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Optimality conditions for vector equilibrium problems ☆

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Abstract

In this paper, we present the necessary and sufficient conditions for weakly efficient solution, Henig efficient solution, globally efficient solution, and superefficient solution to the vector equilibrium problems with constraints. As applications, we give the necessary and sufficient conditions for corresponding solution to the vector variational inequalities and vector optimization problems.

Keywords: Vector equilibrium problems; Optimality condition; Weakly efficient solution; Henig efficient solution; Globally efficient solution; Superefficient solution

1. Introduction

Throughout this paper, let X, Z be real Hausdorff topological vector spaces, and Y be a real locally convex Hausdorff topological vector space. Let X_0 be a nonempty convex subset of X, $g : X_0 \to Z$ be a mapping, and that $F : X_0 \times X_0 \to Y$ be a mapping. Let K be a closed convex pointed cone in Z with $\operatorname{int} K \neq \emptyset$, where $\operatorname{int} K$ denotes the interior of the set K. We define the constraint set

$$A = \{x \in X_0\colon\ g(x) \in K\},$$

and consider the vector equilibrium problems with constraints (for short, VEPC): find $x \in A$ such that

$$F(x, y) \notin -P \quad \text{for all } y \in A,$$

where $P \cup \{0\}$ is a convex cone in Y.

A number of papers have been devoted to the existence of solutions (see [1–19]). But so far, there are few papers which deal with the properties of the solutions for the vector equilibrium problems. Giannessi, Mastroeni, and Pellegrini [20] turned the vector variational inequalities with constraints into another vector variational inequalities without constraints. They gave sufficient conditions for efficient solution and weakly efficient solution to the vector

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variational inequalities in finite dimensional spaces. By using the concept of subdifferential of the function, Morgan and Romaniello [21] gave scalarization and Kuhn–Tucker-like conditions for weak vector generalized quasivariational inequalities in Hilbert space.

Vector variational inequality problems and vector optimization problems, as well as several other problems, are special realizations of vector equilibrium problems. It is therefore important to give the optimality conditions for the solution to the vector equilibrium problems. As far as we know, this problem remains unstudied.

In this paper, we give the optimality conditions for weakly efficient solution to the vector equilibrium problems with constraints. We then give the optimality conditions for Henig efficient solution, globally efficient solution, and superefficient solution to the vector equilibrium problems with constraints in topological vector spaces, which are important solutions to the vector equilibrium problems (see [22–24]).

We will see that the weakly efficient solution, Henig efficient solution, globally efficient solution, and superefficient solution to the vector equilibrium problems with constraints are equivalent to solution of corresponding scalar optimization problems without constraints, respectively. As applications, we give the necessary and sufficient conditions for corresponding solution to the vector variational inequalities and vector optimization problems.

2. Preliminaries and definitions

Let Y^* be the topological dual space of Y. Let C be a closed convex pointed cone in Y. Let

$$C^* = \{y^* \in Y^*\colon\ y^*(y) \geqslant 0 \text{ for all } y \in C\}$$

be the dual cone of C.

Denote the quasi-interior of C^* by $C^\sharp$, i.e.

$$C^\sharp := \{y^* \in Y^*\colon\ y^*(y) > 0 \text{ for all } y \in C \setminus \{0\}\}.$$

Let D be a nonempty subset of Y. The cone hull of D is defined as

$$\operatorname{cone}(D) = \{td\colon\ t \geqslant 0,\ d \in D\}.$$

Denote the closure of D by $\operatorname{cl}(D)$ and interior of D by $\operatorname{int} D$.

A nonempty convex subset B of the convex cone C is called a base of C, if $C = \operatorname{cone}(B)$ and $0 \notin \operatorname{cl}(B)$. It is easy to see that $C^\sharp \neq \emptyset$ if and only if C has a base.

