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Rational functions associated with double infinite sequences of complex numbers

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Abstract

Let $\{\mu_k\}_{-\infty}^{+\infty}$ be a given double infinite sequence of complex numbers. By defining a linear functional on the space of the Laurent polynomials, certain rational functions are first constructed and some algebraic properties studied.

The hermitian case, i.e. $\mu_{-k} = \bar{\mu}_k$, $k \in \mathbb{Z}$ is separately considered and it is shown how the theory of polynomials orthogonal on the unit circle can be used in order to prove geometric convergence for sequences such as these rational functions.

Keywords: Laurent polynomials; Generating function; Linear functional; Szegö polynomials; Geometric convergence

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1. Introduction

From a purely algebraic point of view, Padé Approximants (and Padé-type too, see e.g. [1, 2]) can be seen as certain rational functions associated with a sequence of complex numbers. Indeed, let $\{c_k\}_{k=0}^{\infty}$ be a given sequence and consider the formal power series,

$$L = \sum_{j=0}^{\infty} c_j z^j \tag{1.1}$$

(which may or may not be the Taylor or asymptotic expansion of a function around the origin). Thus, for m and n nonnegative integers, one tries to find polynomials P_m and Q_n of degree m and n, respectively, such that

$$L(z)Q_n(z) - P_m(z) = \sum_{j=0}^{\infty} d_j z^j \tag{1.2}$$

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with $d_j=0$, $0\leqslant j\leqslant m+n$. (For further details concerning formal power series, see [21]). The rational function $P_m(z)/Q_n(z)$ is called an (m,n) Padé approximant to (1.1).

In this paper, instead of the sequence $\{c_k\}_{k=0}^{\infty}$ we start from a double infinite sequence $\{\mu_k\}_{-\infty}^{+\infty}$, so that rational functions defined in the Padé sense (1.2) are considered and certain algebraic properties studied. In this respect, an immediate question arises: which type of power expansion do we associate with the sequence $\{\mu_k\}_{-\infty}^{+\infty}$?

One might initially think of a Laurent series like $\sum_{-\infty}^{\infty}\mu_j z^j$. This situation was studied by Bultheel [5], giving rise to the theory of the so-called Padé–Laurent approximants. (see also, [20]). In our case, we shall associate with $\{\mu_k\}_{-\infty}^{\infty}$ a pair (L_0,L_∞) of formal power series,

$$L_0=\mu_0+2\sum_{j=1}^{\infty}\mu_j z^j \quad (z\to 0), \tag{1.3a}$$

$$L_\infty=-\mu_0-2\sum_{j=1}^{\infty}\mu_{-j}z^{-j} \quad (z\to\infty). \tag{1.3b}$$

From (1.3), it is not new to infer that the rational functions to be considered are to be Two-point Padé approximants [12]. Approximants like those given by (1.2) have free poles. This fact, can sometimes become a serious drawback [1]. For this reason, we shall mostly restrict our attention to the Padé-type situation where the poles can be given beforehand. (Actually, we could fix part of the denominator and part of the numerator and thus Partial Padé approximant, would arise, see [3, 17].) Definitively, we shall be concerned with Two-point Padé-type approximants to the pair (L_0,L_∞). First, an algebraic approach is presented (Section 2). Such approach allows to connect to the Szegö polynomials and related topics, and deduce convergence properties (Section 3).

In order to fix notations, in the sequel, Π_n will stand for the polynomials of degree n at most and Π for the set of all the polynomials. Furthermore, for every pair (p,q) of integers with $p\leqslant q$, we denote by $\Lambda_{p,q}$ the linear space of all Laurent polynomials (L-polynomials),

$$R(z)=\sum_{j=p}^{q}d_j z^j, \quad d_j\in\mathbb{C}.$$

We shall also write Λ for the linear space of all L-polynomials. (Observe that $\Pi_n=\Lambda_{0,n}$.)

2. Preliminary results

In [2], Brezinski introduces the concept of generating function associated with a formal power series as (1.1). Indeed, the linear functional $C(x^j)=c_j$, $j=0,1,\ldots$ enables us to write, at least formally,

$$L(z)=\sum_{j=0}^{\infty}c_j z^j=C\left(\frac{1}{1-xz}\right). \tag{2.1}$$

(The functional C acts on the variable x, z being a parameter). Brezinski refers to $(1-xz)^{-1}$ as the "generating function" of the sequence $\{c_k\}_{k=0}^{\infty}$ (or of L). According to Jones et al. [22] from $\{\mu_k\}_{-\infty}^{\infty}$ we consider the linear functional μ acting on the L-polynomials on the variable x),

$$\mu(x^j) = \mu_{-j}, \quad j \in \mathbb{Z}. \tag{2.2}$$

Our first aim will be to obtain a generating function for the pair of formal power series (1.3). Let us first consider,

$$L = \sum_{j=0}^{\infty} \mu_j z^j \quad (z \to 0), \tag{2.3a}$$

$$L^* = -\sum_{j=1}^{\infty} \mu_{-j} z^{-j} \quad (z \to \infty) \tag{2.3b}$$

and the linear functional D defined on Λ by $D(x^j) = \mu_j$, $j \in \mathbb{Z}$. Then, one readily gets (at least formally)

$$D((1-xz)^{-1}) = L(z) \quad (z \to 0), \tag{2.4a}$$

$$D((1-xz)^{-1}) = L^*(z) \quad (z \to \infty). \tag{2.4b}$$

Actually, (2.4) should be understood in the sense that if $F(z)$ is a function with Taylor and Laurent series around the origin and infinity given by (2.3), then,

