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On the Yamabe Problem Concerning the Compact Locally Conformally Flat Manifolds

O. GIL-MEDRANO

Departamento de Geometría y Topología, Facultad de Matemáticas, Universidad de Valencia, Burjasot, Valencia, Spain

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For all known locally conformally flat compact Riemannian manifolds (M_n, g) $(n > 2)$, with infinite fundamental group, we give the complete proof of Aubin's conjecture on scalar curvature. That solves the Yamabe Problem for these manifolds. There exists a metric g' conformal to g, such that $\mathrm{vol}_{g'} = 1$ and whose scalar curvature R' is constant and satisfies $R' < n(n-1)\,\omega_n^{2/n}$, where ω_n is the volume of the sphere S_n with radius 1.

1. INTRODUCTION

Let (M, g) be a smooth, n-dimensional, compact Riemannian manifold with scalar curvature R. Let J be Yamabe's functional, defined by

$$J(\varphi) = \left[\frac{4(n-1)}{n-2}\int_M |\nabla\varphi|^2\,dv + \int_M R\varphi^2\,dv\right]\cdot\left[\int_M \varphi^{2n/(n-2)}\,dv\right]^{-1+2/n}$$

and set $\mu = \operatorname{Inf} J(\varphi)$, for all $\varphi \in H_1(M)$, $\varphi \geqslant 0$, $\varphi \not\equiv 0$.

THEOREM 1 [1, p. 289; 2, p. 129]. *For every Riemannian manifold $\mu \leqslant n(n-1)\,\omega_n^{2/n}$. If $\mu < n(n-1)\,\omega_n^{2/n}$ there exists a strictly positive solution $\varphi_0 \in C^\infty(M)$ of*

$$\frac{4(n-1)}{(n-2)}\Delta\varphi + R\varphi = R'\varphi^{(n+2)/(n-2)}$$

with $R' = \mu$ and $\|\varphi_0\|_N = 1$. Then, $g' = \varphi_0^{4/(n-2)}g$ is a metric of volume 1 whose scalar curvature R' is constant and $R' < n(n-1)\,\omega_n^{2/n}$. Here $N = 2n/(n-2)$ and ω_n is the volume of the unit n-dimensional sphere.

It is also established in [1, 2] that $\mu = n(n-1)\,\omega_n^{2/n}$ for the sphere, and some sufficient conditions are given for μ to be strictly smaller than

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$n(n-1)\,\omega_n^{2/n}$. According to this results, the Yamabe Problem remains open only in the following cases:

(a) Nonlocally conformally flat manifolds of dimension 3, 4, or 5.

(b) Locally conformally flat manifolds, with infinite fundamental group and admitting a metric of everywhere strictly positive scalar curvature.

The only known examples of the last class of manifolds are

(i) $\tilde{S}_{n-1} \times S_1$, where $\tilde{S}_{n-1}$ is a quotient of the sphere S_{n-1}.

(ii) $\tilde{S}_p(c) \times \tilde{H}_q(-c)$, $(p > q)$, where $\tilde{H}_q(-c)$ is a compact quotient of the hyperbolic space, whose sectional curvature is $-c$.

(iii) Fibre spaces over $\tilde{S}_p(c)$ (resp. $\tilde{H}_q(-c)$) with fibre $\tilde{H}_q(-c)$ (resp. $\tilde{S}_p(c)$).

(iv) The connected sums of manifolds above with a compact locally conformally flat manifold.

We are going to prove here that $\mu < n(n-1)\,\omega_n^{2/n}$ for these particular manifolds. The method was announced in [3]. For another approach covering case (a) and (b) see [6].

2. The Products

(1) We shall consider first the manifold $(S_n(\alpha) \times S_1(\beta), g)$, where $S_i(\rho)$ is the sphere of radius $1/\rho$ and dimension i, and g is the Riemannian product metric; α and β will be taken such that its volume, $\mathrm{Vol}_g(S_n(\alpha) \times S_1(\beta)) = 1$, that is $2\pi\beta^{-1}\alpha^{-n}\omega_n = 1$.