Let B be a base of C. Set

$$C^\Delta(B) = \{y^* \in C^\sharp\colon \text{ there exists } t > 0 \text{ such that } y^*(b) \geqslant t \text{ for all } b \in B\}.$$

By the separation theorem of convex sets, we know $C^\Delta \neq \emptyset$. It is clear that $C^\Delta(B) \subset C^\sharp$. Let B be a base of C. Then $0 \notin \operatorname{cl} B$. By the separation theorem of convex sets, there exists $y^* \in Y^* \setminus \{0\}$ such that

$$r = \inf\{y^*(b)\colon\ b \in B\} > y^*(0) = 0.$$

Set

$$V_B = \{y \in Y\colon\ |y^*(y)| < r/2\}.$$

Then V_B is an open convex circled neighborhood of 0 in Y. The notion V_B will be used throughout this paper.

It is clear that

$$\inf\{y^*(y)\colon\ y \in B + V_B\} \geqslant r/2.$$

It is easy to see that for each convex neighborhood U of 0 with $U \subset V_B$, $B + U$ is a convex set and $0 \notin \operatorname{cl}(B + U)$, and therefore $C_U(B) := \operatorname{cone}(U + B)$ is a pointed convex cone, and $C \setminus \{0\} \subset \operatorname{int} C_U(B)$.

If $\operatorname{int} C \neq \emptyset$, a vector $x \in A$ satisfying

$$F(x, y) \notin -\operatorname{int} C \quad \text{for all } y \in A,$$

is called a weakly efficient solution to the VEPC.

For each $x \in X_0$, we denote

$$F(x, A) = \bigcup_{y \in A} F(x, y).$$

Definition 2.1. (See [24].) A vector $x \in A$ is called a globally efficient solution to the VEPC if there exists a point convex cone $H \subset Y$ with $C \setminus \{0\} \subset \operatorname{int} H$ such that

$$F(x, A) \cap \big((-H) \setminus \{0\}\big) = \emptyset.$$

Definition 2.2. (See [22,24].) A vector $x \in A$ is called a Henig efficient solution to the VEPC if there exists some neighborhood U of 0 with $U \subset V_B$ such that

$$\operatorname{cone}\big(F(x, A)\big) \cap \big(-\operatorname{int} C_U(B)\big) = \emptyset.$$

It is clear that a vector $x \in A$ is a Henig efficient solution if and only if

$$F(x, A) \cap \big(-\operatorname{int} C_U(B)\big) = \emptyset.$$

Definition 2.3. (See [23,24].) A vector $x \in A$ is called a superefficient solution to the VEPC if for each neighborhood V of 0, there exists some neighborhood U of 0 such that

$$\operatorname{cone}\big(F(x, A)\big) \cap (U - C) \subset V.$$

Let $L(X, Y)$ be the space of all bounded linear mapping from X to Y. We denote by (h, x) the value of $h \in L(X, Y)$ at x.

VEPC includes as a special case a vector variational inequality with constraints (for short, VVIC) involving

$$F(x, y) = (Tx, y - x),$$

where T is a mapping from A to $L(X, Y)$.

Definition 2.4. If $F(x, y) = (Tx, y - x)$, $x, y \in A$, and if $x \in A$ is a weakly efficient solution, or a Henig efficient solution, or a globally efficient solution, or a superefficient solution to the VEPC, then $x \in A$ is called a weakly efficient solution, or a Henig efficient solution, or a globally efficient solution, or a superefficient solution to the VVIC, respectively.

Another special case of VEPC is a vector optimization problem with constraints (for short, VOPC) involving

$$F(x, y) = f(y) - f(x), \quad x, y \in A,$$

where $f : A \to Y$ is a mapping.

Definition 2.5. If $F(x, y) = f(y) - f(x)$, $x, y \in A$, and if $x \in A$ is a weakly efficient solution, or a Henig efficient solution, or a globally efficient solution, or a superefficient solution to the VEPC, then $x \in A$ is called a weakly efficient solution, or a Henig efficient solution, or a globally efficient solution, or a superefficient solution to the VOPC, respectively.

We denote the set of weakly efficient solutions, the set of Henig efficient solutions, the set of globally efficient solutions, and the set of superefficient solutions to the VOPC, by V_W, V_H, V_G, and V_S, respectively.

