\cdots\geqslant n_2\geqslant 1}\frac{q^{n_2+\cdots+n_K}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2\cdots(1-q^{n_2})^2}\left(1-\frac{(q)_{n_2}(q/\rho_1\rho_2)_{n_2}}{(q/\rho_1)_{n_2}(q/\rho_2)_{n_2}}\right)$$
$$+\sum_{r=1}^{\infty}\frac{q^{(K-1)r}(\rho_1)_r(\rho_2)_r(q/\rho_1\rho_2)^r\alpha_r}{(1-q^r)^{2K-2}(q/\rho_1)_r(q/\rho_2)_r}.$$

We divide both sides by $(1-\rho_1)(1-\rho_2)$, let $\rho_1\to 1$, $\rho_2\to 1$, and use (3.1) to obtain

$$\sum_{n_K\geqslant n_{K-1}\geqslant\cdots\geqslant n_1\geqslant 1}\frac{(q)_{n_1}^2q^{n_1+n_2+\cdots+n_K}\beta_{n_1}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2\cdots(1-q^{n_1})^2}$$
$$=\sum_{n_K\geqslant n_{K-1}\geqslant\cdots\geqslant n_1\geqslant 1}\frac{q^{n_1+n_2+\cdots+n_K}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2\cdots(1-q^{n_1})^2}+\sum_{r=1}^{\infty}\frac{q^{Kr}\alpha_r}{(1-q^r)^{2K}},$$

which is the result for $k=K$. The general result follows by induction. □

Corollary 3.4.

$$\sum_{n_k\geqslant n_{k-1}\geqslant\cdots\geqslant n_1\geqslant 1}\frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2}$$
$$=\sum_{n=1}^{\infty}(-1)^{n-1}q^{n(n-1)/2+kn}\frac{(1+q^n)}{(1-q^n)^{2k}}. \tag{3.2}$$

Proof. The result follows from Theorem 3.3 using the well-known Bailey pair [1, pp. 27–28]

$$\alpha_n=\begin{cases}1, & n=0,\\ (-1)^nq^{n(n-1)/2}(1+q^n), & n\geqslant 1,\end{cases}\qquad \beta_n=\begin{cases}1, & n=0,\\ 0, & n\geqslant 1.\end{cases}$$ □

We note that we can rewrite (3.2) as

$$\sum_{n=1}^{\infty}\mu_{2k}(n)q^n=\frac{1}{(q)_\infty}\sum_{n_k\geqslant n_{k-1}\geqslant\cdots\geqslant n_1\geqslant 1}\frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2}, \tag{3.3}$$

after using (2.2).

Corollary 3.5.

$$\sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{(q)_{n_1} q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2}$$
$$= \sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2}$$
$$+ \sum_{n=1}^{\infty}(-1)^n q^{n(3n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}}. \tag{3.4}$$

Proof. The result follows from Theorem 3.3 using the well-known Bailey pair [1, p. 28]

$$\alpha_n = \begin{cases} 1, & n=0, \\ (-1)^n q^{n(3n-1)/2}(1+q^n), & n \geqslant 1, \end{cases} \qquad \beta_n = \frac{1}{(q)_n}. \qquad \square$$

Corollary 3.6.

$$\sum_{n=1}^{\infty}\big(\mu_{2k}(n) - \eta_{2k}(n)\big)q^n$$
$$= \sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2(q^{n_1+1};q)_\infty}. \tag{3.5}$$

Proof. After dividing both sides of (3.4) by $(q)_\infty$ and using (3.2) we have

$$\sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2\cdots(1-q^{n_1})^2(q^{n_1+1};q)_\infty}$$
$$= \frac{1}{(q)_\infty}\left(\sum_{n=1}^{\infty}(-1)^{n-1} q^{n(n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}} - \sum_{n=1}^{\infty}(-1)^{n-1} q^{n(3n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}}\right)$$
$$= \sum_{n=1}^{\infty}\big(\mu_{2k}(n) - \eta_{2k}(n)\big)q^n,$$

by (2.2) and (2.1). $\square$

4. Rank and crank moment inequalities

In this section we prove the conjectured inequality (1.2) for rank and crank moments. We need to relate ordinary and symmetrized moments. This is achieved by defining an analog of Stirling numbers of the second kind. This approach was suggested by Mike Hirschhorn.

We define a sequence of polynomials

$$g_k(x) = \prod_{j=0}^{k-1} \left(x^2 - j^2\right),$$

for $k \geqslant 1$. We want a sequence of numbers $S^*(n,k)$ such that

$$x^{2n} = \sum_{k=1}^{n} S^*(n,k) g_k(x),$$

for $n \geqslant 1$.

Definition 4.1. We define the sequence $S^*(n,k)$ $(1 \leqslant k \leqslant n)$ recursively by

(1) $S^*(1,1) = 1$,
(2) $S^*(n,k) = 0$ if $k \leqslant 0$ or $k > n$, and
(3) $S^*(n+1,k) = S^*(n,k-1) + k^2 S^*(n,k)$, for $1 \leqslant k \leqslant n+1$.

Below is a table of $S^*(n,k)$ for small n:

1					
1	1				
1	5	1			
1	21	14	1		
1	85	147	30	1	
1	341	1408	627	55	1

We note that if we replace k^2 by k in the recurrence we obtain the Stirling numbers of the second kind. The numbers $S^*(n,k)$ first occur in a paper of MacMahon [20, p. 106]. Miklós Bóna reminded me that Neil Sloane's On-Line Encyclopedia of Integer Sequences [23] can also handle 2-dimensional sequences. One just needs to input the first few terms of

$$\left\{\left\{S^*(n,k)\right\}_{k=1}^{n}\right\}_{n=1}^{\infty} = 1, 1, 1, 1, 5, 1, 1, 21, 14, 1, 1, 85, 147, 30, 1, \ldots, \tag{4.1}$$

to find the sequence labeled `A036969` [24], where more references can be found.

