A new and very powerful set theoretic axiom, the axiom of determinacy (AD), was introduced by Mycielski [22] in 1964 and has since been the subject of a great deal of research. One area in which AD has proved to be very fruitful is that of the theory of thin sets of reals. AD implies a regularity property: any set of reals that is thin (that is, has no perfect subset) is countable [7, 22]. The axiom also has many consequences in descriptive set theory; in particular it yields a rich structure theory for thin sets in the various analytical pointclasses [12]. And finally, there is a connection between thin sets of reals and models of set theory [13, 16, 25]. This use of AD, or of some strong axiom, is necessary; it is known that many of the natural and interesting questions about thin sets and descriptive set theory are undecidable in ZFC.

Let $\lambda$ be an infinite ordinal. A $\lambda$-set is a subset of $\lambda$; thus an $\omega$-set is a 'real'. The purpose of this paper is to generalize several known theorems about sets of reals to sets of $\lambda$-sets, for projective ordinals $\lambda$. We will always be assuming AD plus the axiom of dependent choice (DC).

We show that any set of $\lambda$-sets either has a perfect subset or else is well-orderable. For some $\lambda$'s, including $\lambda = \delta^1_\omega$, if the set is well-orderable it has cardinality at most $\text{card}(\lambda)$. Much of the descriptive set theory of thin sets of reals also goes through for $\lambda$-sets. For example we will prove that there is a largest thin $\Pi^1_{2n+1}(\lambda)$ set and a largest thin $\Sigma^1_{2n+2}(\lambda)$ set of $\lambda$-sets. For some $\lambda$'s, all of the structure theory developed by Kechris [12] for the largest thin $\Sigma^1_{2n+2}$ set of reals, generalizes to the largest thin $\Sigma^1_{2n+2}(\lambda)$ set of $\lambda$-sets.

We also study the inner models of set theory, $H_{2n+1}$, introduced by Moschovakis [18, 8G]. These models are analogs of $L$ for the pointclasses $\Pi^1_{2n+1}$ and $\Sigma^1_{2n+2}$: $H_1 = L$. These models are closely connected to the theory of thin sets of $\lambda$-sets, since as Moschovakis [21] showed, for $\lambda < \delta^1_{2n+1}$, the largest thin $\Sigma^1_{2n+2}(\lambda)$
set of $\lambda$-sets is exactly the power set of $\lambda$ in $H_{2n+1}$. For certain $\lambda$, we prove that $H_{2n+1} \vDash 2^\lambda = \lambda^+$, and in fact there is a ‘good’ well-ordering (resembling the $L$ well-ordering) that witnesses this. This fact is then used to study the internal structure of the largest thin $\Sigma^1_{2n+2}(\lambda)$ set.

AD is, of course, false, since it contradicts the axiom of choice. However, the weaker axiom, that every definable game on $\omega$ is determined, is generally thought to be consistent with choice. In fact, many mathematicians believe that it is probably true, and that therefore, consequences of this axiom are probably true. Since every set in $L[\mathbb{R}]$ is ordinal definable from a real, this weaker axiom implies that every game in $L[\mathbb{R}]$ is determined (in $V$), and since a strategy is a real, it is determined in $L[\mathbb{R}]$; that is, $L[\mathbb{R}] \vDash AD$. The axiom of choice implies that $L[\mathbb{R}] \vDash DC$. So by working inside $L[\mathbb{R}]$, we can assume AD+DC, and prove theorems under this assumption. Most of the theorems in this paper are actually absolute for $L[\mathbb{R}]$. Thus, by the above analysis they are true in $V$, assuming that definable games are determined. Even those consequences of AD+DC that are not absolute, since true in $L[\mathbb{R}]$, imply corresponding theorems in $V$ by changing ‘every set’ to ‘every set in $L[\mathbb{R}]$. For more information on AD as an axiom, and on some of the points touched on here, the reader should consult [18, particularly sections 7D and 8I]. In this paper we will no longer explicitly mention $L[\mathbb{R}]$. We just assume AD for the rest of the paper.

0. Preliminaries

We work in ZF+DC+AD. In fact a few of the results in the paper need only projective determinacy (PD); therefore, we have put explicitly in the statements of the theorems all assumptions needed to prove them, beyond ZF+DC.

Throughout this paper, $\lambda$ denotes an infinite ordinal. Capital Latin letters $A, B, C, \ldots$ are used for sets of ordinals, while script letters $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$ are used for collections of sets or ordinals. We use $\alpha, \beta, \gamma, \ldots$ as variables over the reals. $\mathcal{N}$, that is the set $\omega_\omega$. Other notation and terminology follows Moschovakis, Descriptive Set Theory [18]. The reader should be familiar with the basic facts from descriptive set theory, which can be found in [18].

Since it is awkward to keep referring to ‘sets of sets’ of ordinals, we call a subset of $\lambda$ a $\lambda$-set.

We will state our results for the pointclass $\Pi^1_{2n+1}$ and the ordinal $\delta^1_{2n+1}$ associated with that class. Actually the results can be generalized to an abstract pointclass $\Gamma$ sufficiently like $\Pi^1_{2n+1}$.

Code ordinals less than $\delta^1_{2n+1}$ via some fixed $\Pi^1_{2n+1}$-norm $\varphi$ on a complete $\Pi^1_{2n+1}$ set $\mathcal{S}$; $\varphi$ has length $\delta^1_{2n+1}$. The elements of $\mathcal{S}$ are ordinal codes. A real $\alpha$ in $\mathcal{S}$ encodes the ordinal $\varphi(\alpha)$; $|\alpha|$ denotes this ordinal. We code countable sets of ordinals in the obvious way; the real $\beta$ encodes $\{(|(\beta)_0|, |(\beta)_1|, \ldots \}$. The collection of all countable sets of ordinals less than $\lambda$ is denoted by $\mathcal{P}_\mathcal{S}(\lambda)$. 
Sets of reals that are $\Delta^1_{2n+1}$ can be encoded by reals in such a way that ‘$\alpha$ encodes a $\Delta^1_{2n+1}$ set $D_\alpha$’ is $\Pi^1_{2n+1}$, and ‘$\beta \in D_\alpha$’ is $\Delta^1_{2n+1}$, assuming $\alpha$ is a code. This follows from the prewell-ordering theory developed in [18, Chapter 4]. The coding is defined as follows: Let $G(\gamma, \beta)$ be universal $\Pi^1_{2n+1}$, let $\psi$ be a $\Pi^1_{2n+1}$-norm on $G$ and let $\alpha$ be a code if $\alpha = \langle \gamma_0, \beta_0, \gamma_1 \rangle$ where $G(\gamma_0, \beta_0)$; $\alpha$, then encodes the $\Delta^1_{2n+1}$ set \{ $\beta : \psi(\gamma_1, \beta) < \psi(\gamma_0, \beta_0)$ \}. If $\lambda < \delta^1_{2n+1}$, then by the coding lemma [19, 18, Theorem 7D.6], every $\lambda$-set $B$ is $\Delta^1_{2n+1}$-i-$\gamma$-the-codes. This means that $B^* = \{ \alpha : |\alpha| \in B \}$ is $\Delta^1_{2n+1}$. If $\mathcal{C}$ is a set of $\lambda$-sets, with $\lambda < \delta^1_{2n+1}$, then let 
\[ \mathcal{C}^* = \{ \beta : (\exists B \in \mathcal{C}) (\beta \text{ is a } \Delta^1_{2n+1}\text{-code for } B^*) \}. \]

Thus $\lambda$-sets are coded via $\varphi$ and sets of $\lambda$-sets are coded via $G$ and $\psi$, as above. A $\lambda$-set $B$ is in a pointclass $\Gamma$ if $B^*$ is in $\Gamma$. A set of $\lambda$-sets $\mathcal{C}$ is in a pointclass $\Gamma$ if $\mathcal{C}^*$ is in $\Gamma$. Note that countable sets are coded differently than arbitrary sets. This will cause no confusion, as it will be clear from the context (in Section 4) when we are referring to countable sets.

Let $\xi_0, \ldots, \xi_k$ be ordinals less than $\delta^1_{2n+1}$, and let $\Gamma$ be a pointclass containing $\Delta^1_{2n+1}$ (e.g., $\Sigma^1_{2n+2}$). A set of reals $\mathcal{A}$ is $\Gamma(\xi_0, \ldots, \xi_k)$ if there is a $\Gamma$ relation $\mathcal{R}(\alpha, \beta_0, \ldots, \beta_k)$ such that for all $\beta_0, \ldots, \beta_k$, if $|\beta_0| = \xi_0, \ldots, |\beta_k| = \xi_k$, then for all $\alpha \in \mathcal{N}$,
\[ \mathcal{R}(\alpha, \beta_0, \ldots, \beta_k) \leftrightarrow \alpha \in \mathcal{A}. \]

That is, $\Gamma(\xi_0, \ldots, \xi_k)$ always means uniformly in all codes for $\xi_0, \ldots, \xi_k$.

In pointclass computations we will frequently use the following theorem.

**Theorem 0.1** (PD; Harrington–Kechris [8], [18, Theorem 8G.20]). Let $\varphi : \mathcal{G} \rightarrow \lambda$ be a $\Delta^1_{2n+1}(\lambda)$-norm onto an ordinal $\lambda < \delta^1_{2n+1}$. Let $P(\xi, \beta) \subseteq \lambda \times \mathcal{N}$ be $\Sigma^1_{2n+2}(\lambda)$, i.e. if $P^*(\alpha, \beta) \iff |\alpha| = \xi \& P(\xi, \beta)$, then $P^*$ is $\Sigma^1_{2n+2}(\lambda)$. Let $Q(\beta) \equiv (\forall \xi < \lambda) P(\xi, \beta)$. Then $Q$ is $\Sigma^1_{2n+2}(\lambda)$.

Theorem 0.1 says, in effect, that $\Sigma^1_{2n+2}(\lambda)$ is closed under quantification of the form $(\forall \xi < \lambda)$. Note that for any $\Pi^1_{2n+1}$-norm onto $\delta^1_{2n+1}$, the initial segments of the norm satisfy the hypothesis of Theorem 0.1. One consequence of Theorem 0.1 is that ordinals coded via two different $\Pi^1_{2n+1}$-norms can be compared in a $\Delta^1_{2n+2}$ way; see [8] or [18, Theorem 8G.21] for details.

We use two types of ordinal games.

For $\lambda < \delta^1_{2n+1}$ and $A \subseteq \omega^\lambda$, let 
\[ A^* = \{ (\alpha_0, \alpha_1, \ldots) \in \omega^\mathcal{N} : (|\alpha_0|, |\alpha_1|, \ldots) \in A \}. \]

The game $G_A^*$ is defined as follows. Players I and II alternately play reals, and thus pick a sequence in $\omega^\mathcal{N}$. If either player plays a real that does not code an ordinal less than $\lambda$, the first one to do so loses. Otherwise, I wins if and only if the sequence is in $A^*$. Note that the payoff set for $G_A^*$ depends only on the ordinals played, not on the codes.
Theorem 0.2 (AD; Harrington–Kechris [8]). For any projective ordinal $\lambda$ and for any $A \subset \omega\lambda$, $G_A$ is determined.

A strategy $\sigma$ for this type of game can be identified with a function $F_\sigma$ from $\mathcal{N}$ to $\mathcal{N}$; a strategy $\sigma$ is in a pointclass if the graph of $F_\sigma$ is in the pointclass. The games are not only determined, but have definable winning strategies.

Theorem 0.3 (AD; Harrington–Kechris [8]). For all $\lambda < \delta_{2n+1}^1$, for any $A \subset \omega\lambda$, either $I$ or $II$ has a $\Delta_{2n+1}^1$ winning strategy for $G_A$.

The above theorems are also valid for games on tuples of ordinals and integers.

Definition 0.4 Let $\Gamma$ be a pointclass with ordinal $\delta$ (e.g. $\Gamma = \Pi_n^1$ or $\Sigma_n^1$ and $\delta = \delta_n^1$), and let $\lambda < \delta$. A subset of $\omega(\lambda^k \times \omega')$ is pseudo-$\Gamma$ if it is in the smallest class containing
(a) open and closed relations that are $\Gamma(\xi_0, \ldots, \xi_k)$-in-the-codes, for some $\xi_0, \ldots, \xi_k < \delta$, and
(b) $\Gamma$ relations with arguments in $\omega$ and $\mathcal{N}$ (no ordinals!)
and closed under $\exists^\omega$ and $\forall^\omega$.

Theorem 0.5 (PD; Martin, Moschovakis [21]). Let $\lambda < \delta_{2n+1}^1$, let $\mathcal{B} \subset \omega(\lambda^k \times \omega')$, and let $G_{\mathcal{B}}$ be the game on $(\lambda^k \times \omega')$ with payoff set $\mathcal{B}$ (for $I$). If $\mathcal{B}$ is pseudo-$\Pi_{m}^1$ for some $m < \omega$, then $G_{\mathcal{B}}$ is determined.

Theorem 0.5 was originally proved by Martin for a smaller class of ordinal games; the general theorem is due to Moschovakis. Here, too, we can compute the complexity of winning strategies.

Theorem 0.6 (PD; Moschovakis [21]). Let $\lambda < \delta_{2n+1}^1$, let $\mathcal{B} \subset \omega(\lambda^k \times \omega')$, and let $G_{\mathcal{B}}$ be the game on $(\lambda^k \times \omega')$ with payoff set $\mathcal{B}$.

(a) If $\mathcal{B}$ is pseudo-$\Pi_{2n+1}^1$ and $I$ has a winning strategy, then $I$ has a winning strategy $\sigma$ that is $\Delta_{2n+2}^1(\nu_0, \ldots, \nu_m)$ for some ordinals $\nu_0, \ldots, \nu_m < \delta_{2n+1}^1$.

(b) If $\mathcal{B}$ is pseudo-$\Sigma_{2n+1}^1$ and $II$ has a winning strategy, then $II$ has a winning strategy $\tau$ that is $\Delta_{2n+2}^1(\nu_0, \ldots, \nu_m)$ for some ordinals $\nu_0, \ldots, \nu_m < \delta_{2n+1}^1$.

1. The perfect set theorem for collections of $\lambda$-sets

In this section we generalize the definition of 'thin' from collections of reals to collections of $\lambda$-sets. We then prove a generalized perfect set theorem, that is we prove that thin sets are well-orderable. And finally we get some upper bounds on the cardinality of thin sets.
**Definition 1.1.** A non-empty set $C$ of $\lambda$-sets is **perfect** if there is a countable set $W$ of ordinals less than $\lambda$ such that

1. for all $A, B \in C$, if $A \cap W = B \cap W$, then $A = B$,
2. If

$$C' = \{ f \in \omega_2 : (\exists A \in C)(f \text{ is the characteristic function of } A \cap W) \},$$

then $C'$ is a perfect set in the space $\omega_2$, with the product topology, taking $2$ discrete (that is, $C'$ is closed and has no isolated points).

A set of $\lambda$-sets is **thin** if it has no perfect subset.

**Proposition 1.2** A perfect set of $\lambda$-sets is equinumerous with the continuum.

**Proposition 1.3** (AD). If $C$ is a set of $\lambda$-sets and $C$ can be well-ordered, then $C$ is thin.

We will prove the converse of Proposition 1.3 below (Theorem 1.8): every thin set is well-orderable.

**Proposition 1.4** (AD). A well-ordered union of thin sets is thin.

**Proof.** Suppose not and let $\mathcal{C} = \bigcup_{\xi < \eta} \mathcal{C}_\xi$, with each $\mathcal{C}_\xi$ thin. Let $\mathcal{C} \subseteq \mathcal{C}$ be perfect and let $W$ be a countable subset of $\lambda$ such that $\mathcal{C}$ and $W$ satisfy Definition 1.1. Let $\mathcal{C}' = \{ A \cap W : A \in C \}$. Let $\mathcal{C}'_\xi = \{ A \cap W : A \in C_\xi \cap \mathcal{C}_\xi \}$. Then $\mathcal{C}' = \bigcup_{\xi < \eta} \mathcal{C}'_\xi$. A well-ordered union of thin sets in $\omega_2$ (in the ordinary sense) is thin, by [12]. But each $\mathcal{C}'_\xi$ is thin and $\mathcal{C}'$ is not, a contradiction.

**Definition 1.5.** An ordinal $\alpha$ is **reliable** if there is a projective scale $\{ \varphi_i \}_{i \in \omega}$ on a projective set $\mathcal{P}$ such that each norm $\varphi_i$ maps onto an initial segment of the ordinals and such that

$$\sup \{ \varphi_i(\alpha) : i \in \omega, \alpha \in \mathcal{P} \} = \lambda.$$

An ordinal $\alpha$ is **semireliable** if there is a reliable $\lambda'$ of the same cardinality as $\lambda$.

'Reliable' essentially means that every ordinal less than $\lambda$ can be coded so that all witnesses (with respect to a scale) are less than $\lambda$. Consider, for example, a scale such that the first norm $\varphi_0$ has length $\lambda$. We can then code ordinals less than $\lambda$ via $\varphi_0$; that is, $\alpha$ encodes the ordinal $\varphi_0(\alpha)$. And for every ordinal code (element of $\mathcal{P}$) $\alpha$, there are ordinals $\eta_1, \eta_2, \ldots$ all less than $\lambda$, such that for all $i \geq 1$, $\varphi_i(\alpha) = \eta_i$. It may happen that $\operatorname{cof}(\lambda) = \omega$ and the scale has length $\lambda$, but each norm $\varphi_i$ has length less than $\lambda$; an example of this is the Martin–Solovay scale on $\kappa_\omega$ (see [18, Section 8H]). In this case, ordinals less than $\lambda$ must be coded by a pair consisting of an integer and a real.

A $\Pi_{2n+1}$-scale on a complete $\Pi_{2n+1}^1$ set has length $\delta_{2n+1}$. Hence $\delta_{2n+1}$ is
Lemma 1.6 (PD; Moschovakis [21]). Let $\lambda < \delta^1_{2n+1}$ be reliable and let $\mathcal{A}$ be a $\Pi^1_{2n+1}$ set of $\lambda$-sets. If $\mathcal{A}$ is thin, then $\text{card}(\mathcal{A}) \leq \text{card}(\lambda)$.

A proof of Lemma 1.6 is given in [21]. We will repeat that proof in this paper. Our reason for including the proof here is that the basic idea of Moschovakis' proof will be used later in this paper, in a more complicated and confusing situation; the reader will have to understand the proof of Lemma 1.6 in order to follow the proofs of Theorems 2.5 and 4.9.