We denote the strong topology on Y^* by $\beta(Y^*, Y)$. The sets

$$\omega = \left\{ \bigcap_{i=1}^{n} \Big\{ y^* \in Y^*\colon \sup_{y \in A_i} \big|y^*(y)\big| < \varepsilon \Big\}\colon A_i \ (i = 1, \ldots, n) \text{ are bounded subsets of } Y,\ \varepsilon > 0,\ n \in N \right\}$$

form a base of neighborhoods of zero of Y^* with respect to $\beta(Y^*, Y)$.

Lemma 2.1. *(See [25].) Assume that pointed convex cone C has a base B.*

(i) *For any open convex circled neighborhood U of zero in Y with $U \subset V_B$, we have*

$$\big(C_U(B)\big)^* \setminus \{0\} \subset C^{\Delta}(B).$$

(ii) *For any $f \in C^{\Delta}(B)$, there exists an open convex circled neighborhood U of zero in Y with $U \subset V_B$ such that $f \in (C_U(B))^* \setminus \{0\}$.*
(iii) *If closed convex cone C has a bounded closed base B, then* $\operatorname{int} C^* = C^{\Delta}(B)$*, where* $\operatorname{int} C^*$ *is the interior of C^* with respect to $\beta(Y^*, Y)$.*

3. Optimality condition

In this section, we give the optimality conditions for weakly efficient solution, Henig efficient solution, globally efficient solution, and superefficient solution to the vector equilibrium problems with constraints. We will see that the conditions are not only necessary but also sufficient.

A mapping $f : X_0 \to Y$ is said to be C-convex, if for any $x_1, x_2 \in X_0$, $t \in [0, 1]$,

$$tf(x_1) + (1-t)f(x_2) \in f\big(tx_1 + (1-t)x_2\big) + C.$$

A mapping $g : X_0 \to Z$ is said to be K-concave on X_0, if for any $x_1, x_2 \in X_0$, and $t \in [0, 1]$,

$$tg(x_1) + (1-t)g(x_2) \in g\big(tx_1 + (1-t)x_2\big) - K.$$

We make an assumption:

(A) *For each $x \in X_0$, $F(x, x) = 0$, and $F(x, y)$ is C-convex in y; g is K-concave on X_0, and there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$.*

If g is K-concave on X_0, by assumption (A), we can see that $A = \{x \in X_0\colon\ g(x) \in K\}$ is a nonempty convex set.

Theorem 3.1. *Let assumption* (A) *be satisfied, and that* $\operatorname{int} C \neq \emptyset$*. Then $x \in A$ is a weakly efficient solution to the VEPC if and only if there exist $y^* \in C^* \setminus \{0\}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(F(x, x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big(F(x, y)\big) + z^*\big(g(y)\big)\big\}.$$

Proof. Assume that $x \in A$ is a weakly efficient solution to the VEPC. Define the set

$$M = \big\{(y, z) \in Y \times Z\colon \text{ there exists } y' \in X_0 \text{ such that } y - F(x, y') \in \operatorname{int} C,\ g(y') - z \in \operatorname{int} K\big\}.$$

It is clear that $M \neq \emptyset$. By the C-convexity of F in second variable, and the K-concaveness of g, we can see that M is a convex set. It is clear that M is an open set. We claim that $(0, 0) \notin M$. If not, then there exists $y' \in X_0$ such that

$$0 - F(x, y') \in \operatorname{int} C, \qquad g(y') - 0 \in \operatorname{int} K.$$

Then $F(x, y') \in -\operatorname{int} C$, and $y' \in A$. This contradicts that x is a weakly efficient solution to the VEPC. Thus $(0, 0) \notin M$. By the separation theorem of convex sets, there exists $(0, 0) \neq (y^*, z^*) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$0 < y^*(y) + z^*(z) \quad \text{for all } (y, z) \in M. \tag{1}$$

