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AVERAGE HEIGHT IN A PARTIALLY ORDERED SET

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The average height of an element x in a finite poset P is the expected number of elements below x in a random linear extension of P. We prove a number of theorems about average height, some intuitive and some not, using a recent result of L.A. Shepp.

Let P be an arbitrary finite poset having n elements, and let $\Phi(P)$ be the set of one-to-one order-preserving maps from P onto $\{0, 1, 2, \ldots, n-1\}$. Let $L(P)$ be the cardinality of $\Phi(P)$, i.e. the number of linear extensions of P.

If x and y are elements of P, we use the expression $P \cup \{x<y\}$ to denote the poset with the same underlying set as P whose order relation is the smallest one containing the order relation of P and the pair (x, y). The *probability that x is smaller than y*, denoted $\Pr(x<y)$, is defined as $L(P\cup\{x<y\})/L(P)$. Thus if a linear extension $f \in \Phi(P)$ is chosen at random, $\Pr(x<y) = \Pr(f(x)<f(y))$. We assume throughout that all linear extensions are equally likely.

Any collection of inequalities involving elements of P may be regarded as an *event* and identified with the set of linear extensions in $\Phi(P)$ in which all the inequalities hold. The usual notation for conditional probabilities then obtains, so that, for example,

$$\Pr(a<b \mid c<d) = L(P\cup\{a<b, c<d\})/L(P\cup\{c<d\}).$$

Ivan Rival and Bill Sands [7] proposed the following conjecture: is it true for any three elements x, y and z of a finite poset P that $\Pr(x>y \mid x>z) \geqslant \Pr(x>y)$? It seems reasonable intuitively that placing x above z can only improve its probability of being above y, but no elementary proof is known. Recently Larry Shepp [9] proved the conjecture, which will be referred to here as the 'xyz inequality', using an ingenious application of the FKG inequality [4]. (Shepp [8] had already used a similar approach to settle a problem involving comparisons between elements of different components of a poset; see also Graham, Yao and Yao [5] and Kleitman and Schearer [6].)

The *height* H_x of an element x of P is a random variable whose value is $f(x)$, f a random member of $\Phi(P)$. It was conjectured by Chung, Fishburn and Graham

[1] and recently proved by Stanley [10] that the distribution of H_x is always log concave, i.e. for any i with $0<i<n-1$,

$$\Pr(H_x = i-1)\Pr(H_x = i+1) \leq (\Pr(H_x = i))^2.$$

The *average height* $h(x)$ of x is defined to be the expected value $E(H_x)$ of the height of x. The relation $h(x)<h(y)$ is a partial ordering which extends the ordering on P, and is often linear, thus providing a canonical linear extension. Such a linear extension is not obtainable in general from the relation $\Pr(x<y)<\frac{1}{2}$, as Fishburn [2, 3] has shown that the latter relation is not always transitive.

Let $h(x \mid x>y) = E(H_x \mid x>y)$, where $H_x \mid x>y$ is the height of x in $P \cup \{x>y\}$. It seems reasonable that $h(x \mid x>y) \geq h(x)$, and in fact more is true.

Theorem 1. *Let x and y be a pair of incomparable elements of an arbitrary finite poset P. Then $h(x \mid x>y) \geq 1 + h(x \mid x<y)$.*

Proof. Let x and y be incomparable elements of P, and for each $z \neq x$ let G_z be the random variable which takes the value 1 if $f(x)>f(z)$ and 0 otherwise, where f is as usual a random linear extension. Then

$$H_x = \sum\{G_z : z \neq x\}$$

and thus

$$h(x) = E(H_x) = \sum\{E(G_z) : z \neq x\} = \sum\{\Pr(x>z) : z \neq x\}.$$

Repeating the argument for $P \cup \{x<y\}$ and $P \cup \{x>y\}$ respectively, we obtain

$$h(x \mid x<y) = \sum\{\Pr(x>z \mid x<y) : z \neq x\},$$
$$h(x \mid x>y) = \sum\{\Pr(x>z \mid x>y) : z \neq x\}.$$

Now for any two events E and F, we have $\Pr(E \mid F) \geq \Pr(E)$ iff $\Pr(E$ and $F) \geq \Pr(E)\Pr(F)$ iff $\Pr(E \mid F) \geq \Pr(E \mid \text{not } F)$. Thus it follows from the xyz inequality that

$$\Pr(x>z \mid x<y) \leq \Pr(x>z \mid x>y)$$

for each $z \neq x$. Moreover in the case $z = y$ we have

$$\Pr(x>z \mid x<y) = \Pr(x>y \mid x<y) = 0$$

and

$$\Pr(x>z \mid x>y) = \Pr(x>y \mid x>y) = 1.$$

Thus, by comparing sums, we have the desired result:

$$h(x \mid x<y) + 1 \leq h(x \mid x>y).$$

This, of course, implies also the weaker inequality

$$h(x \mid x>y) > h(x).$$

Another way to improve on this observation is to show that the entire distribution of H_x is shifted upwards when the inequality $x > y$ is introduced. To do this a stronger form of the xyz inequality is needed, but fortunately the stronger form is an immediate consequence of Shepp's proof.

Theorem 2 (Shepp). *Let x be a fixed member of a finite poset P and let A and B be disjuncts of statements of the form $x < y$, $y \in P$. Then* $\Pr(A \text{ and } B) \geq \Pr(A)\Pr(B)$.