Proof of Lemma 1.6. Let $\bar{\phi} = \{\phi_i\}_{i<\omega}$ be the scale satisfying Definition 1.5 for the reliable ordinal $\lambda$. To simplify the proof, let us assume that $\bar{\phi}$ is a scale such that the length of $\phi_0$ is $\lambda$, and we will code ordinals via $\phi_0$. The modifications in the proof needed to obtain the general result will be obvious. Let $\mathcal{T}$ be the tree on $\omega \times \lambda$ associated with the scale. Let $G_{\mathcal{A}}$ be the game on ordinals less than $\lambda$ described below.

\begin{align*}
\text{I} & \quad d_0, \eta_0 \quad d_1, \eta_1 \quad d_2, \eta_2 \quad \cdots \\
& \quad (\alpha_0) \quad (\alpha_1) \quad (\alpha_2) \\
\text{II} & \quad t_0 \quad t_1 \quad t_2 \quad \cdots
\end{align*}

On the $i$th move, I plays $d_i \in \omega$ and II plays $t_i \in 2$. Also on the $i$th move, I plays an ordinal $\eta_i < \lambda$, and a real $\alpha_i$ (played one integer at a time) that is an alleged code for $\eta_i$ (as described below in more detail). Player I also plays witnesses (with respect to $\mathcal{T}$) for his codes, and II plays alleged codes to keep I's codes honest.

For example, on his first move, I will play an ordinal $\eta_0$. On subsequent moves of the game II will play $c_0, c_1, \ldots$ and $\xi_1, \xi_2, \ldots, \xi_i \in \omega$. If $\xi_i < \lambda$. Then $(c_0, \eta_0, c_1, \xi_1, c_2, \xi_2, \ldots)$ will be a branch through $\mathcal{T}$; if not I loses. Thus since we are coding ordinals via the first norm $\phi_0$ of the scale, if $\alpha_0 = (c_0, c_1, \ldots)$, then $\alpha_0$ is an ordinal code and $|\alpha_0| \leq \eta_0$; this follows directly from the definition of scale, in [18, 4E]. Thus $\alpha_0$ is an alleged code for $\eta_0$. It may be fake, that is, it may be that $|\alpha_0| < \eta_0$. To keep this code honest, on subsequent moves of the game II will play $c_0', c_1', \ldots$ and $\xi_1', \xi_2', \ldots, c_i' \in \omega$ and $\xi_i' < \lambda$. Then $(c_0', \eta_0, c_1', \xi_1', c_2, \xi_2', \ldots)$ will be a branch through $\mathcal{T}$; if not II loses. If $\gamma = (c_0', c_1', \ldots)$, then $\gamma$ also is a code and $|\gamma| \leq \eta_0$. If $|\gamma| > |\alpha_0|$, II has thus proved that I has not played an honest code, and hence will win the game.

The above description also holds for all the other $\eta_i$. All of this has to be
weaved together in some reasonable way. We only showed the \( \eta_i \)'s and \( \alpha_i \)'s on the above diagram, since if both players play honest codes (which is the interesting case), then that is all that really matters. The rest of the moves are witnesses that become irrelevant in this case. Note that all ordinals played are less than \( \lambda \) (including witnesses). Note also that at any position of the game only finitely many ordinals and integers have been played.

The payoff set is defined as follows:

(i) Both players must play into the tree \( T \); the first player to fail to do loses.

(ii) If neither player loses because of (i), and one of \( I \)'s codes is proved dishonest by \( II \), then I loses.

(iii) Suppose neither player loses because of conditions (i) or (ii). Let \( \delta = (d_0, d_1, \ldots) \). I wins if and only if

\[
\delta \text{ is a } \Delta^1_{2n+1}\text{-code for a } \lambda \text{-set } D \in \mathcal{B} \text{ for all } t_i (i = 1 \leftrightarrow |\alpha_i| \in D).
\]

Claims. (1) If I has a winning strategy for \( G_{\mathcal{B}} \), then \( \mathcal{B} \) has a perfect subset.

(2) If II has a winning strategy for \( G_{\mathcal{B}} \), then \( \text{card}(\mathcal{B}) \leq \text{card}(\lambda) \).

Playing into the tree (condition (i)) is a closed condition, and comparing ordinal codes \( \alpha \) and \( \gamma \) to see whether \( |\gamma| > |\alpha| \) (condition (ii)) is a projective condition on reals (for example, if the scale is a \( \Pi^1_{2n+1} \)-scale, (ii) is a \( \Delta^1_{2n+1} \)-condition on reals. So by Theorem 0.5 the game \( G_{\mathcal{B}}\) is determined. So the lemma follows from the claims. The proof of the claims is just a modification of the usual argument in the perfect set theorem, which we outline below.

Proof of Claim 1. Let \( \sigma \) be a winning strategy for I for \( G_{\mathcal{B}} \). Let W be a countable subset of \( \lambda \) such that the following two conditions are satisfied:

(a) W is closed under \( \sigma \), i.e. if \( p \) is any position in the game \( G_{\mathcal{B}} \) in which it is I's turn to move, and all ordinals occurring in \( p \), including witnesses, are in \( W \), then \( \sigma \) has I play \( s \), a finite sequence, such that all ordinals in \( s \) are in \( W \).

(b) For all \( \nu \in W \), there is an honest code for \( \nu \) that can be witnessed with ordinals in \( W \); i.e. for all \( \nu \in W \), there is a real \( \alpha \) and ordinals \( \nu_1, \nu_2, \ldots \) in \( W \) such that \( \varphi_0(\alpha) = \nu \), and for all \( i \geq 1 \), \( \varphi_i(\alpha) = \nu_i \).

Such a \( W \) exists since \( \lambda \) is reliable; for using dependent choice, for each \( \nu \) one can choose a code, and thus build up \( W \). Fix once and for all, for each \( \nu \in W \), a code and witnesses satisfying (b).

For every \( X \in \omega^2 \), let \( \delta^X, (\alpha^X_0, \alpha^X_1, \ldots) \) be the play of I, assuming I plays via \( \sigma \) and II plays \( (t_0, t_1, \ldots) = X \), and when I plays \( \eta_0 \), II plays (for his challenge to I's code for \( \eta_0 \)) the fixed code and witnesses for \( \eta_0 \) chosen above. Let \( D^X \) be the \( \lambda \)-set coded by \( \delta^X \); it exists since I wins. If \( X \neq Y \) and \( k \) is the least integer such that \( X(k) \neq Y(k) \), then \( \eta_k = |\alpha^X_k| = |\alpha^Y_k| \), but \( |\alpha^X_k| \in D^X \leftrightarrow |\alpha^Y_k| \notin D^Y \). So if \( X \neq Y \), then \( D^X \cap W \neq D^Y \cap W \). Hence \( \mathcal{B}' = \{D^X \cap W : X \in \omega^2 \} \) is uncountable. By the ordinary perfect set theorem for \( \omega^2 \), \( \mathcal{B}' \) has a perfect subset \( \mathcal{C}' \) (in the sense of sets in \( \omega^2 \)). Let \( \mathcal{C} = \{D^X : D^X \cap W \in \mathcal{C}' \} \). Then \( \mathcal{C} \) is a perfect set (in the sense of \( \lambda \)-sets). By definition of the payoff, \( \mathcal{C} \subset \mathcal{B} \).
Proof of Claim 2. Let \( \tau \) be a winning strategy for II. Let \( p \) be a position in the game in which it is I’s turn to play an \( \eta_i \). (That is, I has already played \( d_i \) and all witnesses to be played on the \( i \)th move; when \( \eta_i \) is played, I’s \( i \)th move is done and it is II’s turn to play.) Let \( D \) be a \( \lambda \)-set. Define \( \tau \) rejects \( D \) at \( p \) if

1. \( p \) is consistent with \( \tau \).
2. for all \( j < i \), \( (\eta_j \in D \iff t_j = 1) \).
3. for all \( \nu < \lambda \), if I plays \( \eta_i = \nu \) at \( p \), then \( \tau \) calls for II to play \( t_i \) such that \( (\eta_i \in D \iff t_i = 0) \).

Since each position \( p \) is a finite sequence of integers and ordinals less than \( \lambda \) and \( \lambda \) is infinite, there are only \( \text{card}(\lambda) \) many such positions. If \( \tau \) rejects \( D \) at \( p \), then \( D \) is totally determined by \( p \) and \( \tau \). So to complete the proof, it is enough to show that for all \( D \in \mathbb{A} \), there is a \( p \) such that \( \tau \) rejects \( D \) at \( p \).

Given \( D \in \mathbb{A} \), suppose that there is no \( p \) such that \( \tau \) rejects \( D \) at \( p \). Let \( \delta \) be a \( \Delta^1_{2n-1} \)-code for \( D \). Consider the run of \( G_\delta \) in which II plays according to \( \tau \). I plays \( \delta = (d_0, d_1, \ldots) \) and plays ordinals, codes, and witnesses as follows. On the \( i \)th move, I plays for \( \eta_i \) the least \( \nu \) such that \( \nu \in D \iff \tau \) has II respond to \( \nu \) by playing \( t_i = 1 \).

Such a \( \nu \) exists; otherwise \( D \) is rejected by \( \tau \) here. Then I chooses an honest code for \( \eta_i \) and witnesses (using dependent choice), and plays them on succeeding moves. Player I will win the game playing against \( \tau \). But \( \tau \) is a winning strategy.

We have stated Lemma 1.6 for \( \mathbb{A}, \mathbb{II}^{1}_{2n-1} \), since that is all we will need to prove Theorem 1.8, below. However, the proof we gave clearly goes through for any \( \mathbb{A} \) which is projective (and in fact hyperprojective); it is in this more general form that Moschovakis [21] gives the theorem. A version of Lemma 1.6 was independently proved by Harrington and Sami [9], at least for some \( \lambda \)'s. Their proof is entirely different, and it is not clear whether the \( \lambda \)'s for which their proof works are the same as those for which the above proof works, i.e., reliable \( \lambda \). Before Lemma 1.6 was proved, a similar theorem using the axiom of choice had been proved, in one version by Moschovakis [21] and others, in another version by Burgess [5]. That is, they essentially proved in the theory

\[ \text{ZFC} + L[\mathbb{R}] \models AD \]

that if \( \mathbb{A} \) is a set of \( \lambda \)-sets and \( \mathbb{A} \) is projective, then either \( \mathbb{A} \) has a perfect subset or else \( \text{card}(\mathbb{A}) \leq \text{card}(\lambda) \). The trouble with this theorem is that this cardinality is in \( V \), not in \( L[\mathbb{R}] \); that \( \text{card}(\mathbb{A}) \leq \text{card}(\lambda) \) in \( V \) does not even imply that \( \mathbb{A} \) is well-orderable in \( L[\mathbb{R}] \).

In Lemma 1.6 and its variations, with or without choice, the hypothesis of the theorem is that \( \mathbb{A} \) is definable. To generalize the perfect set theorem from reals to \( \lambda \)-sets, we need a way of lifting the result from definable collections of \( \lambda \)-sets to arbitrary collections. Lemma 1.7, below, is one way of doing it. Another approach was discovered by Sami [23], independently. We wish to thank A. Kechris for informing us of Sami’s work.
Lemma 1.7 (AD). Let $\lambda < \delta^1_{2n+1}$ and let $\mathcal{A}$ be a thin set of $\lambda$-sets. There is a set $\mathcal{B}$ of $\lambda$-sets such that

1. $\mathcal{A} \subset \mathcal{B}$
2. $\mathcal{B}$ is thin
3. $\mathcal{B}$ is $\Pi^1_{2n+1}$.

Proof. Consider the following game:

$$
\begin{array}{ccccccc}
1 & d_0, \beta_0 & d_1, \beta_1 & d_2, \beta_2 & \cdots \\
11 & m_0 & m_1 & m_2 & \cdots \\

d_i \in \omega, |\beta_i| < \lambda, m_i \in 2.
\end{array}
$$

Let $\delta = (d_0, d_1, \ldots)$. I wins if and only if

$\delta$ is a $\Delta^1_{2n+1}$-code for a $\lambda$-set $D \in \mathcal{A}$ and $\forall \beta_i \in D$ $(|\beta_i| \in D \leftrightarrow m_i = 1)$.

Suppose I has a winning strategy. The same type of argument used in proving Claim 1 of Lemma 1.6 then shows that $\mathcal{A}$ has a perfect subset. This contradicts the fact that $\mathcal{A}$ is thin.

This is a Harrington–Kechris game on $\lambda$, with $\lambda < \delta^1_{2n+1}$; note that the payoff set depends only on the ordinals played, not on the codes. So since I has no winning strategy, by Theorem 0.3 (and the remark following it) II must have a $\Delta^1_{2n+1}$ winning strategy. Call it $\tau$.

Let $\mathcal{C}$ be a countable set of codes for ordinals less than $\lambda$. Let $p$ be a finite sequence of the form

$$(d_0, \beta_0, m_0, d_1, \beta_1, m_1, \ldots, d_{k-1}, \beta_{k-1}, m_{k-1}, d_k),$$

that is, a position in the game when it is I’s turn to play an ordinal code. Let $B$ be a $\lambda$-set. Define $\tau$ rejects $B$ at $p$ relative to $\mathcal{C}$ if

(a) all ordinal codes occurring in $p$ are in $\mathcal{C}$,
(b) $p$ is consistent with $\tau$.
(c) for all $i < k$, $(|\beta_i| \in B \leftrightarrow m_i = 1)$,
(d) for all $\beta_k \in \mathcal{C}$, if I plays $\beta_k$ at $p$, then $\tau$ calls for II to play $m_k$ such that $(|\beta_k| \in B \leftrightarrow m_k = 0)$.

Let $\mathcal{B} = \{B : (\forall \text{countable set } \mathcal{C} \text{ of codes for ordinals } < \lambda)(\exists p)(\text{\tau rejects } B \text{ at } p \text{ relative to } \mathcal{C})\}$.

To prove the lemma:

1. $\mathcal{A} \subset \mathcal{B}$. Let $D \in \mathcal{A}$ and choose a code $\delta$ for $D$. Let $\mathcal{C}$ be a countable set of ordinal codes. If there is no $p$ such that $\tau$ rejects $B$ at $p$ relative to $\mathcal{C}$, I can win the game playing against $\tau$ by always playing $\delta = (d_0, d_1, \ldots)$ and playing the least element of $\mathcal{C}$ (with respect to some fixed enumeration) for which $\tau$ does not give the ‘wrong answer’. But $\tau$ is a winning strategy, so there must be such a $p$. Hence $\mathcal{A} \subset \mathcal{B}$.

2. $\mathcal{B}$ is thin. Suppose not; then there is a countable subset $W$ of $\lambda$ such that $\mathcal{B}' = \{B \cap W : B \in \mathcal{B}\}$ is uncountable. Let $\mathcal{C}$ contain one code for each element of
W. By definition of $\mathcal{B}$, such $B \cap W$ in $\mathcal{B}$ is totally determined by $\tau$ and by a finite sequence of integers and elements of $\mathcal{C}$. So $\mathcal{B}$ is countable, a contradiction.

(3) $\mathcal{B}$ is $\Pi^1_{2n+1}$.

\[ \delta \in \mathcal{B}^* \iff \]

\[ ((\delta \text{ is a } \Delta^1_{2n+1}-\text{code for a subset } D \text{ of } \lambda) \& \]

\[ \forall \beta (\forall i (|\beta_i| < \lambda) \rightarrow \exists s \text{ a finite sequence from } \omega) \&
\]

\[ s = (d_0, j_0, m_0, \ldots, d_k, j_k, m_k, \ldots, d_k) \& \]

\[ (\forall i < k) (|\tau (d_0, (\beta)_0, m_0, \ldots, d_i, (\beta)_i) = m_i) \& \]

\[ (\forall i < k) ([|\beta_i| \in D \iff m_i = 1) \& \]

\[ \forall i \forall m (\tau (d_0, (\beta)_0, m_0, \ldots, d_k, (\beta)_k, m_k, \ldots, d_k, (\beta)_k) = m \rightarrow (|\beta_i| \in D \iff m = 0)) \]}

Since $\tau$ is $\Delta^1_{2n+1}$, this is $\Pi^1_{2n+1}$.

We can now generalize the perfect set theorem from sets of reals to sets of $\lambda$-sets, for semireliable $\lambda$.

**Theorem 1.8 (AD).** Let $\lambda < \delta^1_{2n+1}$ be semireliable and let $\mathcal{A}$ be a set of $\lambda$-sets. If $\mathcal{A}$ is thin, then $\text{card}(\mathcal{A}) \leq \text{card}(\lambda)$.

**Proof.** Lemmas 1.6 and 1.7.

Lemmas 1.6 and 1.7 which we have stated for the pointclass $\Pi^1_{2n+1}$, actually are true for any class sufficiently like $\Pi^1_{2n+1}$, in particular for the class of inductie sets, $\text{IND}$. Therefore Theorem 1.8 holds for a set of $\lambda$'s that is closed unbounded in $\kappa^\omega$, the least non-hyperprojective ordinal. The obstacle to generalizing it to classes larger than $\text{IND}$ is that scales are needed, and $\text{IND}$ is the largest pointclass known to admit scales, by [20]; see also [18, Section 7C]. It can be shown that $\kappa^\omega$ satisfies Theorem 1.8. Whether every thin collection of subsets of $(\kappa^\omega)^+$ is well-orderable is open.

**Corollary 1.9 (AD).** Let $\lambda < \delta^1_{2n+1}$ be semireliable and let $\mathcal{A}$ be a set of $\lambda$-sets. If $\mathcal{A}$ is well-orderable, then $\text{card}(\mathcal{A}) \leq \text{card}(\lambda)$.

**Corollary 1.10 (AD).** Let $\lambda < \delta^1_{2n+1}$ and let $\mathcal{A}$ be a thin set of $\lambda$-sets.

(a) There is a $\Delta^1_{2n+1}$ set of reals $\mathcal{A}^\ast$ such that $\mathcal{A}^\ast \subset \mathcal{A}^w$ and every element of $\mathcal{A}$ has at least one code in $\mathcal{A}^\ast$.

(b) $\mathcal{A}$ is $\Sigma^1_{2n+2}$.

**Proof.** (a) The set of $\Delta^1_{2n+1}$-codes for $\lambda$-sets is $\Pi^1_{2n+1}$. Let $\psi$ be a $\Pi^1_{2n+1}$-norm on
thin collections of sets

For \( \xi < \delta_{2n+1} \) and \( \eta < \lambda \), let

\[
\mathcal{C}_{\eta}^\xi = \{ \beta : \beta \text{ codes a set } B \subseteq \lambda \land \psi(\beta) < \xi \land \eta \in B \}.
\]

\[
\hat{\mathcal{C}}_{\eta}^\xi = \{ \beta : \beta \text{ codes a set } B \subseteq \lambda \land \psi(\beta) < \xi \land \eta \notin B \}.
\]

The \( \mathcal{C}_{\eta}^\xi \)'s and \( \hat{\mathcal{C}}_{\eta}^\xi \)'s are all \( \Delta_2^{1+} \).