We have

Lemma 4.2. *For $n \geqslant 1$,*

$$x^{2n} = \sum_{k=1}^{n} S^*(n,k) g_k(x).$$

Proof. We proceed by induction on n. The result is true for $n = 1$ since $S^*(1,1) = 1$ and $g_1(x) = x^2$. We now suppose the result is true for $n = m$, so that

$$x^{2m} = \sum_{k=1}^{m} S^*(m,k) g_k(x).$$

We have $g_{k+1}(x) = (x^2 - k^2)g_k(x)$ and

$$x^2 g_k(x) = g_{k+1}(x) + k^2 g_k(x),$$

for $k \geqslant 1$. Thus

$$\begin{aligned} x^{2m+2} &= \sum_{k=1}^{m} S^*(m,k) x^2 g_k(x) \\ &= \sum_{k=1}^{m} S^*(m,k)\bigl(g_{k+1}(x) + k^2 g_k(x)\bigr) \\ &= \sum_{k=1}^{m+1} \bigl(S^*(m,k-1) + k^2 S^*(m,k)\bigr) g_k(x) \\ &= \sum_{k=1}^{m+1} S^*(m+1,k) g_k(x), \end{aligned}$$

and the result is true for $n = m+1$ and true for all n by induction. □

We can now express ordinary moments in terms of symmetrized moments.

Theorem 4.3. *For $k \geqslant 1$*

$$\mu_{2k}(n) = \frac{1}{(2k)!} \sum_{m=-n}^{n} g_k(m) M(m,n), \tag{4.2}$$

$$\eta_{2k}(n) = \frac{1}{(2k)!} \sum_{m=-n}^{n} g_k(m) N(m,n), \tag{4.3}$$

$$M_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k,j) \mu_{2j}(n), \tag{4.4}$$

$$N_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k,j) \eta_{2j}(n). \tag{4.5}$$

Proof. Suppose $k \geqslant 1$. Then

$$\begin{aligned} \mu_{2k}(n) &= \sum_{m=-n}^{n} \binom{m+k-1}{2k} M(m,n) \\ &= \frac{1}{(2k)!} \sum_{m=-n}^{n} (m+k-1)(m+k-2)\cdots(m-k) M(m,n) \end{aligned}$$

$$= \frac{1}{(2k)!} \sum_{m=-n}^{n} \bigl(m^2 - (k-1)^2\bigr)\bigl(m^2 - (k-2)^2\bigr) \cdots \bigl(m^2 - 1\bigr) m(m-k) M(m,n)$$

$$= \frac{1}{(2k)!} \sum_{m=-n}^{n} g_k(m) M(m,n),$$

since $M(-m,n) = M(m,n)$ for all m. This gives (4.2) and similarly (4.3). Using Lemma 4.2 and (4.2) we see that

$$\begin{aligned} M_{2k} &= \sum_{m=-n}^{n} m^{2k} M(m,n) \\ &= \sum_{m=-n}^{n} \left(\sum_{j=1}^{k} S^*(k,j) g_j(m) \right) M(m,n) \\ &= \sum_{j=1}^{k} (2j)! S^*(k,j) \mu_{2j}(n), \end{aligned}$$

which is (4.4). Eq. (4.5) follows similarly. □

We can now deduce our crank–rank moment inequality.

Corollary 4.4. *For all $k \geqslant 1$ and $n \geqslant 1$,*

$$M_{2k}(n) > N_{2k}(n).$$

Proof. Suppose $k \geqslant 1$. Then from (3.5) we have

$$\sum_{n=1}^{\infty} \bigl(\mu_{2j}(n) - \eta_{2j}(n)\bigr) q^n = \frac{q^j}{(1-q)^{2j}(q^2;q)_\infty} + \cdots,$$

and we see that

$$\mu_{2j}(n) > \eta_{2j}(n),$$

for all $n \geqslant j \geqslant 1$. Now using (4.4), (4.5) and the fact that the coefficients $S^*(k,j)$ are positive integers we have

$$M_{2k}(n) - N_{2k}(n) = \sum_{j=1}^{k} (2j)! S^*(k,j) \bigl(\mu_{2j}(n) - \eta_{2j}(n)\bigr) \geqslant 2\bigl(\mu_2(n) - \eta_2(n)\bigr) > 0,$$

for all $n \geqslant 1$. □

5. Higher order spt-functions

In this section we define a higher order spt-function $\mathrm{spt}_k(n)$ so that

$$\mathrm{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n),$$

for all $k \geqslant 1$ and $n \geqslant 1$. The idea is to interpret the right side of (3.5) in terms of partitions.

Definition 5.1. For a partition π with m different parts

$$n_1 < n_2 < \cdots < n_m,$$

we define $f_j = f_j(\pi)$ to be the frequency of part n_j for $1 \leqslant j \leqslant m$.

We note that $f_1 = f_1(\pi)$ is the number of smallest parts in the partition π and Andrews' function

$$\mathrm{spt}(n) = \sum_{\pi \vdash n} f_1(\pi).$$

Definition 5.2. Let $k \geqslant 1$. For a partition π we define a weight

$$\omega_k(\pi) = \sum_{\substack{m_1+m_2+\cdots+m_r=k \\ 1\leqslant r\leqslant k}} \binom{f_1+m_1-1}{2m_1-1} \times \sum_{2\leqslant j_2<j_3<\cdots<j_r} \binom{f_{j_2}+m_2}{2m_2}\binom{f_{j_3}+m_3}{2m_3}\cdots\binom{f_{j_r}+m_r}{2m_r},$$

and

$$\mathrm{spt}_k(n) = \sum_{\pi \vdash n} \omega_k(\pi).$$

We note that the outer sum above is over all compositions $m_1 + m_2 + \cdots + m_r$ of k.