Theorem 2 [3]. *For every $\alpha, \beta \in \mathbb{R}^+$, the infimum of Yamabe's functional for the Riemannian manifold $(S_n(\alpha) \times S_1(\beta), g)$ is strictly lower than $n(n+1)\,\omega_{n+1}^{2/(n+1)}$. There is a metric g' conformal to g, such that its volume $\mathrm{Vol}_{g'}(S_n(\alpha) \times S_1(\beta)) = 1$ and whose scalar curvature R' is constant and satisfies $R' < n(n+1)\,\omega_{n+1}^{2/(n+1)}$.*

Proof. Fix y_0 an element of $S_1(\beta)$ and take $r(x, y) = d_{S_1}(y, y_0)$ for $(x, y) \in S_n(\alpha) \times S_1(\beta)$; we define $u(x, y) = u(r) = (\cosh \alpha r)^{(1-n)/2}$, then

$$J(u) = \left[\frac{4n}{n-1} \int_{S_n(\alpha) \times S_1(\beta)} (u')^2\, dv + R \int_{S_n(\alpha) \times S_1(\beta)} u^2\, dv\right] \times \left[\int_{S_n(\alpha) \times S_1(\beta)} u^{2(n+1)/(n-1)}\, dv\right]^{-(n-1)/(n+1)}$$

$$= (2\alpha^{-n}\omega_n)^{2/(n+1)} \left[\frac{4n}{n-1} \int_0^{\pi/\beta} (u')^2 \, dr + R \int_0^{\pi/\beta} u^2 \, dr \right] \times \left[\int_0^{\pi/\beta} u^{2(n+1)/(n-1)} \, dr \right]^{-(n-1)/(n+1)}.$$

It is easy to see that u satisfies the equation

$$-\frac{4n}{n-1} u'' + Ru = n(n+1)\,\alpha^2 u^{(n+3)/(n-1)}$$

and consequently

$$J(u) = (2\alpha^{-n}\omega_n)^{2/(n+1)}\, n(n+1)\,\alpha^2 \left[\int_0^{\pi/\beta} u^{2(n+1)/(n-1)} \, dr \right]^{2/(n+1)} + \frac{4n}{n-1} (2\alpha^{-n}\omega_n)^{2/(n+1)} \, |u'u|_0^{\pi/\beta} \left[\int_0^{\pi/\beta} u^{2(n+1)/(n-1)} \, dr \right]^{-(n-1)/(n+1)}.$$

But,

$$|u'u|_0^{\pi/\beta} = \frac{\alpha(1-n)}{2} \, |(\cosh \alpha r)^{-n} \sinh \alpha r|_0^{\pi/\beta} < 0$$

and

$$\begin{aligned} \int_0^{\pi/\beta} u^{2(n+1)/(n-1)} \, dr &= \int_0^{\pi/\beta} (\cosh \alpha r)^{-(n+1)} \, dr \\ &= 2^{n+1} \int_0^{\pi/\beta} (e^{\alpha r} + e^{-\alpha r})^{-(n+1)} \, dr \\ &= \frac{2^n}{\alpha} \int_1^{e^{2\alpha\pi/\beta}} (t+1)^{-(n+1)} \, t^{(n-1)/2} \, dt. \end{aligned}$$

Using now that

$$\int_0^1 (t+1)^{-(n+1)} \, t^{(n-1)/2} \, dt = \int_1^\infty (t+1)^{-(n+1)} \, t^{(n-1)/2} \, dt$$

we have

$$\begin{aligned} J(u) &< (2\alpha^{-n}\omega_n)^{2/(n+1)} \, n(n+1)\, \alpha^2 2^{2(n-1)/(n+1)} \alpha^{-2/(n+1)} \times \left[\int_0^\infty (t+1)^{-(n+1)} \, t^{(n-1)/2} \, dt \right]^{2/(n+1)} \\ &= n(n+1) \left[2^n \omega_n \int_0^\infty (t+1)^{-(n+1)} \, t^{(n-1)/2} \, dt \right]^{2/(n+1)} \\ &= n(n+1)\, \omega_{n+1}^{2/(n+1)}. \end{aligned}$$

That proves the first part of the theorem. The second part follows now directly from Theorem 1.

Let us consider now the Riemannian manifold (M_{n+1}, g), where $M_{n+1} = \tilde{S}_n(\alpha) \times S_1(\beta)$, $\tilde{S}_n(\alpha)$ being a quotient of the sphere of radius $1/\alpha$ and g the product metric. Using the fact that $\mathrm{vol}(\tilde{S}_n(\alpha)) \leqslant \alpha^{-n}\omega_n$ we can obtain, by a similar argument to that used above, the following:

THEOREM 3. *For (M_{n+1}, g) the infimum of Yamabe's functional is strictly lower than $n(n+1)\,\omega_{n+1}^{2/(n+1)}$ and there is a conformal metric g' on M_{n+1} whose scalar curvature is constant and satisfies $R' < n(n+1)\,\omega_{n+1}^{2/(n+1)}$, and such that* $\mathrm{Vol}_{g'}(M_{n+1}) = 1$.