By a theorem of Martin, \( \Delta_2^{1+} \) is closed under unions and intersections of length less than \( \delta_{2n+1} \) (see [18, Theorem 7D.9]). So for any fixed \( \lambda \)-set \( B \) and \( \xi < \delta_{2n+1} \), the set

\[
\{ \beta : \beta \text{ is a } \Delta_2^{1+}\text{-code for } B \land \psi(\beta) < \xi \}
\]

is \( \Delta_2^{1+} \), since it's the intersection of \( \mathcal{C}_{\eta}^\xi \)'s and \( \hat{\mathcal{C}}_{\eta}^\xi \)'s. By Theorem 1.8, \( \text{card}(\mathcal{A}) < \delta_{2n+1} \). So for all \( \xi < \delta_{2n+1} \), the set

\[
\mathcal{A}_\xi = \{ \beta : \beta \text{ is a } \Delta_2^{1+}\text{-code for a } \lambda \text{-set } B \land \psi(\beta) < \xi \land \eta \in B \}
\]

is \( \Delta_2^{1+} \). Since \( \delta_{2n+1} \) is a regular cardinal and \( \text{card}(\mathcal{A}) < \delta_{2n+1} \), there is a \( \xi_0 < \delta_{2n+1} \) such that every \( \mathcal{A} \subseteq \mathcal{A}_\xi \) has at least one code in \( \mathcal{A}_\xi \). Let \( \mathcal{A}' = \mathcal{A}^{\xi_0} \). Then \( \mathcal{A}' \) satisfies (a).

(b) \( \alpha \in \mathcal{A} \Rightarrow [\exists \alpha' (\alpha' \in \mathcal{A}' \land \alpha \text{ and } \alpha' \text{ code the same } \lambda \text{-set})] \). This is \( \Sigma_2^{1+} \).

Definition 1.11. An infinite cardinal \( \lambda \) has the perfect set property if every thin set of \( \lambda \)-sets has cardinality at most \( \text{card}(\lambda) \).

We do not know exactly which cardinals have the perfect set property. We conjecture that all projective cardinals do. Of course by Theorem 1.8, every thin set of \( \lambda \)-sets has cardinality at most \( \lambda^+ \), where \( \lambda^+ \) is the least semireliable ordinal greater than or equal to \( \lambda \), and hence is well-orderable. So in Definition 1.11, 'thin' can be replaced by 'well-orderable'. Besides the semireliable cardinals, there are three other classes of cardinals that are known to satisfy the perfect set property, namely:

1. \( \{ \aleph_m : m \in \omega \} \),
2. \( \{ \lambda : \lambda^+ \text{ is Rowbottom} \} \),
3. \( \{ \delta_{2n+2} : n \in \omega \} \).

We will not prove (1) in this paper, but see the remark after Theorem 3.13 for more details on this point. Case (1) is due to Martin. A cardinal \( \kappa > \aleph_1 \) is a Rowbottom cardinal if given any structure \( \mathfrak{A} = \langle A; U, . . . \rangle \) (in the sense of model theory), with \( U \) a subset of \( A \), if \( \text{card}(A) = \kappa \) and \( \text{card}(U) < \kappa \), then there is an \( \mathfrak{A}' = \langle A'; U', . . . \rangle \) such that \( \mathfrak{A}' \) is an elementary substructure of \( \mathfrak{A} \), \( \text{card}(A') = \kappa \), and \( U' \) is countable. Since no set of reals can have cardinality \( \lambda^+ \), it is easy to see
that $\lambda^+$ Rowbottom implies that no set of $\lambda$-sets has cardinality $\lambda^+$, which takes care of case (2). Measurable cardinals are Rowbottom by [6, Theorems 7.3.14 and 7.3.16]. The only projective cardinals $\lambda$ that we know for a fact are semireliable, have the property that $\lambda^+$ is measurable. So (2) may well include the class of uncountable semireliable cardinals. Martin [17] has proved that $\aleph_{\omega_2+1}$ and $\aleph_{\omega+1}$ are both measurable cardinals and both lie properly between $(\delta_4^+)$ and $\kappa_5$; most likely, neither $\aleph_{\omega_2}$ nor $\aleph_{\omega^+}$ is semireliable. (One non-measurable cardinal, $\aleph_\omega$, is also known to be Rowbottom, by [15].) Case (3) is proved in the following theorem.

**Theorem 1.12 (AD).** Let $n \in \omega$ and let $\mathcal{A}$ be a set of $\delta_{2n+2}^+$-sets. If $\mathcal{A}$ is thin, then $\text{card}(\mathcal{A}) \leq \delta_{2n+2}^+$. 

**Proof.** For each $\xi < \delta_{2n+2}^+$, let $\mathcal{A}_\xi$ be the set of $\xi$-sets $\{B \cap \xi : B \in \mathcal{A}\}$. Since $\mathcal{A}$ is thin, by Theorem 1.8 it is well-orderable, hence (without using the axiom of choice) we can simultaneously well-order each $\mathcal{A}_\xi$. Let $<_{\xi}$ denote this well-ordering of $\mathcal{A}_\xi$. Let $\nu_\xi$ be its order type. For all $B \in \mathcal{A}$ and all $\xi < \delta_{2n+2}^+$, let $\nu_\xi'$ be the ordinal which is the rank of $B \cap \xi$ in the well-ordering $<_{\xi}$ of $\mathcal{A}_\xi$.

There is a normal measure $\mu$ on $\delta_{2n+2}^+$ such that the ultrapower of $\delta_{2n+2}^+$ with respect to this measure, $\Pi \delta_{2n+2}^+ / \mu$, has length $(\delta_{2n+2}^+)$ (Kunen—see [10]). For $B, C \in \mathcal{A}$ define $B <_{\mu} C$ if for $\mu$-a.e. $\xi$, $(B \cap \xi) <_{\xi} (C \cap \xi)$. Then $<_{\mu}$ is a well-ordering of $\mathcal{A}$, let $\nu$ be its order type, and for $B \in \mathcal{A}$ let $\nu_B'$ denote the rank of $B$ in this well-ordering. To complete the proof it is enough to show that $\nu < (\delta_{2n+2}^+)$.

Now $\delta_{2n+2}^+ = (\delta_{2n+1}^+)^+$, and $\delta_{2n+1}^+$ satisfies the perfect set property by Theorem 1.8, so for all $\xi < \delta_{2n+2}^+$, $\nu_\xi < \delta_{2n+2}^+$. There is an obvious map from $\nu$ into the ultraproduct $\prod_{\xi \in \delta_{2n+2}^+} \nu_\xi / \mu$, namely the map that for each $B \in \mathcal{A}$, takes $\nu_B'$ to $[\xi \mapsto \nu_B']$. This map is one-to-one and order preserving. Thus we have

$$\nu \leq \prod_{\xi \in \delta_{2n+2}^+} \nu_\xi / \mu \leq \prod \delta_{2n+2}^+ / \mu = (\delta_{2n+2}^+)^{+}. $$

Theorem 1.12 was also proved by Sami [23] and Martin, independently of each other and of the author.

The perfect set property for $\omega$ completely solves the problem of the possible cardinalities of sets of reals—every set of reals is either countable or equinumerous with the continuum. It does not completely solve the problem for uncountable cardinals $\lambda$, although it is a first step in that direction. Consider the simplest case, $\aleph_1$. The four sets $\omega, \omega_1, \text{Pow}(\omega)$, and $\text{Pow}(\omega_1)$ are each a set of $\aleph_1$-sets, and all four have different cardinalities (in the sense of injections, not surjections). They are ordered (by injection) as follows:

```
      Pow(ω₁)
     /     \
Pow(ω) --- ω₁
     \
      ω
```


The sets $\omega_1$ and $\text{Pow}(\omega)$ are incomparable, and there are no sets that go between $\omega$ and $\text{Pow}(\omega)$ or between $\omega$ and $\omega_1$ on the above diagram. So there are, a priori, four other types of sets $\mathcal{A} \subset \text{Pow}(\omega_1)$ that could occur. They are the four listed below, where $\|\mathcal{A}\| < \|\mathcal{B}\|$ means there is an injection from $\mathcal{A}$ into $\mathcal{B}$:

1. $\mathcal{A}$ is incomparable with either $\omega_1$ or $\text{Pow}(\omega)$.
2. $\|\omega_1\| < \|\mathcal{A}\|$ and $\mathcal{A}$ is incomparable with $\text{Pow}(\omega)$.
3. $\|\text{Pow}(\omega)\| < \|\mathcal{A}\|$ and $\mathcal{A}$ is incomparable with $\omega_1$.
4. $\|\omega_1\| < \|\mathcal{A}\|$ and $\|\text{Pow}(\omega)\| < \|\mathcal{A}\|$.

The perfect set property for $\mathbb{N}_1$ shows that no set of type (1) or (2) exists. Whether any sets of type (3) or (4) exist is open.

2. The largest thin $\Pi_{2n+1}^1(\lambda)$ and $\Sigma_{2n+2}^1(\lambda)$ sets

In this section we show that there is a largest thin $\Pi_{2n+1}^1(\lambda)$ set and a largest thin $\Sigma_{2n+2}^1(\lambda)$ set of subsets of $\lambda$.

Theorem 2.1 (PD). Let $\mathcal{R}(\alpha, \beta)$ be a $\Sigma_{2n+1}$ relation. Let

$$\mathcal{T}(\alpha) \iff \forall \beta(\mathcal{R}(\alpha, \beta) \rightarrow \exists \beta(\exists \gamma(\mathcal{R}(\alpha, \beta) \& \beta \text{ codes } B) \text{ is thin}).$$

Then $\mathcal{T}$ is $\Pi_{2n+1}^1$.

Proof. Let

$$Q(\alpha) \iff \forall \beta(\mathcal{R}(\alpha, \beta) \rightarrow \exists \beta(\exists \gamma(\mathcal{R}(\alpha, \beta) \& \beta \text{ codes } B) \text{ is thin}).$$

Clearly $Q$ is $\Pi_{2n+1}^1$. So it is enough to show that if $Q(\alpha)$, then '$\mathcal{R}_\alpha$ is not thin' is $\Sigma_{2n+1}^1(\alpha)$, uniformly in $\alpha$. So consider only $\alpha$'s such that $Q(\alpha)$. Let $B_\beta$ be the $\Delta_{2n+1}^1$ set coded by $\beta$.

For any countable $A \subset \delta_{2n+1}^1$, let $\mathcal{R}_\alpha^A = \{B \cap A : B \in \mathcal{R}_\alpha\}$. Then by definition of thin (1.1), $\mathcal{R}_\alpha^A$ is not thin if and only if there is an $A \in \mathcal{P}_{\mathbb{N}_1}(\delta_{2n+1}^1)$ such that $\mathcal{R}_\alpha^A$ is not thin (as a subset of $\wedge 2$, i.e. in the ordinary sense). So

$$\mathcal{R}_\alpha \text{ is not thin } \iff \exists \gamma(\forall i \exists \beta(\mathcal{R}(\alpha, \beta) \& (\gamma_i) \in B_\beta) \& \text{ the set } \mathcal{S}_\gamma^\alpha = \{X \in \wedge 2 : \exists \beta(\mathcal{R}(\alpha, \beta) \& \forall i(X(i) = 1 \leftrightarrow (\gamma)_i \in B_\beta))\} \text{ is not thin}].$$

$S_\gamma^\alpha$ is $\Sigma_{2n+1}^1(\alpha, \gamma)$, uniformly in $\alpha, \gamma$. So by a theorem of Kechris [12], '$S_\gamma^\alpha$ is not thin' is $\Sigma_{2n+1}^1(\alpha, \gamma)$, uniformly in $\alpha, \gamma$.

Theorem 2.2 (PD; Kechris [11, 12] for $\lambda \leq \mathfrak{N}_1$). Let $\lambda < \delta_{2n+1}^1$. There is a largest thin $\Pi_{2n+1}^1(\lambda)$ set of $\lambda$-sets.
Proof. Let $\mathcal{U}(m, \alpha, \beta)$ be universal $\Pi^1_{2n+1}$. Let $\psi$ be a $\Pi^1_{2n+1}$-norm on $\mathcal{U}$. Let

$$F(m, \alpha, \beta) \iff |\alpha| = \lambda \land \mathcal{U}(m, \alpha, \beta) \land \forall \alpha' \forall \beta'[(|\alpha'| = \lambda \land \psi(m, \alpha', \beta') \leq \psi(m, \alpha, \beta) \rightarrow \beta' \text{ codes a } \lambda \text{-set}] \land \alpha' = \beta \land \forall \alpha' \forall \beta'[(|\alpha'| = \lambda \land \psi(m, \alpha', \beta') \leq \psi(m, \alpha, \beta))] \text{ is thin.}$$

By Theorem 2.1, $F$ is $\Pi^1_{2n+1}(\lambda)$. Let $\mathcal{T}_m = \{\beta : \exists \alpha(|\alpha| = \lambda \land F(m, \alpha, \beta))\}$, and let $\mathcal{T}'_m = \{B \subseteq \lambda : \exists \beta \in \mathcal{T}_m \text{ such that } \beta \text{ codes } B\}$. $\mathcal{T}_m$ is $\Pi^1_{2n+1}(\lambda)$, uniformly in $m$, and by Proposition 1.4, each $\mathcal{T}'_m$ is thin. Since $\mathcal{U}$ is universal, every thin $\Pi^1_{2n+1}(\lambda)$ set of $\lambda$-sets is a $\mathcal{T}_m$, for some $m$. So $\bigcup_m \mathcal{T}'_m$ is the largest thin $\Pi^1_{2n+1}(\lambda)$ set of $\lambda$-sets.

Although Theorems 2.1 and 2.2 are consequences of PD, i.e. full AD is not needed, they seem to be useless without the coding lemma, which requires AD.

Let $\mathcal{C}^\lambda_{2n+1}$ denote the largest thin $\Pi^1_{2n+1}(\lambda)$ set of $\lambda$-sets. Thus $\mathcal{C}^\lambda_{2n+1}$ is the largest thin $\Pi^1_{2n+1}$ set of reals, which is usually denoted by $\mathcal{C}_{2n+1}$ (see [12] for more on $\mathcal{C}_{2n+1}$). The usual method [12] of obtaining the largest thin $\Sigma^1_{2n+2}$ set of reals from the largest thin $\Pi^1_{2n+1}$ set, does not seem to work for sets of $\lambda$-sets. The problem is that we have no uniformization theorem for $\lambda$-sets. The largest thin $\Sigma^1_{2n+2}(\lambda)$ set has to be characterized in another manner, namely be the definability of its members. This is the subject of the rest of Section 2.

Lemma 2.3 (AD). Let $\lambda < \delta^1_{2n+1}$ and let $A$ be a $\lambda$-set. The following are equivalent:

(a) There are ordinals $\xi_0, \ldots, \xi_k < \delta^1_{2n+1}$ such that $A$ is $\Delta^1_{2n+1}(\xi_0, \ldots, \xi_k)$.
(b) There are ordinals $\xi_0, \ldots, \xi_k < \delta^1_{2n+1}$ such that $A$ is $\Sigma^1_{2n+2}(\xi_0, \ldots, \xi_k)$.
(c) There is an ordinal $\xi < \delta^1_{2n+1}$ such that $A$ is $\Sigma^1_{2n+2}(\xi)$.

Proof. That (a) $\Rightarrow$ (b) is trivial, and since sequence coding (via the Gödel ordering) is $\Delta^1_{2n+2}$, clearly (b) $\Rightarrow$ (c).

To prove that (c) $\Rightarrow$ (a), let $\mathcal{B}(\alpha, \beta)$ be $\Sigma^1_{2n+2}$ such that for all $\alpha \in N$, if $|\alpha| = \xi$, then $(\mathcal{B}(\alpha, \beta) \iff |\beta| \in A)$. Let $\mathcal{F}(\alpha, \beta, \gamma)$ be $\Pi^1_{2n+1}$ such that $\mathcal{B}(\alpha, \beta) \iff \exists \gamma \gamma(\alpha, \beta, \gamma)$. Since $\mathcal{F}$ is $\Pi^1_{2n+1}$ there is a recursive function $f : N \times N \times N \rightarrow N$ such that $(\alpha, \beta, \gamma) \in \mathcal{F} \iff f(\alpha, \beta, \gamma) \in \mathcal{F}$, where $\mathcal{F}$ is the complete $\Pi^1_{2n+1}$ set used in coding ordinals less than $\delta^1_{2n+1}$, via the norm $\phi$. Since $\delta^1_{2n+1}$ is a regular cardinal, there is an $\eta < \delta^1_{2n+1}$ such that for all $\nu \in A$, there are reals $\alpha, \beta, \gamma$ such that $|\alpha| = \xi, |\beta| = \nu, \mathcal{B}(\alpha, \beta, \gamma)$, and $\phi(f(\alpha, \beta, \gamma)) < \eta$. Thus

$$|\beta| \in A \iff \exists \alpha \exists \beta' \exists \gamma(|\alpha| = \xi \land |\beta'| = |\beta| \land \phi(f(\alpha, \beta, \gamma)) < \eta).$$

This is a $\Sigma^1_{2n+1}(\xi, \eta)$ definition. So $A$ is $\Sigma^1_{2n+1}(\xi, \eta, \lambda)$. Hence $(\lambda \setminus A)$ is $\Pi^1_{2n+1}(\xi, \eta, \lambda)$ and since (b) $\Rightarrow$ (c) it is $\Sigma^1_{2n+2}(\xi', \eta')$, for some $\xi' < \delta^1_{2n+1}$. Now by the above argument applied to $(\lambda \setminus A)$ rather than to $A$, $(\lambda \setminus A)$ is $\Sigma^1_{2n+1}(\xi', \eta')$ for some $\eta' < \delta^1_{2n+1}$. Hence $A$ is $\Delta^1_{2n+1}(\lambda, \xi, \eta, \xi', \eta')$. 

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Let $e_{2n+2}^\lambda = \{ A \subseteq \lambda : \exists \xi < \delta_{2n+1}^1 \text{ such that } A \text{ is } \Sigma_{2n+2}^1(\xi) \}.$

**Lemma 2.4 (AD).** For all $\lambda < \delta_{2n+1}^1,$ $e_{2n+2}^\lambda$ is $\Sigma_{2n+2}^1(\lambda).$

**Proof.** Let $e^* = \{ \alpha : \alpha \text{ is a } \Delta_{2n+1}^1 \text{-code for a subset of } \lambda \}.$ It must be shown that $e^*$ is $\Delta_{2n+2}^1(\lambda).$

Let $W^* = \{ \alpha_0 : \alpha_0 \text{ is an } \Delta_{2n+1}^1 \text{-code for a subset of } \lambda \}.$ It must be shown that $W^*$ is $\Delta_{2n+2}^1(\lambda).$

$\alpha \in e^* \Leftrightarrow \alpha$ is a $\Delta_{2n+1}^1$-code for a subset of $\lambda$ and

$(\exists \xi_0, \ldots, \xi_k < \delta_{2n+1}^1)(A \text{ is } \Delta_{2n+1}^1(\xi_0, \ldots, \xi_k))$

$\Leftrightarrow \alpha$ is a $\Delta_{2n+1}^1$-code for a set $D_0 \subseteq \lambda$ and

$(\exists \xi_0, \ldots, \xi_k < \delta_{2n+1}^1)(\exists \eta < \delta_{2n+1}^1)(\exists \psi \text{ a recursive function}$

$f : \mathbb{N}^{\xi^2} \to \mathbb{N}((\forall \beta_0, \ldots, \beta_k)(|\beta_0| = \xi_0 \& \cdots \& |\beta_k| = \xi_k) \Rightarrow$

$\forall \psi(\gamma \in D_0 \Leftrightarrow (f(\cdot, \beta_0, \ldots, \beta_k) \in \mathcal{S} \& \psi(f(\cdot, \beta_0, \ldots, \beta_k) < \eta))).$

The first equivalence is Lemma 2.3; the second equivalence is a standard boundedness argument. It gives a $\Sigma_{2n+2}^1(\lambda)$ definition of $e^*.$

**Theorem 2.5 (AD; Moschovakis [21], Kechris [11, 12] for $\lambda \leq \aleph_1).$** Let $\lambda < \delta_{2n+1}^1.$

$\mathcal{C}^\lambda_{2n+2}$ is the largest thin $\Sigma_{2n+2}^1(\lambda)$ set of $\lambda$-sets.