$$F(z) = D((1-xz)^{-1}). \tag{2.5}$$

In this case, $(1-xz)^{-1}$ is said to be the generating function of the pair (L, L^*) (For an approach on Two-point Padé approximation based upon this generating function, see e.g. [13, 14, 16, 18]). In order to find a generating function for the pair (L_0, L_∞), one should first take into account that the functionals μ and D are related by

$$\mu(R(x)) = D(R(x^{-1})), \quad R \in \Lambda_{p,q}. \tag{2.6}$$

Assume that now there exists a function $F(z)$ as defined by (2.5), and set $H(z) = 2F(z) - \mu_0$. Thus, one has

$$\begin{aligned} H(z) &= 2F(z) - \mu_0 = 2D((1-xz)^{-1}) - \mu_0 \\ &= D(2(1-xz)^{-1} - 1) = D\left(\frac{1+xz}{1-xz}\right) = \mu\left(\frac{x+z}{x-z}\right). \end{aligned}$$

Strictly speaking, we should assume that the functional D has been extended to a larger space than Λ containing, at least, the function $(1-xz)^{-1}$. This causes taking as generating function for the pair (L_0, L_∞), the following,

$$g(x,z) = \frac{x+z}{x-z}.$$

We have, formally,

$$\mu\left(\frac{x+z}{x-z}\right) = L_0(z) \quad (z \to 0),$$

$$\mu\left(\frac{x+z}{x-z}\right) = L_\infty(z) \quad (z \to \infty).$$

In order to make the paper self-contained, we shall briefly recall the definition of Two-point Padé-type approximant. Thus, let m and k be nonnegative integers such that $0 < k \leqslant m$ and $Q_{km}(z)$ a given polynomial of degree m with $Q_{km}(0) \neq 0$, then there exists a unique polynomial $P_{km} \in \Pi_m$ so that,

$$L_0 - (k/m)_{(L_0,L_\infty)}(z) = \mathbb{O}(z^k) \quad (z \to 0), \tag{2.7a}$$

$$L_\infty - (k/m)_{(L_0,L_\infty)}(z) = \mathbb{O}((z^{-1})^{m-k+1}) \quad (z \to \infty), \tag{2.7b}$$

where we have written

$$P_{km}(z)/Q_{km}(z) = (k/m)_{(L_0,L_\infty)}(z).$$

This rational function (with prescribed denominator Q_{km}) associated with the pair (L_0, L_∞) is called a Two-point Padé-type approximant (2PTA) of order $(k, m-k+1)$ (for other definitions of 2PTA see, e.g. [12, 16]). It should also be remarked that since $0 < k \leqslant m$, we are implicitly imposing that the order of correspondence either at the origin or infinity is greater than one. Certainly, this does not mean a loss of generality but we are properly dealing with the two-point situation.

On the other hand, for every pair (p,q) of integers with $p \leqslant q$, the system $\{x^j\}_{j=p}^q$ satisfies the Haar condition on any set $X \subset \mathbb{C}$ such that $0 \notin X$ (see [10]). Therefore, any Hermite interpolation problem has always a unique solution. Thus, we have the following:

Theorem 1. (a) *Let* $\{x_j\}_{j=1}^r$, *be r distinct given complex numbers such that* $x_j \neq 0$. *Let* $\{m_j\}_{j=1}^r$ *be natural numbers with* $m_1 + \cdots + m_r = m$. *Denote by* $R_{km}(x,z)$ *the L-polynomial (in x, z is a parameter) in* $\Lambda_{-(k-1),m-k}$ *interpolating* $g(x,z)$ *at the nodes* x_j *with multiplicity* m_j. *Then,*

$$\mu(R_{km}(x,z)) = (k/m)_{(L_0,L_\infty)}(z)$$

with denominator $Q_{km}(z) = Q_m(z) = \Pi_1^r (z - x_j)^{m_j}$.

(b) *Let* $H(z)$ *be a function admitting* L_0 *and* L_∞ *as Taylor and Laurent expansion around* $z = 0$ *and* $z = \infty$, *respectively. Then,*

$$H(z) - (k/m)_{(L_0,L_\infty)}(z) = \frac{2z^k}{Q_m(z)}\mu\left(\frac{V(x)}{x-z}\right),$$

where $V(x) = x^{1-k}Q_m(x)$.

Proof. (a) First, it is a simple matter to check that $R_{km}(x,z)$ can be expressed by

$$R_{km}(x,z) = 1 + \frac{2z}{x-z}\left(1 - \frac{V(x)}{V(z)}\right), \tag{2.8}$$

where $V(x) = x^{-k+1}Q_m(x)$.

On the other hand,

$$\mu(R_{km}(x,z)) = \mu_0 - \frac{2z}{V(z)}\mu\left(\frac{V(x)-V(z)}{x-z}\right) = \mu_0 - \frac{2z^k}{Q_m(z)}\mu\left(\frac{V(x)-V(z)}{x-z}\right).$$