Example 5.3 ($k = 1$). There is only one composition of 1, $\omega_1(\pi) = f_1(\pi)$ and

$$\mathrm{spt}_1(n) = \mathrm{spt}(n).$$

Example 5.4 ($k = 2$). There are two compositions of 2, namely 2 and $1 + 1$,

$$\omega_2(\pi) = \binom{f_1+1}{3} + f_1 \sum_{2\leqslant j} \binom{f_j+1}{2},$$

and

$$\mathrm{spt}_2(n) = \sum_{\pi \vdash n} \omega_2(\pi).$$

We calculate $\mathrm{spt}_2(4)$. There are five partitions of 4:

4	$f_1 = 1$	$\omega_2 = 0$
$3+1$	$f_1 = f_2 = 1$	$\omega_2 = 1$
$2+2$	$f_1 = 2$	$\omega_2 = 1$
$2+1+1$	$f_1 = 2,\ f_2 = 1$	$\omega_2 = 1+2 = 3$
$1+1+1+1$	$f_1 = 4$	$\omega_2 = 10$

Hence $\mathrm{spt}_2(4) = 0+1+1+3+10 = 15$.

Example 5.5 ($k = 3$). There are four compositions of 3, namely 3, $2+1$, $1+2$ and $1+1+1$. Hence the definition of $\omega_3(\pi)$ has four terms:

$$\omega_3(\pi) = \binom{f_1+2}{5} + \binom{f_1+1}{3} \sum_{2 \leqslant j} \binom{f_j+1}{2} + f_1 \sum_{2 \leqslant j} \binom{f_j+2}{4} + f_1 \sum_{2 \leqslant j < k} \binom{f_j+1}{2}\binom{f_k+1}{2},$$

and

$$\mathrm{spt}_3(n) = \sum_{\pi \vdash n} \omega_3(\pi).$$

To illustrate, we calculate $\mathrm{spt}_3(5)$. There are seven partitions of 5:

5	$f_1 = 1$	$\omega_3 = 0$
$4+1$	$f_1 = f_2 = 1$	$\omega_3 = 0$
$3+2$	$f_1 = f_2 = 1$	$\omega_3 = 0$
$3+1+1$	$f_1 = 2,\ f_2 = 1$	$\omega_3 = 1$
$2+2+1$	$f_1 = 1,\ f_2 = 2$	$\omega_3 = 1$
$2+1+1+1$	$f_1 = 3,\ f_2 = 1$	$\omega_3 = 1+4 = 5$
$1+1+1+1+1$	$f_1 = 5$	$\omega_3 = 21$

Hence $\mathrm{spt}_3(5) = 0+0+0+1+1+5+21 = 28$.

Our goal in this section is to prove

Theorem 5.6. *For* $1 \leqslant k \leqslant n$

$$\mathrm{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n).$$

Proof. First we need the elementary identities

$$\sum_{n=j}^{\infty}\binom{n+j-1}{2j-1}x^n=\frac{x^j}{(1-x)^{2j}}\quad\text{and}\quad\sum_{n=j}^{\infty}\binom{n+j}{2j}x^n=\frac{x^j}{(1-x)^{2j+1}}.$$

To give the idea of the proof we first consider the case $k=4$. From (3.5) we have

$$\begin{aligned}
&\sum_{n=4}^{\infty}\bigl(\mu_8(n)-\eta_8(n)\bigr)q^n\\
&\quad=\sum_{1\leqslant m\leqslant j\leqslant k\leqslant n}\frac{q^{m+j+k+n}}{(1-q^m)^2(1-q^j)^2(1-q^k)^2(1-q^n)^2(q^{m+1};q)_\infty}\\
&\quad=\sum_{1\leqslant m=j=k=n}+\sum_{1\leqslant m=j=k<n}+\sum_{1\leqslant m=j<k=n}+\sum_{1\leqslant m<j=k=n}+\sum_{1\leqslant m=j<k<n}+\sum_{1\leqslant m<j=k<n}\\
&\qquad+\sum_{1\leqslant m<j<k=n}+\sum_{1\leqslant m<j<k<n}\frac{q^{m+j+k+n}}{(1-q^m)^2(1-q^j)^2(1-q^k)^2(1-q^n)^2(q^{m+1};q)_\infty}\\
&\quad=\sum_{m=1}^{\infty}\frac{q^{4m}}{(1-q^m)^8}\prod_{i>m}\frac{1}{(1-q^i)}+\sum_{1\leqslant m<n}\frac{q^{3m}}{(1-q^m)^6}\frac{q^n}{(1-q^n)^3}\prod_{\substack{i>m\\ i\neq n}}\frac{1}{(1-q^i)}\\
&\qquad+\sum_{1\leqslant m<n}\frac{q^{2m}}{(1-q^m)^4}\frac{q^{2n}}{(1-q^n)^5}\prod_{\substack{i>m\\ i\neq n}}\frac{1}{(1-q^i)}+\sum_{1\leqslant m<n}\frac{q^{m}}{(1-q^m)^2}\frac{q^{3n}}{(1-q^n)^7}\prod_{\substack{i>m\\ i\neq n}}\frac{1}{(1-q^i)}\\
&\qquad+\sum_{1\leqslant m<k<n}\frac{q^{2m}}{(1-q^m)^4}\frac{q^k}{(1-q^k)^3}\frac{q^n}{(1-q^n)^3}\prod_{\substack{i>m\\ i\neq k,n}}\frac{1}{(1-q^i)}\\
&\qquad+\sum_{1\leqslant m<k<n}\frac{q^{m}}{(1-q^m)^2}\frac{q^{2k}}{(1-q^k)^5}\frac{q^n}{(1-q^n)^3}\prod_{\substack{i>m\\ i\neq k,n}}\frac{1}{(1-q^i)}\\
&\qquad+\sum_{1\leqslant m<k<n}\frac{q^{m}}{(1-q^m)^2}\frac{q^k}{(1-q^k)^3}\frac{q^{2n}}{(1-q^n)^5}\prod_{\substack{i>m\\ i\neq k,n}}\frac{1}{(1-q^i)}\\
&\qquad+\sum_{1\leqslant m<j<k<n}\frac{q^{m}}{(1-q^m)^2}\frac{q^j}{(1-q^j)^3}\frac{q^k}{(1-q^k)^3}\frac{q^n}{(1-q^n)^3}\prod_{\substack{i>m\\ i\neq j,k,n}}\frac{1}{(1-q^i)}.
\end{aligned}$$

