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On the equivariant cohomology of rotation groups and Stiefel manifolds

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ABSTRACT

In this paper, we compute the $RO(\mathbb{Z}/2)$-graded equivariant cohomology of rotation groups and Stiefel manifolds with particular involutions.

0. Introduction

Lewis, May, and McClure extend ordinary Bredon cohomology theories to $RO(G)$-graded theories in [7]. Equivariant cohomology and homotopy theories have had a wide array of applications, in part because of their close connections with Voevodsky's motivic cohomology theory. In spite of this, there have been very few published computations of the equivariant cohomology of specific spaces. Some examples of computations are in [5,6,12]. This paper provides computations of an important class of $\mathbb{Z}/2$-spaces. In particular, we compute the cohomology of the special orthogonal groups and Stiefel manifolds with particular $\mathbb{Z}/2$ actions. The main results are stated in Theorem 2.17 and Theorem 3.2.

Section 1 provides some background information on $RO(\mathbb{Z}/2)$-graded cohomology and establishes some definitions and notation which will be used throughout the paper.

In Section 2 we introduce an equivariant cell structure on $SO(p,q)$, the group of rotations of p-dimensional Euclidean space endowed with a particular action of $\mathbb{Z}/2$, and use this cell structure to determine the cohomology of $SO(p,q)$ as an algebra over the cohomology of a point with constant $\underline{\mathbb{Z}/2}$ Mackey functor coefficients.

In Section 3 we use the cell structure on $SO(p,q)$ from Section 2 to put an equivariant cell structure on $V_q(\mathbb{R}^{p,q})$, the Stiefel manifold of q-frames in the $\mathbb{Z}/2$-representation $\mathbb{R}^{p,q}$ with action inherited from $\mathbb{R}^{p,q}$. The cell structure on the Stiefel manifold is compatible with the one on $SO(p,q)$ and allows for the cohomology algebra structure of $V_q(\mathbb{R}^{p,q})$ to be deduced from that of $SO(p,q)$.

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1. Preliminaries

This section contains some of the background information and notations that will be used throughout the paper. In this section, G can be any finite group unless otherwise specified.

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Given a G-representation V, let $D(V)$ and $S(V)$ denote the unit disk and unit sphere, respectively, in V with action induced by that on V. A Rep(G)-**complex** is a G-space X with a filtration $X^{(n)}$ where $X^{(0)}$ is a disjoint union of G-orbits and $X^{(n)}$ is obtained from $X^{(n-1)}$ by attaching cells of the form $D(V_\alpha)$ along maps $f_\alpha : S(V_\alpha) \to X^{(n-1)}$ where V_α is an n-dimensional real representation of G. The space $X^{(n)}$ is referred to as the n-**skeleton** of X, and the filtration is referred to as a **cell structure**. In addition, S^V will denote the one-point compactification of V with fixed base point at infinity. If X is a Rep(G)-complex, then $X^{(n)}/X^{(n-1)} \cong \bigvee_i S^{V_i}$ where each V_i is an n-dimensional G-representation.

The reader is referred to May [9] for details on $RO(G)$-graded cohomology theories. Briefly, these are theories graded on the Grothendieck ring of virtual representations of the group G. The natural coefficients for such theories are Mackey functors. In fact, $RO(G)$-graded cohomology theories can naturally be thought of as ($RO(G)$-graded) Mackey functor-valued. However, this paper will focus on abelian group-valued cohomology theories.

In this paper, the group G will be $\mathbb{Z}/2$. For the precise definition of a $\mathbb{Z}/2$-Mackey functor, the reader is referred to Lewis et al. [7] or Dugger [3]. The data of a $\mathbb{Z}/2$-Mackey functor M are encoded in a diagram like the one below.

$$\overset{t^*}{\circlearrowleft} M(\mathbb{Z}/2) \underset{i^*}{\overset{i_*}{\rightleftarrows}} M(e)$$

A p-dimensional real $\mathbb{Z}/2$-representation V decomposes as $V = (\mathbb{R}^{1,0})^{p-q} \oplus (\mathbb{R}^{1,1})^q = \mathbb{R}^{p,q}$ where $\mathbb{R}^{1,0}$ is the trivial 1-dimensional real representation of $\mathbb{Z}/2$ and $\mathbb{R}^{1,1}$ is the nontrivial 1-dimensional real representation of $\mathbb{Z}/2$. Thus the $RO(\mathbb{Z}/2)$-graded theory is a bigraded theory, one grading measuring dimension and the other measuring the number of "twists". In this case, we write $H^V(X; M) = H^{p,q}(X; M)$ for the Vth graded component of the $RO(\mathbb{Z}/2)$-graded equivariant cohomology of X with coefficients in a Mackey functor M. Similarly, we will write $S^{p,q}$ for S^V when $V = \mathbb{R}^{p,q}$.