(2) We are going to prove the same result for the manifolds (ii).

Let M_n be $\tilde{S}_p \times \tilde{H}_q$, $p+q=n$ and $p>q$. Here $\tilde{S}_p$ (resp. $\tilde{H}_q$) is a compact Riemannian manifold of constant sectional curvature c (resp. $-c$), $c>0$. Let g be the product metric on M_n, then

THEOREM 4. *For (M_n, g) the infimum of Yamabe's functional is strictly lower than $n(n-1)\,\omega_n^{2/n}$ and there is a conformal metric g' on M_n with* $\mathrm{Vol}_{g'}(M_n) = 1$, *whose scalar curvature is constant and satisfies $R' < n(n-1)\,\omega_n^{2/n}$.*

Proof. As above we are going to show that $\mu < n(n-1)\,\omega_n^{2/n}$ and then to apply Theorem 1. μ being a conformal invariant [2, p. 126], and in particular invariant by homotheties, we can then suppose that $c=1$. Furthermore, $\mathrm{Vol}(\tilde{S}_p) \leqslant \omega_p$ and we will see that it suffices to make the proof when $M_n = S_p \times \tilde{H}_q$.

Let y_0 be a fixed element of $\tilde{H}_q$ and for $(x, y) \in S_p \times \tilde{H}_q$ set $r(x, y) = d_{\tilde{H}_q}(y, y_0)$. We can define the function

$$u(x, y) = u(r) = (\cosh r)^{(2-n)/2}.$$

This function is uniformly lipschitzian, then $u \in H_1(M_n)$.

Now the scalar curvature of M_n is equal to $p(p-1) - q(q-1)$, so

$$J(u) = \left[\frac{4(n-1)}{(n-2)} \int_{M_n} |\nabla u|^2\, dv + [p(p-1) - q(q-1)] \int_{M_n} u^2\, dv\right] \times \left[\int_{M_n} u^{2n/(n-2)}\, dv\right]^{-(n-2)/n}. \tag{1}$$

If f is a function which depends only on r we have

$$\int_{M_n} f(r)\, dv = \omega_p \int_{S_{q-1}(1)} \left[\int_0^{\rho(\theta)} f(r)(\sinh r)^{q-1}\, dr\right] d\theta,$$

where $d\theta$ is the volume element of the unit sphere and, for $\theta \in S_{q-1}(1)$, $\rho(\theta)$ is such that the set $\exp_{y_0}(\rho(\theta)\,\theta)$ is the cut-locus of $y_0 \in \tilde{H}_q$.

Let us compute $J(u)$.

$$|\nabla u|^2 = \frac{(n-2)^2}{4}(\cosh r)^{-n}(\sinh r)^2$$

and then

$$\int_{M_n} |\nabla u|^2\, dv = \frac{(n-2)^2}{4}\omega_p \int_{S_{q-1}} \int_0^{\rho(\theta)} (\cosh r)^{-n}(\sinh r)^{q+1}\, dr\, d\theta$$
$$= \frac{(n-2)^2}{4}\omega_p \int_{S_{q-1}} \int_0^{\rho(\theta)} (\tanh r)^{q+1}(\cosh r)^{1-p}\, dr\, d\theta. \quad (2)$$

We have also

$$\int_{M_n} u^2\, dv = \omega_p \int_{S_{q-1}} \int_0^{\rho(\theta)} (\tanh r)^{q-1}(\cosh r)^{1-p}\, dr\, d\theta \quad (3)$$

and

$$\int_{M_n} u^{2n/(n-2)}\, dv = \omega_p \int_{S_{q-1}} \int_0^{\rho(\theta)} (\tanh r)^{q-1}(\cosh r)^{-1-p}\, dr\, d\theta \quad (4)$$