If X and Y are random variables, then X will be said to *majorize* Y if for any real number r, $\Pr(X > r) \geq \Pr(Y > r)$.

Theorem 3. *Let x and y be incomparable elements of a finite poset P. Then $H_x \mid x > y$ majorizes H_x (which in turn majorizes $H_x \mid x < y$).*

Proof. Let r be a real number and let F be the set of subsets of $P - \{x\}$ whose cardinality is greater than r. For each set $S \in F$, let A_S be the conjunction of the statements $x > z$ for $z \in S$; and let A be the disjunction of the statements A_S.

Let B be the statement $x > y$; then $\Pr(H_x > r) = \Pr(A)$ and $\Pr((H_x \mid x > y) > r) = \Pr(A \text{ and } B)/\Pr(B)$, and the result then follows from Theorem 2.

It is obvious that the average height of a minimal element is at most $\frac{1}{2}(n-1)$. If x is the *only* minimal element of P, then of course $h(x) = 0$. One consequence of the next theorem is the much less obvious fact that if there are precisely two minimal elements, then the product of their average heights is at most one.

If U is a non-null subset of a poset P, define the height of U to be the height of its lowest element, i.e. $H_U = \min\{H_u : u \in U\}$.

Theorem 4. *Let U and V be subsets of P with $P = U \cup V$. Then $h(U)h(V) \leq 1$.*

Proof. The following lemma is trivial but useful, and supports the naturality of the definition of average height.

Lemma. *Let the poset P^* be obtained from P by adding a new element w to P with the relations $\{w < u;\ u \in U\}$, and let $h(U)$ be the average height of U in P as defined above. Then the number of linear extensions of P^* in which $H_w \neq 0$ is $h(U)L(P)$.*

The following notation is convenient: if x, y and z are members of a poset then $L(xy)$ is the number of linear extensions in which y is the immediate successor of x, and $L(y-z)$ the number in which at least one other element lies above y and below z; $L(xy-z)$ will then be the number in which both events occur. We make use also of the standard convention that if A and B are disjoint posets, then the sum $A \oplus B$ is the ordinal sum of A and B, with A below B.

Now let P' be a copy of the dual of P, with corresponding subsets U' and V', and assume P and P' are disjoint. Let x, y, and z be elements not in P or P' and set

$$Q=(P'\oplus\{x\}\oplus P)\cup(U'\oplus\{y\}\oplus U)\cup(V'\oplus\{z\}\oplus V)$$

(see Fig. 1 below).

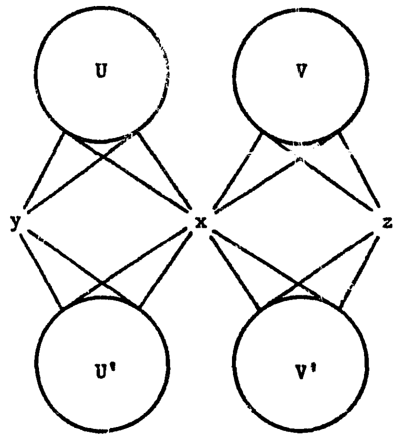

Fig. 1.

Notice that in linear extensions of Q, y and z cannot both penetrate P (or P') since the element at the bottom of P must belong to U or V. Thus,

$$\begin{aligned}\Pr(x<y \text{ and } x<z)&=[L(xyz)+L(xzy)+L(xy-z)+L(xz-y)]/L(Q)\\&=[L(P')L(P)+L(P')L(P)+L(P')h(V)L(P)\\&\quad+L(P')h(U)L(P)]/L(Q)\\&=(2+h(U)+h(V))(L(P))^2/L(Q)\end{aligned}$$

and similarly,

$$\begin{aligned}L(Q)&=6L(xyz)+2L(xy-z)+2L(xz-y)\\&\quad+2L(y-xz)+2L(z-xy)+2L(y-x-z)\\&=(L(P))^2(6+4h(U)+4h(V)+2h(V)h(U)).\end{aligned}$$

Now $\Pr(x<y)=\Pr(x<z)=\frac{1}{2}$ by symmetry, so Theorem 2 implies that $\Pr(x<y$ and $x<z)\geq\frac{1}{4}$; thus

$$4(2+h(U)+h(V))\geq 6+4h(U)+4h(V)+2h(U)h(V),$$

from which we obtain $h(U)h(V)\leq 1$ as desired.

It should perhaps be noted that by letting $P=U+V$, where U and V are long chains, $h(U)$ and $h(V)$ may each be made arbitrarily close to 1.

In the case where U is the singleton $\{u\}$ and $V=P-\{u\}$, the conclusion of Theorem 2 reduces to $h(u)\Pr(H_u=0)<1$; in fact it seems that the following stronger statement may hold: for any poset P and any element u of P, $(1+h(u))\Pr(H_u=0)\leq 1$. Since this is interesting only if u is minimal, we call this the *minimal element conjecture.* Even though only the distribution of H_u for a single element is involved, it seems not to be directly connected to the log concavity property.

The property cited above for posets with just two minimal elements u and v is derived from Theorem 4 by setting $U = \{x \in P: u \leq x\}$ and $V = \{x \in P: v \leq x\}$.

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