In this paper, the Mackey functor will always be constant $M = \underline{\mathbb{Z}/2}$ which has the following diagram.

$$\overset{id}{\circlearrowleft} \mathbb{Z}/2 \underset{id}{\overset{0}{\rightleftarrows}} \mathbb{Z}/2$$

Because this paper only considers this constant Mackey functor, the coefficients will be suppressed from the notation and we write $H^{p,q}(X)$ for $H^{p,q}(X; \underline{\mathbb{Z}/2})$. With these constant coefficients, the $RO(\mathbb{Z}/2)$-graded cohomology of a point has the following description:

$$H^{*,*}(pt) \cong \mathbb{Z}/2\left[\tau, \rho, \frac{\theta}{\tau^n \rho^m}\right] \Big/ \sim$$

The relations among the generators give $H^{*,*}(pt)$ the following structure. The **top cone** is a polynomial algebra on the nonzero elements $\rho \in H^{1,1}(pt)$ and $\tau \in H^{0,1}(pt)$. The nonzero element $\theta \in H^{0,-2}(pt)$ in the **bottom cone** is infinitely divisible by both ρ and τ, and $\theta^2 = 0$. It is important to note that neither τ nor ρ have multiplicative inverses in $H^{*,*}(pt)$, yet we write $\frac{\theta}{\tau^n \rho^m}$ for the unique nonzero element of $H^{-m,-2-n-m}(pt)$ since $\tau^n \rho^m \frac{\theta}{\tau^n \rho^m} = \theta$. The cohomology of $\mathbb{Z}/2$ is easier to describe: $H^{*,*}(\mathbb{Z}/2) = \mathbb{Z}/2[t, t^{-1}]$ where $t \in H^{0,1}(\mathbb{Z}/2)$. Details can be found in [3] and [2]. From here on out, we will denote $H^{*,*}(pt; \underline{\mathbb{Z}/2})$ by $H\underline{\mathbb{Z}/2}$.

Note that the suspension axioms completely determine the cohomology for the spheres $S^{p,q}$. If $(p,q) \neq (1,1)$, then $H^{*,*}(S^{p,q}) = H\underline{\mathbb{Z}/2}[x]/x^2$ where x is in bidegree (p,q). In the special case of $S^{1,1}$ we have the following proposition.

Proposition 1.1. *As a $H\underline{\mathbb{Z}/2}$-module, $H^{*,*}(S^{1,1})$ is free with a single generator a in degree $(1,1)$. As an algebra, $H^{*,*}(S^{1,1}) \cong H\underline{\mathbb{Z}/2}[a]/(a^2 = \rho a)$.*

Proof. See [5]. □

A useful tool is the following exact sequence of [1].

Lemma 1.2 *(Forgetful long exact sequence). Let X be a based $\mathbb{Z}/2$-space. Then for every q there is a long exact sequence*

$$\cdots \longrightarrow H^{p,q}(X) \xrightarrow{\cdot\rho} H^{p+1,q+1}(X) \xrightarrow{\psi} H^{p+1}(X) \xrightarrow{\delta} H^{p+1,q}(X) \to \cdots.$$

The map $\cdot\rho$ is multiplication by $\rho \in H^{1,1}(pt)$ and ψ is the forgetful map to non-equivariant cohomology with $\mathbb{Z}/2$ coefficients.

Given a filtered $\mathbb{Z}/2$ space X, for each fixed q there is a long exact sequence

$$\cdots H^{*,q}\big(X^{(n+1)}/X^{(n)}\big) \to H^{*,q}\big(X^{(n+1)}\big) \to H^{*,q}\big(X^{(n)}\big) \to H^{*+1,q}\big(X^{(n+1)}/X^{(n)}\big) \cdots$$

and so there is an Atiyah–Hirzebruch spectral sequence for each integer q.

Proposition 1.3. *Let X be a filtered $\mathbb{Z}/2$-space. Then for each $q \in \mathbb{Z}$ there is a spectral sequence with*

$$E_1^{p,n} = H^{p,q}\big(X^{(n+1)}, X^{(n)}\big)$$

converging to $H^{p,q}(X)$.

The construction of the spectral sequence is completely standard. (See, for example, Proposition 5.3 of [10].) When the space X is filtered in such a way that $X^{(n+1)}$ is obtained from $X^{(n)}$ by attaching cells, the collection of the above spectral sequences for all q will be called the **cellular spectral sequence**. The cellular spectral sequence will be used extensively in the computations below.

2. Rotation groups

In this section, we define a $\mathrm{Rep}(\mathbb{Z}/2)$-complex structure on the group of rotations $SO(n)$ with a particular action of $\mathbb{Z}/2$. The construction closely follows that of [4], which in turn is inspired by Miller [11] and Whitehead [13]. The key is to introduce the correct action of $\mathbb{Z}/2$ on the rotation groups so that the standard constructions are equivariant.

Let $O(p,q)$ denote the group of orthogonal transformations of $\mathbb{R}^p$ with the following $\mathbb{Z}/2$-action. Let g denote the non-identity element of $\mathbb{Z}/2$. Denote by $I_{p,q}$ the block matrix

$$I_{p,q} = \begin{pmatrix} I_{p-q} & 0 \\ 0 & -I_q \end{pmatrix},$$

where I_n is the $n \times n$-identity matrix. Each orthogonal transformation of $\mathbb{R}^p$ can be represented as an orthogonal matrix A with $\det A = \pm 1$. Define

$$g \cdot A = I_{p,q} A I_{p,q}.$$

That is, g acts on A by changing the sign on the last q entries of each row and each column. If we represent A as a block matrix

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix},$$

where A_1 is $(p-q) \times (p-q)$ and A_4 is $q \times q$, then

$$g \cdot A = \begin{pmatrix} A_1 & -A_2 \\ -A_3 & A_4 \end{pmatrix}.$$

Notice that $\det(g \cdot A) = \det(I_{p,q})\det(A)\det(I_{p,q}) = \det(A)$ and so g preserves determinant. Also, it is easy to check that if $A, B \in O(p,q)$, then $g \cdot (AB) = (g \cdot A)(g \cdot B)$. In particular, this action of g induces an action on the subgroup of rotations which will be denoted by $SO(p,q)$.