Now, on the one hand

$$\int (\tanh r)^{q-1}(\cosh r)^{1-p}\, dr = \int (\tanh r)^{q-1}(\cosh r)^{-1-p}\, dr$$
$$+ \int (\tanh r)^{q-1}(\sinh r)^2(\cosh r)^{-1-p}\, dr$$
$$= \int (\tanh r)^{q-1}(\cosh r)^{-1-p}\, dr + \int (\tanh r)^{q+1}(\cosh r)^{1-p}\, dr, \quad (5)$$

and on the other hand

$$\int (\tanh r)^{q+1}(\cosh r)^{1-p}\, dr = -\frac{(\tanh r)^q(\cosh r)^{1-p}}{(p-1)}$$
$$+\frac{q}{p-1}\int (\tanh r)^{q-1}(\cosh r)^{-1-p}\, dr. \quad (6)$$

Using (5) and (6) in (2), (3), (4) we see that equality (1) can be written

$$J(u)=\left[\omega_p\left\{-\frac{(n-1)(n-2)+p(p-1)-q(q-1)}{p-1}\right.\right.$$

$$\times\int_{S_{q-1}}(\tanh\rho(\theta))^q(\cosh\rho(\theta))^{1-p}\,d\theta$$

$$\left.\left.+n(n-1)\int_{S_{q-1}}\int_0^{\rho(\theta)}(\tanh r)^{q-1}(\cosh r)^{-1-p}\,dr\,d\theta\right\}\right]$$

$$\times\left[\omega_p\int_{S_{q-1}}\int_0^{\rho(\theta)}(\tanh r)^{q-1}(\cosh r)^{-1-p}\,dr\,d\theta\right]^{-(n-2)/2}$$

and hence we have

$$J(u)<n(n-1)\left[\omega_p\int_{S_{q-1}}\int_0^{\rho(\theta)}(\tanh r)^{q-1}(\cosh r)^{-1-p}\,dr\,d\theta\right]^{n/2}.$$

To finish the proof we use the fact that for each $\theta\in S_{q-1}$,

$$\int_0^{\rho(\theta)}(\tanh r)^{q-1}(\cosh r)^{-1-p}\,dr=\frac{1}{2}\int_{(\cosh\rho(\theta))^{-2}}^1(1-t)^{(q-2)/2}\,t^{(p-1)/2}\,dt$$
$$<\frac{1}{2}\frac{\Gamma(q/2)\,\Gamma((p+1)/2)}{\Gamma((n+1)/2)}$$

and then

$$J(u)<n(n-1)\left[\omega_p\omega_{q-1}\frac{1}{2}\frac{\Gamma(q/2)\,\Gamma((p+1)/2)}{\Gamma((n+1)/2)}\right]^{n/2}$$
$$=n(n-1)\,\omega_n^{n/2}.$$

The last equality holds from the well known expression of the volume of the unit sphere in terms of the Γ functions

$$\omega_n=2\frac{\Gamma(1/2)^{n+1}}{\Gamma((n+1)/2)}.$$

3. The Fibre Bundles

Next, let us consider the known examples of compact, locally conformally flat manifolds with infinite fundamental group and admitting a metric of positive scalar curvature, which are fibre spaces over $\tilde{S}_p$ (resp.

$\tilde{H}_q$) with fibre $\tilde{H}_q$ (resp. $\tilde{S}_p$) where $\tilde{S}_p$ (resp. $\tilde{H}_q$) is a compact Riemannian manifold of constant sectional curvature c (resp. $-c$).

Let M_n be such a manifold, with $p+q=n$, $p>q$ and let g be the fibre metric.

THEOREM 5. *The infimum of Yamabe's functional for* (M_n, g) *is strictly lower than* $n(n-1)\,\omega_n^{2/n}$ *and there is a conformal metric* g' *on* M_n, *of volume* 1, *whose scalar curvature is constant and satisfies* $R' < n(n-1)\,\omega_n^{2/n}$.

Proof. The second part follows directly from the first one and Theorem 1. As in Theorem 4, we can assume $c=1$.

First Case. (M_n, g) is a fibre bundle over $\tilde{H}_q$ with fibre $\tilde{S}_p$.

Let $\pi: M_n \to \tilde{H}_q$ be the projection, there exists a finite open cover $\{U_\alpha\}$ of $\tilde{H}_q$ such that $\pi^{-1}(U_\alpha)$ is isometric to $U_\alpha \times \tilde{S}_p$ with the product metric. If y_0 is a fixed element of $\tilde{H}_q$, we can define the function on M_n, $u(x)= u(r)=(\cosh r)^{(2-n)/2}$, where $r(x)=d_{\tilde{H}_q}(\pi(x), y_0)$. Then

$$J(u)=\left[\frac{4(n-1)}{(n-2)}\int_{M_n}|\nabla u|^2\,dv+[p(p-1)-q(q-1)]\int_{M_n}u^2\,dv\right]\times\left[\int_{M_n}u^{2n/(n-2)}\,dv\right]^{-(n-2)/n}.$$