There are eight compositions of 4: 4, $3+1$, $2+2$, $1+3$, $2+1+1$, $1+2+1$, $1+1+2$, and $1+1+1+1$. Each of the eight sums above has the form

$$\sum_{1\leqslant n_1<n_{j_2}<\cdots<n_{j_r}}\frac{q^{m_1n_1}}{(1-q^{n_1})^{2m_1}}\frac{q^{m_2n_{j_2}}}{(1-q^{n_{j_2}})^{2m_2+1}}\cdots\frac{q^{m_rn_{j_r}}}{(1-q^{n_{j_r}})^{2m_r+1}}\prod_{\substack{i>n_1\\ i\notin\{n_{j_2},\dots,n_{j_r}\}}}\frac{1}{(1-q^i)},$$

where $m_1 + m_2 + \cdots + m_r$ is a composition of $k = 4$. This sum can be written as

$$\sum_{\substack{1 \leqslant n_1 < n_{j_2} < \cdots < n_{j_r} \\ f_1 \geqslant m_1,\, f_{j_2} \geqslant m_2,\, \ldots,\, f_{j_r} \geqslant m_r}} \binom{f_1 + m_1 - 1}{2m_1 - 1}\binom{f_{j_2} + m_2}{2m_2} \cdots \binom{f_{j_r} + m_r}{2m_r} \times q^{f_1 n_1 + f_{j_2} n_{j_2} + \cdots + f_{j_r} n_{j_r}} \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \ldots, n_{j_r}\}}} \frac{1}{(1 - q^i)}.$$

We see that this is the generating function for certain weighted partitions in which n_1 is the smallest part, $n_1 < n_{j_2} < \cdots < n_{j_r}$ is an r-subset of the parts of the partition, and f_j is the frequency of part n_j for each j. It follows that

$$\sum_{n=4}^{\infty} \big(\mu_8(n) - \eta_8(n)\big) q^n = \sum_{n=4}^{\infty} \Bigg(\sum_{\pi \vdash n} \omega_4(\pi) \Bigg) q^n = \sum_{n=4}^{\infty} \mathrm{spt}_4(n) q^n.$$

The proof of the general case is completely analogous. Now suppose $k \geqslant 1$. From (3.5) we have

$$\sum_{n=1}^{\infty} \big(\mu_{2k}(n) - \eta_{2k}(n)\big) q^n = \sum_{1 \leqslant n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k} \frac{q^{n_1 + n_2 + \cdots + n_k}}{(1 - q^{n_1})^2 (1 - q^{n_2})^2 \cdots (1 - q^{n_k})^2 (q^{n_1 + 1}; q)_\infty}.$$

We partition this sum into 2^{k-1} subsums by changing each "$\leqslant$" in the general inequality $n_1 \leqslant n_2 \leqslant \cdots \leqslant n_k$ to either "$=$" or "$<$". In this way each subsum corresponds to a unique composition $m_1 + m_2 + \cdots + m_r$ of k (where $1 \leqslant r \leqslant k$). We proceed just as in the case $k = 4$ and the general result follows. □

6. Congruences for higher order spt-functions

In [10] it was shown that given any prime $\ell > 3$ with k and j fixed there are infinitely many arithmetic progressions $An + B$ such that

$$\eta_{2k}(An + B) \equiv 0 \pmod{\ell^j}.$$

Using known results for crank moments [10, §7] and standard techniques [10,8] we may deduce the analog of this result for higher order spt-functions. In this section we prove a number of nice explicit congruences for higher order spt-functions. Many of the congruences follow from known results for rank and crank moments [6].

Theorem 6.1.

$$\mathrm{spt}_2(n) \equiv 0 \pmod 5, \quad \textit{if } n \equiv 0, 1, 4 \pmod 5, \tag{6.1}$$

$$\mathrm{spt}_2(n) \equiv 0 \pmod 7, \quad \textit{if } n \equiv 0, 1, 5 \pmod 7, \tag{6.2}$$

$$\mathrm{spt}_2(n) \equiv 0 \pmod{11}, \quad \textit{if } n \equiv 0 \pmod{11}. \tag{6.3}$$

Proof. By definition,

$$\mathrm{spt}_2(n) = \mu_4(n) - \eta_4(n) = \frac{1}{24}\big(M_4(n) - M_2(n) - N_4(n) + N_2(n)\big).$$

From [6, (5.6)] we have

$$N_4(n) = -\frac{2}{3}(3n+1)M_2(n) + \frac{8}{3}M_4(n) + (1-12n)N_2(n),$$

and

$$24\,\mathrm{spt}_2(n) = \left(2n - \frac{1}{3}\right)M_2(n) - \frac{5}{3}M_4(n) + 12nN_2(n). \tag{6.4}$$

The congruence (6.1) now follows from

$$\begin{aligned} &M_2(n) = 2np(n) \quad [6, (1.27)],\\ &N_2(n) \equiv (n+4)p(n), \quad \text{for } n \not\equiv 0, 3 \pmod 5 \quad [16, \text{p. } 285],\\ &p(5n+4) \equiv 0 \pmod 5. \end{aligned}$$

To begin the proof of (6.2) we use (6.4) to obtain

$$\mathrm{spt}_2(n) \equiv M_4(n) + 3(n+1)M_2(n) + 4nN_2(n) \pmod 7.$$

From [16, p. 285]

$$N_2(n) \equiv (6n+1)p(n) \pmod 7, \quad \text{for } n \not\equiv 0, 2, 6 \pmod 7, \tag{6.5}$$

so that

$$\mathrm{spt}_2(n) \equiv M_4(n) + 3(n+1)M_2(n) \pmod 7, \quad \text{for } n \equiv 0, 1, 5 \pmod 7. \tag{6.6}$$

From [6, (1.21)] we have

$$M_4(7n+5) \equiv M_2(7n+5) \equiv 0 \pmod 7, \quad \text{and} \quad \mathrm{spt}_2(7n+5) \equiv 0 \pmod 7.$$

From [6, (6.5)]

$$(n+2)M_4(n) \equiv -\big(6n^2 + 4n + 1\big)M_2(n) \pmod 7, \tag{6.7}$$

so that

$$M_4(7n) \equiv 3M_2(7n) \equiv 0 \pmod 7 \quad \big(\text{since } M_2(n) = 2np(n)\big), \tag{6.8}$$

$$M_4(7n+1) \equiv M_2(7n+1) \pmod 7, \tag{6.9}$$

and

$$\mathrm{spt}_2(7n) \equiv \mathrm{spt}_2(7n+1) \equiv 0 \pmod 7,$$

by (6.6).