Let $(y, z) \in M$. Then there exists $y' \in X_0$ such that $y - F(x, y') \in \operatorname{int} C$, $g(y') - z \in \operatorname{int} K$. Hence, for every $c \in \operatorname{int} C$, $k \in \operatorname{int} K$, $t > 0$, $t' > 0$, we have $(y + tc, z) \in M$, and $(y, z - t'k) \in K$. By (1), we have

$$0 < y^*(y + tc) + z^*(z) \quad \text{for all } c \in \operatorname{int} C \text{ and } t > 0.$$

Thus,

$$\big(-z^*(z) - y^*(y)\big)/t < y^*(c) \quad \text{for all } c \in \operatorname{int} C \text{ and } t > 0.$$

Letting $t \to \infty$, we get

$$0 \leqslant y^*(c) \quad \text{for all } c \in \operatorname{int} C.$$

Since C is a closed convex cone, $C = \operatorname{cl}(\operatorname{int} C)$. By the continuity of y^*, we can see that $0 \leqslant y^*(c)$ for all $c \in C$. That is, $y^* \in C^*$. Similarly, we can show that $z^* \in -K^*$. We also have $y^* \neq 0$. In fact, if $y^* = 0$, from (1) we get

$$0 < z^*(z) \quad \text{for all } (y, z) \in M.$$

By assumption (A), there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$. Thus, we have

$$\big(F(x, x_0) + c, g(x_0) - z\big) \in M \quad \text{for all } c \in \operatorname{int} C,\ z \in \operatorname{int} K.$$

Hence,

$$0 < z^*\big(g(x_0) - z\big),$$

and hence

$$z^*(z) < z^*\big(g(x_0)\big).$$

In particular, we have

$$z^*\big(g(x_0)\big) < z^*\big(g(x_0)\big).$$

This is a contradiction. Thus, $y^* \neq 0$. It is clear that

$$\big(F(x, y) + c, g(y) - k\big) \in M \quad \text{for all } y \in X_0,\ c \in \operatorname{int} C \text{ and } z \in \operatorname{int} K.$$

By (1), we can obtain

$$0 \leqslant y^*\big(F(x, y)\big) + z^*\big(g(y)\big) \quad \text{for all } y \in X_0. \tag{2}$$

It is clear that $(F(x, x) + tc, g(x) - tk) \in M$ for all $c \in \operatorname{int} C$, $z \in \operatorname{int} K$, $t > 0$. By (1) and assumption (A), we have

$$0 < y^*\big(F(x, x) + tc\big) + z^*\big(g(x) - tk\big) = ty^*(c) + z^*\big(g(x)\big) - tz^*(k).$$

Letting $t \to 0$, we obtain $0 \leqslant z^*(g(x))$. Noting that $x \in A$, and $z^* \in -K^*$, we have $z^*(g(x)) \leqslant 0$. Thus,

$$z^*\big(g(x)\big) = 0. \tag{3}$$

From (2) and (3), and $F(x, x) = 0$, we get

$$y^*\big(F(x, x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} y^*\big(F(x, y)\big) + z^*\big(g(y)\big). \tag{4}$$

Conversely, let $x \in A$, and suppose that there exist $y^* \in C^* \setminus \{0\}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and

$$y^*\big(F(x, x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(F(x, y)\big) + z^*\big(g(y)\big)\big\}. \tag{5}$$

We will show that x is a weakly efficient solution to the VEPC. If not, then there exists $y_0 \in A$ such that

$$F(x, y_0) \in -\operatorname{int} C.$$

Since $y^* \in C^* \setminus \{0\}$, we have

$$y^*\big(F(x, y_0)\big) < 0.$$

Notice $y_0 \in A$, we have $g(y_0) \in K$, and we have $z^*(g(y_0)) \leqslant 0$ because of $z^* \in -K^*$. This together with (5) give us

$$0 = y^*\big(F(x, x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(F(x, y)\big) + z^*\big(g(y)\big)\big\} \leqslant y^*\big(F(x, y_0)\big) + z^*\big(g(y_0)\big) < 0.$$