Now, standard arguments allow to see that

$$\mu\left(\frac{V(x)-V(z)}{x-z}\right) \in \Lambda_{-(k-1),m-k}.$$

Hence, $\mu(R_{km}(x,z))$ is a rational function of type (m,m) i.e. numerator and denominator of degree m at most. Furthermore,

$$\begin{aligned} L_0(z) - \mu(R_{km}(x,z)) &= \mu\left(\frac{x+z}{x-z}\right) - \mu\left[1 + \frac{2z}{x-z}(1 - V(x)/V(z))\right] \\ &= \frac{2z}{V(z)}\mu\left(\frac{V(x)}{x-z}\right) = \frac{2z^k}{Q_m(z)}\mu\left(\frac{V(x)}{x-z}\right) = \mathbb{O}(z^k) \quad (z \to 0), \end{aligned} \tag{2.9}$$

since $Q_m(0) \neq 0$. Finally,

$$\begin{aligned} L_\infty(z) - \mu(R_{km}(x,z)) &= -\frac{2z^k}{Q_m(z)}\mu\left(\frac{z^{-1}V(x)}{1-xz^{-1}}\right) \\ &= \mathbb{O}((z^{-1})^{m-k})\mathbb{O}(z^{-1}) = \mathbb{O}((z^{-1})^{m-k+1}) \quad (z \to \infty) \end{aligned} \tag{2.10}$$

since Q_m has exact degree m. □

(b) Set $(k/m)_{(L_0,L_\infty)}(z) = P_m(z)/Q_m(z)$. It can be deduced that the numerator P_m can be rewritten as,

$$P_m(z) = \mu_0 Q_m(z) - 2z\hat{P}_m(z),$$

where

$$\hat{P}_m(z) = z^{k-1}\mu\left(\frac{V(x)-V(z)}{x-z}\right) \in \Pi_{m-1}.$$

Then, one has,

$$P_m(z) = \mu_0 Q_m(z) - 2z^k\mu\left(\frac{V(x)}{x-z}\right) + 2z^k\mu\left(\frac{V(z)}{x-z}\right).$$

Now, having in mind that $Q_m(z) = z^{k-1}V(z)$, it follows that

$$P_m(z) = \mu\left(1 + \frac{2z}{x-z}\right)Q_m(z) - 2z^k\mu\left(\frac{V(x)}{x-z}\right) = H(z)Q_m(z) - 2z^k\mu\left(\frac{V(x)}{x-z}\right). \tag{2.11}$$

Dividing both members in (3.4) by $Q_m(z)$ the proof follows. □

Next, an integral representation for the (k/m) 2PTA will be given. Indeed, assume $H(z)$ analytic in a domain $\mathbb{V}$ containing the origin and infinity (We can think of both $\mathbb{V}$, as a connected domain, for example, $\mathbb{V} = \bar{\mathbb{C}}\backslash K$, K a compact such that $0 \notin K$ or as an unconnected domain of the form $\mathbb{V} = \mathbb{V}_0 \cup \mathbb{V}_\infty = \text{Int}(\Gamma_0) \cup \text{Ext}(\Gamma_\infty)$, Γ_0 and Γ_∞ being closed Jordan curves such that $0 \in \text{Int}(\Gamma_0) \subset \text{Int}(\Gamma_\infty)$. Here, $\bar{\mathbb{C}}$ will denote the extended complex plane.

Assume that

$$H(z)=\mu_0+2\sum_{j=1}^{\infty}\mu_j z^j \quad (z\to 0), \tag{2.12}$$

$$H(z)=-\mu_0-2\sum_{j=1}^{\infty}\mu_{-j} z^{-j} \quad (z\to \infty). \tag{2.13}$$

Theorem 2. *Under the previous conditions, we have*

$$H(z)-(k/m)_{(L_0,L_\infty)}(z)=\frac{z^k}{2\pi i Q_m(z)}\int_\Gamma \frac{Q_m(x)H(x)}{x^k(x-z)}\mathrm{d}x, \tag{2.14}$$

where the denominator $Q_m(z)$ *is a polynomial of degree exact* m *with* $Q_m(0)\neq 0$ *and* Γ *a closed Jordan curve contained in* $\mathbb{V}$.

Proof. See [24, pp. 186–187], taking in formulas (4) and (5)

$$\beta_1=\beta_2=\cdots=\beta_k=0 \quad\text{and}\quad \beta_{k+1}=\cdots=\beta_{m+1}=\infty.$$

Remark 1. We can deduce which type of conditions are required on the polynomial $Q_m(z)$ so that the order of correspondence given by (2.7) and (2.8) can be increased as much as possible. (This is what Brezinski calls, higher-order Padé-type approximants [2]). Indeed by imposing to $V(z)=z^{-k+1}Q_m(z)$ that

$$\mu(x^{-i}V(x))=0, \quad i=1,\ldots,n\leqslant h,$$

$$\mu(x^{i}V(x))=0, \quad i=1,\ldots,p-1\leqslant k-1,$$

(where $h+k=m$ and $n+p\leqslant m$) or equivalently,

$$\mu(x^{i}V(x))=0, \quad -n\leqslant i\leqslant p-1 \tag{2.16}$$

from Theorem 2(b), one has, on the one hand,

$$\begin{aligned} L_0(z)-(k/m)_{(L_0,L_\infty)}(z) &= \frac{2z^{k-1}}{Q_m(z)}\mu[x^{-(n+1)}V(x)z^{n+1}+\cdots] \\ &= \frac{2z^{k-1}}{Q_m(z)}\mu\left[\frac{x^{-n}V(x)}{x-z}\right]=\mathbb{O}(z^{k+n}), \quad z\to 0 \end{aligned}$$

and on the other,

$$\begin{aligned} L_\infty(z)-(k/m)_{(L_0,L_\infty)}(z) &= \frac{2z^{k-1}}{Q_m(z)}\mu[x^{p}V(x)z^{-p}+x^{p+1}V(x)z^{-p-1}+\cdots] \\ &= -\frac{2z^{k-1}}{Q_m(z)}\mu\left[\frac{x^{p}V(x)}{x-z}\right]=\mathbb{O}((z^{-1})^{h+p+1}), \quad z\to \infty. \end{aligned}$$