Let $\mathbb{P}(\mathbb{R}^{p,q})$ denote the space of lines in $\mathbb{R}^{p,q}$ with action inherited from the action on $\mathbb{R}^{p,q}$ which fixes the first $p-q$ coordinates and acts as multiplication by -1 on the last q coordinates. Define a map $\omega : \mathbb{P}(\mathbb{R}^{p,q}) \to SO(p,q)$ as follows:

$$\omega(v) = r(v)r(e_1)$$

where $r(v)$ denotes reflection across the hyperplane orthogonal to v and $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^{p,q}$. Notice that $\omega(v)$ is the product of two reflections, whence a rotation. In addition, the map ω is a $\mathbb{Z}/2$-equivariant map.

Now, choose a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_p = \mathbb{R}^{p,q}$ such that $g \cdot V_i = V_i$ and $\dim(V_i \cap V_{i+1}) = 1$ for all $i = 0, 1, \dots, p-1$. For example, declaring $V_i = \{(x_1, x_2, \dots, x_i, 0, 0, \dots, 0)\}$ is such a flag. This flag gives rise to a sequence of inclusions $\mathbb{P}(V_1) \subset \mathbb{P}(V_2) \subset \cdots \subset \mathbb{P}(V_p)$. In addition, we can define equivariant maps $\omega : \mathbb{P}(V_i) \to SO(p,q)$ by restricting the map ω above. For the sake of brevity, write P^i for $\mathbb{P}(V_i)$ and P^I for $P^{i_1} \times \cdots \times P^{i_m}$ where I is a sequence $(i_1, \dots, i_m)$ with each $i_j < p$. Then we have an equivariant map $\omega : P^I \to SO(p,q)$ given by $\omega(v_1, \dots, v_m) = \omega(v_1) \cdots \omega(v_m)$. (The action of $\mathbb{Z}/2$ on P^I is diagonal.) Sequences $I = (i_1, \dots, i_m)$ for which $p > i_1 > \cdots > i_m > 0$ and the sequence consisting of a single 0 will be called **admissible**.

If $\varphi^i : D^i \to P^i$ is the characteristic map for the i-cell of P^i, then the product $\varphi^I : D^I \to P^I$ of the appropriate φ^{i_j}'s is a characteristic map for the top-dimensional cell of P^I.

Proposition 2.1. *The maps $\omega\varphi^I : D^I \to SO(p,q)$, for I ranging over all admissible sequences, are the characteristic maps of a $\mathrm{Rep}(\mathbb{Z}/2)$-complex structure on $SO(p,q)$ for which the map $\omega : P^{n-1} \times \cdots \times P^1 \to SO(p,q)$ is cellular.*

Proof. The proof is a matter of adapting the proof of the non-equivariant statement in [4, Proposition 3D.1]. Consider $SO(p-1, q-1)$ the subset of $SO(p,q)$ which fixes e_p. Then Hatcher's maps $p : SO(p,q) \to S(\mathbb{R}^{p,p-q})$, $h : (P^{p-1} \times SO(p-1,q-1), P^{p-2} \times SO(p-1,q-1)) \to (SO(p,q), SO(p-1,q-1))$, and $h^{-1} : SO(p,q) - SO(p-1,q-1) \to (P^{p-1} - P^{p-2}) \times SO(p-1,q-1)$, given by $p(\alpha) = \alpha(e_p)$ and $h(v,\alpha) = \rho(v)\alpha$, are equivariant with the above defined action of $\mathbb{Z}/2$ on $SO(p,q)$.

For the induction, notice that $SO(p-q, q-q)$ has the trivial $\mathbb{Z}/2$-action and so has a Rep($\mathbb{Z}/2$)-complex structure using cells which arise from trivial representations. Thus the inductive process begins with $SO(p-q, p-q)$ and continues as in [4], and so $SO(p,q)$ has a Rep($\mathbb{Z}/2$)-complex structure.

The statement about the map ω being cellular is also immediate. □

The freeness theorem from Kronholm [5] and the previous proposition give the corollary below. (Recall that the coefficient Mackey functor is $\underline{\mathbb{Z}/2}$.)

Theorem 2.2 *(Freeness theorem). If X is a connected, locally finite, finite dimensional* Rep($\mathbb{Z}/2$)*-complex, then $H^{*,*}(X)$ is a free $H\underline{\mathbb{Z}/2}$-module.*

Corollary 2.3. *$H^{*,*}(SO(p,q))$ is a free $H\underline{\mathbb{Z}/2}$-module.*

Remark 2.4. Varying the flag $0 = V_0 \subset V_1 \subset \cdots \subset V_p = \mathbb{R}^{p,q}$ will alter the cell structure of the projective spaces involved, and hence the cell structure on $SO(p,q)$.

In light of this remark, it will be convenient to impose a standard cell structure on the real projective spaces. For further convenience, we will restrict attention to the case where $p = n$ and $q = \lfloor \frac{n}{2} \rfloor$ and, following the notation in [5], let $\mathbb{RP}^n_{tw} = \mathbb{P}(\mathbb{R}^{n+1, \lfloor \frac{n+1}{2} \rfloor})$ denote the equivariant space of lines in $\mathbb{R}^{n+1, \lfloor \frac{n+1}{2} \rfloor}$. For example, $\mathbb{RP}^3_{tw} = \mathbb{P}(\mathbb{R}^{4,2})$, $\mathbb{RP}^4_{tw} = \mathbb{P}(\mathbb{R}^{5,2})$, and $\mathbb{RP}^1_{tw} = S^{1,1}$. Considering a Schubert cell decomposition of $\mathbb{RP}^n_{tw}$ yields the following lemma.