This is a contradiction. Hence, x is a weakly efficient solution to the VEPC. □

Theorem 3.2. *Assume that the assumption* (A) *is satisfied, and that C has a base B. Then $x \in A$ is a Henig efficient solution to the VEPC if and only if there exist $y^* \in C^{\Delta}(B)$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(F(x, x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(F(x, y)\big) + z^*\big(g(y)\big)\big\}.$$

Proof. Assume that $x \in A$ is a Henig efficient solution to the VEPC. By definition, there exists some neighborhood U of 0 with $U \subset V_B$ such that

$$F(x, A) \cap \big(-\operatorname{int} C_U(B)\big) = \emptyset. \tag{6}$$

Define the set

$$M = \{(y,z) \in Y \times Z\colon \text{ there exists } y' \in X_0 \text{ such that } y - F(x,y') \in \operatorname{int} C_U(B),\ g(y') - z \in \operatorname{int} K\}.$$

It is clear that $M \neq \emptyset$. By the C-convexity of F in second variable, the K-concaveness of g, $C \setminus \{0\} \subset \operatorname{int} C_U(B)$, and that $C_U(B)$ is a convex cone, we know that M is a convex set. It is clear that M is an open set. We claim that $(0,0) \notin M$. If not, then there exists $y' \in X_0$ such that

$$0 - F(x,y') \in \operatorname{int} C_U(B), \qquad g(y') - 0 \in \operatorname{int} K.$$

Then $F(x,y') \in -\operatorname{int} C_U(B)$, and $y' \in A$. This contradicts (6). Thus $(0,0) \notin M$. By the separation theorem of convex sets, there exists $(0,0) \neq (y^*, z^*) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$0 < y^*(y) + z^*(z) \quad \text{for all } (y,z) \in M. \tag{7}$$

Let $(y,z) \in M$. Then there exists $y' \in X_0$ such that $y - F(x,y') \in \operatorname{int} C_U(B)$, $g(y') - z \in \operatorname{int} K$. Hence, for every $c \in \operatorname{int} C_U(B)$, $k \in \operatorname{int} K$, $t > 0$, $t' > 0$, we have $(y + tc, z) \in M$, and $(y, z - t'k) \in M$, this implies that $y^* \in (C_U(B))^*$ and $z^* \in -K^*$. In a way similar to the proof of Theorem 3.1, we have $y^* \neq 0$. By Lemma 2.1, we can see that $y^* \in C^\Delta(B)$. It is clear that

$$\big(F(x,y) + c, g(y) - k\big) \in M \quad \text{for all } y \in X_0,\ c \in \operatorname{int} C_U(B) \text{ and } k \in \operatorname{int} K.$$

We can obtain

$$0 \leqslant y^*\big(F(x,y)\big) + z^*\big(g(y)\big) \quad \text{for all } y \in X_0. \tag{8}$$

It is clear that $(F(x,x) + tc, g(x) - tk) \in M$ for all $c \in \operatorname{int} C_U(B)$, $k \in \operatorname{int} K$, $t > 0$. By (7) and assumption (A), we have

$$0 < y^*\big(F(x,x) + tc\big) + z^*\big(g(x) - tk\big) = ty^*(c) + z^*\big(g(x)\big) - tz^*(k).$$

Letting $t \to 0$, we obtain $0 \leqslant z^*(g(x))$. Noting that $x \in A$, and $z^* \in -K^*$, we have $z^*(g(x)) \leqslant 0$. Thus,

$$z^*\big(g(x)\big) = 0. \tag{9}$$

From (8) and (9), we get

$$y^*\big(F(x,x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} y^*\big(F(x,y)\big) + z^*\big(g(y)\big). \tag{10}$$

Conversely, let $x \in A$, and suppose that there exist $y^* \in C^\Delta(B)$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and

$$y^*\big(F(x,x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(F(x,y)\big) + z^*\big(g(y)\big)\big\}. \tag{11}$$