Hence, we see the order of correspondence has been increased up to $n+p$. The highest order is reached when $n+p=m$. In this case Two-point Padé-type become Two-point Padé approximants (for a precise definition of Two-point Padé approximant, see e.g. [12, 17]). Now, (2.16) is translated into

$$\mu^{-(n+k-1)}(x^j Q_m(x)) = 0, \quad j = 0, 1, \ldots, m-1. \tag{2.17}$$

That is, $Q_m(x)$ represents the mth orthogonal polynomial with respect to the functional $\mu^{-(n+k-1)}$. Where for a given integer s, we set

$$\mu^{(s)}(x^j) = \mu(x^{s+j}) = \mu_{-(s+j)}, \quad j \in \mathbb{Z}.$$

This property of orthogonality was already given by Draux [14]

Remark 2. Let K and m be nonnegative integers such that $0 < K \leqslant 2m$ and denote by $[K/m](z)$ the two-point Padé approximant to (L_0, L_∞), i.e. a rational function of type (m, m) such that

$$\begin{aligned} L_0(z) - [K/m](z) &= \mathbb{O}(z^K) && (z \to 0), \\ L_\infty(z) - [K/m](z) &= \mathbb{O}((z^{-1})^{2m-K+1}) && (z \to \infty). \end{aligned} \tag{2.18}$$

Then, if $[K/m](z)$ exists, it is unique (see e.g. [12]). Thus, it should be observed that its construction from the orthogonal polynomial Q_m respect to $\mu^{-(k-1)}$ is completely independent of $(k/m)_{(L_0, L_\infty)}(z)$, for any k $(0 < k \leqslant m)$ and for any generating polynomial of $(k/m)(z)$ (see [14]).

3. Convergence

In this section we are mainly concerned with certain results about convergence of sequences of 2PAs and 2PTAs under the assumption that the coefficients $\{\mu_k\}_{-\infty}^{+\infty}$, $k \in \mathbb{Z}$ admit the following integral representation

$$\mu_k = \int_{-\pi}^{\pi} \exp(-\mathrm{i}k\theta) w(\theta)\, \mathrm{d}\theta, \tag{3.1}$$

where $w(\theta)$ is a function, possibly complex, and L_1-integrable on $[-\pi, \pi]$. In this case, the power series

$$\begin{aligned} L_0(z) &= \mu_0 + 2\sum_{j=1}^{\infty} \mu_j z^j, \quad (z \to 0), \\ L_\infty(z) &= -\mu_0 - 2\sum_{j=1}^{\infty} \mu_{-j} z^{-j}, \quad (z \to \infty), \end{aligned} \tag{3.2}$$

represent the Taylor expansions of the function

$$H(z) = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\theta} + z}{\mathrm{e}^{\mathrm{i}\theta} - z} w(\theta)\, \mathrm{d}\theta, \tag{3.3}$$

around $z = 0$ and $z = \infty$, respectively. Observe that (3.3) defines an analytic function on the extended complex plane $\bar{\mathbb{C}}$, unless the unit circle $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$ A function $H(z)$ given by (3.3) is said to be the Herglotz–Riesz transform of the function w.

In the sequel and for the sake of simplicity, we will write $(k/m)_H(z)$ instead of $(k/m)_{L_0,L_\infty}(z)$ when referring to 2PTA to the function $H(z)$ and the same for 2PAs. Now the integral representation (2.14) for the error can be expressed as follows.

Theorem 3. *Let $R_m(z)=P_m(z)/Q_m(z)=(k/m)_H(z)$ be a 2PTA to $H(z)$ with denominator $Q_m(z)$ $(\deg(Q_m)=m)$ and $Q_m(0)\neq 0$. Then,*

$$H(z)-R_m(z)=\frac{2z^k}{Q_m(z)}\int_{-\pi}^{\pi}\frac{e^{-i(k-1)\theta}}{z-e^{i\theta}}Q_m(e^{i\theta})w(\theta)\,d\theta. \tag{3.4}$$

Proof. Let G be a region (closed and connected) in $\mathbb{C}$ so that $\mathbb{T}\subset G$ and $0\notin G$. Suppose $\Gamma=\partial G$ is a finite union of Jordan curves (G can be taken as the annulus $\{z\in\mathbb{C}: r\leqslant|z|\leqslant R,\ r<1<R\}$).

By (2.14), one can write $(z\notin\mathbb{T})$

$$\begin{aligned}H(z)-R_m(z)&=\frac{z^k}{2\pi i Q_m(z)}\int_\Gamma\frac{Q_m(x)H(x)}{x^k(x-z)}\,dx\\&=\frac{z^k}{2\pi i Q_m(z)}\int_\Gamma\frac{Q_m(x)}{x^k(x-z)}\left[\int_{-\pi}^{\pi}\frac{e^{i\theta}+x}{e^{i\theta}-x}w(\theta)\,d\theta\right]dx.\end{aligned}$$

By using Fubini's Theorem, one has,

$$H(z)-R_m(z)=\frac{z^k}{2\pi i Q_m(z)}\int_{-\pi}^{\pi}\left[\int_\Gamma\frac{Q_m(x)}{x^k(x-z)}\frac{e^{i\theta}+x}{e^{i\theta}-x}\,dx\right]w(\theta)\,d\theta.$$

By taking the boundary Γ sufficiently close to the unit circle so that $\mathbb{T}$ is contained in the interior of Γ, we can apply the Cauchy integral formula to get (3.4). □

In [6], connection between quadrature formulas on the unit circle and Two-point Pade approximants was established along with convergence results for sequences of 2PTA with poles on the unit circle $\mathbb{T}$, w being a complex function.