Let $\{\tilde{\varphi}_\alpha\}$ be a partition of unity subordinate to the cover $\{\pi^{-1}(U_\alpha)\}$, where $\tilde{\varphi}_\alpha=\varphi_\alpha\circ\pi$ and $\{\varphi_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$. If f is a function on $\tilde{H}_q$ we have

$$\int_{M_n} f\circ\pi\,dv=\sum_\alpha\int_{\pi^{-1}(U_\alpha)}\tilde{\varphi}_\alpha(f\circ\pi)\,dv=\sum_\alpha \operatorname{vol}(\tilde{S}_p)\int_{U_\alpha}\varphi_\alpha f\,dv$$
$$=\operatorname{vol}(\tilde{S}_p)\int_{\tilde{H}_q} f\,dv_{\tilde{H}_q}.$$

Using now this formula to compute $J(u)$, we obtain that it is strictly lower than $n(n-1)\,\omega_n^{2/n}$, by an argument similar to that used in Theorem 4.

Second Case. (M_n, g) is a fibre bundle over $\tilde{S}_p$ with fibre $\tilde{H}_q$.

Let $\pi: M_n \to \tilde{S}_p$ be the projection and let G be the group of the fibre bundle, which is a group of transformations of $\tilde{H}_q$, suppose that G has k elements. There exists an open cover $\{V_\alpha\}$ of $\tilde{S}_p$ and a family of isometries $\{\psi_\alpha\}$ from $\pi^{-1}(V_\alpha)$ onto $V_\alpha\times\tilde{H}_q$. The application $\tilde{\pi}: M_n\to\tilde{H}_q/G$ given by $\tilde{\pi}(x)=\tilde{q}\circ p_2\circ\psi_\alpha(x)$ if $x\in\pi^{-1}(V_\alpha)$ is well defined. Here p_2 is the second projection of the product $V_\alpha\times\tilde{H}_q$ and $\tilde{q}$ is the quotient map $\tilde{q}: \tilde{H}_q\to\tilde{H}_q/G$. $\tilde{\pi}$ is a Riemannian submersion with totally geodesic fibres which is a sufficient condition for $\tilde{\pi}: M_n\to\tilde{H}_q/G$ to be a fibre bundle [5].

The quotient Riemannian manifold $\tilde{H}_q/G$ is also a compact manifold of constant curvature equal to -1 and we are almost in case one above, the only difference being now that the fibre is not $\tilde{S}_p$ but a disjoint union of k copies of that manifold. So, to finish the proof we must only verify that, under our conditions, $k\,\mathrm{vol}(\tilde{S}_p) \leqslant \omega_p$.

Let $S_p \to \tilde{S}_p$ be the universal covering of $\tilde{S}_p$ and assume that it is a k'-fold covering, then $k'\,\mathrm{vol}(\tilde{S}_p) = \omega_p$. But, G being finite, the principal bundle associated to $(M_n, \tilde{S}_p, \pi, \tilde{H}_q, G)$ is a k-fold covering of $\tilde{S}_p$ and then $k \leqslant k'$ and the result holds.

4. The Connected Sums

(1) We are going to prove that the infimum of the Yamabe functional decreases when taking connected sums of locally conformally flat Riemannian manifolds. To do that we need

Lemma 6. *Let (M, g) be a compact Riemannian manifold of dimension n $(n > 2)$. For each $p \in M$ and $\varepsilon > 0$, there exists a function $u \in C^\infty(M)$, $u \not\equiv 0$, $u \geqslant 0$ such that $J(u) < \mu + \varepsilon$ and such that u vanishes on a neighborhood of p.*

Proof. By the definition of μ, there is a smooth function $\bar{u}$, $\bar{u} \not\equiv 0$, $\bar{u} \geqslant 0$ such that $J(\bar{u}) < \mu + \varepsilon$.

Now, let $f_\delta\colon M \to \mathbb{R}$ be a family of smooth functions $0 \leqslant f_\delta \leqslant 1$, vanishing on a geodesic ball of radius δ and centre p which are equal to 1 outside the geodesic ball of radius 2δ, and which satisfy $|\nabla f_\delta| < 2/\delta$.

Define $u_\delta = f_\delta \bar{u}$, we have

$$\lim_{\delta \to 0} J(u_\delta) = J(\bar{u}).$$

So we can choose δ small enough so that $J(u_\delta) < \mu + \varepsilon$.