The proof of (6.3) is similar to that of (6.1) and (6.2). From (6.4) we have

$$\mathrm{spt}_2(n) \equiv M_4(n) + (n+9)M_2(n) + 6nN_2(n) \pmod{11}.$$

From [6, (6.6)]

$$(n+5)^3 M_4(n) \equiv \left(5n^4 + 10n^3 + 8n^2 + 8n + 9\right) M_2(n) \pmod{11},$$

so that

$$M_4(11n) \equiv M_2(11n) \equiv 0 \pmod{11},$$

and

$$\mathrm{spt}_2(11n) \equiv 0 \pmod{11}. \qquad \square$$

Theorem 6.2.

$$\mathrm{spt}_3(n) \equiv 0 \pmod 7, \quad \text{if } n \not\equiv 3, 6 \pmod 7, \tag{6.10}$$

$$\mathrm{spt}_3(n) \equiv 0 \pmod 2, \quad \text{if } n \equiv 1 \pmod 4. \tag{6.11}$$

Proof. From [6, (5.6)–(5.7)] and the definition of $\mathrm{spt}_3(n)$ we have

$$\begin{aligned}\mathrm{spt}_3(n) = {} & -\frac{7}{7920} M_6(n) + \frac{1}{1584}(60n+13)M_4(n) + \frac{1}{3960}\left(7 - 78n - 108n^2\right) M_2(n) \\ & - \frac{1}{20} n(1+3n)N_2(n),\end{aligned} \tag{6.12}$$

and

$$\mathrm{spt}_3(n) \equiv n(5n+4)M_2(n) + (3+2n)M_4(n) + n(3n+1)N_2(n) \pmod 7. \tag{6.13}$$

This implies that

$$\mathrm{spt}_3(7n+2) \equiv 0 \pmod 7.$$

Known results for the rank and crank [6, (1.18), (1.21)] imply that

$$\mathrm{spt}_3(7n+5) \equiv 0 \pmod 7.$$

The congruences (6.5), (6.8), (6.9) and (6.13) imply that

$$\mathrm{spt}_3(7n) \equiv \mathrm{spt}_3(7n+1) \equiv 0 \pmod 7.$$

The congruences (6.7) and (6.13) imply that

$$\mathrm{spt}_3(7n+4) \equiv 2M_2(7n+4) + 3N_2(7n+4) \pmod 7.$$

From (6.5) and the fact that $M_2(n) = 2np(n)$ we have

$$M_2(7n+4) \equiv p(7n+4), \qquad N_2(7n+4) \equiv 4p(7n+4) \pmod 7$$

and

$$\mathrm{spt}_3(7n+4) \equiv 0 \pmod 7.$$

We now turn to the congruence (6.11). First we note that the term

$$\frac{1}{20}n(1+3n)N_2(n) \equiv 0 \pmod 2,$$

when $n \equiv 1 \pmod 4$ since $N_2(n) \equiv 0 \pmod 2$. We define

$$s_3(n) = -\frac{7}{7920}M_6(n) + \frac{1}{1584}(60n+13)M_4(n) + \frac{1}{3960}\big(7 - 78n - 108n^2\big)M_2(n)$$

so that

$$\mathrm{spt}_3(4n+1) \equiv s_3(4n+1) \pmod 2.$$

By [6, Theorem 4.2], the function

$$S_3(q) := \sum_{n=1}^{\infty} s_3(n)q^n \in P\mathcal{W}_3,$$

where $\mathcal{W}_n$ is a space of quasimodular forms of weight bounded by $2n$ defined in [6, (3.27)], and

$$P = P(q) = \frac{1}{(q)_\infty}. \tag{6.14}$$

We define the functions

$$P_3(q) = \sum_{n=1}^{\infty} p_3(n)q^n := P(q)\sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

and

$$P_5(q) = \sum_{n=1}^{\infty} p_5(n)q^n := P(q)\sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

As usual $\sigma_k(n) = \sum_{d|n} d^k$. Let $\delta_q = q\frac{d}{dq}$. By [6, (3.29) and Lemma 4.1] the functions $\delta_q(P)$, $\delta_q^2(P)$, $\delta_q^3(P)$, P_3, $\delta_q(P_3)$, and $P_5 \in P\mathcal{W}_3$. Since $\dim \mathcal{W}_3 = 6$ by [6, Cor. 3.6], there is a linear relation between these functions and $S_3(q)$. A calculation gives that

$$s_3(n) = \frac{n}{270}\big(5 - 12n - 147n^2\big)p(n) + \frac{1}{12}(6n+1)p_3(n) - \frac{7}{540}p_5(n)$$

and

$$4s_3(n) \equiv 6n\big(1+n^2\big)p(n) + (3+2n)p_3(n) + 7p_5(n) \pmod 8.$$

Since $d^3 \equiv d^5 \pmod 8$ it follows that

$$\sigma_3(n) \equiv \sigma_5(n) \pmod 8 \quad \text{and} \quad p_3(n) \equiv p_5(n) \pmod 8.$$

Hence

$$4s_3(n) \equiv 6n\big(1+n^2\big)p(n) + (10+2n)p_3(n) \pmod 8,$$

and

$$s_3(4n+1) \equiv p(4n+1) + p_3(4n+1) \pmod 2.$$

It is well known that

$$\delta_q(P) = \sum_{n=1}^{\infty} np(n)q^n = P(q)\sum_{n=1}^{\infty}\sigma(n)q^n.$$