Lemma 2.5. *$\mathbb{RP}^n_{tw}$ has a* Rep($\mathbb{Z}/2$)*-structure with cells in dimension* $(0,0)$, $(1,1)$, $(2,1)$, $(3,2)$, $(4,2)$, $\ldots$, $(n, \lceil \frac{n}{2} \rceil)$.

Proof. See [5]. □

With this cell structure, there is an additive basis for $H^{*,*}(\mathbb{RP}^n_{tw})$ where the bidegrees of the generators agree with the dimensions of the cells.

Proposition 2.6. *As a $H\underline{\mathbb{Z}/2}$-module, $H^{*,*}(\mathbb{RP}^n_{tw})$ is free with a single generator in each degree $(k, \lceil \frac{k}{2} \rceil)$ for $k = 0, 1, \ldots, n$.*

Proof. See [5]. □

In this particular case, Proposition 2.1 indicates that $SO(n, \lfloor \frac{n}{2} \rfloor)$ has a cell structure with one cell for each admissible sequence $I = (i_1, \ldots, i_m)$. These cells are the top cells of the spaces $P^I = \mathbb{RP}^{i_1}_{tw} \times \cdots \times \mathbb{RP}^{i_m}_{tw}$. Thus, $SO(n, \lfloor \frac{n}{2} \rfloor)$ has cells in bijection with the cells of $S^{1,1} \times S^{2,1} \times S^{3,2} \times S^{4,2} \times \cdots \times S^{n-1, \lceil \frac{n-1}{2} \rceil}$.

Example 2.7. Consider $SO(5,2)$. The admissible sequences provide a cell structure with cells in dimensions $(0,0)$, $(1,1)$, $(2,1)$, $(3,2)$, $(3,2)$, $(4,2)$, $(4,3)$, $(5,3)$, $(5,3)$, $(6,3)$, $(6,4)$, $(7,4)$, $(7,4)$, $(8,5)$, $(9,5)$, and $(10,6)$. By considering the forgetful long exact sequence, we see that $H^{*,*}(SO(5,2))$ is additively generated by generators with bidegrees matching the dimensions of these cells.

In general, we cannot rely on the forgetful long exact sequence to determine an additive basis for $H^{*,*}(SO(p,q))$. A general Rep($\mathbb{Z}/2$)-complex X may have "dimension shifting" differentials in the cellular spectral sequence. These dimension shifting differentials are nontrivial maps (with nontrivial kernels and cokernels) which cause $H^{*,*}(X)$ to have generators in bidegrees which do not match the dimensions of the cells. (See [5] for examples of this behavior in Grassmann manifolds.) However, the following theorem tells us that there is an additive basis for $H^{*,*}(SO(p,q))$ with generators in bijection with the cells in the above cell structure, and with bidegree agreeing with the dimensions of the cells.

Theorem 2.8. *The cellular spectral sequence for $SO(n, \lfloor \frac{n}{2} \rfloor)$, with the* Rep($\mathbb{Z}/2$)*-complex structure given above, has no nontrivial differentials, hence collapses at the E_1 page, and $H^{*,*}(SO(n, \lfloor \frac{n}{2} \rfloor)) \cong H^{*,*}(S^{1,1} \times S^{2,1} \times \cdots \times S^{n-1, \lceil \frac{n-1}{2} \rceil})$, as $H\underline{\mathbb{Z}/2}$-modules.*

Proof. First note that there are no nontrivial differentials in the cellular spectral sequence for the product of projective spaces since there are no nontrivial differentials in the spectral sequences for each individual projective space.

The map ω from the construction of the cell structure allows for a comparison of the cellular spectral sequence for $SO(n, \lfloor \frac{n}{2} \rfloor)$ with that of the product of projective spaces P^I. This comparison implies there are no nontrivial differentials in the cellular spectral sequence for $SO(n, \lfloor \frac{n}{2} \rfloor)$. Explicitly, we can consider the following commutative diagram.

$$\begin{array}{ccc} H^{*,*}(SO(n, \lfloor \frac{n}{2} \rfloor)^{(k)}) & \xrightarrow{\omega^*} & H^{*,*}((P^I)^{(k)}) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta=0} \\ H^{*+1,*}(\bigvee_i S^{V_i}) & \xrightarrow{\omega^*} & H^{*+1,*}(\bigvee_j S^{V_j}). \end{array}$$

Here, $X^{(k)}$ denotes the k-skeleton, δ is the connecting homomorphism in the long exact sequence of the pair $(X^{(k+1)}, X^{(k)})$, and ω^* is the map induced by ω. The V_i's and V_j's are k-dimensional $\mathbb{Z}/2$-representations. The construction of the cell structure ensures that the lower ω^* is an isomorphism and the specific choice of cell structure on P^I ensures that the right-hand δ is zero. Thus, the left-hand δ must also be zero. These δ's determine the differentials in the cellular spectral sequence, hence all differentials are zero in the cellular spectral sequence for $SO(n, \lfloor \frac{n}{2} \rfloor)$.