It is known that the connected sum of two locally conformally flat manifolds, can be endowed with a locally conformally flat metric. We outline the construction.

(2) If (M, g) is a n-dimensional, locally conformally flat Riemannian manifold, for each $p \in M$ there is a chart defined on an open set U, $p \in U$, $\varphi\colon (U, g) \to (\mathbb{R}^n, g_0)$, with $\varphi(p) = 0$, and there is a $\sigma \in C^\infty(U)$, $\sigma > 0$, such that $\sigma\varphi^* g_0 = g$, where g_0 is the Euclidean metric of $\mathbb{R}^n$.

The metric on $U \sim \{p\}$ can be conformally deformed onto a metric which agrees with g near the boundary of U, and which finishes with the product metric $\mathbb{R} \times S_{n-1}(b)$ (b, to be choosen) on a neighborhood of p.

To do so, we must only to take a smooth function $h\colon [b, \infty) \to \mathbb{R}$, $b > 0$, which is zero on $[a, \infty)$ for some $a > b$ and such that the curve $t = h(r)$

joins smoothly the half line $r=b$, $t\geqslant t_0$, at the point (b, t_0). Let G be the subset of $\mathbb{R}^2$ which is the union of the graph of h and the set $\{(b, t); t\geqslant t_0\}$.

Let $\tilde{U}$ be the hypersurface of $\mathbb{R}\times\mathbb{R}^n\sim\{0\}$ defined by

$$\tilde{U}=\{(t, r, \theta)\in\mathbb{R}\times\mathbb{R}^n\sim\{0\}; (r, t)\in G\}.$$

That manifold can be diffeomorphically mapped onto $\mathbb{R}^n\sim\{0\}$ by $\tilde{\varphi}\colon \tilde{U}\to\mathbb{R}^n\sim\{0\}$ given by

$$\tilde{\varphi}(t, r, \theta)=(re^{-t/r}, \theta).$$

Hence, $\varphi^{-1}\tilde{\varphi}$ is a diffeomorphism of $\tilde{U}$ onto $U\sim\{p\}$.

Let now $f(t)$ be a smooth function which is equal to 1 for $t\leqslant t_0$ and that vanishes for $t\geqslant t_1$. We define $\tilde{\sigma}\colon\tilde{U}\to\mathbb{R}$

$$\tilde{\sigma}(t, r, \theta)=1-f(t)+f(t)\,\sigma(\varphi^{-1}\circ\tilde{\varphi}(t, r, \theta)).$$

Let $\tilde{g}$ be the conformally flat metric on $\tilde{U}$ given by $\tilde{g}=\tilde{\sigma}e^{2t/r}\tilde{\varphi}^*g_0$.

If (t, r, θ) is an element of $\tilde{U}$ with $r\geqslant a$, t must be zero, consequently $f(t)=1$ and $\tilde{\varphi}(0, r, \theta)=(r, \theta)$. Hence

$$\tilde{g}_{(0,r,\theta)}=\tilde{\sigma}(0, r, \theta)\,\tilde{\varphi}^*g_0=\sigma(\varphi^{-1}(r, \theta))\,\tilde{\varphi}^*g_0.$$

On the other hand

$$\begin{aligned}(\varphi^{-1}\circ\tilde{\varphi})^*_{(0,r,\theta)}\,g&=\tilde{\varphi}^*\varphi^{-1*}g=\tilde{\varphi}^*(\sigma(\varphi^{-1}(r, \theta)))\,g_0\\ &=\sigma(\varphi^{-1}(r, \theta))\,\tilde{\varphi}^*g_0.\end{aligned}$$

Let us suppose now $t\geqslant t_1$, then $r=b$. In that case $f(t)=0$ and $\tilde{\varphi}(t, r, \theta)=(be^{-t/b}, \theta)$, and so

$$\tilde{g}_{(t,b,\theta)}=e^{2t/b}\tilde{\varphi}^*g_0,$$

but a straightforward computation shows that

$$\tilde{\varphi}^*g_0=e^{-2t/b}\,dt^2+b^2e^{-2t/b}\,d\theta^2$$

and then $\tilde{g}_{(t,b,\theta)}=dt^2+b^2\,d\theta^2$.

The argument above proves that g agrees with $(\varphi^{-1}\circ\tilde{\varphi})^*\,g$ for $r\geqslant a$ and with the metric of a cylinder for $t\geqslant t_1$.