**Proof.** It is clearly well-orderable, hence thin. It is $\Sigma_{2n+2}^1(\lambda)$ by Lemma 2.4. To complete the proof it is enough to show that if $\mathcal{B}$ is any thin $\Sigma_{2n+2}^1(\lambda)$ set of $\lambda$-sets, then for every $A \in \mathcal{B},$ there exist $\xi_0, \ldots, \xi_k < \delta_{2n+1}^1$ such that $A$ is $\Sigma_{2n+2}^1(\xi_0, \ldots, \xi_k).$

Fix such a $\mathcal{B}.$ Recall the proof of Lemma 1.6; what is shown there is that if $A \in \mathcal{B},$ there is a finite sequence of ordinals $p$ such that $A$ is totally determined by $p$ and a fixed winning strategy $\tau.$ That is, $A$ is definable from $p.$ To prove Theorem 2.5, following Moschovakis [21], we modify the proof of Lemma 1.6 to make $A$ definable from $p$ in a $\Sigma_{2n+2}^1$ way.

Let $\mathcal{C}(\gamma, \delta)$ be $\Sigma_{2n+2}^1$ such that if $\mathcal{C}(\gamma, \delta),$ then $\gamma$ codes an ordinal less than $\delta_{2n+1}^1$ and $\delta$ is a $\Delta_{2n+1}^1$-code for a $|\gamma|$-set, and such that for any $\gamma \in \mathcal{N},$ if $|\gamma| = \lambda,$ then $\forall \delta[\mathcal{C}(\gamma, \delta) \Leftrightarrow (\delta$ is a $\Delta_{2n+1}^1$-code for a $\lambda$-set $D_0 \& D_0 \subseteq \mathcal{B})].$

Let $\mathcal{C}(\gamma, \delta, \varepsilon)$ be $\Pi_{2n+1}^1$ such that $\mathcal{C}(\gamma, \delta, \varepsilon) \Leftrightarrow \exists \varepsilon \mathcal{C}(\gamma, \delta, \varepsilon).$ Let $S$ be the tree associated with a $\Pi_{2n+1}^1$-scale on $\mathcal{S}.$ Let $T$ be the tree associated with a $\Pi_{2n+1}^1$-scale used to code ordinals, as in Lemma 1.6. Consider the following game on ordinals less than $\delta_{2n+1}^1$:

<table>
<thead>
<tr>
<th></th>
<th>$\lambda$</th>
<th>$d_0, e_0, \xi_0, \eta_0$</th>
<th>$d_1, e_1, \xi_1, \eta_1$</th>
<th>$d_2, e_2, \xi_2, \eta_2$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$\alpha_0$</td>
<td>$\alpha_1$</td>
<td>$\alpha_2$</td>
<td>$\cdots$</td>
<td></td>
</tr>
<tr>
<td>$\tau_0$</td>
<td>$t_0$</td>
<td>$t_1$</td>
<td>$t_2$</td>
<td>$\cdots$</td>
<td></td>
</tr>
</tbody>
</table>
The above diagram is to be interpreted in the same way as the diagram used in the proof of Lemma 1.6. That is, I plays an ordinal \( \eta_i < \delta_{2n+1}^1 \) and a real \( \alpha_i \) (played one integer at a time) that is an alleged code for \( \eta_i \). Player I plays witnesses (with respect to \( T \)) for his codes and II plays alleged codes to keep I's codes honest. I also plays \( \lambda \) on his first move and gives an alleged code \( \gamma_0 \) for \( \lambda \) which II will challenge. In addition, I plays \( d_i \in \omega, e_i \in \omega, \xi_i < \delta_{2n+1}^1 \) and II plays \( t_i \in \mathbb{2} \).

The payoff set is defined as follows:

(i) Both players must play into the tree \( T \), as in Lemma 1.6; the first player to fail to do so loses.

(ii) As in Lemma 1.6, if one of I's codes is proved dishonest by II, then I loses.

(iii) For all \( i, (\gamma(0), d_0, e_0, \xi_0, \ldots, \gamma(i), d_i, e_i, \xi_i) \) must be in the tree \( S \) — if not I loses.

(iv) Suppose neither player loses because of (i), (ii), or (iii). Let \( \delta = (d_0, d_1, \ldots) \). Condition (iii) implies that the \( \gamma \) played by I is actually an ordinal code and that \( \delta \) is a \( \Delta^1_{2n+1} \) code for a subset \( D_\delta \) of \( \{\gamma, \} \). I wins the game if and only if

\[
\forall i (t_i = 1 \iff |\alpha_i| \in D_\delta).
\]

The same argument used in the proof of Lemma 1.6 will show that II has a winning strategy. Note that (i) and (iii) in the definition of the payoff are clopen conditions on sequences of ordinals, easily \( \Sigma^1_{2n+2}(\lambda) \), hence by Lemma 2.3, \( \Delta^1_{2n+1} \) in ordinals less than \( \delta_{2n+1}^1 \), and that (ii) and (iv) are \( \Delta^1_{2n+1} \) conditions on reals. So the game is pseudo-\( \Sigma^1_{2n+2}(\lambda) \), hence by Theorem 0.6, II has a winning strategy \( \tau \) such that there exist ordinals \( \nu_0, \ldots, \nu_i < \delta_{2n+1}^1 \) such that \( \tau \) is \( \Delta^1_{2n+2}(\nu_0, \ldots, \nu_i) \).

Let \( A \in \mathcal{B} \). Again by the same proof as in Lemma 1.6, there is a position \( p \) in the game such that \( \tau \) rejects \( A \) at \( p \). Say \( p \) contains the ordinals \( \theta_0, \ldots, \theta_k \). Then \( A \) is easily \( \Sigma^1_{2n+2}(\nu_0, \ldots, \nu_i, \theta_0, \ldots, \theta_k) \), which proves the theorem.

Thus \( \mathcal{C}^\lambda_{2n+2} = \mathcal{C}^\lambda_{2n+2} \), the largest thin \( \Sigma^1_{2n+2} \) set of reals (see [12]).

**Proposition 2.6.** If \( \lambda < \eta < \delta_{2n+1}^1 \), then \( \mathcal{C}^\lambda_{2n+2} \) is a proper subset of \( \mathcal{C}^\eta_{2n+2} \). In fact, \( \mathcal{C}^\lambda_{2n+2} = \{A \in \mathcal{C}^\lambda_{2n+2}; A \subset \lambda \} \).

**Corollary 2.7 (AD; Moschovakis [21]).** Let \( \lambda < \delta_{2n+1}^1 \). \( \mathcal{C}^\lambda_{2n+2} \) is the largest thin set of \( \lambda \)-sets which is \( \Sigma^1_{2n+2}(\xi_0, \ldots, \xi_k) \) for any \( \xi_0, \ldots, \xi_k \) less than \( \delta_{2n+1}^1 \).

**Proof.** Let \( \mathcal{C} \) be a thin \( \Sigma^1_{2n+2}(\xi_0, \ldots, \xi_k) \) set of \( \lambda \)-sets. Let \( \eta < \delta_{2n+1}^1 \) code the sequence \((\lambda, \xi_0, \ldots, \xi_k)\). Sequence coding is \( \Delta^1_{2n+2} \) and \( \lambda < \eta \), so \( \mathcal{C} \) is a thin \( \Sigma^1_{2n+2}(\eta) \) set of \( \eta \)-sets. By Theorem 2.5, \( \mathcal{C} \subset \mathcal{C}^\eta_{2n+2} \), hence by Proposition 2.6, \( \mathcal{C} \subset \mathcal{C}^\lambda_{2n+2} \).

We do not know whether Proposition 2.6 or Corollary 2.7 is true for the \( \mathcal{C}^\lambda_{2n+1} \)'s. We have not been able to prove that for all \( \lambda, \mathcal{C}^\lambda_{2n+1} \neq \mathcal{C}^\lambda_{2n+2} \), but
conjecture that it is true, and that in fact $\mathcal{C}_{2n+2}^\lambda$ is not $II_{2n+2}^1(\lambda)$ and $\mathcal{C}_{2n+1}^\lambda$ is not $\Sigma_{2n+1}^1(\lambda)$.

The results of Section 2 all relativize to an arbitrary real $\alpha$ in a routine manner. Thus for all $\lambda < \delta_{2n+1}^1$ and all $\alpha \in \mathcal{N}$, there are largest $t_i$ in $II_{2n+1}^1(\alpha, \lambda)$ and $\Sigma_{2n+2}^1(\alpha, \lambda)$ sets of $\lambda$-sets, which we denote by $\mathcal{C}_{2n+1}^\lambda(\alpha)$ and $\mathcal{C}_{2n+2}^\lambda(\alpha)$, respectively.

**Theorem 2.8 (AD).** Let $\lambda < \delta_{2n+1}^1$ and let $\mathcal{A}$ be a thin set of $\lambda$-sets. Then there is an $\alpha \in \mathcal{N}$ such that $\mathcal{A} \subseteq \mathcal{C}_{2n+2}^\lambda(\alpha)$.

**Proof.** Corollary 1.10 and the relativized version of Theorem 2.5.

Every theorem in this section through Corollary 2.7 is absolute for $L[\mathbb{R}]$. Theorem 2.8, of course, is not.

### 3. The inner model $H_{2n+1}$

There is a close relationship between the constructible universe $L$ and the pointclasses $II_1^1$ and $\Sigma_2^1$. This fact has led to several different attempts to find analogs of $L$ for the higher analytical pointclasses, that is, inner models that are related to the pointclass $\Sigma_{2n+2}^1$ in the same way that $L$ is related to $\Sigma_2^1$. However, there is more than one theorem connecting $L$ and $\Sigma_2^1$, and the attempt to generalize two different properties of $L$ may lead to two distinct models. In fact, this turns out to be the case. In [2], two different analogs of $L$, known as $M_{2n+2}$ and $L_{2n+2}$, are presented. While both of these models are interesting, and in fact are quite good analogs of $L$ with regard to sets of reals, they are worthless with regard to sets of $\lambda$-sets. For both are models of $V = L[\mathbb{R}]$, with countably many reals, and thus no interesting subset of $\lambda$ (e.g. the set of ordinals of cofinality $\omega$ in $V$) can be in them.

Since our interest in this paper is in sets of $\lambda$-sets, we have no use for either $M_{2n+2}$ or $L_{2n+2}$. What we wish to study here is a third analog, which is similar to $L$ with regard to $\lambda$-sets. It is a classical theorem of Shoenfield that every $\Sigma_2^1$ subset of $\omega$ is in $L$. This was later strengthened by Solovay, Kechrís, and Moschovakis, who showed that any subset of $\mathbb{N}$ that is $\Sigma_2^1$-in-the-codes is in $L$ (see [14] or [4]). Thus $L$ is the smallest model containing all ordinals and containing every subset of $\delta_1^1(= \mathbb{N})$ that is $\Sigma_2^1$-in-the-codes. Our analog, $H_{2n+1}$, is the smallest model containing every subset of $\delta_{2n+1}^1$ that is $\Sigma_{2n+2}^1$-in-the-codes; a more precise definition is given below.

**Definition 3.1.** Let $G \subseteq \omega \times \mathcal{N}$ be a good (in the sense of [18, 3H.1]) universal set in $\Sigma_{2n+2}^1$. Let $G' \subseteq \omega \times \check{\delta}_{2n+1}^1$ be the relation

$$G'(m, \xi) \Leftrightarrow \exists \beta [\beta \in \mathcal{F} \& \varphi(\beta) = \xi \& G(m, \beta)],$$
where \( \varphi \) is the \( H^{1}_{2n+1} \)-norm on the \( H^{1}_{2n+1} \)-set \( \mathcal{G} \) that is used in coding ordinals less than \( \delta^{1}_{2n+1} \) (see Section 0). Let \( H^{1}_{2n+1} = L[G'] \).

The model \( H^{1}_{2n+1} \) is independent of the choice of \( \mathcal{G} \), \( \varphi \), and \( G \). For if \( \mathcal{G}_0 \), \( \varphi_0 \), \( G_0 \) and \( \mathcal{G}_1 \), \( \varphi_1 \), \( G_1 \) are different, then for all \( m \), the set \( \{ \xi : G_0'(m, \xi) \} \) equals the set \( \{ \xi : G_1'(m, \xi) \} \) for some \( \hat{m} \), where the map \( m \mapsto \hat{m} \) is recursive, since two norms can be compared in a \( \Delta^{2}_{2n+2} \)-way using Theorem 0.1 (see [18, 8G.21]). For the sequel we fix some \( \mathcal{G} \), \( \varphi \), \( G \). It is clear from the above definition that \( H^{1}_{2n+1} \) is actually the smallest transitive model of ZFC containing all ordinals and containing every subset of \( \delta^{1}_{2n+1} \) that is \( \Sigma^{1}_{2n+2} \)-in-the-codes, as promised. Therefore \( \mathcal{H} = L \).

The model \( H^{1}_{2n+1} \) is absolute for \( L[\mathbb{R}] \), and therefore so are the theorems about \( H^{1}_{2n+1} \) that follow. But recall that, for the purposes of this paper, we have embedded ourselves in \( L[\mathbb{R}] \) and are assuming that AD is true. Since we are dealing with models, it will be convenient to use the phrase ‘in \( V \)’; hence, in this paper, ‘\( V \)’ is a universe in which AD is true.

The models \( H^{n}_{n+1} \) were first defined and studied by Moschovakis in [18, Section 8G]. Here in Section 3 of this paper we will state many of the basic properties of \( H^{n}_{n+1} \), often without proof; whenever no proof is given here, it can be found in [18] or in Moschovakis [21].

First of all, the tree \( T^{2n+1} \) associated with a \( H^{1}_{2n+1} \)-scale on a \( H^{1}_{2n+1} \)-set is \( \Delta^{1}_{2n+1} \)-in-the-codes, hence in \( H^{n}_{n+1} \). Any model containing \( T^{2n+1} \) is \( \Sigma^{1}_{2n+2} \)-correct. Hence \( H^{n}_{n+1} \) and every extension of \( H^{n}_{n+1} \) is \( \Sigma^{1}_{2n+2} \)-correct; in this it is indeed analogous to \( L \). This implies that \( \Delta^{1}_{2n+1} \)-determinacy is true in \( H^{n}_{n+1} \) – \( H^{n}_{n+1} \) is a ‘partially playful universe’. However, \( H^{1}_{2n+1} \)-determinacy is false in \( H^{n}_{n+1} \).

\( H^{n}_{n+1} \) also bears the same relationship to regularity properties of \( \Sigma^{1}_{2n+2} \) that \( L \) bears to \( \Sigma^{1}_{2} \). For the results connecting \( H^{n}_{n+1} \) with Lebesgue measurability and the property of Baire, see [18, 8G]; in this paper we are not concerned with either of these subjects. We are, however, concerned with a third regularity property, the perfect set property. Every \( \Sigma^{1}_{2n+2} \) set of reals is either contained in \( H^{n}_{n+1} \) or else has a perfect subset. For \( L \) this is due to Solovay [25]; the general theorem is due to Kechris and Moschovakis [13]. We will generalize this result from sets of reals to sets of \( \lambda \)-sets.

It is well known that if \( \xi, \eta < \aleph_1 \), \( A \subset \eta \), then membership in \( L[\mathbb{R}] \) is \( \Delta^{1}_{2} \)-uniform. This generalizes to higher levels of the analytical hierarchy. (Actually they are \( \Delta^{1}_{2} \)-uniform, this does not generalize.) By \( I(\alpha) \), where \( A \subset \lambda \), we of course mean \( I(\alpha) \), uniformly for all codes \( \alpha \) for \( A \).

**Theorem 3.2** (AD). Let \( \xi, \eta < \varepsilon^{1}_{2n+1} \), and let \( A \subset \eta \).

(a) Membership in \( L[\mathbb{R}] \) is \( \Delta^{1}_{2n+2} \)-uniform, i.e. the set \( \{ \xi : B \in L[\mathbb{R}] \} \) is \( \Delta^{1}_{2n+2} \)-uniform, uniformly in \( \xi, A \).
(b) Truth in $L_\xi[A]$ is $\Delta^1_{2n+2}(\xi, A)$, i.e. the set
\[ \{ (k, B) \in \omega \times \text{Pow}(\xi) : B \in L_\xi[A] \land L_\xi[A] \models \varphi_k(B) \} \]
is $\Delta^1_{2n+2}(\xi, A)$, uniformly in $\xi, A$, where $\varphi_k$ denotes the formula of set theory with G"odel number $k$.

The idea of the proof is similar to the proofs for $L$. Each $L_\xi[A]$ has cardinality less than $\delta^1_{2n+1}$ and $\delta^1_{2n+1} = (\kappa_{2n+1})^\ast$, so it is isomorphic to a structure $(M; E)$ where $M$ is a subset of $\kappa_{2n+1}$ and $E$ is a binary relation on $\kappa_{2n+1}$. Thus, e.g. truth in $L_\xi[A]$, involves quantification only over $\kappa_{2n+1}$, just as truth in $L_\xi (\xi$ countable) involves only quantification over $\omega$. And by Theorem 0.1, $\Delta^1_{2n+2}$ is closed under quantification over $\kappa_{2n+1}$. Detailed proofs can be found in [2], [8], and [18]. We wish to warn the reader that the codings and computations involved are very complicated and very long — it is not as simple as the above remarks make it appear to be.

**Corollary 3.3 (AD).** Let $\xi, \eta < \delta^1_{2n+1}$ and let $A \subseteq \eta$. If $A$ is $\Delta^1_{2n+2}(\eta)$, then:

(a) membership in $L_\xi[A]$ is $\Delta^1_{2n+2}(\xi, \eta)$.

(b) truth in $L_\xi[A]$ is $\Delta^1_{2n+2}(\xi, \eta)$.

**Definition 3.4.** Let $\psi$ be the $\Sigma^1_{2n+2}$-norm on $G$ that is obtained from a $\Pi^1_{2n+1}$ norm in the canonical way (that is, if $G = \exists x F$ and $\psi'$ is a $\Pi^1_{2n+1}$-norm on $F$, then $\psi(\beta) = \inf \{ \psi'(\alpha, \beta) : F(\alpha, \beta) \}$). Let $G^* \subseteq \omega \times \delta^1_{2n+1} \times \delta^1_{2n+1}$ be the relation
\[ G^*(m, \xi, \eta) \Leftrightarrow \exists \beta (\beta \in : \alpha \in \exists x F(\alpha, \beta) \land G(m, \beta) \land \psi(m, \beta) < \eta). \]

**Lemma 3.5 (PD).** (a) $G^*$ is $\Delta^1_{2n+2}$.

(b) $H_{2n+1} = L[G^*]$.