Here, we will restrict our attention to the case $w(\theta)>0$ almost everywhere on $[-\pi,\pi]$, or more generally we will assume that there exists a distribution function ϕ on $[-\pi,\pi]$, i.e. a real bounded nondecreasing funcion with infinitely many points of increase on $[-\pi,\pi]$ such that

$$d\phi(\theta)=w(\theta)\,d\theta. \tag{3.5}$$

Obviously, the sequence (3.1) is now hermitian, i.e. $\mu_{-k}=\bar{\mu}_k$, $k\in\mathbb{Z}$. Under these conditions we will consider sequences of 2PTA of higher order and 2PA introduced by Jones et al. [22]. These authors established the uniform convergence of such sequences to the function $H(z)$ on appropriate domains of $\bar{\mathbb{C}}-\mathbb{T}$. Here, we will improve such results by showing that geometric convergence also holds. For this purpose, let us introduce, the so-called Szegö polynomials or polynomials orthogonal on the unit circle [22]. Indeed, from the distribution ϕ we have the following inner product:

$$\langle f,g\rangle=\int_{-\pi}^{\pi}f(e^{i\theta})\overline{g(e^{i\theta})}\,d\phi(\theta).$$

Orthogonalization process when applied to the basis $\{1, z, \dots, z^n\}$ of Π_n produces an orthogonal basis $\{\rho_0, \rho_1, \dots, \rho_n\}$ of monic polynomials. The sequence $\{\rho_n(z)\}$ is called the sequence of monic Szegö polynomials with respect to the distribution ϕ. These polynomials along with the trigonometric moment problem and quadrature on the unit circle have received much attention recently as a result of their applications in the rapidily growing field of digital signal processing (see e.g. [4, 9, 11]).

Next, we will see that according to the ideas introduced in Section 2, Szegö polynomials appear, in a natural way, as denominators of certain 2PA. Indeed, let us consider in (2.18) $K=m$ (recall that $0<K\leqslant 2m$) i.e. we are concerned with $[m/m]_H(z)$ 2PA denoted by $P_m(z)/Q_m(z)$. Since Q_m is orthogonal with respect to $\mu^{-(K-1)}$, then making use of the classical writing of this polynomial as a determinant (see Theorem 2.1 in [2]), one has

$$Q_m(z)=\lambda_m \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-m} \\ \mu_1 & \mu_0 & \cdots & \mu_{1-m} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{m-1} & \mu_{m-2} & & \mu_{-1} \\ 1 & z & & z^m \end{vmatrix}, \quad (\lambda_m \neq 0). \tag{3.6}$$

Now, from (3.1) and (3.5) it is known that the Toeplitz determinants

$$\Delta_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_{-1} & \mu_0 & \cdots & \mu_{n-1} \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{-n} & \mu_{-n+1} & \cdots & \mu_0 \end{vmatrix}, \quad n=0,1,\dots (\Delta_{-1}=1)$$

are positive for $n=0,1,\dots$. Thus, $Q_m(z)$ has exact degree m and taking $\lambda_n = 1/\Delta_{m-1}$, Q_m is monic. Furthermore, $Q_m(z)$ trivially satisfies, $\langle Q_m(x), x^j\rangle = 0$, $j=0,1,\dots,m-1$ $(x=e^{i\theta})$ which amounts to say that the denominator $Q_m(z)$ of the $[m/m]_H(z)$ 2PA coincides with the mth monic Szegö polynomial.

On the other hand, if we set $K=m+1$, in (2.18), then the $[m+1/m]_H(z)$ 2PA results, so that if we write $[m+1/m]_H(z) = \hat{P}_m(z)/\hat{Q}_m(z)$, by similar arguments as given above, we have

$$\hat{Q}_m(z) = z^m \overline{\rho_m(1/\bar{z})} = \rho_m^*(z).$$

The polynomial $\rho_m^*(z)$ is used to be called the reciprocal polynomial of $\rho_m(z)$, satisfying

$$\langle \rho_n^*(x), x^j\rangle = 0,\ \ j=1,2,\dots,m,\ \ \langle \rho_n^*(x), 1\rangle \neq 0, \quad \text{with } x=e^{i\theta}.$$

Thus, we see that the denominator of the $[m/m]_H$ 2PA has exact degree m, but $Q_m(0)=\rho_m(0)$, in general, cannot be guaranteed to be different from zero. On the other hand, the denominator of the $[m+1/m]_H$ 2PA satisfies $\hat{Q}_m(0)=1$, but in general, $\deg(\hat{Q}_m)\leqslant m$. On the other hand, it is well known see e.g. [15] that the zeros of the Szegö polynomials $\rho_n(z)$ all lie inside the unit disk, $\mathbb{D}=\{z\colon |z|<1\}$ and, consequently, $\rho_m^*(z)$ has all its zeros in $\mathbb{E}=\{z\colon |z|>1\}$. Thus, $[m/m]_H = P_m(z)/Q_m(z)$, $m=1,2,\dots$ represents a sequence of analytic functions on $\mathbb{D}$. In [22], Jones et al., making use of the so-called Hermitian–Perron Caratheodory continued fraction (HPC-fraction), proved the uniform convergence of the sequences $\{P_m(z)/Q_m(z)\}$ and $\{\hat{P}_m(z)/\hat{Q}_m(z)\}$ to the function $H(z)$ on compacts of $\mathbb{E}$ and $\mathbb{D}$, respectively. Now, an estimate of the rate of convergence is also got, as assured in the following.