Furthermore, if the scalar curvature of g is positive, the construction above can be made in a suitable way to obtain a new metric whose scalar curvature is also positive [4].

Let (M_i, g_i) $i=1, 2$, be two manifolds as above, then on the connected sum $M_1\#M_2$, a family of locally conformally flat Riemannian metrics can be constructed, depending on the choice of the application φ_i, the reals a_i, b_i, the function h_i $(i=1,2)$ and on the length of the piece of cylinder used to glue the two manifolds together, or equivalently on the radii δ_i $(i=1,2)$ of the removed balls.

Nevertheless, each metric agrees with g_i on $M_i \sim \varphi_i^{-1}(B_{a_i}(0))$ and is conformal to g_i on $M_i \sim \varphi_i^{-1}(B_{\delta_i}(0))$. So, the conformal class of the constructed metric depends only on the reals δ_1, δ_2.

Let us fix one of these metrics $(M_1 \# M_2, g)$. By considering the natural differentiable structure on the connected sum of two manifolds, we know that the sphere where M_1 joins M_2 has an open neighborhood $W = U_1 \sim \varphi_1^{-1}(B_{\delta_1}(0)) \# U_2 \sim \varphi_2^{-1}(B_{\delta_2}(0))$ diffeomorphical to the cylinder by the application $\psi\colon W \to \mathbb{R} \times S_{n-1}$ defined as follows

$$\psi(\varphi_1^{-1}(r, \theta)) = (\log(\delta_1/r), \theta),$$

$$\psi(\varphi_2^{-1}(r, \theta)) = (\log(r/\delta_2), \theta).$$

ψ is also a conformal transformation, when the cylinder is supposed endowed with the standard product metric, g_c.

Let $\tilde{\psi}$ be the application of $\mathbb{R} \times S_{n-1}$ onto $\mathbb{R}^n \sim \{0\}$ given by $\tilde{\psi}(t, \theta) = (\delta_1 e^{-t}, \theta)$, then $\tilde{\psi} \circ \psi = \varphi_1$ on U_1 and, as it is easy to see, $g_c = (e^t/\delta_1)^2 \tilde{\psi}^* g_0$. Hence

$$(\psi^* g_c)_{\varphi_1^{-1}(r,\theta)} = r^2 \psi^* \tilde{\psi}^* g_0 = r^2(\sigma(\varphi_1^{-1}(r, \theta)))^{-1} g_{\varphi_1^{-1}(r,\theta)}$$

and the restriction of $\psi^* g_c$ to $U_1 \sim \varphi_1^{-1}(B_{\delta_1}(0))$ is conformal to the metric considered in this set. By a similar argument we obtain the same result for the restriction of $\psi^* g_c$ to $U_2 \sim \varphi_2^{-1}(B_{\delta_2}(0))$.

For any $\varepsilon_1 > 0$ we can construct a new locally conformally flat metric $\bar{g}$ on $M_1 \# M_2$ by removing $\varphi_1^{-1}(B_{\varepsilon_1}(0))$ from M_1 and $\varphi_2^{-1}(B_{\varepsilon_2}(0))$ from M_2.

Proposition 7. *If $\varepsilon_2 \cdot \varepsilon_1 = \delta_2 \cdot \delta_1$ then $\bar{g}$ is conformal to g.*

Proof. Take now, $V = U_1 \sim \varphi_1^{-1}(B_{\varepsilon_1}(0)) \# U_2 \sim \varphi_2^{-1}(B_{\varepsilon_2}(0))$ and $\bar{\psi}\colon V \to \mathbb{R} \times S_{n-1}$ the conformal transformation, which in this case is given by

$$\bar{\psi}(\varphi_1^{-1}(r, \theta)) = (\log(\varepsilon_1/r), \theta),$$

$$\bar{\psi}(\varphi_2^{-1}(r, \theta)) = (\log(r/\varepsilon_2), \theta).$$

Let Φ be the following conformal transformation of the cylinder

$$\Phi\colon \quad \mathbb{R} \times S_{n-1} \to \mathbb{R} \times S_{n-1}$$

$$(t, \theta) \mapsto (t + \log(\delta_1/\varepsilon_1), \theta),$$

then $\psi^{-1} \circ \Phi \circ \bar{\psi}$ is a conformal transformation of V on W. But, for $x \in V$ and $x = \varphi_1^{-1}(r, \theta)$ with $r > \delta_1$