**Proof.** (a) This is a routine computation.

(b) Since $G'(m, \xi) \Leftrightarrow \exists \eta G^*(m, \xi, \eta)$, clearly $G' \in L[G^*]$. By (a), there is a $k \in \omega$ such that $G^*(m, \xi, \eta) \Leftrightarrow G'(k, (m, \xi, \eta))$, so $G^* \in L[G'] = H_{2n+1}$.

Although $L[G'] = L[G^*]$, their canonical constructibility orderings are different. Let $\leq^2_{n+1}$ denote the canonical constructibility ordering of $L[G^*]$. For $\xi$ an ordinal, $H_{2n+1} \upharpoonright \xi$ denotes the set of all elements of $H_{2n+1}$ whose rank in the well-ordering $\leq^2_{n+1}$ is less than $\xi$.

**Theorem 3.6 (AD; Moschovakis [18]).** Let $\xi, \eta < \delta^1_{2n+1}$.

(a) Membership in $H_{2n+1} \upharpoonright \xi$ is $\Delta^1_{2n+2}(\xi)$.

(b) Truth in $H_{2n+1} \upharpoonright \xi$ is $\Delta^1_{2n+2}(\xi)$.

(c) The ordering $\leq^2_{n+1} \upharpoonright (\text{Pow}(\eta) \cap H_{2n+1} \upharpoonright \xi)$ is $\Delta^1_{2n+2}(\xi, \eta)$.

**Proof.** Both (a) and (b) follow easily from Corollary 3.3 and Lemma 3.5, while (c) is a special case of (a) and (b).
Corollary 3.7 (AD; Moschovakis [18]). (a) Let $\lambda < \delta^{1}_{2n+1}$.\footnote{Footnote, not shown here.} and let $A$ be a $\lambda$-set. Then $A \in H_{2n+1}$ if and only if there is a $\xi < \delta^{1}_{2n+1}$ such that $A$ is $\Sigma^{1}_{2n+2}(\xi)$.

(b) Let $A$ be a $\delta^{1}_{2n+1}$-set. If there is a $\xi < \delta^{1}_{2n+1}$ such that $A$ is $\Sigma^{1}_{2n+2}(\xi)$, then $A \in H_{2n+1}$.

Proof. (a) Suppose $A$ is $\Sigma^{1}_{2n+2}(\xi)$. Then there is a $k \in \omega$ such that for all $\eta < \lambda$, $\eta \in A \iff G'(k, (\eta, \xi))$, so $A \in H_{2n+1}$. Conversely suppose $A \in H_{2n+1}$. Since $A \in L[G^*]$, $A \subset \lambda < \delta^{1}_{2n+1}$. $G^* \subset \omega \times \delta^{1}_{2n+1} \times \delta^{1}_{2n+1}$, and $\delta^{1}_{2n+1}$ is an uncountable regular cardinal, a standard collapsing argument shows that $A \in L_{\nu}[G^* \cap (\omega \times \nu \times \nu)]$ for some $\nu < \delta^{1}_{2n+1}$. So $A$ is in $H_{2n+1} \upharpoonright \theta$ for some $\theta < \delta^{1}_{2n+1}$. Say $A$ is the $\eta$th element of $H_{2n+1}$ with respect to $\leq^{2n+1}_{\eta}$; then $\eta < \delta^{1}_{2n+1}$. So $\nu' \in A \iff \nu' < \lambda \& (H_{2n+1} \upharpoonright \nu) \nu'$ is in the $\eta$th subset of $\lambda$ (w.r.t. $\leq^{2n+1}_{\eta}$).

By Theorem 3.6 this is $\Sigma^{1}_{2n+2}(\lambda, \theta, \eta)$. Letting $\xi < \delta^{1}_{2n+1}$ encode the triple $(\lambda, \theta, \eta)$. $A$ is $\Sigma^{1}_{2n+2}(\xi)$.

(b) The proof is similar to the proof for (a).

It is not true that every $\delta^{1}_{2n+1}$-set in $H_{2n+1}$ is $\Sigma^{1}_{2n+2}(\xi)$ for some $\xi < \delta^{1}_{2n+1}$. For the set of $\delta^{1}_{2n+1}$-sets that are can be well-ordered in $H_{2n+1}$ with order type $\delta^{1}_{2n+1}$, and hence cannot be the power set in $H_{2n+1}$.

Corollary 3.8 (AD). Let $\lambda < \delta^{1}_{2n+1}$ and let $A \subset \lambda$. If $A \in H_{2n+1}$, then there exist $\xi_{0}, \ldots, \xi_{k} \lambda$ less than $\delta^{1}_{2n+1}$ such that $A$ is $\Delta^{1}_{2n+1}(\xi_{0}, \ldots, \xi_{k})$.

Proof. Lemma 2.3 and Corollary 3.7.

Corollary 3.9 (AD; Moschovakis [21]). Let $\lambda < \delta^{1}_{2n+1}$. Then $\text{Pow}(\lambda) \cap H_{2n+1}$ is $\gamma^{1}_{2n+2}$, the largest thin $\Sigma^{1}_{2n+2}(\lambda)$ set of $\lambda$-sets.

Proof. Corollary 3.7(a), the definition of $\gamma^{1}_{2n+2}$, and Theorem 2.5.

As shown in the proof of Corollary 3.7, $A \subset \lambda < \delta^{1}_{2n+1}$ is in $H_{2n+1}$ if and only if there is a $\xi < \delta^{1}_{2n+1}$ such that $A \in H_{2n+1} \upharpoonright \xi$. Thus by Theorem 3.6(a), $\text{Pow}(\lambda) \cap H_{2n+1}$ is $\Sigma^{1}_{2n+2}(\lambda)$. This gives an alternative $\Sigma^{1}_{2n+2}(\lambda)$ definition of the set $\gamma^{1}_{2n+2}$. By Corollary 3.9, every $\Sigma^{1}_{2n+2}(\lambda)$ set of $\lambda$-sets is either contained in $H_{2n+1}$, or else has a perfect subset; but for sets of $\lambda$-sets, unlike sets of reals, it is not true that the perfect subset must have a code in $H_{2n+1}$, in any sense. For example, let $\lambda$ be the least ordinal such that no code for $\lambda$ is in $H_{2n+1}$. Consider the set of reals that code $\lambda$. This set of $\omega$-sets (hence of $\lambda$-sets) is $\Sigma^{1}_{2n+2}(\lambda)$ (in fact $\Delta^{1}_{2n+1}(\lambda)$), and it has a perfect subset; but no tree corresponding to such a perfect set can be in H2n+1, since no branch of it is.

Most of the subsets of $\delta^{1}_{2n+1}$ that are likely to occur in practice are $\Sigma^{1}_{2n+2}(\xi)$ for some $\xi < \delta^{1}_{2n+1}$, hence are in $H_{2n+1}$. For example, the set of ordinals of cofinality
Thin collections of sets

... (in V) is in $H_{2n+1}$. (Not all of these ordinals have cofinality $\omega$ in $H_{2n+1}$. This follows from Theorem 3.11, below.) Also in $H_{2n+1}$ are the set of ordinals of cofinality $\kappa$ (in V), for any $\kappa < \delta^1_{2n+1}$, the uniform indiscernibles (i.e. indiscernibles of $L[\alpha]$ for all $\alpha \in \mathcal{N}$), the ordinals with $\Delta^1_{2n+1}$ codes, the reliable ordinals, etc. Also, the tree associated with any $\Pi^1_{2n+1}$-scale will be in $H_{2n+1}$ (since it is $\Sigma^1_{2n+2}$-in-the-codes). Pow($\lambda$) $\cap H_{2n+1}$ is a very rich set. Theorem 0.6 shows that pseudo-$\Pi^1_{2n+1}$ games on ordinals have $\Delta^1_{2n+2}$ winning strategies; combining this theorem with Theorem 3.7, we see that another important collection of sets consists of elements of $H_{2n+1}$.

**Corollary 3.10** (PD; Moschovakis [21]). Let $\lambda < \delta^1_{2n+1}$ and let $G$ be an ordinal game on tuples of ordinals less than $\lambda$ and integers.

(a) If $G$ is pseudo-$\Pi^1_{2n+1}$ and $I$ has a winning strategy, then some winning strategy for $I$ lies in $H_{2n+1}$.

(b) If $G$ is pseudo-$\Sigma^1_{2n+1}$ and $II$ has a winning strategy, then some winning strategy for $II$ lies in $H_{2n+1}$.

Recall the proof of Theorem 2.5. We showed there that if $A$ is a thin $\Sigma^1_{2n+2}(\lambda)$ set of $\lambda$-sets, there is a pseudo-$\Delta^1_{2n+1}$ game such that if $\tau$ is any winning strategy for $II$ for the game, then for all $A \in A$ there is a position $p$ such that $\tau$ rejects $A$ at $p$, and in addition we proved that $II$ does have a winning strategy. This, plus Corollary 3.10(b), gives another proof that thin $\Sigma^1_{2n+2}(\lambda)$ sets of $\lambda$-sets are contained in $H_{2n+1}$.

The well-ordering $\leq_{2n+1}$ gives a canonical mapping from Pow($\lambda$) $\cap H_{2n+1}$ into ordinals: since $H_{2n+1}$ is rich in sets of ordinals, it must also be rich in sets of $\lambda$-sets. We will give only one example of a set of $\lambda$-sets that can be proved to be in $H_{2n+1}$ by this technique. By [10], the $\omega$-closed unbounded filter on $\delta^1_m$ is an ultrafilter, for all $m$, odd or even.

**Theorem 3.11** (AD; Moschovakis [3]). Let $m < 2n+1$ and let $\mathcal{U}$ be the $\omega$-closed unbounded filter on $\delta^1_m$. Then $\mathcal{U} \upharpoonright H_{2n+1}$ is in $H_{2n+1}$. Hence $H_{2n+1} \upharpoonright (\delta^1_m)^V$ is a measurable cardinal'.

Moschovakis [3] has in fact proved that, in $H_{2n+1}$, there are infinitely many measurable cardinals, with measures of high order. These cardinals are all between $\mathcal{N}^V$ and $(\delta^1_{2n+1})^V$. By Scott’s Theorem, no $\kappa \geq (\delta^1_{2n+1})^V$ is measurable in $H_{2n+1}$, and by [3], no $\kappa < \mathcal{N}^V$ is.

The models $H_{2n+1}$ can be relativized to an arbitrary real $\alpha$ in the following way:

**Definition 3.12.** Let $G_\alpha = \omega \times \mathcal{N}$ be a good universal set in $\Sigma^1_{2n+2}(\alpha)$. Let $G'_\alpha \subset \omega \times \delta^1_{2n+1}$ be the relation

$G'_\alpha(m, \xi) \iff \exists \beta[\beta \in \mathcal{I} \land \varphi(\beta) = \xi \& G_\alpha(m, \beta)]$.

Let $H_{2n+1}[\alpha] = L[G'_\alpha]$. 

It is not known whether $L[G'_\alpha] = L[G', \alpha]$, although the latter is clearly a subclass of the former. It is also not known whether $H_{2n+1}$ is $L[T^{2n+1}]$, where $T^{2n+1}$ is the tree associated with a $\Pi^1_{2n+1}$-scale (which is in $H_{2n+1}$), nor is it even known whether $L[T^{2n+1}]$ is independent of the choice of the scale. Incidentally, $L[T^{2n+1}]$ is another analog of $L$, the smallest model containing the analog of the Shoenfield tree. It can be shown, however, that for any $\alpha$, $L[T^{2n+1}, \alpha]$, $L[G', \alpha]$, and $H_{2n+1}[\alpha]$ all agree on sets of rank less than $\aleph_1^V$.

The results of this section all relativize to $H_{2n+1}[\alpha]$ in a straightforward way. By Theorem 2.8 and the relativized version of Corollary 3.9, for $\lambda < \delta_{2n+1}$, every thin set of $\lambda$-sets is contained in $H_{2n+1}[\alpha]$, for some $\alpha \in \mathcal{N}$. We can actually get a slightly stronger result.

**Theorem 3.13 (AD).** Let $\mathcal{A}$ be a thin set of $\delta_{2n+1}^1$-sets. Then there is an $\alpha \in \mathcal{N}$ such that $\mathcal{A} \in H_{2n+1}[\alpha]$.

**Proof.** By Theorem 1.8, $\text{card}(\mathcal{A}) \leq \delta_{2n+1}^1$. Let $F : \delta_{2n+1}^1 \to \mathcal{A}$ be a surjection. Let $E = \{(\xi, \eta) \in \delta_{2n+1}^1 \times \delta_{2n+1}^1 : \eta \in F(\xi)\}$. It will suffice to prove that $E$ is in some $H_{2n+1}[\alpha]$. By [18, 7D.20] $E$ is $\Pi^1_{2n+1}$-in-the-codes; let $\alpha$ be a real such that $E$ is $\Pi^1_{2n+1}(\alpha)$. Then $E \in H_{2n+1}[\alpha]$.

Martin has proved that for any $m \geq 1$, and any $\alpha \in \mathcal{N}$,

$$H_{2n+1}[\alpha] \models \text{'There exists a measurable cardinal } \kappa \text{ such that } \aleph_m^\mathcal{N} \lessdot \kappa < \aleph_{m+1}^\mathcal{V}.'$$

Therefore,

$$(2^{\aleph_m})^{H_{2n+1}[\alpha]} < \aleph_{m+1}^\mathcal{V},$$

and so in $\mathcal{V}$,

$$\text{card}(\text{Pow}(\aleph_m) \cap H_{2n+1}[\alpha]) = \aleph_m.$$  

So by Theorem 3.13, $\aleph_m$ satisfies the perfect set property. Martin's unpublished proof uses the Kunen measures (see [24]) and therefore probably cannot be generalized to cardinals above $\aleph_\omega$.

We have shown that $H_{2n+1}$ is closely connected to the pointclass $\Sigma^1_{2n+2}$, with respect to sets of $\lambda$-sets. To summarize, $\text{Pow}(\lambda) \cap H_{2n+1}$ is the set of all $\lambda$-sets definable in a $\Sigma^1_{2n+2}$ way from ordinals less than $\delta_{2n+1}^1$, and it is also the largest thin $\Sigma^1_{2n+2}(\lambda)$ set of $\lambda$-sets. In this respect, $H_{2n+1}$ is very $L$-like. But it is unlike $L$ in other respects, for example in the existence of measurable cardinals or in the fact that collapsing arguments do not work. Some more $L$-like properties of $H_{2n+1}$ will be proved in the next two sections.
4. On the generalized continuum hypothesis in $H_{2n+1}$

Suppose there is a $\lambda < \delta^1_{2n+1}$ and a set $\mathcal{A}$ of $\lambda$-sets such that $\text{card}(\mathcal{A}) = \lambda^{++}$; this possibility is not ruled out by anything in Section 1. If it occurs, by Theorem 3.13 $\mathcal{A} \subseteq H_{2n+1}[\alpha]$, for some $\alpha$, and so clearly $H_{2n+1}[\alpha] \models 2^\lambda > \lambda^+$. Hence if this situation occurs a relativized $H_{2n+1}$ violates the GCH, and in fact, using a basis theorem it is easy to show that $H_{2n+1}$ itself violates it, for some $m$. We don't know whether such an $\mathcal{A}$ can exist, but since we are unable to prove it cannot, we clearly cannot prove the GCH. We will, however, prove that the GCH holds for certain $\lambda$'s.

Call an ordinal $\lambda \Pi^1_{2n+1}$-reliable if it satisfies Definition 1.5 for a scale $\{\psi_i\}_{i=\omega}$ that is $\Pi^1_{2n+1}(\xi)$, for some ordinal $\xi < \delta^1_{2n+1}$.

A $\Pi^1_{2n+1}$-scale of length $\delta^1_{2n+1}$ satisfies the above definition, as does that scale cut off at some $\xi < \delta^1_{2n+1}$. Hence there is a set of $\Pi^1_{2n+1}$-reliable $\lambda$'s that is closed unbounded in $\delta^1_{2n+1}$. We will prove that $H_{2n+1} \models 2^\lambda = \lambda^+$, for all $\Pi^1_{2n+1}$-reliable $\lambda$. We, of course, already know that the perfect set property holds for these $\lambda$'s, so the problem mentioned above cannot occur. While the perfect set property for $\lambda$ is related to whether $H_{2n+1} \models 2^\lambda = \lambda^+$, neither fact is known to imply the other.

The proof for $\Pi^1_{2n+1}$-reliable $\lambda$ is the subject of this section. We first define a well-ordering $\leq^\lambda$ of $\text{Pow}(\lambda) \cap H_{2n+1}$, then prove it is 'good', and finally that it has order type $(\lambda^+)^{\text{bd}_{2n+1}}$. We do not know whether $\leq^\lambda$ is the constructibility ordering $\leq^{2n+1}$. In Section 5 we will apply these results to the study of the structure of $\mathcal{P}_{2n+2}$.

First we must relativize $H_{2n+1}$ to an arbitrary element of $\mathcal{P}_{2n+1}(\lambda)$, the family of countable subsets of $\lambda$.

**Definition 4.1.** For $\lambda < \delta^1_{2n+1}$ and $A \in \mathcal{P}_{2n+1}(\lambda)$, let $H_{2n+1}[A] = L[G^*, A]$ (where $G^*$ is as in Definition 3.4). Let $\leq^\lambda_A$ denote the canonical constructibility ordering, and let $\leq^\lambda$ be the restriction of $\leq^\lambda_A$ to $\text{Pow}(A)$.

Thus $\leq^\lambda$ is a well-ordering of $(\text{Pow}(A) \cap H_{2n+1}[A])$. This notation is ambiguous, since a '$2n + 1$' should appear somewhere, but this will cause no problem in practice. As before, for $\xi$ an ordinal, $H_{2n+1}[A] \upharpoonright \xi$ denotes the set of all elements of $H_{2n+1}[A]$ whose rank in the well-ordering $\leq^\lambda_A$ is less than $\xi$.

**Lemma 4.2** (AD). Let $\lambda, \xi < \delta^1_{2n+1}$ and let $A \in \mathcal{P}_{2n+1}(\lambda)$.

(a) Membership in $H_{2n+1}[A] \upharpoonright \xi$ is $\Delta^1_{2n+2}(\xi, A)$, uniformly in $\xi, A$.

(b) Truth in $H_{2n+1}[A] \upharpoonright \xi$ is $\Delta^1_{2n+2}(\xi, A)$, uniformly in $\xi, A$.

(c) The ordering $\leq^\lambda \upharpoonright (H_{2n+1}[A] \upharpoonright \xi)$ is $\Delta^1_{2n+2}(\xi, A)$, uniformly in $\xi, A$.