Since the stated product of spheres has a cell structure with cells of the same dimension as $SO(n, \lfloor \frac{n}{2} \rfloor)$, the result follows. □

Remark 2.9. For general p and q, projective spaces other than $\mathbb{RP}^n_{tw}$ are involved (e.g. $\mathbb{P}(\mathbb{R}^{8,2})$, etc.). However, these projective spaces can also be given cell structures for which the cellular spectral sequence has no nontrivial differentials. (See [5].) Thus, an analogous statement is true about the spaces $SO(p, q)$, although it is a little more cumbersome to describe the appropriate product of spheres in the general case.

Theorem 2.10. *For all p and q, the cellular spectral sequence for $SO(p, q)$, with the* $\mathrm{Rep}(\mathbb{Z}/2)$*-complex structure given above, has no nontrivial differentials, hence collapses at the E_1 page.*

The isomorphism in the above theorem determines the $H\underline{\mathbb{Z}/2}$-module structure of $H^{*,*}(SO(n, \lfloor \frac{n}{2} \rfloor))$. We wish to say something about the $H\underline{\mathbb{Z}/2}$-algebra structure. For this, we will need to compare $H^{*,*}(SO(n, \lfloor \frac{n}{2} \rfloor))$ with $H^{*,*}(P^I)$ where $I = (n-1, n-2, \ldots, 1)$ using the map ω above. We recall the following result from Kronholm [5] which describes the $H\underline{\mathbb{Z}/2}$-algebra structure of $\mathbb{RP}^\infty_{tw} = \mathbb{P}(\mathcal{U})$ where $\mathcal{U} \cong (\mathbb{R}^{2,1})^\infty$ is a complete $\mathbb{Z}/2$-universe in the sense of [9].

Theorem 2.11. $H^{*,*}(\mathbb{RP}^\infty_{tw}) = H\underline{\mathbb{Z}/2}[a, b]/(a^2 = \rho a + \tau b)$*, where* $\deg(a) = (1, 1)$ *and* $\deg(b) = (2, 1)$.

Proof. See [5]. □

The natural inclusions $\mathbb{RP}^n_{tw} \to \mathbb{RP}^\infty_{tw}$ determine the algebra structure for the cohomology of the finite projective spaces.

Theorem 2.12. *Let $n \geqslant 2$. Let* $\deg(a) = (1, 1)$ *and* $\deg(b) = (2, 1)$*. If n is odd, then $H^{*,*}(\mathbb{RP}^n_{tw}) = H\underline{\mathbb{Z}/2}[a, b]/\sim$ where the generating relations are $a^2 = \rho a + \tau b$ and $b^k = 0$ for $k \geqslant \frac{n+1}{2}$. If n is even, then $H^{*,*}(\mathbb{RP}^n_{tw}) = H\underline{\mathbb{Z}/2}[a, b]/\sim$ where the generating relations are $a^2 = \rho a + \tau b$, $b^k = 0$ for $k > \frac{n}{2}$, and $a \cdot b^{n/2} = 0$.*

If $n = 1$, then $H^{*,*}(\mathbb{RP}^1_{tw}) = H^{*,*}(S^{1,1}) \cong H\underline{\mathbb{Z}/2}[a]/(a^2 = \rho a)$.

Proof. See [5]. □

The space P^I above is the product of projective spaces $P^I = \mathbb{RP}^{n-1}_{tw} \times \cdots \times \mathbb{RP}^1_{tw}$. In the non-equivariant setting (with constant $\mathbb{Z}/2$ coefficients), $H^*(\mathbb{RP}^k)$ is free as a $H^*(pt)$-module ($H^*(pt) \cong \mathbb{Z}/2$), and so the Künneth theorem tells us that $H^*(\mathbb{RP}^{n-1} \times \cdots \times \mathbb{RP}^1) \cong H^*(\mathbb{RP}^{n-1}) \otimes \cdots \otimes H^*(\mathbb{RP}^1)$ as $H^*(pt)$-algebras. (Here the tensor products are taken over $H^*(pt)$.) We would like a similar statement about the $RO(\mathbb{Z}/2)$-graded cohomology of the product of projective spaces P^I. To do this, we need the following version of the Künneth spectral sequence from [8]. This Künneth theorem is phrased in terms of homological algebra of Mackey functors and views $RO(G)$-graded cohomology as Mackey functor-valued, rather than abelian group-valued.

Theorem 2.13 *(Künneth theorem). Let $\underline{R}_*$ be an $RO(G)$-graded Mackey functor ring which represents the $RO(G)$-graded cohomology theory $\underline{E}^*$. Let X and Y be G-spectra indexed on the same universe. There is a natural conditionally convergent cohomology spectral sequence of $\underline{R}^*$-modules*

$$E_2^{s,\tau} = \underline{\mathrm{Ext}}^{s,\tau}_{\underline{R}^*}\left(\underline{R}_{-*}X, \underline{R}^*Y\right) \Rightarrow \underline{R}^{s+\tau}(X \wedge Y).$$

In particular, if either $\underline{R}^*X$ or $\underline{R}^*Y$ is projective, then the spectral sequence collapses at the E_2-page and the E_2-page can be identified as $\underline{R}^*X \,\square_{\underline{R}^*}\, \underline{R}^*Y$. The edge homomorphism is then an isomorphism.