Theorem 4. *Set* $E_m(z)=H(z)-P_m(z)/Q_m(z)$ *and* $\hat{E}_m(z)=H(z)-\hat{P}_m(z)/\hat{Q}_m(z)$. *In the conditions above, we have*
(1) $\lim_{m\to\infty}\sup|E_m(z)|^{1/m}\leqslant 1/|z|<1, \quad z\in\mathbb{E}$
(2) $\lim_{m\to\infty}\sup|E_m^*(z)|^{1/m}\leqslant |z|<1, \quad z\in\mathbb{D}$.
Limits (1) *and* (2) *hold uniformly on compact sets of* $\mathbb{E}$ *and* $\mathbb{D}$, *respectively.*

Proof. Making use of Theorem 4.1 in [22] we have for the numerators of both approximants the following:

$$P_m(z)=-\int_{-\pi}^{\pi}\left(\frac{x+z}{x-z}\right)\left[Q_m(x)\frac{z^k}{x^k}-Q_m(z)\right]\mathrm{d}\phi(\theta),\quad x=\exp(\mathrm{i}\theta),\quad 0\leqslant k\leqslant m-1, \tag{3.6}$$

$$\hat{P}_m(z)=-\int_{-\pi}^{\pi}\left(\frac{x+z}{x-z}\right)\left[\hat{Q}_m(z)-\hat{Q}_m(x)\frac{z^k}{x^k}\right]\mathrm{d}\phi(\theta),\quad x=\exp(\mathrm{i}\theta),\quad 1\leqslant k\leqslant m. \tag{3.7}$$

Now, since part (2) follows from (1) by considering the transform $z\to 1/\bar{z}$, it suffices to prove (1). Thus, by (3.6) we can immediately deduce an integral representation for $E_m(z)$, $z\in\mathbb{E}$. Indeed,

$$H(z)Q_m(z)-P_m(z)=\int_{-\pi}^{\pi}\left(\frac{x+z}{x-z}\right)\left[Q_m(x)\frac{z^k}{x^k}\right]\mathrm{d}\phi(\theta),\quad x=\exp(\mathrm{i}\theta),\quad 0\leqslant k\leqslant m-1.$$

Since $Q_m(z)\neq 0$, for any $z\in\mathbb{E}$, one can write

$$\begin{aligned}E_m(z)&=\frac{z^k}{Q_m(z)}\int_{-\pi}^{\pi}\left(\frac{x+z}{x-z}\right)Q_m(x)\frac{\mathrm{d}\phi(\theta)}{x^k}=\frac{z^k}{Q_m(z)}\int_{-\pi}^{\pi}\left(1+\frac{2z}{x-z}\right)Q_m(x)\frac{\mathrm{d}\phi(\theta)}{x^k}\\&=\frac{2z^{k+1}}{Q_m(z)}\int_{-\pi}^{\pi}\frac{Q_m(x)}{x^k(x-z)}\,\mathrm{d}\phi(\theta).\end{aligned} \tag{3.8}$$

(3.8) follows from the fact that

$$\int_{-\pi}^{\pi}x^{-k}Q_m(x)\,\mathrm{d}\phi(\theta)=\int_{-\pi}^{\pi}\rho_m(x)\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\phi(\theta)=\langle\rho_m,x^k\rangle=0,\quad 0\leqslant k\leqslant m-1,$$

because of the orthogonality property for $Q_m(z)=\rho_m(z)$.
On the other hand, $[Q_m(x)-Q_m(z)]/(x-z)=x^{m-1}+P(x)$, $P\in\Pi_{m-2}$. Therefore,

$$\int_{-\pi}^{\pi}Q_m(x)\left[\frac{Q_m(x)-Q_m(z)}{(x-z)x^{m-1}}\right]\mathrm{d}\phi(\theta)=\int_{-\pi}^{\pi}Q_m(x)\frac{x^{m-1}+P(x)}{x^{m-1}}\mathrm{d}\phi(\theta)=0 \tag{3.9}$$

by the orthogonality of $Q_m(z)$.
Hence, from (3.8) taking $k=m-1$ and (3.9) the following expression for the error results in,

$$E_m(z)=\frac{2z^m}{Q_m^2(z)}\int_{-\pi}^{\pi}\frac{Q_m^2(x)}{x^{m-1}(x-z)}\mathrm{d}\phi(\theta)=\mathbb{O}((z^{-1})^{m+1}). \tag{3.10}$$

Let us next consider the sequence $\{\psi_m(z)\}$ of orthonormal polynomials, i.e.

$$\langle\psi_m,\psi_m\rangle=1,\quad m=0,1,2,\ldots,$$

which is uniquely determined by taking appropriate positive numbers α_m such that $\psi_m(z) = \alpha_m Q_m(z) = \alpha_m \rho_m(z)$. Furthermore, one needs the following known result (see, e.g. [19]):

$$\lim_{m\to\infty} |\psi_m(z)|^{1/m} = |z| \quad \text{uniformly on } \mathbb{T} \cup \mathbb{E}. \tag{3.11}$$

Now, we are in a situation to reach the proof. Indeed, from (3.10), it is clear that we can replace $Q_m(z)$ by $\psi_m(z)$, yielding

$$|E_m(z)| \leqslant \frac{2|z|^m}{|\psi_m^2(z)|} \int_{-\pi}^{\pi} \frac{|\psi_m^2(x)|}{|x-z|} \mathrm{d}\phi(\theta), \quad z \in \mathbb{E}, \quad x = \mathrm{e}^{\mathrm{i}\theta}. \tag{3.12}$$

Let F be a compact set in $\mathbb{E}$. Then, there exists $R > 1$ (depending on F) such that for any $z \in F$:

$$|x - z| \geqslant R - 1 = M > 0.$$

So, by (3.12), it now follows, for any $z \in F$

$$|E_m(z)| \leqslant \frac{2|z|^m}{|\psi_m^2(z)|} \frac{1}{M} \int_{-\pi}^{\pi} |\psi_m^2(x)| \, \mathrm{d}\phi(\theta) = \frac{2}{M} \frac{|z|^m}{|\psi_m^2(z)|}. \tag{3.13}$$

By (3.11)–(3.13) the proof is readily concluded. □

Remark 3. Observe that now from (3.10) it is clearly demonstrated that the approximant has order of correspondence $(m+1)$ (in the strong sense) at infinity, which is not immediately clear from (3.8).