**Proof.** Parts (a) and (b) follow easily from Theorem 3.2. The coding is different, since in Theorem 3.2 $A$ is coded as a set of ordinals (i.e. via a $\Delta^1_{2n+1}$-code),
whereas here $A$ is coded as an element of $\mathcal{P}_\kappa(A)$. But going from one method of coding to the other is $\Delta^1_{n+2}$. Part (c) is a special case of (a) and (b).

Lemma 4.2 is clearly just the relativized version of Theorem 3.6. The only reason for introducing these models is to get the canonical well-ordering $\leq^A$, which will be used to construct a well-ordering $\leq^A$, of $\text{Pow}(\lambda) \cap H_{2n+1}$. In order to define $\leq^A$, we use an ultrafilter $\mathcal{U}^A$ on $\mathcal{P}_\kappa(\lambda)$ which is defined and studied in [1]; we list below without proof the facts from [1] which we will need. 'Almost every' will always mean with respect to $\mathcal{U}^A$.

(\(\mathcal{U}1\)) The ultrafilter $\mathcal{U}^A$ is countably complete, fine, and normal. Fine means that for all $\eta < \lambda$, \(\{S \in \mathcal{P}_\kappa(\lambda) : \eta \in S\} \in \mathcal{U}^A\). Normal means that whenever $F : \mathcal{P}_\kappa(\lambda) \to \lambda$ is a function such that $\{S : F(S) \in S\} \in \mathcal{U}^A$, then there is an $\eta < \lambda$ such that $\{S : F(S) = \eta\} \in \mathcal{U}^A$, i.e. a choice function on $\mathcal{P}_\kappa(\lambda)$ is constant a.e. ($\kappa$ is $\lambda$-supercompact means, by definition, that such an ultrafilter exists.)

(\(\mathcal{U}2\)) Let $A \subset \mathcal{P}_\kappa(\lambda)$. The collection $A$ is unbounded if for any set $B$ in $\mathcal{P}_\kappa(\lambda)$ there is a $C$ in $A$, such that $B \subset C$. We call $A$ strongly-closed if for every $B \subset A$, if $\bigcup B$ is countable and for all finite $F \in \lambda$, whenever $F \subset (\bigcup B)$ there exists an $A \in B$ such that $F \subset A$, then $(\bigcup B) \in A$. Every strongly-closed unbounded set is in $\mathcal{U}^A$.

(\(\mathcal{U}3\)) Let $\nu < \lambda^+$ and let $\leq$ be a well-ordering of a subset of $\lambda$ of order type $\nu$. Let $f_\nu : \mathcal{P}_\kappa(\lambda) \to \kappa_1$ be the function $f_\nu(A) =$ order type of $\leq \restriction A$ (i.e. $\leq \cap (A \times A)$). Then $[f_\nu]$ is the $\nu$th element of the ultrapower $H_{\kappa_1}/\mathcal{U}^A$.

(\(\mathcal{U}4\)) The pointclasses $\Sigma^1_{2n+2}(\lambda)$, $\Pi^1_{2n+2}(\lambda)$, and $\Delta^1_{2n+2}(\lambda)$ are closed under quantification of the form 'for a.e. $A \in \mathcal{P}_\kappa(\lambda)'.'

**Definition 4.3.** For $\lambda < \delta^1_{2n+1}$, let $B$ and $C$ in $\text{Pow}(\lambda) \cap H_{2n+1}$. define $B \leq^A C$ if and only if

\[ \forall A \in \mathcal{P}_\kappa(\lambda), \quad B \cap A \leq^A C \cap A. \]

**Proposition 4.4 (AD).** Let $\lambda < \delta^1_{2n+1}$.

(a) The relation $\leq^A$ is a well-ordering of $\text{Pow}(\lambda) \cap H_{2n+1}$.

(b) There are $\Sigma^1_{2n+2}(\lambda)$ and $\Pi^1_{2n+2}(\lambda)$ relations $\leq_\Sigma$ and $\leq_\Pi$ such that for all $B \subset \lambda$, if $B \in H_{2n+1}$, then

\[ (\forall B' \subset \lambda)[(B' \in H_{2n+1} \& B' \leq^A B) \iff (B' \leq_\Sigma B) \iff (B' \leq_\Pi B)]. \]

(c) The well-ordering $\leq^A$ is in $H_{2n+1}$.

**Proof.** (a) This follows easily from the fact that $\mathcal{U}^A$ is fine and countably additive.

(b) This follows from Lemma 4.2(c) and (\(\mathcal{U}4\)).

(c) By (b), $\leq^A$ is $\Sigma^1_{2n+2}(\lambda)$, hence in $H_{2n+1}$.
Let \( F: \mathcal{P}_{\kappa_1}(\lambda) \rightarrow \mathcal{P}_{\kappa_1}(\lambda) \) be any map such that for a.e. \( A \), \( F(A) \subseteq A \). Let \( I_F \) denote \( \{ \xi < \lambda : \text{for a.e. } A, \xi \in F(A) \} \).

**Proposition 4.5 (AD).** Let \( F: \mathcal{P}_{\kappa_1}(\lambda) \rightarrow \mathcal{P}_{\kappa_1}(\lambda) \) be any map such that for a.e. \( A \), \( F(A) \subseteq A \). Then for a.e. \( A \), \( F(A) = I_F \cap A \).

**Proof.** Normality.

Proposition 4.5 is a special case of the fact that if \( \kappa \) is \( \lambda \)-supercompact, then \( \kappa \) is \( \lambda \)-ineffable'. The point of the next theorem is that the ineffability property of Proposition 4.5 respects the models \( H_{2n+1} \) and \( H_{2n+1}[A] \).

**Theorem 4.6 (AD).** Let \( \lambda < \delta_{2n+1} \), \( \lambda \) reliable, and let \( F: \mathcal{P}_{\kappa_1}(\lambda) \rightarrow \mathcal{P}_{\kappa_1}(\lambda) \) be a function such that for a.e. \( A \), \( F(A) \subseteq A \).

(a) If for a.e. \( A \), \( F(A) \in H_{2n+1}[A] \), then \( I_F \in H_{2n+1} \).

(b) In fact, if for a.e. \( A \), \( F(A) \) is the \( \nu_\lambda \)th subset of \( A \), with respect to \( \leq^\lambda \), and \( [A \mapsto \nu_\lambda] \) is the \( \nu \)th element of the ultrapower \( \mathcal{P}_{\kappa_1}/\mathcal{U}_{\lambda} \), then \( I_F \) is the \( \nu \)th element of \( \text{Pow}(\lambda) \cap H_{2n+1} \) with respect to \( \leq^\lambda \).

**Proof.** Fix \( F \) such that for a.e. \( A \), \( F(A) \) is the \( \nu_\lambda \)th subset of \( A \) and \( [A \mapsto \nu_\lambda] \) is the \( \nu \)th element of the ultrapower. We show that \( I_F \) is in \( H_{2n+1} \). This trivially implicates (a), and by definition of the ordering \( \leq^\lambda \) it implies (b). Since \( \lambda \) is reliable, by Theorem 1.8 there are at most \( \text{card}(\lambda) \) subsets of \( \lambda \) in \( H_{2n+1} \). So if the above claim is false, the least \( \nu \) for which it fails is less than \( \lambda^+ \). So without loss of generality, assume \( \nu < \lambda^+ \).

Let \( g \) be the map \( A \mapsto \nu_\lambda \). Since each \( A \) is countable (in \( V \)), any well-orderable collection of subsets of \( A \) is countable. Hence each \( \nu_\lambda \) is less than \( \kappa_1 \). Since \( \nu < \lambda^+ \), by (\#3), for any well ordering \( \leq \) of a subset of \( \lambda \) of order type \( \nu \), for a.e. \( A \), \( g(A) = (\text{order type of } \leq \uparrow A) \). Therefore

\[
\xi \in I_F \iff
\]

\[
\iff \text{for a.e. } A, \xi \text{ is in the } \nu_\lambda \text{th subset of } A \text{ in } H_{2n+1}[A]
\]

\[
\iff \xi < \lambda \land \exists \gamma [ \gamma \text{ codes a } \Delta^1_{2n+1} \text{ binary relation } D, \&
\]

\[
D_\gamma \text{ is a well-ordering of a subset of } \lambda \text{ of order type } \nu \land
\]

\[
(\text{for a.e. } A \in \mathcal{P}_{\kappa_1}(\lambda))(\exists B_A \subset A)(\exists \nu_A < \kappa_1)(\nu_A \text{ is the order type of } D_\nu \uparrow A \& B_A, \text{ the } \nu_\lambda \text{th subset of } A \text{ in } H_{2n+1}[A] \text{ w.r.t. } \leq^\lambda \& \xi \in B_A]
\]

By Lemma 4.2 and (\#4), the above definition of \( I_F \) is \( \Sigma^1_{n+3}(\lambda, \nu) \), where \( \lambda, \nu < \delta^1_{2n+1} \). We are using the fact that the two order types, \( \nu_\lambda \) and the type of \( D_\nu \uparrow A \), can be compared in a \( \Delta^1_{n+2} \) way; this is a consequence of Theorem 0.1. Hence \( I_F \in H_{2n+1} \).
There is an alternative way of looking at Theorem 4.6. Let $M^\lambda$ be the transitive collapse of the ultraproduct

$$\left( \prod_{A \in \mathcal{P}^\lambda} H_{2n+1}[A] \right)/\mathcal{U}^\lambda.$$ 

It then follows from Theorem 4.6 that $\text{Pow}(\lambda) \cap H_{2n+1}$ equals $\text{Pow}(\lambda) \cap M^\lambda$.

We remark in passing that Theorem 4.6 holds for $L[T^{2n+1}, A]$ and $L[T^{2n+1}]$ if and only if $L[T^{2n+1}] = H_{2n+1}$.

**Definition 4.7.** A $\Delta^1_{2n+2}(\lambda)$ well-ordering (in the sense of Proposition 4.4(b)), $\leq$, of a set $\mathcal{A} \subseteq \text{Pow}(\lambda)$, is $\Delta^1_{2n+2}(\lambda)$-good if for any $\Sigma^1_{2n+2}(\lambda)$ relation $\mathcal{R}(X, Y)$ on $\mathcal{A}$, the set $\{ Y \in \mathcal{A} : (\forall B < Y) \mathcal{R}(B, Y) \}$ is $\Sigma^1_{2n+2}(\lambda)$.

**Theorem 4.8 (AD).** For all $\lambda < \delta_{2n+1}$, if $\lambda$ is reliable, then $\leq^A$ is $\Delta^1_{2n+2}(\lambda)$-good.

That $\leq^A$ is good is not needed to prove that $H_{2n+1} \vDash 2^\lambda = \lambda^+$. It is needed for all the pointclass computations involving $\leq^A$, such as those in the next section. Constructibility orderings are good; the point of Theorem 4.8 is that $\leq^A$ behaves like a constructibility ordering. The ordering $\leq^A$ is $\Delta^1_{2n+2}(A)$-good, i.e. $\Sigma^1_{2n+2}(A)$ is closed under $\leq^A$-bounded quantification. The idea of the proof is to use the methods of Theorem 4.6 to lift this property from a.e. $\leq^A$ to $\leq^A$.

**Proof of Theorem 4.8.** Fix $\mathcal{R} \subseteq \Sigma^1_{2n+2}(\lambda)$, and let $C \in \text{Pow}(\lambda) \cap H_{2n+1}$. Using Theorem 4.6 and (U3), $(\forall B < C) \mathcal{R}(B, C)$ is equivalent to the following formula, which is $\Sigma^1_{2n+2}(\lambda)$ by Lemma 4.2, (U4), and Theorem 0.1.

$$\exists \nu \exists \gamma \exists P[\gamma \text{ codes a } \Delta^1_{2n+1} \text{ binary relation } D_\gamma \text{ & }$$

$$D_\gamma \text{ is a well-ordering of a subset of } \lambda \text{ of order type } \nu \text{ & }$$

(for a.e. $A \in \mathcal{P}_N(\lambda)$)$\exists P_A \prec N_1(\nu_A \text{ is the order type of } D_\gamma \upharpoonright A \text{ & }$

$(C \cap A)$ is the $\nu_A$th subset of $A \text{ in } H_{2n+1}[A] \text{ w.r.t. } \leq^A) \text{ & }$

$(\forall \xi < \nu)(\exists B < \lambda)(\exists \delta)(\exists \mathcal{R}[\delta \text{ codes a } \Delta^1_{2n+1} \text{ binary relation } D_\delta \text{ & } D_\delta \text{ is a well-ordering of a subset of } \lambda \text{ of order type } \xi \text{ & (for a.e. }$

$A \in \mathcal{P}_N(\lambda))(\exists P_A \prec N_1(\nu_A \text{ is the order type of } D_\delta \upharpoonright A \text{ & (B \cap A) is the } \xi_A \text{th subset of } A \text{ in } H_{2n+1}[A] \text{ w.r.t. } \leq^A) \text{ & } \mathcal{R}(B, C))].$

**Theorem 4.9 (AD).** Let $\lambda < \delta^1_{2n+1}$. If $\lambda$ is $\Pi^1_{2n+1}$-reliable, then $H_{2n+1} \vDash \leq^A$ has order type $\lambda^+$.

**Proof.** Let $\phi = \{ \varphi_i \}_{i \in \omega}$ be the scale that witnesses that $\lambda$ is $\Pi^1_{2n+1}$-reliable, and let $T$ be the tree on $\omega \times \lambda$ associated with $\phi$. As in Lemma 1.6, to simplify the proof, we assume that the first norm of the scale has length $\lambda$, and we code ordinals via this norm.
Fix $D_0 \subseteq \lambda$, $D_0 \in H_{2n+1}$, and let $\mathcal{D}_0 = \{ C \subseteq \lambda : C \in H_{2n+1} \& C <^\lambda D_0 \}$. To prove the theorem it will suffice to show that in $H_{2n+1}$, $\operatorname{card}(\mathcal{D}_0) \leq \operatorname{card}(\lambda)$.

Let

$$\mathcal{F}(\alpha, \beta, \delta) \iff$$

$$[\alpha, \beta, \delta, \text{ and all } (\beta)_i \text{ code countable subsets } A, D, \text{ and } B_i \text{ of }$$

$$\delta_{2n+1}^1 \& D \subseteq A \& \forall i (B_i \subseteq A) \& D \in H_{2n+1}[A] \& \forall i (B_i \in H_{2n+1}[A] \&$$

$$B_i <^\lambda D) \& (\forall B <^\lambda D) \exists (B = B_i)].$$

By Lemma 4.2, $\mathcal{F}$ is $\Sigma^1_{2n+2}$. Let $\mathcal{R}(\alpha, \beta, \delta, \varepsilon)$ be a $\Pi^1_{2n+1}$ relation such that $\mathcal{F}(\alpha, \beta, \delta) \iff \exists \varepsilon' \mathcal{R}(\alpha, \beta, \delta, \varepsilon')$. Let

$$\mathcal{P}(\alpha, \beta, \delta, \varepsilon) \iff$$

$$[(\varepsilon)_0 \text{ and } (\varepsilon)_1 \text{ code the same countable sets of ordinals as do }$$

$$\alpha \text{ and } \delta, \text{ respectively } \& \mathcal{R}((\varepsilon)_0, \beta, (\varepsilon)_1, (\varepsilon)_2)].$$

Then $\mathcal{P}$ is $\Pi^1_{2n+1}$, $\mathcal{F}(\alpha, \beta, \delta) \iff \exists \varepsilon \mathcal{P}(\alpha, \beta, \delta, \varepsilon)$, and moreover $\mathcal{P}$ is uniform on codes for the same countable set of ordinals, that is, if $\alpha$ and $\alpha'$ code the same countable set of ordinals, and $\delta$ and $\delta'$ also do, and $\mathcal{P}(\alpha, \beta, \delta, \varepsilon)$, then $\mathcal{P}(\alpha', \beta, \delta', \varepsilon)$.

Consider the following game on finite sequences from $\lambda$ and $\omega$:

I

$$(\xi_0, \nu_0, \eta_0, \xi_2, \nu_2, \eta_1, \ldots) \quad \quad (\alpha_0)(\delta_0)(\gamma_0) \quad \quad (\alpha_2)(\delta_2)(\gamma_1)$$

II

$$(\xi_1, \nu_1, b_0, e_0, t_0, \xi_3, \nu_3, b_1, e_1, t_1, \ldots) \quad \quad (\alpha_1)(\delta_1) \quad \quad (\alpha_3)(\delta_3)$$

The above diagram is to be interpreted in the same way as the diagrams used in the proofs of Lemma 1.6 and of Theorem 2.5. That is, I plays alleged codes $\alpha_{2i}$, $\delta_{2i}$, and $\gamma_i$ for $\xi_{2i}$, $\nu_{2i}$, and $\eta_i$ and II plays alleged codes $\alpha_{2i+1}$ and $\delta_{2i+1}$ for $\xi_{2i+1}$ and $\nu_{2i+1}$. Each plays witnesses with respect to $T$ for his own codes, and each player gets to challenge all of the other player's codes, as in Lemma 1.6. Player II also plays $b_i, e_i \in \omega$ and $t_i \in \mathbb{E}$.

Let $\alpha$ and $\delta$ be the reals such that for all $i$, $(\alpha)_i = \alpha_i$ and $(\delta)_i = \delta_i$. Let $\beta = (b_0, b_1, \ldots)$ and let $e = (e_0, e_1, \ldots)$. For all $k \in \omega$, let $\beta^k = (\beta)_k$.

(Let $A = \{ |\alpha_0|, |\alpha_1|, \ldots \}$ and $D = \{ |\delta_0|, |\delta_1|, \ldots \}$. The basic idea here is that this is a perfect set game for $\hat{\mathcal{P}} = \{ \hat{C} \subseteq A : \hat{C} \in H_{2n+1}[A] \& \hat{C} <^\lambda \hat{D} \}$, the initial segment of $\leq^A$ below $\hat{D}$; I wins if and only if there is a $\hat{C}$ in $\hat{\mathcal{P}}$ such that $\forall i (t_i = 1 \iff |\gamma_i| \in \hat{C})$. Think of $\beta$ as encoding $\hat{\mathcal{P}}$. Of course $A$ and $\hat{D}$ are not given in advance, but are determined by the outcome of the game.) Formally, the payoff is defined as follows:

(i) If either player fails to play into $T$, the first player to fail loses.

(ii) All ordinals $\nu_i$ must be in $D_0$ (but the witnesses for them need not be). If neither player loses because of (i), the first player to play a $\nu_i \notin D_0$ loses.

(iii) If neither player loses by (i) or (ii), and one player has an alleged code
proved dishonest by the other player (as in the proof of Lemma 1.6), the first player to fail in this manner loses.