The box product $\square$ for Mackey functors satisfies $(M \,\square\, N)(G/G) \cong M(G/G) \otimes N(G/G)$. Since each of $H^{*,*}(\mathbb{RP}^k_{tw})$ is a free $H\underline{\mathbb{Z}/2}$-module, hence projective, the Künneth theorem yields that $H^{*,*}(P^I) \cong H^{*,*}(\mathbb{RP}^{n-1}_{tw}) \otimes \cdots \otimes H^{*,*}(\mathbb{RP}^1_{tw})$ as $H\underline{\mathbb{Z}/2}$-algebras. (Here the tensor products are taken over $H\underline{\mathbb{Z}/2}$.)

In principal, we can use the map $\omega : P^I \to SO(p,q)$ to determine the algebra structure of $H^{*,*}(SO(p,q))$ from the algebra structure of $H^{*,*}(P^I)$. We begin with an example.

Example 2.14. Consider $SO(4,2)$. We have the map $\omega : \mathbb{RP}^3_{tw} \times \mathbb{RP}^2_{tw} \times \mathbb{RP}^1_{tw} \to SO(4,2)$ which is cellular. Because ω determines the Rep($\mathbb{Z}/2$)-structure on $SO(4,2)$ and there are no dimension shifting differentials in the cellular spectral sequence for $SO(4,2)$, the map $\omega^* : H^{*,*}(SO(4,2)) \to H^{*,*}(\mathbb{RP}^3_{tw} \times \mathbb{RP}^2_{tw} \times \mathbb{RP}^1_{tw})$ is injective. In addition, ω can be thought of as giving an embedding $\mathbb{RP}^3_{tw} \to SO(4,2)$.

Write $H^{*,*}(\mathbb{RP}^3_{tw}) = H\underline{\mathbb{Z}/2}[a_3, b_3]/{\sim}$, $H^{*,*}(\mathbb{RP}^2_{tw}) = H\underline{\mathbb{Z}/2}[a_2, b_2]/{\sim}$, and $H^{*,*}(\mathbb{RP}^3_{tw}) = H\underline{\mathbb{Z}/2}[a_1]/{\sim}$ where each a_i has bidegree $(1,1)$, each b_i has bidegree $(2,1)$, and $\sim$ denotes the appropriate equivalence relation as stated in Theorem 2.12.

For $i = 1,2,3$, let β_i be the cohomology generator corresponding to the embedded i-dimensional cell of $\mathbb{RP}^3_{tw}$. Then $\omega^*(\beta_1) = a_1 + a_2 + a_3$, $\omega^*(\beta_2) = b_2 + b_3$, and $\omega^*(\beta_3) = a_3 b_3$. From Theorem 2.8 we know that $H^{*,*}(SO(4,2))$ is freely generated with generators in bidegrees $(0,0)$, $(1,1)$, $(2,1)$, $(3,2)$, $(3,2)$, $(4,3)$, $(5,3)$, and $(6,4)$. We wish to see that each of these generators can be expressed as products of the β_i's. Already we have that β_1 is a free generator in bidegree $(1,1)$, β_2 is a free generator in bidegree $(2,1)$, and β_3 is a free generator in bidegree $(3,2)$. Computing yields the following:

- $\beta_1^2 = \rho\beta_1 + \tau\beta_2$,
- $\beta_1^3 = \rho\beta_1^2 + \tau\beta_1\beta_2 \neq 0$, from which it can be inferred that $\beta_1\beta_2$ is a free generator in bidegree $(3,2)$,
- $\beta_2^2 = 0$,
- $\beta_1\beta_3$ is a free generator in bidegree $(4,3)$,
- $\beta_2\beta_3$ is the free generator in bidegree $(5,3)$,
- $\beta_3^2 = 0$,
- $\beta_1\beta_2\beta_3$ is the free generator in bidegree $(6,4)$.

These results are summarized in the proposition below.

Proposition 2.15. *As an $H\underline{\mathbb{Z}/2}$-algebra, $H^{*,*}(SO(4,2)) \cong H\underline{\mathbb{Z}/2}[\beta_1, \beta_2, \beta_3]/{\sim}$, where β_i is in bidegree $(i, \lceil i/2 \rceil)$ and $\sim$ is generated by $\beta_1^2 = \rho\beta_1 + \tau\beta_2$, $\beta_2^2 = 0$, and $\beta_3^2 = 0$.*

Remark 2.16. It is worthwhile to notice that $\psi(H^{*,*}(SO(4,2))) \cong \mathbb{Z}/2[\beta_1, \beta_3]/(\beta_1^3, \beta_3^2)$ and this is precisely $H^*(SO(4); \mathbb{Z}/2)$. (Notice that $\psi(\beta_2) = \psi(\beta_1^2)$.)

In general, we have the following result.

Theorem 2.17. *Let $p > 1$ and $q = \lfloor p/2 \rfloor$. Then as an $H\underline{\mathbb{Z}/2}$-algebra,*

$$H^{*,*}\big(SO(p,q)\big) \cong \big(H\underline{\mathbb{Z}/2}[\beta_1, \beta_2]/\langle \beta_1^2 = \rho\beta_1 + \tau\beta_2,\ \beta_2^{n_2}\rangle\big) \otimes \bigotimes_{i \geqslant 3,\, i\,\mathrm{odd}} H\underline{\mathbb{Z}/2}[\beta_i]/\langle \beta_i^{n_i}\rangle$$

where β_i has bidegree $(i, \lceil i/2 \rceil)$ and n_i is the smallest power of 2 such that $i \cdot n_i \geqslant p$ for $i \geqslant 2$.