Since $\sigma(n) \equiv \sigma_3(n) \pmod 2$ it follows that

$$np(n) \equiv p_3(n) \pmod 2,$$
$$p(4n+1) \equiv p_3(4n+1) \pmod 2,$$

and

$$s_3(4n+1) \equiv 0 \pmod 2,$$

which completes the proof of (6.11). □

Theorem 6.3.

$$\operatorname{spt}_4(3n) \equiv 0 \pmod 3. \tag{6.15}$$

Proof. From (2.1) and (2.2) we have

$$\sum_{n=1}^{\infty} \operatorname{spt}_4(n)q^n = \frac{1}{(q)_\infty}\left(\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2+4n}}{(1-q^n)^8} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^{n-1}q^{n(3n+1)/2+4n}}{(1-q^n)^8}\right)$$

$$\equiv \frac{1}{(q)_\infty}\left(\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^{n-1}q^{n(n+1)/2+4n}(1-q^n)}{(1-q^{9n})} - \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^{n-1}q^{n(3n+1)/2+4n}(1-q^n)}{(1-q^{9n})}\right) \pmod 3.$$

Before we can proceed we need some results for the rank and crank mod 9. We define

$$S_k(b) = S_k(b,t) := \sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} \frac{(-1)^n q^{n(kn+1)/2+bn}}{(1-q^{tn})},$$

so that

$$\sum_{n=1}^{\infty} \operatorname{spt}_4(n)q^n \equiv \frac{1}{(q)_\infty}\bigl(-S_1(4,9)+S_1(5,9)+S_3(4,9)-S_3(5,9)\bigr) \pmod 3.$$

Now let $M(r,t,n)$ denote the number of partitions of n with crank congruent to r mod t and let $N(r,t,n)$ denote the number of partitions of n with rank congruent to r mod t. Then by [7, (2.13)] and [13, (2.5)] we have

$$\sum_{n=0}^{\infty} N(r,t,n)q^n = \frac{1}{(q)_\infty}\bigl(S_3(r,t)+S_3(t-r,r)\bigr)$$

and

$$\sum_{n=0}^{\infty} M(r,t,n)q^n = \frac{1}{(q)_\infty}\bigl(S_1(r,t)+S_1(t-r,r)\bigr).$$

From [13, (2.3)] and [7, (6.2)]

$$S_k(b,t) = -S_k(t-1-b,t),$$

for $k=1,3$. Hence

$$\sum_{n=0}^{\infty} M(4,9,n)q^n = \frac{1}{(q)_\infty}\bigl(S_1(4,9)+S_1(5,9)\bigr) = \frac{1}{(q)_\infty}S_1(5,9)$$

and

$$\sum_{n=0}^{\infty} N(4,9,n)q^n = \frac{1}{(q)_\infty}\big(S_3(4,9)+S_3(5,9)\big) = \frac{1}{(q)_\infty} S_3(5,9),$$

since

$$S_1(4,9) = S_3(4,9) = 0.$$

It follows that

$$\mathrm{spt}_4(n) \equiv M(4,9,n) - N(4,9,n) \pmod 3.$$

Lewis [19, (1a)] has shown that

$$M(4,9,3n) = N(4,9,3n)$$

and our congruence (6.15) follows. □

If we try the approach of using quasimodular forms to prove the congruence (6.15) we are led to a congruence for the Ramanujan tau-function.

Corollary 6.4.

$$\begin{aligned}\tau(n) \equiv{}& \big(588+297n+258n^2+9n^3+108n^4+486n^5\big)\sigma_1(n)\\ &+ \big(60+255n+189n^2+612n^3+162n^4\big)\sigma_3(n)\\ &+ \big(306+297n+540n^2+180n^3\big)\sigma_5(n) + \big(177+576n+454n^2\big)\sigma_7(n)\\ &+ (201+690n)\sigma_9(n) + 117\sigma_{11}(n) \pmod{3^6}.\end{aligned} \tag{6.16}$$

Proof. From [6, (5.6)–(5.8)] and the definition of $\mathrm{spt}_3(n)$ we see that

$$\begin{aligned}\mathrm{spt}_4(n) ={}& -\frac{67}{7\,362\,432} M_8(n) + \frac{1}{2\,629\,440}(491+1176n)M_6(n)\\ &- \frac{1}{1\,051\,776}\big(1309+8400n+5856n^2\big)M_4(n)\\ &+ \frac{1}{3\,067\,680}\big(-851+10\,966n+21\,204n^2+12\,162n^3\big)M_2(n)\\ &+ \frac{1}{140}\big(n+4n^2+3n^3\big)N_2(n).\end{aligned} \tag{6.17}$$

We define

$$\begin{aligned} s_4(n) = &-\frac{67}{7\,362\,432} M_8(n) + \frac{1}{2\,629\,440}(491 + 1176n) M_6(n) \\ &- \frac{1}{1\,051\,776}\big(1309 + 8400n + 5856n^2\big) M_4(n) \\ &+ \frac{1}{3\,067\,680}\big(-851 + 10\,966n + 21\,204n^2 + 12\,162n^3\big) M_2(n), \end{aligned}$$

so that

$$\mathrm{spt}_4(3n) \equiv s_4(3n) \pmod 3.$$

By [6, Theorem 4.2], the function

$$S_4(q) := \sum_{n=1}^{\infty} s_4(n) q^n \in P\mathcal{W}_4,$$

$$S_4^*(q) := \big(\delta_q^2 - 1\big) S_4(q) = \sum_{n=1}^{\infty} \big(n^2 - 1\big) s_4(n) q^n \in P\mathcal{W}_6,$$

and

$$S_4^*(q) \equiv 0 \pmod 3, \tag{6.18}$$

by Theorem 6.3. By [6, (3.29)] the functions $\delta_q^j(\Phi_{2k+1})$ $(0 \leqslant j \leqslant 5-k, 0 \leqslant k \leqslant 5)$, and $\Delta \in \mathcal{W}_6$, where

$$\Phi_j = \Phi_j(q) = \sum_{n=1}^{\infty} \frac{n^j q^n}{1 - q^n} = \sum_{m,n \geqslant 1} n^j q^{nm} = \sum_{n=1}^{\infty} \sigma_j(n) q^n,$$

and

$$\Delta = \Delta(q) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} \big(1 - q^n\big)^{24}.$$