We are next interested in considering 2PTA to $H(z)$ of higher order (actually as high as possible) to approximate $H(z)$ on $\bar{\mathbb{C}} - \mathbb{T}$. This means that the poles of such approximants, if exist, should be on $\mathbb{T}$. Let us denote them by $R_m(z) = T_m(z)/S_m(z)$ with required order m both at the origin and at infinity.

Furthermore, we will assume that $\deg(S_m) = m$; and $S_m(0) \neq 0$, so that it holds

$$H(z) - R_m(z) = \mathbb{O}(z^m) \quad (z \to 0), \tag{3.14}$$

$$H(z) - R_m(z) = \mathbb{O}((z^{-1})^m) \quad (z \to \infty). \tag{3.15}$$

These approximants were first considered in [22] in connection with the trigonometric moment problem and uniform convergence proved in compacts of $\mathbb{C} - \mathbb{T}$. This will be completed here by showing that geometric convergence can be reached. For this, we need some polynomials related to Szegö polynomials called para-orthogonal (see [22]) or also quasi-orthogonal polynomials [8]. Thus, $X_n \in \Pi_n$ is said to be a para-orthogonal polynomial with respect to $\mathrm{d}\phi$ if X_n has exact degree n and satisfies

$$\langle X_n, 1\rangle \neq 0, \quad \langle X_n, z^j\rangle = 0, \quad 1 \leqslant j \leqslant n-1 \quad \text{and} \quad \langle X_n, z^n\rangle \neq 0.$$

Furthermore, it is known (see [22]) that any para-orthogonal polynomial X_n is of the form $X_n(z) = \rho_n(z) + w\rho_n^*(z)$; $|w| = 1$ and that has exactly n distinct zeros on $\mathbb{T}$.

With these considerations, the existence of the approximant $R_m(z)$ satisfying (3.14) and (3.15) is inferred from the following:

Theorem 5. (1) *$R_m(z)$ has exactly m distinct poles on $\mathbb{T}$.*

(2) *There cannot exist a 2PA to $H(z)$ of order $(m, m+1)$ or $(m+1, m)$ with poles on the unit circle.*

Proof. (1) Set $R_m(z) = T_m(z)/S_m(z)$. Then, by (2.16) $S_m(z)$ must satisfy

$$\mu^{-(m-1)}(x^j S_m(x)) = 0, \quad j = 0, 1, \ldots, m-2. \tag{3.16}$$

Since we have order of correspondence m both at the origin and at infinity, it follows, $\mu^{-(m-1)}(x^{-1}S_m(x)) \neq 0$ and $\mu^{-(m-1)}(x^{m-1}S_m(x)) \neq 0$.

Therefore, $\langle S_m, 1\rangle = \mu(S_m(x)) = \mu^{-(m-1)}(x^{m-1}S_m(x)) \neq 0$, and

$$\langle S_m, x^m\rangle = \mu(S_m(x)x^{-m}) = \mu^{-(m-1)}(x^{-1}S_m(x)) \neq 0.$$

For j such that $1 \leqslant j \leqslant m-1$, we have

$$\begin{aligned}\langle S_m(x), x^j\rangle &= \mu(S_m(x)x^{-j}) = \mu(S_m(x)x^{-j+m-m+1-1})\\ &= \mu^{-(m-1)}(x^{m-j-1}S_m(x)) = \mu^{-(m-1)}(x^r S_m(x)) = 0, \quad 0 \leqslant r \leqslant m-2.\end{aligned}$$

Thus, by (3.16) we see that S_m is a para-orthogonal polynomial and (1) follows.

(2) Let us first assume that there exists a 2PA of order $(m, m+1)$ with poles on the unit circle. As has already been seen, the denominator of such approximant coincides, up to a multiplicative factor with the mth Szegö polynomial. However, this is impossible since because, the zeros of $\rho_m(z)$ lie inside the unit disk. A similar contradiction arises if we suppose that there exists a 2PA of order $(m+1, m)$ with poles on $\mathbb{T}$. □

In order to estimate the rate of convergence for the sequence $\{R_m(z)\}$ from (3.4) one can see that the nth root asymptotic behavior for the denominator is now required. Since these are para-orthogonal polynomials, we make use of

Theorem 6 (González-Vera et al. [19]). *Let ϕ be a distribution function and $\{X_n(z)\}$ a sequence of para-orthogonal polynomials with respect to $\mathrm{d}\phi$. Set $\|X_n\| = \max\{|X_n(x)|: x \in \mathbb{T}\}$.*

Then, the following holds:

(a) $\lim_{n\to\infty} |X_n(z)|^{1/n} = |z|$, *uniformly on compact subsets of* $\mathbb{E} = \{z: |z| > 1\}$.

(b) $\lim_{n\to\infty} |X_n(z)|^{1/n} = 1$, *uniformly on compact subsets of* $\mathbb{D} = \{z: |z| < 1\}$.