Proof. Write $H^{*,*}(\mathbb{RP}^k_{tw}) \cong H\underline{\mathbb{Z}/2}[a_k, b_k]/{\sim}$ as in Theorem 2.12. For $i = 1, \ldots, p-1$, let β_i be the cohomology generator corresponding to the embedded i-dimensional cell of $\mathbb{RP}^{p-1}_{tw}$. Then $\omega^*(\beta_i) = \sum_{j=i}^{p-1} a_j b_j^{(i-1)/2}$ if i is odd and $\omega^*(\beta_i) = \sum_{j=i}^{p-1} b_j^{i/2}$ if i is even. In particular, $\omega^*(\beta_1) = \sum_{j=1}^{p-1} a_j$ and so $\omega^*(\beta_1^2) = \sum_{j=i}^{p-1} (a_j)^2 = \sum_{j=i}^{p-1} \rho a_j + \tau b_j = \omega^*(\rho\beta_1 + \tau\beta_2)$. Since ω^* is an injection, $\beta_1^2 = \rho\beta_1 + \tau\beta_2$. Similarly, if $i > 1$ then $\beta_i^2 = \beta_{2i}$ if $2i < p$ and $\beta_i^2 = 0$ if $2i \geqslant p$.

We shall see that the monomials $\beta_I \in H^{*,*}(SO(p,q))$ corresponding to admissible sequences are linearly independent. Suppose $\sum \lambda_I \beta_I = 0$ with each $\lambda_I \in H\underline{\mathbb{Z}/2}$. Rewrite this equation as $x\beta_1 + y = 0$ where neither x nor y has a factor of β_1. Then after applying ω^* to this equation, we can write $xa_1 + z = 0$ where z has no factor of a_1, since a_1 only appears in $\omega^*(\beta_1)$. From this we conclude that $x = 0$ and so β_1 does not appear in $\sum \lambda_I \beta_I$. Since b_2 only appears in $\omega^*(\beta_2)$, a similar argument shows that β_2 does not appear in the linear dependency $\sum \lambda_I \beta_I$. Continuing, $a_3 b_3$ appears only in $\omega^*(\beta_3)$, again implying that β_3 does not appear in the linear dependency. Continuing in this way, we see that $\lambda_I = 0$ for all I, and so the monomials corresponding to admissible sequences are linearly independent in $H^{*,*}(SO(p,q))$.

Let A be the $H\underline{\mathbb{Z}/2}$-algebra $H\underline{\mathbb{Z}/2}[\beta_1, \beta_2, \ldots]/{\sim}$ where the relations are $\beta_1^2 = \rho\beta_1 + \tau\beta_2$, $\beta_i^2 = \beta_{2i}$ if $2i < p$, and $\beta_i^2 = 0$ if $2i \geqslant p$. The preceding observations are enough to see that there is a surjective map $A \to H^{*,*}(SO(p,q))$ sending $\beta_i \mapsto \beta_i$.

The relations in A allow each element of A to be expressed as a linear combination of monomials $\beta_I = \beta_{i_1}\beta_{i_2}\cdots\beta_{i_n}$ for admissible sequences I. These monomials are linearly independent in $H^{*,*}(SO(p,q))$, hence also in A, and so the map $A \to H^{*,*}(SO(p,q))$ is an isomorphism.

The relations $\beta_i^2 = \beta_{2i}$ and $\beta_1^2 = \rho\beta_1 + \tau\beta_2$ allow the admissible monomials β_I of A to be uniquely expressed in terms of β_1, β_2, and β_i for $i \geqslant 3$ and odd. The relation $\beta_j = 0$ for $j \geqslant p$ can be written as $\beta_{in_i} = \beta_i^{n_i} = 0$ where $j = in_i$ with i odd and n_i a power of 2. This relation holds if and only if $i \cdot n_i \geqslant p$. Hence, A can be expressed as the tensor product in the statement of the theorem. □

Remark 2.16 generalizes as well.

Corollary 2.18. *Let $p > 1$ and $q = \lfloor p/2 \rfloor$. Then the forgetful map $\psi : H^{*,*}(SO(p,q) \to H^*(SO(p))$ is surjective and $\psi(\beta_i) = \beta_i$.*

Proof. We have the following commutative diagram:

$$\begin{array}{ccc} H^{*,*}(SO(p,q)) & \xrightarrow{\omega^*} & H^{*,*}(P^I) \\ \big\downarrow \psi & & \big\downarrow \psi \\ H^*(SO(p)) & \xrightarrow{\omega^*} & H^*(P^I) \end{array}$$

From Kronholm [5] we have that $\psi(a_i)$ is the generator $\alpha \in H^1(\mathbb{RP}^i)$ and $\psi(b_i) = \alpha^2$. In addition, $\psi(\tau) = 1$ and $\psi(\rho) = 0$ in $H^0(pt)$. Commutativity of the above diagram assures that the definition of the β_i's in equivariant cohomology is consistent with the corresponding β_i's in non-equivariant cohomology, and so $\psi(\beta_i) = \beta_i$ for all i. □

3. Stiefel manifolds

Let $V_k(\mathbb{R}^{p,q})$ denote the Stiefel manifold of k-frames in $\mathbb{R}^{p,q}$ with action inherited from the one on $\mathbb{R}^{p,q}$. There is an equivariant projection $\pi : O(p,q) \to V_q(\mathbb{R}^{p,q})$ sending $A \in O(p,q)$ to the q-frame consisting of the last q columns of A.