(iv) Assuming neither player loses due to any of the conditions (i)-(iii), above, II wins if and only if

\[ [\mathcal{P}(\alpha, \beta, \delta, \varepsilon) \& \neg \exists k \forall i (t_i = 1 \leftrightarrow \exists m (|y_i| = |(\beta^k)_m|))] \]

Both \( T \) and \( D_0 \) are in \( H_{2n+1} \), so by Corollary 3.8 are definable in a \( \Delta^1_{2n+1} \) way from ordinals less than \( \delta^1_{2n+1} \). And \( \mathcal{P} \) is \( H^1_{2n+1} \). So the game is pseudo-\( \Sigma^1_{2n+1} \). Hence by Theorem 0.5 it is determined, and by Corollary 3.10(b), if II has a winning strategy then some winning strategy for II lies in \( H_{2n+1} \).

**Claim 1.** I does not have a winning strategy.

**Proof of Claim 1.** Suppose I has a winning strategy \( \sigma \). Let \( A \) be a countable subset of \( \lambda \) such that \( A \) is closed under \( \sigma \) and for all \( \nu \in A \) there is an honest code for \( \nu \) that can be witnessed with ordinals in \( A \) (as in the proof of Claim 1 of Lemma 1.6). Consider runs in which I plays according to \( \sigma \) and II plays as described below. For \( \xi_1, \xi_2, \ldots \), II enumerates the countable set \( A \), and for \( \nu_1, \nu_3, \ldots \), he enumerates \( D = D_0 \cap A \). Player II plays honest codes for all his own ordinals and for his challenges to I's ordinals, with all witnesses in \( A \). Let \( \mathcal{J} = \{ \tilde{C} \subseteq A : \tilde{C} \in H_{2n+1}[A] \& \tilde{C} <^A \tilde{\mathcal{D}} \} \). Since \( A \) is countable and \( \mathcal{J} \subseteq \text{Pow}(A) \) is well-orderable, \( \mathcal{J} \) is countable. Player II plays \( \beta \) that enumerates \( \mathcal{J} \). He plays \( \varepsilon \) such that \( \mathcal{P}(\alpha, \beta, \delta, \varepsilon) \); this \( \varepsilon \) depends only on \( A \) and \( D \), not on their codes. since \( \mathcal{J} \) is uniform.

In any run of the game as above, \( \alpha \) will code this set \( A \) and \( \delta \) will code this set \( \mathcal{D} \). Hence \( \mathcal{P}(\alpha, \beta, \delta, \varepsilon) \). Since \( \sigma \) is a winning strategy, by definition of the payoff set, for any run of the game as above (regardless of how II plays the \( t_i \)'s), it will be the case that there is a \( \tilde{C} \in \mathcal{J} \) such that \( \forall i (t_i = 1 \leftrightarrow \eta_i \in \tilde{C}) \). This implies that \( \mathcal{J} \) is uncountable (as in Lemma 1.6). This contradiction proves the claim.

So II has a winning strategy \( \tau \) which lies in \( H_{2n+1} \). Let

\[ \mathcal{B} = \{ A \in \mathcal{P}_{\mathcal{J}}(\lambda) : A \text{ is closed under } \tau \& \text{ for all } \nu \in A, \text{ there is an honest code for } \nu \text{ that can be witnessed with ordinals in } A \} \]

The set \( \mathcal{B} \) is strongly-closed unbounded. That it is unbounded follows easily from the reliability of \( \lambda \). To see that it is strongly-closed, note that the property that for all \( \nu \in A \) there is an honest code for \( \nu \) that can be witnessed with ordinals in \( A \), is preserved under arbitrary unions. Being closed under \( \tau \) is not preserved under arbitrary unions, but it is preserved under the type of union in the definition of 'strongly-closed', since any position \( p \) in the game contains only finitely many ordinals. So by (U2), \( \mathcal{B} \in \mathcal{U}^\lambda \).

Let \( A \in \mathcal{P}_{\mathcal{J}}(\lambda) \), let \( p \) be a position in the game in which it is I's turn to play an \( \eta_i \), and let \( \tilde{C} \subseteq A \). Define \( \tau \) rejects \( \tilde{C} \) at \( p \) relative to \( A \) if

1. all ordinals occurring in \( p \) are in \( A \),
(2) \( p \) is consistent with \( \tau \),
(3) \( (\forall j < \gamma)(\eta_j \in \hat{C} \leftrightarrow t_i = 1) \),
(4) \( (\forall \nu \in A)(\text{if } I \text{ plays } \eta_i = \nu \text{ at position } p, \tau \text{ calls for } II \text{ to play } \nu \text{ such that} \)
\( (\nu \in \hat{C} \leftrightarrow t_i = 0) \).

Fix \( A \in \mathcal{B} \) and let \( \hat{D} = D_0 \cap A \). Let \( \hat{\mathcal{D}} = \{ \hat{C} \subseteq A : \hat{C} \in H_{2n+1}[A] \& \hat{C} \subseteq \hat{D} \} \).

**Claim 2.** For all \( \hat{C} \in \hat{\mathcal{D}} \), there exists a \( p \) such that \( \tau \) rejects \( \hat{C} \) at \( p \) relative to \( A \).

**Proof of Claim 2.** Consider runs of the game in which \( II \) plays according to \( \tau \) and \( I \) plays as described below. All ordinals played by \( I \) are in \( A \). Player \( I \) always plays honest codes for the \( \eta_i \)'s. He plays so that in the outcome, \( \alpha \) codes \( A \) and \( \delta \) codes \( \hat{D} \) (as \( II \) played against \( \sigma \) in the proof of Claim 1, above). Since \( \tau \) is a winning strategy, \( \beta \) codes \( \hat{\mathcal{D}} \) and so some \( \beta^k \) codes \( \hat{C} \). And since \( II \) wins, it is not the case that \( \langle t_i = 1 \leftrightarrow \eta_i \in \hat{C} \rangle \). So if the claim was false, \( I \) could win playing against \( \tau \), as in Lemma 1.6, by always playing for \( \eta_i \) the least ordinal in \( A \) for which \( \tau \) 'gives the right answer', and choosing an honest code for it with witnesses in \( A \), which he subsequently will play. This proves the claim.

Now fix \( C \in \mathcal{C}_0 \). By definition of \( \leq^\lambda \) (Definition 4.3) and \( \mathcal{C}_0 \), for a.e. \( A \), \( C \cap A \leq^\lambda D_0 \cap A \). Let \( \mathcal{B}_C = \{ A \in \mathcal{B} : C \cap A \leq^\lambda D_0 \cap A \} \). Then \( \mathcal{B}_C \subseteq \mathcal{U}^A \). By Claim 2, for each \( A \in \mathcal{B}_C \), there exists a \( p^A \) such that \( \tau \) rejects \( (C \cap A) \) at \( p^A \) relative to \( A \). Recall that \( p^A \) is a finite sequence of ordinals from \( A \) and integers. So by normality and countable additivity, there is a fixed position \( p \) such that \( p = p^A \) for a.e. \( A \). Say it is \( I \)'s turn to play \( t \) at \( p \).

So for a.e. \( A \), \( \tau \) rejects \( C \cap A \) at \( p \) relative to \( A \). Since \( \mathcal{U}^A \) is fine, this means that for every \( \nu < \lambda \), if \( I \) plays \( \eta_i = \nu \) at position \( p \), \( \tau \) calls for \( II \) to play \( t \) such that \( (\nu \in C \leftrightarrow t_i = 0) \). Thus we have shown that for every \( C \in \mathcal{C}_0 \) there is a position \( p \) in the game such that \( \tau \) rejects \( C \) at \( p \) (as defined in the proof of Lemma 1.6); let \( p_C \) be the least such \( p \). Since \( \tau \in H_{2n+1} \), the map \( C \mapsto p_C \) is in \( H_{2n+1} \). And this map is clearly one-to-one. Since the \( p_C \)'s are finite sequences from \( \lambda \) and \( \omega \) and \( \lambda \) is infinite, \( H_{2n+1} \vdash \text{card}(\mathcal{C}_0) \leq \text{card}(\lambda) \).

It follows from Theorem 4.6, or Los' Theorem and the remark following Theorem 4.6, that for \( \lambda < \delta^1_{2n+1} \) (reliable or not), if for a.e. \( A \in \mathcal{P}_n(\lambda) \),

\[ H_{2n+1}[A] \models \text{card}(\text{Pow}(A)) = (\text{card}(A))^+ , \]

then \( H_{2n+1} \models 2^\lambda = \lambda^+ \). This method of trying to prove the GCH in \( H_{2n+1} \) seems promising, since we need only understand sets of countable ordinals in \( H_{2n+1}[A] \) (or sets of reals in a forcing extension of \( H_{2n+1}[A] \)) to prove it. However, at the present time, \( H_{2n+1}[A] \) is not understood even that well.

That \( H_{2n+1} \models 2^\omega = \omega^+ \) follows from the fact that \( \mathcal{N} \cap H_{2n+1} = \mathcal{C}_{2n+2} \) (Corollary 3.9), together with the structure theory of \( \mathcal{C}_{2n+2} \) in [12]. For \( \lambda \leq \aleph_1 \), Theorem 4.9 is due to Moschovakis. For \( \lambda < \aleph_1 \), he proved it by forcing and using the above method relativized to \( \mathcal{C}_{2n+2}(\gamma) \), where \( \gamma \) is the generic code for \( \lambda \). For \( \aleph_1 \), it follows from the fact that \( \aleph_1^\gamma \) is measurable in \( H_{2n+1} \) (Theorem 3.11). \( \aleph_1 \) a well-known
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Theorem on relative constructibility,

\[ H_{2n+1} \models (\forall \lambda \ni (\delta_{2n+1}^\lambda)^0)(2^\lambda = \lambda^+). \]

The measure \( \mathcal{U}^\lambda \) is not only related to the question of whether \( H_{2n+1} \models 2^\lambda = \lambda^+ \), but also to the perfect set property for \( \lambda \). Call \( \mathcal{U}^\lambda \) exact if every element of the ultrapower \( \Pi \mathcal{N}_1/\mathcal{U}^\lambda \) is an \([f_\prec]\), where \( f_\prec \) is as in (R 3). Clearly by (R 3), \( \mathcal{U}^\lambda \) is exact if and only if the length of the ultrapower \( \Pi \mathcal{N}_1/\mathcal{U}^\lambda \) is exactly \( \lambda^+ \). If \( \mathcal{U}^\lambda \) is exact, then \( \lambda \) has the perfect set property; the proof is similar to that of Theorem 1.12. In [1], the author proved that for semi-reliable \( \lambda \), \( \mathcal{U}^\lambda \) is exact. But this gives no new information about the perfect set property. It is an open question whether or not all \( \mathcal{U}^\lambda \) are exact.

It is not clear whether or not \( \kappa_{2n+1} \) is \( \Pi_{2n+1}^1 \)-reliable (if \( n > 1 \)). However, even if it is not, the GCH still holds for \( \kappa_{2n+1} \).

**Corollary 4.10.** (AD). There is a \( \Delta^1_{2n+2} \)-good well-ordering \( \leq^* \) of \( \text{Pow}(\kappa_{2n+1}^\mathcal{V}) \cap H_{2n+1} \) such that \( H_{2n+1} \models \leq^* \) has order type \( (\kappa_{2n+1}^\mathcal{V})^{*b} \).

**Proof.** There is a \( \Pi_{2n+1}^1 \)-reliable ordinal \( \lambda \) such that \( \text{card}(\lambda) = \kappa_{2n+1} \). Using the basis theorem, \( \lambda \) can be chosen to have a \( \Delta_{2n+2}^1 \)-code. By the coding lemma there is a bijection \( F : \kappa_{2n+1} \to \lambda \) which is \( \Delta_{2n+1}^1 \)-in-the-codes, and by another application of the basis theorem \( F \) can be chosen with a \( \Delta_{2n+1}^1 \)-code that is a \( \Delta_{2n+2}^1 \) real. Hence \( F \) is \( \Delta_{2n+2}^1 \) and so \( F \in H_{2n+1} \).

By Theorems 4.8 and 4.9, \( \leq^\mathcal{U} \) is a \( \Delta_{2n+2}^1 \)-good well-ordering such that \( H_{2n+1} \models \leq^\mathcal{U} \) has order type \( \lambda^{*b} \). Let \( \leq^\lambda \) be the well-ordering of \( \text{Pow}(\kappa_{2n+1}^\mathcal{V}) \cap H_{2n+1} \) obtained from \( \leq^\mathcal{U} \) by \( F \); that is,

\[ \Lambda \equiv^* B \iff \{ F(\xi) : \xi \in A \} \equiv^\mathcal{U} \{ F(\xi) : \xi \in B \}. \]

5. Structure of the largest thin \( \Sigma^1_{2n+2}(\lambda) \) set

In this section we study the internal structure of the largest thin \( \Sigma^1_{2n+2}(\lambda) \) set of \( \lambda \)-sets, \( \mathcal{C}_{2n+2}^\lambda \). For the case \( \lambda = \omega \), Kechris [12] has proved that \( \mathcal{C}_{2n+2}^\lambda \) is a set of \( \Delta_{2n+2}^1 \)-degrees, well-ordered by \( \Delta_{2n+2}^1 \)-in, that the \( \Delta_{2n+2}^1 \)-jump of a degree is its successor in this well-ordering, and that the canonical well-ordering of \( \mathcal{C}_{2n+2}^\lambda \) is a refinement of the ordering of degrees. We will generalize these results from \( \mathcal{C}_{2n+2}^\lambda \) to \( \mathcal{C}_{2n+2}^\lambda \). Even for \( \lambda = \omega \), some of the proofs (e.g. Lemma 5.5) are new, as we derive these theorems from the fact that \( \mathcal{C}_{2n+2}^\lambda = \mathcal{N} \cap H_{2n+1} \), whereas Kechris derived them directly from the fact that \( \mathcal{C}_{2n+2}^\lambda \) is the largest countable \( \Sigma^1_{2n+2} \) set of reals, without any use of models.

Call an ordinal \( \lambda \) nice if for all \( \xi_0, \ldots, \xi_k < \lambda \), the ordinal \( \langle \xi_0, \ldots, \xi_k \rangle \) that encodes the sequence is less than \( \lambda \). We code sequences by the Gödel ordering (see [18, 8G.24]). This sequence coding is \( \Delta_{2n+2}^1 \). In order to keep the technical...
details simple, we will study the structure of $\mathcal{C}_{2n+2}$ only for nice $\lambda$; but with a little more work, everything would go through for arbitrary $\lambda$. Every cardinal of $L$, hence of $H_{2n+1}$, is nice.

**Definition 5.1.** Let $\lambda < \delta_{2n+1}^1$ be nice. For $A, B \subset \lambda$, let $A \leq_{2n+2}^\lambda B$ if $A$ is $\Delta_{2n+2}^1(B, \lambda, \xi_0, \ldots, \xi_k)$ for some $\xi_0, \ldots, \xi_k < \lambda$. (This means $\Delta_{2n+2}^1$ uniformly in all codes for $B, \lambda, \xi_0, \ldots, \xi_k$.) Define

$$A \leq_{2n+2}^\lambda B \iff A \leq_{2n+2}^\lambda B \& \neg(B \leq_{2n+2}^\lambda A),$$

$$A =_{2n+2}^\lambda B \iff A \leq_{2n+2}^\lambda B \& B \leq_{2n+2}^\lambda A.$$ 

The $(\lambda, 2n+2)$-degree of a $\lambda$-set $A$, denoted $[A]_{2n+2}^\lambda$, is the set $\{B \subset \lambda : B \equiv_{2n+2}^\lambda A\}$.

For $\lambda = \omega$, this notion is just that of $\Delta_{2n+2}^1$-reducibility of reals and $\Delta_{2n+2}^1$-degrees.

**Proposition 5.2.** Let $\lambda < \delta_{2n+1}^1$ be nice.

(a) The relation $\leq_{2n+2}^\lambda$ is reflexive and transitive.

(b) The relation $=_{2n+2}^\lambda$ is an equivalence relation. The $(\lambda, 2n+2)$-degrees are the equivalence classes.

(c) Each degree has cardinality $\text{card}(\lambda)$.

(d) For all $\lambda$-sets $A$, there are at most $\text{card}(\lambda)$ degrees below $[A]_{2n+2}^\lambda$.

(e) If $\{A_\xi : \xi < \lambda\}$ is a set of $\lambda$-sets, and $A = \{\langle \xi, \eta \rangle : \eta \in A_\xi\}$, then for all $\xi < \lambda$, $A_\xi \equiv_{2n+2}^\lambda A$.

**Definition 5.3.** Let $[A]_{2n+2}^\lambda \leq_{2n+2}^\lambda [B]_{2n+2}^\lambda$ if $A \leq_{2n+2}^\lambda B$ and let $[A]_{2n+2}^\lambda <_{2n+2}^\lambda [B]_{2n+2}^\lambda$ if $A <_{2n+2}^\lambda B$.

This is clearly well-defined.

**Lemma 5.4** (Kechris [12] for $\lambda = \omega$). Let $\lambda < \delta_{2n+1}^1$ be nice. The set $\mathcal{C}_{2n+2}^\lambda$ is closed under $\leq_{2n+2}^\lambda$, that is

$$A \in \mathcal{C}_{2n+2}^\lambda & B \leq_{2n+2}^\lambda A \Rightarrow B \in \mathcal{C}_{2n+2}^\lambda.$$ 

**Proof.** This follows trivially from Lemma 2.3 and the definition of $\mathcal{C}_{2n+2}^\lambda$. If $A$ is definable from ordinals in a $\Sigma_{2n+2}^1$ way, and $B \leq_{2n+2}^\lambda A$, then so is $B$.

So $\mathcal{C}_{2n+2}^\lambda$ is a set of $(\lambda, 2n+2)$-degrees. We next consider how the degrees in $\mathcal{C}_{2n+2}^\lambda$ are ordered by their natural partial ordering, $\leq_{2n+2}^\lambda$.

**Lemma 5.5** (AD). Let $\lambda < \delta_{2n+1}^1$ be nice and $\Pi_{2n+1}^1$-reliable.

(a) If $A, B \in \mathcal{C}_{2n+2}^\lambda$ and $A \leq^\lambda B$, then $A \leq_{2n+2}^\lambda B$.

(b) (Kechris [12] for $\lambda = \omega$). The $(\lambda, 2n+2)$-degrees in $\mathcal{C}_{2n+2}^\lambda$ are well-ordered
by \( \leq_{2n+2} \). \( \mathcal{C}_{2n+2} \) admits a \( \Delta^1_{2n+2}(\lambda) \)-good well-ordering, \( \leq^\lambda \), which is a refinement of \( \leq_{2n+2} \).

**Proof.** Recall that \( \leq^\lambda \) is the well-ordering of \( \mathcal{C}_{2n+2} \) defined in Definition 4.3, and observe that (b) follows easily from (a) and Theorem 4.8. To prove (a), let \( A \prec^\lambda B \), where \( A \) and \( B \) are the \( \eta_A \)th and \( \eta_B \)th subsets of \( \lambda \), respectively, with respect to \( \leq^\lambda \). By Theorem 4.9, \( \leq^\lambda \) has length \( \lambda^+ \), in \( H_{2n+1} \). Hence there is a well-ordering of a subset of \( \lambda \) in \( H_{2n+1} \) whose order type is \( \eta_B \). Let \( \leq \) be the \( \leq^\lambda \)-least such well-ordering. (Since \( \leq \) is a subset of \( \lambda \times \lambda \), by coding pairs it can be identified with a subset of \( \lambda \).) Let \( \xi \prec \lambda \) be the ordinal such that \( \eta_A \) is the order type of the initial segment of \( \leq \) below \( \xi \). We can now clearly define \( A \) from \( B \) and \( \xi \). We must show that this definition is \( \mathcal{C}_{2n+2} \).