(c) $\lim_{n\to\infty} \{\|X_n\|\}^{1/n} = 1$.

Now, we can prove the following,

Theorem 7. *Let ϕ be a distribution function on $[-\pi, \pi]$ and consider the function $H(z)$ defined by (3.3). Let $R_m(z) = T_m(z)/S_m(z)$ be a sequence of 2PTA to $H(z)$ of order (m, m), with $S_m(z)$ as defined in Theorem 5, $m = 1, 2, \ldots$. Then, $\{R_m(z)\}$ converges geometrically to $H(z)$ on any compact subset of $\mathbb{D} \cup \mathbb{E}$.*

Proof. Take k a nonnegative integer such that $0 \leqslant k \leqslant m$, then from (3.4) one can write

$$H(z) - R_m(z) = \frac{2z^k}{S_m(z)} \int_{-\pi}^{\pi} \frac{e^{-i(k-1)\theta}}{e^{i\theta} - z} S_m(e^{i\theta})\, d\phi(\theta) \tag{3.16}$$

with $S_m(z) = \rho_m(z) + w_m \rho_m^*(z)$, $|w_m| = 1$.

Because of the para-orthogonality property for the denominator $S_m(z)$ we can improve formula (3.16), yielding:

$$H(z) - R_m(z) = \frac{2z^m}{S_m^2(z)} \left[\int_{-\pi}^{\pi} \frac{e^{-i(m-1)\theta}}{e^{i\theta} - z} S_m^2(e^{i\theta})\, d\phi(\theta) - \lambda_m \right], \tag{3.17}$$

where

$$\lambda_m = \int_{-\pi}^{\pi} S_m(x)\, d\phi(\theta) = \int_{-\pi}^{\pi} [\rho_m(x) + w_m \rho^*(x)] d\phi(\theta) = w_m \int_{-\pi}^{\pi} \rho^*(x)\, d\phi(\theta)$$

$$= w_m \int_{-\pi}^{\pi} x^m \overline{\rho_m(1/\bar{x})}\, d\phi(\theta) \quad \text{with } x = e^{i\theta}.$$

Thus,

$$\lambda_m = w_m \langle x^m, \rho_m \rangle = w_m \langle \rho_m, \rho_m \rangle = w_m \int_{-\pi}^{\pi} |\rho_m(x)|^2\, d\phi(\theta)$$

(recall that ρ_m is a monic polynomial). Setting $\psi_m(z) = \alpha_m \rho_m(z)$ so that ψ_m is orthonormal ($\langle \psi_m, \psi_m \rangle = 1$), then one has

$$\alpha_m = \frac{1}{(\langle \rho_m, \rho_m \rangle)^{1/2}} = \frac{1}{(|\lambda_m|)^{1/2}}.$$

Now, it holds (see e.g. [23]) $\lim_{m\to\infty}(\alpha_{m+1}/\alpha_m) = 1$ and this implies that $\lim_{m\to\infty}(|\lambda_m|)^{1/m} = 1$.

By (3.17) it follows

$$|H(z) - R_m(z)| \leqslant \frac{2|z|^m}{|S_m(z)|^2} \|S_m\|^2 \left[\int_{-\pi}^{\pi} \frac{d\phi(\theta)}{|e^{i\theta} - z|} + \hat{\lambda}_m \right],$$

where $\hat{\lambda}_m = |\lambda_m| / \|S_m\|^2 > 0$. Observe that $\lim_{m\to\infty}(\hat{\lambda}_m)^{1/m} = 1$ (by (c) in Theorem 6). So, for a given arbitrary $\varepsilon > 0$, there exist $m_0 \in \mathbb{N}$ such that for any $m > m_0$,

$$\hat{\lambda}_m \leqslant (1+\varepsilon)^m.$$

Thus, one can write ($m > m_0$):

$$|H(z) - R_m(z)| \leqslant \frac{2|z|^m}{|S_m(z)|^2} \|S_m\|^2 [K + (1+\varepsilon)^m]$$

(K is a constant dependent of z).

Take $z \in \mathbb{D}$, since $S_m(z)$ is para-orthogonal with respect to $d\phi$, by (b) in Theorem 6, one has $\lim_{m\to\infty} |S_m(z)|^{1/m} = 1$; then, using (b) and (c) in Theorem 6 and having in mind that $\lim_{m\to\infty}[K + (1+\varepsilon)^m]^{1/m} = 1 + \varepsilon$, one finally gets

$$\lim_{m\to\infty} \sup |H(z) - R_m(z)|^{1/m} \leqslant \frac{|z| [\lim_{m\to\infty} \sup \|S_m\|^{1/m}]^2}{[\lim_{m\to\infty} \operatorname{Inf} |S_m(z)|^{1/m}]^2} (1+\varepsilon) = (1+\varepsilon)|z|.$$

This yields,

$$\lim_{m\to\infty} \sup |H(z) - R_m(z)|^{1/m} \leqslant |z|. \tag{3.18}$$

Similarly, if $z \in \mathbb{E}$, one can also deduce,

$$\lim_{m\to\infty} \sup |H(z) - R_m(z)|^{1/m} \leqslant \frac{1}{|z|}. \tag{3.19}$$

Finally, from (3.18) and (3.19) the proof follows. □

Remark 4. In a series of recent papers, Bultheel et al. have extended the theory of Szegö polynomials to the rational case, i.e. rational functions with poles given on the exterior of the unit disk; so that when all the poles are equal to infinity the polynomial case is immediately recovered. For further details in connection with Theorems 4 and 7, see [7].

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