Since $\dim \mathcal{W}_6 = 22$ by [6, Cor. 3.6], there is a linear relation between these functions and $S_4^*(q)/P$. In fact, we can write the function $S_4^*(q)/P$ as a linear combination of the 22 functions $\delta_q^j(\Phi_{2k+1})$ $(0 \leqslant j \leqslant 5-k,\ 0 \leqslant k \leqslant 5)$, and $\Delta \in \mathcal{W}_6$. The coefficients in this linear combination are rational numbers, and we find that we need to multiply each coefficient by 3^5 to obtain 3-integral rationals. The congruence (6.18) then implies a congruence mod 3^6 between the arithmetic functions $n^j(\sigma_{2k+1}(n))$ $(0 \leqslant j \leqslant 5-k, 0 \leqslant k \leqslant 5)$, and $\tau(n)$. Solving this congruence for $\tau(n)$ gives the result (6.16). □

Ashworth [5] (see also [18]) has also obtained congruences for $\tau(n)$ mod powers of 3. Ashworth's congruences have a different form and depend on the residue of n mod 3.

7. Concluding remarks

It should be pointed out that Bringmann, Mahlburg and Rhoades [12] have proved that there are positive constants α_k and β_k such that

$$M_{2k}(n) \sim N_{2k}(n) \sim \alpha_k n^k p(n), \tag{7.1}$$

$$M_{2k}(n) - N_{2k}(n) \sim \beta_k n^{k-\frac{1}{2}} p(n), \tag{7.2}$$

as $n \to \infty$ when k is fixed. This implies that

$$\mathrm{spt}_k(n) \sim \frac{\beta_k}{(2k)!} n^{k-\frac{1}{2}} p(n), \tag{7.3}$$

as $n \to \infty$ when k is fixed. It would be interesting to consider whether the new identity (1.3) could lead to an elementary upper bound for $\mathrm{spt}_k(n)$.

Folsom and Ono [14] found nontrivial congruences for Andrews spt-function mod 2 and 3. Ono [21] also found simple explicit congruences for Andrews' spt-function modulo every prime > 3. These congruences are related to the action of a weight $\frac{3}{2}$ Hecke operator. It would be interesting to determine whether such behavior continues for the higher degree spt-functions and higher weight Hecke operators.

The function

$$A_k^*(q) = \sum_{n_k \geqslant n_{k-1} \geqslant \cdots \geqslant n_1 \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2} \tag{7.4}$$

occurs in Eq. (1.4) so that

$$\sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_\infty} A_k^*(q). \tag{7.5}$$

The related function

$$A_k(q) = \sum_{n_k > n_{k-1} > \cdots > n_1 \geqslant 1} \frac{q^{n_1+n_2+\cdots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \cdots (1-q^{n_1})^2} \tag{7.6}$$

was first studied by MacMahon [20] as a generalization of

$$A_1^*(q) = A_1(q) = \sum_{n=1}^{\infty} \sigma_1(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}. \tag{7.7}$$

He conjectured that the coefficients of $A_k(q)$ could be expressed in terms of divisor functions. This conjecture was recently proved by Andrews and Rose [4] by showing that in general $A_k(q)$ is a quasimodular form. We note that $A_k^*(q)$ is also a quasimodular form. This result follows from (7.5), (4.4) and the fact that the generating function for $M_{2k}(n)$ is $P(q)$ times a quasimodular form, which was proved by Atkin and the author [6, Theorem 4.2]. Then Andrews and Rose's result that $A_k(q)$ is quasimodular form follows by induction from the equation

$$A_k(q) = (-1)^{k+1}\left(A_k^*(q) + \sum_{j=1}^{k-1}(-1)^j A_j(q) A_{k-j}^*(q)\right).$$

Andrews and Rose's proof is more direct. Andrews and Rose's results were motivated by a certain curve-counting problem on Abelian surfaces.

8. Table

For reference we include a table of $\mathrm{spt}_k(n)$ for $1 \leqslant k \leqslant 6$, $1 \leqslant n \leqslant 29$.

$n\backslash k$	1	2	3	4	5	6
	0	0	0	0		
2	3	1	0	0	0	0
3	5	5	1	0	0	0
4	10	15	7	1	0	0
5	14	35	28	9	1	0
6	26	75	85	45	11	1
7	35	140	217	166	66	13
8	57	259	497	505	287	91
9	80	435	1036	1341	1013	456
10	119	735	2044	3223	3081	1834
11	161	1155	3787	7149	8372	6293
12	238	1841	6797	14 916	20 824	19 125
13	315	2765	11 648	29 480	48 192	52 781
14	440	4 200	19 558	55 902	105 117	134 643
15	589	6125	31 703	101 892	217 945	321 622
16	801	8975	50 645	180 245	433 017	726 650
17	1048	12 731	78 674	309 297	828 346	1 564 696
18	1407	18 179	120 932	518 859	1 534 271	3 231 635
19	1820	25 235	181 664	849 563	2 759 132	6 432 859
20	2399	35 180	270 600	1 366 441	4 837 638	12 395 504
21	3087	48 055	395 682	2 154 789	8 283 014	23 195 905
22	3998	65 681	574 329	3 348 972	13 894 554	42 287 433
23	5092	88 299	820 834	5 119 981	22 856 717	75 274 166
24	6545	118 895	1 166 109	7 733 835	36 968 045	131 143 033
25	8263	157 690	1 634 668	11 520 100	58 818 578	223 982 780
26	10 486	209 230	2 279 242	16 985 374	92 258 215	375 713 010
27	13 165	274 510	3 142 903	24 746 334	142 699 970	619 712 403
28	16 562	359 779	4 312 063	35 735 413	218 041 302	1 006 599 177
29	20 630	466 970	5 859 616	51 073 008	329 162 610	1 611 563 058