First of all, \( \eta_B \) (i.e. \( \{ \beta : |\beta| = \eta_B \} \) is \( \Delta^1_{2n+2}(B, \lambda) \). To see this, first note that the map \( f : \eta \to \text{Pow}(\lambda) \) that takes each \( \theta \in \eta \) to the \( \theta \)th element of \( \mathcal{C}_{2n+2} \) (with respect to \( \leq^\lambda \)) is \( \text{v}_2 \)-in-the-codes, by the coding lemma [19]. Let \( S_\alpha \) be the \( \Sigma^1_{2n+1} \) subset of \( N \times N \) parametrized by \( \alpha \). Then

\[
|\beta| = \eta_B \iff \\
\beta \text{ codes an ordinal } < \delta^1_{2n+1} \& \exists \alpha [\forall \gamma \forall \delta (S_\alpha(\gamma, \delta) \to |\gamma| < |\beta|) \& \\
(\forall \theta < |\beta|) \exists \gamma \exists \delta (|\gamma| = \theta \& \delta \text{ is a } \Delta^1_{2n+1} \text{-code for a set } D_\alpha \subset \lambda \& \\
D_\alpha \prec^\lambda B \& S_\alpha(\gamma, \delta) \& (\forall D <^\lambda B) \exists \gamma \exists \delta (|\gamma| < |\beta| \& \delta \text{ is a } \Delta^1_{2n+1} \text{-code for } D \& S_\alpha(\gamma, \delta) \& (\forall \theta_0 < |\beta|)(\forall \theta_1 < |\beta|)(\forall D_0 <^\lambda B)(\forall D_1 <^\lambda B) \\
((S_\alpha(\theta_0, D_0) \& S_\alpha(\theta_1, D_1)) \rightarrow (\theta_0 < \theta_1 \iff D_0 <^\lambda D_1))]].
\]

By Theorem 0.1 and the fact that \( \leq^\lambda \) is good (Theorem 4.8), the above formula is \( \Sigma^1_{2n+2}(B, \lambda) \). So \( \eta_B \) is \( \Sigma^1_{2n+2}(B, \lambda) \) and since \( |\beta| = \eta_B \) if and only if \( \forall \beta'(|\beta'| = \eta_B \rightarrow |\beta'| = |\beta|) \), \( \eta_B \) is \( \Delta^1_{2n+2}(B, \lambda) \). Similarly, if \( \eta_D \) is the ordinal such that \( D \) is the \( \eta_D \)th subset of \( \mathcal{C}_{2n+2} \), then \( \eta_D \) is \( \Delta^1_{2n+2}(D, \lambda) \), uniformly in \( D \). So \( \alpha \) is a \( \Delta^1_{2n+1} \)-code for \( A \) if and only if

\[
(\forall D <^\lambda B)(\eta_\alpha = \eta_D \rightarrow \alpha \text{ encodes } D).
\]

So \( A \) is \( \Delta^1_{2n+1}(\lambda, \eta_A, B) \).

Next we compute the complexity of \( \leq \).

\[
\langle \nu_0, \nu_1 \rangle \leq \leq \iff \\
\exists \alpha [\alpha \text{ codes a } \Delta^1_{2n+1} \text{ binary relation } W_\alpha \& W_\alpha \text{ is a well-ordering of a subset of } \lambda \text{ of order type } \eta_\alpha \& W_\alpha \in \mathcal{C}_{2n+2} \& \\
(\forall D <^\lambda W_\alpha) \exists \beta [\beta \text{ codes a } \Delta^1_{2n+1} \text{ binary relation } X_\beta \& \\
(\forall \theta_0 < \lambda)(\forall \theta_1 < \lambda)(\langle \theta_0, \theta_1 \rangle \in X_\beta \leftrightarrow \langle \theta_0, \theta_1 \rangle \in D) \& X_\beta \text{ is not a well-ordering of order type } \eta_B] \& \langle \nu_0, \nu_1 \rangle \in W_\alpha].
\]

Comparing two \( \Delta^1_{2n+1} \) prewell orderings \( (W_\alpha \text{ and the norm used to code...} \right)
ordinals) can be done in a $\Delta^1_{2n+2}$ way. As we have pointed out before, this is a consequence of Theorem 0.1, which is in fact proved in a manner similar to that used to compare $\leq^\lambda$ with the norm, above. So $\preceq$ is $\Sigma^1_{2n+2}(\lambda, \eta_B)$, and since $\langle \nu_1, \nu_0 \rangle \in \preceq$ if and only if $\neg((\nu_1, \nu_0) \in \preceq)$ or $\nu_0 = \nu_1$, it is $\Delta^1_{2n+2}(\lambda, \eta_B)$.

Thus we have shown that $\eta_B$ is $\Delta^1_{2n+2}(B, \lambda)$, $A$ is $\Delta^1_{2n+2}(\lambda, \eta_A, B)$, and $\preceq$ is $\Delta^1_{2n+2}(\lambda, \eta_B)$. And clearly $\eta_A$ is $\Delta^1_{2n+2}(\preceq, \xi)$. Putting this all together we get that $A$ is $\Delta^1_{2n+2}(B, \lambda, \xi)$.

Let $\rho^\lambda_{2n+2}$ denote the order type of the well-ordering $\leq^\lambda_{2n+2}$ of $\langle \lambda, 2n+2 \rangle$-degrees of $\mathcal{C}^\lambda_{2n+2}$. For $\xi < \rho^\lambda_{2n+2}$, let $d^\lambda_{2n+2}(\xi)$ be the $\xi$th degree in this ordering.

Lemma 5.6 (AD). Let $\lambda < \delta^1_{2n+1}$ be nice and $\Pi^1_{2n+1}$-reliable.

(a) $\rho^\lambda_{2n+2} = (\lambda^\omega)^{H_{2n+1}}$

(b) $\rho^\lambda_{2n+2}$ is a limit ordinal.

(c) $d^\lambda_{2n+2}(0) = [\emptyset]^\lambda_{2n+2}$.

Proof. Theorem 4.9 and the fact that Proposition 5.2(c) holds inside $H_{2n+1}$, imply (a). Clearly (b) follows from (a) and (c) follows from Lemmas 5.4 and 5.5.

We do not know what the cofinality of $\rho^\lambda_{2n+2}$ (in $V$) is, even in the case $\lambda = \mathcal{N}_1^\omega$. Next we explain the relationship between the degrees $d^\lambda_{2n+2}(\xi)$ and $d^\lambda_{2n+2}(\xi + 1)$. Recall that $G \subseteq \omega \times \mathcal{N}$ is the universal $\Sigma^1_{2n+2}$ set (which is used to define $H_{2n+1}$ in Definition 3.1).

Definition 5.7. Let $\lambda < \delta^1_{2n+1}$ be nice. Let $\hat{G} \subseteq \text{Pow}(\lambda) \times \lambda \times \lambda$ be the relation

$$
\hat{G}(A, \xi, \eta) \Leftrightarrow \\
\xi = \langle m, \nu \rangle \text{ for some } m \in \omega, \nu < \lambda & \\
\exists \alpha \exists \beta \exists \gamma \exists \delta (\alpha \text{ is a } \Delta^1_{2n+1}\text{-code for } A \& \beta = \nu \& \gamma = \eta \& \\
|\delta| = \lambda \& G(m, \langle \alpha, \beta, \gamma, \delta \rangle)).
$$

For any $\lambda$-set $A$, let $\hat{G}_A(\xi, \eta) \Leftrightarrow \hat{G}(A, \xi, \eta)$.

Since $\lambda$ is nice and sequence coding is $\Delta^1_{2n+2}$, $\hat{G}_A$ is a $\Sigma^1_{2n+2}(\lambda, \lambda)$ subset of $\lambda \times \lambda$ which parametrizes the set

$$
\mathcal{A} = \{B \in \lambda : B \text{ is } \Sigma^1_{2n+2}(\lambda, \xi_0, \ldots, \xi_k) \text{ for some } \xi_0, \ldots, \xi_k < \lambda \}.
$$

That is, if $B \in \mathcal{A}$, then there is a fixed $\xi < \lambda$ such that for all $\eta < \lambda$, $(\eta \in B \iff \hat{G}_A(\xi, \eta))$. 

**Definition 5.8.** Let $\lambda < \delta_{2n+1}$ be nice. Let $A$ be a $\lambda$-set. The $(\lambda, 2n+2)$-jump of $A$ is the $\lambda$-set $\{ (\xi, \eta) : \hat{G}_A (\xi, \eta) \}$.

For $\lambda = \omega$, this is the $\Delta_{2n+2}^\lambda$-jump of a real.

**Proposition 5.9.** Let $\lambda < \delta_{2n+1}$ be nice.

(a) Let $B_0$ and $B_1$ be the $(\lambda, 2n+2)$-jumps of $A_0$ and $A_1$, respectively. If $A_0 = \Delta_{2n+2}^\lambda A_1$, then $B_0 = \Delta_{2n+2}^\lambda B_1$.

(b) If $B$ is the $(\lambda, 2n+2)$-jump of $A$, then $A < \Delta_{2n+2}^\lambda B$.

In light of Proposition 5.9(a), the jump of a degree is well defined. (It is also easy to see that the jump of a degree is independent of the particular universal set $G$ and norm $\varphi$ used in the coding of ordinals.)

**Lemma 5.10 (AD; Kechris [12] for $\lambda = \omega$).** Let $\lambda < \delta_{2n+1}$ be nice. Then $\epsilon_{2n+2}^\lambda$ is closed under the $(\lambda, 2n+2)$-jump.

**Proof.** Let $A \in \epsilon_{2n+2}^\lambda$ and let $B$ be the jump of $A$. Say $A$ is the $\xi$th element of $\epsilon_{2n+2}^\lambda$ with respect to $\leq^A$. It follows from the definition of jump (Definition 5.8) and some computations similar to those in the proof of Lemma 5.5, that membership in $B$ is $\Sigma_{2n+2}^\lambda (\lambda, \xi)$. That is, there is an $m \in \omega$ such that for all $\eta < \lambda$, $\eta \in B \iff G'(m, \langle \lambda, \xi, \eta \rangle)$. Since $H_{2n+1} = L[G']$, $B$ is in $H_{2n+1}$, hence in $\epsilon_{2n+2}^\lambda$.

**Lemma 5.11 (AD; Kechris [12] for $\lambda = \omega$).** Let $\lambda < \delta_{2n+1}$ be nice and $H_{2n+1}$-reliable. For all $\xi < \rho_{2n+2}$, the degree $d_{2n+2}^\lambda (\xi + 1)$ is the $(\lambda, 2n+2)$-jump of the degree $d_{2n+2}^\lambda (\xi)$.

**Proof.** Let $A \in d_{2n+2}^\lambda (\xi)$, let $B$ be the jump of $A$, and let $D$ be the $\leq^A$-least $\lambda$-set in $\epsilon_{2n+2}^\lambda$ such that $A < \Delta_{2n+2}^\lambda D$. Then $D \in d_{2n+2}^\lambda (\xi + 1)$. To prove the lemma, it will suffice to show that $B \leq^A \Delta_{2n+2}^\lambda D$. Then by Lemma 5.4, $B \in \epsilon_{2n+2}^\lambda$ (which reproves Lemma 5.10), and by Lemma 5.5(a) and Proposition 5.9(b), $D \leq_{2n+2}^\lambda B$; hence $B = \Delta_{2n+2}^\lambda D$, which completes the proof.

Let $F$ be $H_{2n+1}$ such that $G = \exists^\forall F$, and let $\psi'$ be a $H_{2n+1}$-norm on $F$. We put a uniform norm $\psi$ on $\hat{G}_\lambda$ as follows:

\[
\psi (\xi, \eta) = \min \{ \psi' (m, \langle \alpha, \beta, \gamma, \delta \rangle, \varepsilon) : (m, \langle \alpha, \beta, \gamma, \delta \rangle, \varepsilon) \in F \\
\text{and } \xi = (m, \nu) \text{ and } \alpha \text{ is a } \Delta_{2n+1}^\lambda\text{-code for } A \text{ and } |\beta| = \nu \\
\text{and } |\gamma| = \eta \text{ and } |\delta| = \lambda \}.
\]

Let $\leq^\psi$ be the induced prewell-ordering of $\hat{G}_\lambda$. Then $\leq^\psi$ is a binary relation on ordered pairs of ordinals less than $\lambda$. Each proper initial segment of $\leq^\psi$ is $\Delta_{2n+2}^\lambda (A, \lambda, \xi, \eta_0)$ for some $\xi_0, \eta_0 < \lambda$ (namely the $(\xi_0, \eta_0)$ that determines the initial segment). Let

\[\mathcal{J} = \{ C < \lambda^d : C \text{ is an initial segment (not necessarily proper) of } \leq^\psi \}. \]
Then $\mathcal{J}$ is $\Sigma^1_{2n+2}(\lambda, A)$. So

$$C = \equiv^* \iff $$

$$\equiv C \in \mathcal{J} \& \neg(C \equiv^*_{2n+2} A)$$

$$\equiv C \in \mathcal{J} \& (\exists C' < \lambda)$$

$$(C' \text{ encodes the set of 4-tuples } C \& \neg(C' \equiv^* \mathcal{J})).$$

The last equivalence is the definition of $D$ plus Lemma 5.5. So by Proposition 4.4, $C$ is a $\Sigma^1_{2n+2}(\lambda, A, D)$-singleton. Hence $C$ is $\Delta^1_{2n+2}(\lambda, A, D)$. Now $B$, the jump of $A$, is easily $\Delta^1_{2n+2}(\lambda, C)$. So $B$ is $\Delta^1_{2n-2}(\lambda, A, D)$, and since $A \equiv^*_{2n+2} D$, $B \equiv^*_{2n+2} D$.

The proof of Lemma 5.11 given above is essentially the proof given in [12] for the case $\lambda = \omega$.

Combining Lemmas 5.4, 5.5, 5.6, 5.10, and 5.11, we get the following theorem which summarizes the structure theory of $\mathcal{C}_{2n+2}^\lambda$.

**Theorem 5.12 (AD).** Let $\lambda < \delta^1_{2n+1}$ be nice (i.e. closed under sequence coding) and $\Pi^1_{2n+1}$-reliable. The set $\mathcal{C}_{2n+2}^\lambda$ is closed under $\equiv^*_{2n+2}$. Hence $\mathcal{C}_{2n+2}^\lambda$ is a set of $(\lambda, 2n+2)$-degrees. The degrees of $\mathcal{C}_{2n+2}^\lambda$ are well-ordered by their natural ordering, $\leq^*_{2n+2}$. It is also closed under the $(\lambda, 2n+2)$-jump. The jump of a degree is the first degree above it in the well-ordering $\leq^*_{2n+2}$ of degrees. Furthermore, $\mathcal{C}_{2n+2}^\lambda$ admits a $\Delta^1_{2n+2}(\lambda)$-good well-ordering $\leq^\lambda$ which is a refinement of the degree ordering $\leq^*_{2n+2}$. The order type of the well-ordering $\leq^\lambda$ equals the order type of the degrees which equals $(\lambda^+)^{H_{2n+1}}$.

A few remarks on Theorem 5.12 are in order. First of all, it follows from Theorem 5.12 that $0$-jump is a minimal degree; this solves Post's problem for this type of reducibility. $\equiv^*_{2n+2}$.

Secondly, the structure theory of $\mathcal{C}_{2n+2}^\lambda$ given in Theorem 5.12 is reflected in the model $H_{2n+1}$. This statement must be properly interpreted. Of course if $A \equiv^\lambda B, H_{2n+1}$ does not know that $A$ is definable in a $\Delta^1_{2n+2}$ way from $B$ and ordinals less than $\lambda$. Nor is this even meaningful in $H_{2n+1}$, since in $H_{2n+1}$, $\lambda$ is a huge ordinal much larger than $\Theta(\text{End})$. But there is an $m \in \omega$ such that for all $\xi, \eta, \lambda$ less than $\delta^1_{2n+1}$,

$$G(m, \langle \xi, \eta, \lambda \rangle) \equiv$$

$(the set $A$ constructed at stage $\xi$ and the set $B$ constructed at stage $\eta$ are both subsets of $\lambda$, and $A$ is definable in a $\Delta^1_{2n+2}$ way from $B, \lambda$, and ordinals less than $\lambda$).

And if $A \equiv^\lambda B$, then $H_{2n+1} \vdash G(m, \langle \xi_A, \xi_B, \lambda \rangle)$, where $A$ and $B$ are constructed at stages $\xi_A$ and $\xi_B$, respectively. Similarly, the other parts of Theorem 5.12 are reflected in $H_{2n+1}$.

Although we have stated Theorem 5.12 for $\Pi^1_{2n+1}$-reliable ordinals $\lambda$, all that
was really used about $\lambda$ was that there is a $\Sigma^1_{2n+2}(\lambda)$-good well-ordering $\leq^{\lambda}$ of $\mathcal{C}_{2n+2}$ of order type $(\lambda^*)^{H_{2n+1}}$. In particular, by Corollary 4.10, Theorem 5.12 is valid for $\kappa_{2n+1}$. This is an important case, since $\kappa_{2n+1}$ is the ordinal that is the equivalent of $\omega$ for the pointclass $\Sigma^1_{2n+1}$; $\kappa_1 = \omega$. Hence $\text{Pow}(\kappa_{2n+1}) \cap H_{2n+1}$ is the true analog of the constructible reals.

For $\omega$, Kechris [12] has shown that $\mathcal{C}_{2n+1}$ has a structure theory quite similar to that of $\mathcal{C}_{2n+2}$. For $\lambda > \omega$ we are not able to develop any non-trivial structure theory for $\mathcal{C}_{2n+1}$. In particular, the following three assertions are open for arbitrary nice $\Pi^1_{2n+1}$-reliable $\lambda$. They are all true for $\lambda = \omega$ by [12].

1. The set $\mathcal{C}_{2n+1}$ admits a $\Delta^1_{2n+1}(\lambda)$ well-ordering.
2. For all $A, B \in \mathcal{C}_{2n+1}$, either $A$ is $\Delta^1_{2n+1}(B, \xi_0, \ldots, \xi_k)$ for some $\xi_0, \ldots, \xi_k < \lambda$, or $B$ is $\Delta^1_{2n+1}(A, \lambda, \xi_0, \ldots, \xi_k)$ for some $\xi_0, \ldots, \xi_k < \lambda$.
3. For all $A \in \mathcal{C}_{2n+1}$, there is a $B$ in $\mathcal{C}_{2n+1}$ and there are $\xi_0, \ldots, \xi_k < \lambda$ such that $A$ is $\Delta^1_{2n+1}(B, \lambda, \xi_0, \ldots, \xi_k)$.

References