$$\begin{aligned}\psi^{-1} \circ \Phi \circ \bar{\psi}(x) &= \psi^{-1} \circ \Phi(\log(\varepsilon_1/r), \theta) = \psi^{-1}(\log(\varepsilon_1/r) + \log(\delta_1/\varepsilon_1), \theta) \\ &= \psi^{-1}(\log(\delta_1/r), \theta) = \varphi_1^{-1}(r, \theta) = x\end{aligned}$$

and similarly for $x \in V$ and $x = \varphi_2^{-1}(r, \theta)$ with $r > \delta_2$

$$\begin{aligned}\psi^{-1} \circ \Phi \circ \bar{\psi}(x) &= \psi^{-1} \circ \Phi(\log(r/\varepsilon_2), \theta) = \psi^{-1}(\log(r/\varepsilon_2) + \log(\delta_1/\varepsilon_1), \theta) \\ &= \psi^{-1}(\log(r\delta_1/\varepsilon_2\varepsilon_1), \theta) = \psi^{-1}(\log(r/\delta_2), \theta) \\ &= \varphi_2^{-1}(r, \theta) = x.\end{aligned}$$

Hence, this conformal transformation can be extended to $M_1 \# M_2$ by defining it to be the identity outside V.

(3) THEOREM 8. *Let (M_i, g_i) $(i = 1, 2)$ be a n-dimensional $(n > 2)$, compact locally conformally flat Riemannian manifold and let μ_i $(i = 1, 2)$ be the infimum of the Yamabe functional for the metric g_i. If $(M_1 \# M_2, \tilde{g})$ is the connected sum then $\mu_{\tilde{g}} \leqslant \min(\mu_1, \mu_2)$.*

Proof. Let $p \in M_1$ be the point involved in the construction of $(M_1 \# M_2, \tilde{g})$. From Lemma 6, for a given $\varepsilon > 0$, there is a $\delta > 0$ and a smooth function u_δ, that vanishes on $\varphi_1^{-1}(B_\delta(0))$ and such that $J_1(u_\delta) < \mu_1 + \varepsilon$.

According to Proposition 7 there is on $M_1 \# M_2$ a metric $\tilde{g}'$, conformal to $\tilde{g}$ and such that $\tilde{g}'$ is conformal to g_1 on $M_1 \sim \varphi_1^{-1}(B_\delta(0))$. Suppose $g_1 = \sigma^{4/(n-2)}\tilde{g}'$, we define $\tilde{u}_\delta = \sigma u_\delta$ on $M_1 \sim \varphi_1^{-1}(B_\delta(0))$ and extend by zero outside.

$$J_{\tilde{g}'}(\tilde{u}_\delta) = J_1(u_\delta) < \mu_1 + \varepsilon$$

and then $\mu_{\tilde{g}} = \mu_{\tilde{g}'} < \mu_1 + \varepsilon$, for all $\varepsilon > 0$.

In a similar way we can prove $\mu_{\tilde{g}} \leqslant \mu_2$.

COROLLARY 9. *Let $(\tilde{M}, \tilde{g})$ be the Riemannian manifold obtained by taking the connected sum of one of the manifolds* (i), (ii), *or* (iii) *with any compact, n-dimensional $(n > 2)$, locally conformally flat Riemannian manifold. Then $\mu_{\tilde{g}} < n(n-1)\,\omega_n^{2/n}$ and there is a conformal metric $\tilde{g}'$ on $\tilde{M}$ such that* $\mathrm{vol}_{\tilde{g}'} = 1$ *and whose scalar curvature is constant and lower than $n(n-1)\,\omega_n^{2/n}$.*

Proof. Follows directly from Theorem 8 and the results in Section 1.

References

1. T. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* **55** (1976), 269–296.
2. T. Aubin, "Nonlinear Analysis on Manifolds, Monge-Ampère Equations," Springer, New York, 1982.
3. O. Gil-Medrano, Sur le problème de Yamabe concernant les variétés compactes localement conformément plates, *C. R. Acad. Sci. Paris, Sér. I* **300** (1985), 75–77.
4. M. Gromov and H. B. Lawson, Jr., The classification of simply connected manifolds of positive scalar curvature, *Ann. of Math.* **111** (1980), 423–434.
5. R. Hermann, A sufficient condition that a map of Riemannian manifolds be a fibre bundle, *Proc. Amer. Math. Soc.* **11** (1960), 236–242.
6. R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature, preprint.