For simplicity, we will restrict to the case where $p > 1$ and $q = \lfloor \frac{p}{2} \rfloor$. The projection $\pi : SO(p,q) \to V_q(\mathbb{R}^{p,q})$ is surjective and we can view $V_q(\mathbb{R}^{p,q})$ as the coset space $SO(p)/SO(p-q)$ with action inherited from $SO(p,q)$. From this viewpoint, we can give $V_q(\mathbb{R}^{p,q})$ a $\mathrm{Rep}(\mathbb{Z}/2)$-complex structure. The cells are the sets of cosets corresponding to admissible sequences $I = (i_1, \ldots, i_m)$ where $p > i_1 > \cdots > i_m \geqslant p - q$. Since the Stiefel manifold has a $\mathrm{Rep}(\mathbb{Z}/2)$-complex structure, its cohomology is free as a $H\underline{\mathbb{Z}/2}$-module. The following theorem shows there are no dimension shifting differentials.

Theorem 3.1. *Let $p > 1$ and $q = \lfloor \frac{p}{2} \rfloor$. Then the cellular spectral sequence for $V_q(\mathbb{R}^{p,q})$ with the above $\mathrm{Rep}(\mathbb{Z}/2)$-complex structure, collapses at the E_1 page. In particular, $H^{*,*}(V_q(\mathbb{R}^{p,q})) \cong H^{*,*}(S^{p-q,\lceil \frac{p-q}{2} \rceil} \times \cdots \times S^{p-1,\lceil \frac{p-1}{2} \rceil})$, as $H\underline{\mathbb{Z}/2}$-modules.*

Proof. The additive cohomology generators of $H^{*,*}(S^{p-q,\lceil \frac{p-q}{2} \rceil} \times \cdots \times S^{p-1,\lceil \frac{p-1}{2} \rceil})$ are in bijection with the admissible sequences $I = (i_1, \ldots, i_m)$ where $p > i_1 > \cdots > i_m \geqslant p - q$. Comparison with $SO(p,q)$ shows that there are no nontrivial differentials in the cellular spectral sequence for $V_q(\mathbb{R}^{p,q})$. Thus, $H^{*,*}(V_q(\mathbb{R}^{p,q}))$ is free with generators in bijection with the cells and with bidegrees agreeing with the dimensions of these cells. □

Following Miller [11], we will denote by $[i_1, \ldots, i_n]$ the cohomology generator corresponding to the admissible sequence $I = (i_1, \ldots, i_n)$. The previous theorem implies that these classes form an additive basis for $H^{*,*}(V_q(\mathbb{R}^{p,q}))$. We further make the conventions that $[i_1, \ldots, i_n] = [i_{\lambda(1)}, \ldots, i_{\lambda(n)}]$ for any permutation λ of the indices, and that $[i_1, \ldots, i_n] = 0$ if some $i_k < p - q$, if some $i_k \geqslant p$, or if some $i_k = i_j$ for $k \neq j$. We also denote by $[0]$ the generator corresponding to the admissible sequence $I = (0)$.

Theorem 3.2. *Let $p > 2$ and $q = \lfloor p/2 \rfloor$. As an $H\underline{\mathbb{Z}/2}$-algebra, $H^{*,*}(V_q(\mathbb{R}^{p,q}))$ is multiplicatively generated by $[0]$ and all $[i]$ with $p > i \geqslant p - q$ subject only to the relations*

- *$[0]$ is the unit, and*
- $[i] \cup [j] = \begin{cases} [i,j] & \text{if } i + j < p, \\ 0 & \text{if } i + j \geqslant p. \end{cases}$

Proof. The map $\pi : SO(p,q) \to V_q(\mathbb{R}^{p,q})$ is by definition cellular and we can compare the cellular spectral sequences for each space to see that π induces an injection $\pi^* : H^{*,*}(V_q(\mathbb{R}^{p,q})) \to H^{*,*}(SO(p,q))$. For each admissible sequence $I = (i_1, \ldots, i_n)$, $\pi^*([i_1, \ldots, i_n]) = \beta_{i_1} \cdots \beta_{i_n}$. (Since we are assuming $p > 2$, none of the i_k's are 1's and we need not concern ourselves with the relation $\beta_1^2 = \rho\beta_1 + \tau\beta_2$.) In particular, $\pi^*([i] \cup [j]) = \beta_i\beta_j = \pi^*([i, j])$, and so $[i] \cup [j] = [i, j]$. This class is zero if and only if $i + j \geqslant p$. The assumptions on p and q force $2i \geqslant p$ since $i \geqslant p - \lfloor p/2 \rfloor$, so $[i] \cup [i] = 0$ for all i. □

Notice that Miller [11] provides more detail about the product structure in the non-equivariant setting. Namely, $[i] \cup [j_1, \ldots, j_n] = [i, j_1, \ldots, j_n] + \sum_k [j_1, \ldots, j_k + i, \ldots, j_n]$ in $H^*(V_q(\mathbb{R}^p))$. However, in the setting of the theorem, i and all of the j_k's are between p and $p - \lfloor p/2 \rfloor$. Thus $i + j_k \geqslant p$ and so the summation term above is zero. In addition we necessarily have that $[i] \cup [i] = [2i] = 0$ in the non-equivariant setting, as mentioned in the proof above.

If $p = 2$, then the Stiefel manifold $V_1(\mathbb{R}^{2,1})$ can be identified with $S(\mathbb{R}^{2,1}) \cong S^{1,1}$ and the cohomology of this space has already been determined.

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