Acknowledgments

Firstly, I would like to thank Richard McIntosh for showing me Alexander Patkowski's paper [22]. It was Patkowski's idea of using a limiting form of Bailey's Lemma to derive spt-like results that first got me started on the way to generalizing Andrews' spt-function. Secondly, I would like to thank Mike Hirschhorn for hosting my stay at UNSW in June 2010, and suggesting to me that I take a look at Stirling numbers of the second kind. This was quite helpful in relating ordinary and symmetrized moments. Thirdly, I would like to thank George Andrews, Miklós Bóna, Kathrin

Bringmann, Jeremy Lovejoy and Karl Mahlburg for their comments and suggestions. Finally, I would like to thank the referee for making numerous corrections and suggestions.

References

[1] G.E. Andrews, q-Series: Their Development and Application in Analysis, Number Theory, Combinatorics, Physics, and Computer Algebra, CBMS Reg. Conf. Ser. Math., vol. 66, AMS, Providence, RI, 1986.

[2] G.E. Andrews, Partitions, Durfee symbols, and the Atkin–Garvan moments of ranks, Invent. Math. 169 (2007) 37–73, http://dx.doi.org/10.1007/s00222-007-0043-4.

[3] G.E. Andrews, The number of smallest parts in the partitions of n, J. Reine Angew. Math. 624 (2008) 133–142, http://dx.doi.org/10.1515/CRELLE.2008.083.

[4] G.E. Andrews, S.C.F. Rose, MacMahon's sum-of-divisors functions, Chebyshev polynomials, and quasi-modular forms, preprint, http://front.math.ucdavis.edu/1010.5769.

[5] M.H. Ashworth, Congruence properties of coefficients of modular forms using sigma functions, PhD thesis, Oxford University, 1966.

[6] A.O.L. Atkin, F.G. Garvan, Relations between the ranks and cranks of partitions, Ramanujan J. 7 (2003) 343–366, http://dx.doi.org/10.1023/A:1026219901284.

[7] A.O.L. Atkin, P. Swinnerton-Dyer, Some properties of partitions, Proc. Lond. Math. Soc. (3) 4 (1954) 84–106, http://dx.doi.org/10.1112/plms/s3-4.1.84.

[8] K. Bringmann, On the explicit construction of higher deformations of partition statistics, Duke Math. J. 144 (2008) 195–233, http://dx.doi.org/10.1215/00127094-2008-035.

[9] K. Bringmann, K. Mahlburg, Inequalities between ranks and cranks, Proc. Amer. Math. Soc. 137 (2009) 2567–2574, http://dx.doi.org/10.1090/S0002-9939-09-09806-2.

[10] K. Bringmann, F. Garvan, K. Mahlburg, Partition statistics and quasiharmonic Maass forms, Int. Math. Res. Not. IMRN 2009 (1) (2009) 63–97, http://dx.doi.org/10.1093/imrn/rnn124.

[11] K. Bringmann, J. Lovejoy, R. Osburn, Automorphic properties of generating functions for generalized rank moments and Durfee symbols, Int. Math. Res. Not. IMRN YR 2010 (2010) 238–260, http://imrn.oxfordjournals.org/content/2010/2/238.full.pdf#page=1&view=FitH.

[12] K. Bringmann, K. Mahlburg, R. Rhoades, Asymptotics for rank and crank moments, Bull. London Math. Soc. (2011) http://dx.doi.org/10.1112/blms/bdq129.

[13] A.B. Ekin, Some properties of partitions in terms of crank, Trans. Amer. Math. Soc. 352 (2000) 2145–2156, http://dx.doi.org/10.1090/S0002-9947-00-02306-0.

[14] A. Folsom, K. Ono, The spt-function of Andrews, Proc. Natl. Acad. Sci. USA 105 (2008) 20152–20156, http://mathcs.emory.edu/~ono/publications-cv/pdfs/111.pdf.

[15] F.G. Garvan, New combinatorial interpretations of Ramanujan's partition congruences mod 5, 7 and 11, Trans. Amer. Math. Soc. 305 (1988) 47–77, http://dx.doi.org/10.2307/2001040.

[16] F.G. Garvan, Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank, Int. J. Number Theory 6 (2010) 1–29, http://dx.doi.org/10.1142/S179304211000296X.

[17] G. Gasper, M. Rahman, Basic Hypergeometric Series, Encyclopedia Math. Appl., Cambridge Univ. Press, Cambridge, 2004.

[18] N. Koblitz, 2-adic and 3-adic ordinals of $(1/j)$-expansion coefficients for the weight 2 Eisenstein series, Bull. Lond. Math. Soc. 9 (1977) 188–192, http://www.ams.org/leavingmsn?url=http://dx.doi.org/10.1112/blms/9.2.188.

[19] R. Lewis, Relations between the rank and the crank modulo 9, J. Lond. Math. Soc. (2) 45 (1992) 222–231, http://dx.doi.org/10.1016/0097-3165(92)90101-Y.

[20] P.A. MacMahon, Divisors of numbers and their continuations in the theory of partitions, Proc. Lond. Math. Soc. (2) 19 (1921) 75–113, http://dx.doi.org/10.1112/plms/s2-19.1.75.

[21] K. Ono, Congruences for the Andrews spt-function, Proc. Natl. Acad. Sci. USA 108 (2011) 473–476, http://dx.doi.org/10.1073/pnas.1015339107.

[22] A.E. Patkowski, A strange partition theorem related to the second Atkin–Garvan moment, preprint.

[23] N.J.A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically, 2010, http://oeis.org.

[24] N.J.A. Sloane, Sequence A036969, in: The On-Line Encyclopedia of Integer Sequences, published electronically, 2010, http://oeis.org/A036969.