We will show that x is a Henig efficient solution to the VEPC, that is, there exists some neighborhood U of 0 with $U \subset V_B$,

$$F(x,A) \cap \big(-\operatorname{int} C_U(B)\big) = \emptyset. \tag{12}$$

Suppose to the contrary that for any neighborhood U of 0 with $U \subset V_B$, we have that

$$F(x,A) \cap \big(-\operatorname{int} C_U(B)\big) = \emptyset \tag{13}$$

does not hold, that is,

$$F(x,A) \cap \big(-\operatorname{int} C_U(B)\big) \neq \emptyset.$$

Thus, for each neighborhood U of 0 with $U \subset V_B$, there exists $y_U \in A$ such that

$$F(x, y_U) \in -\operatorname{int} C_U(B). \tag{14}$$

Since $y^* \in C^\Delta(B)$, by Lemma 2.1, there exists some $V \subset V_B$ such that $y^* \in (C_V(B))^* \setminus \{0\}$. For this V, by (14), there exists $y_V \in A$ such that

$$F(x, y_V) \in -\operatorname{int} C_V(B). \tag{15}$$

By $y^* \in (C_V(B))^* \setminus \{0\}$ and (15), we have that

$$y^*\big(F(x, y_V)\big) < 0. \tag{16}$$

Notice $y_V \in A$, we have $g(y_V) \in K$. By $z^* \in -K$, we have

$$z^*\big(g(y_V)\big) \leqslant 0. \tag{17}$$

From (16) and (17), we obtain that

$$y^*\big(F(x, y_V)\big) + z^*\big(g(y_V)\big) < 0.$$

But by (11), we have

$$0 = y^*\big(F(x,x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big(F(x,y)\big) + z^*\big(g(y)\big)\big\}.$$

This is a contradiction. Hence, x is a Henig efficient solution to the VEPC. $\square$

If C has a bounded closed base B, in view of Lemma 2.1, we have $\operatorname{int} C^* = C^{\Delta}(B)$. Moreover, by Proposition 2 of [23], $x \in A$ is a superefficient solution to the VEPC if and only if $x \in A$ is a Henig efficient solution to the VEPC. Hence, by Theorem 3.2, we have the following corollary.

Corollary 3.1. *Assume that the assumption* (A) *is satisfied, and that C has a bounded closed base B. Then $x \in A$ is a superefficient solution to the VEPC if and only if there exist $y^* \in \operatorname{int} C^*$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(F(x,x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big(F(x,y)\big) + z^*\big(g(y)\big)\big\},$$

where $\operatorname{int} C^*$ *is the interior of C^* with respect to $\beta(Y^*, Y)$.*

Theorem 3.3. *Assume that the assumption* (A) *is satisfied, and that C has a base B. Then $x \in A$ is a globally efficient solution to the VEPC if and only if there exist $y^* \in C^\sharp$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(F(x,x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big(F(x,y)\big) + z^*\big(g(y)\big)\big\}.$$

Proof. Assume that $x \in A$ is a globally efficient solution to the VEPC. By definition, there exists pointed convex cone $H \subset Y$ such that $C \setminus \{0\} \subset \operatorname{int} H$ and

$$F(x, A) \cap \big((-H) \setminus \{0\}\big) = \emptyset. \tag{18}$$

Define the set

$$M = \big\{(y,z) \in Y \times Z\text{: there exists } y' \in X_0 \text{ such that } y - F(x,y') \in \operatorname{int} H,\ g(y') - z \in \operatorname{int} K\big\}.$$

It is clear that $M \neq \emptyset$. By the C-convexity of F in second variable, the K-concaveness of g, and $C \setminus \{0\} \subset \operatorname{int} H$, we can see that M is a convex set. It is clear that M is an open set. We claim that $(0,0) \notin M$. If not, then there exists $y' \in X_0$ such that

$$0 - F(x,y') \in \operatorname{int} H, \qquad g(y') - 0 \in \operatorname{int} K.$$

Then $F(x,y') \in -\operatorname{int} H$, and $y' \in A$. Since H is a pointed cone, $F(x,y') \neq 0$. This contradicts (18). Thus $(0,0) \notin M$. By the separation theorem of convex sets, there exists $(0,0) \neq (y^*, z^*) \in (Y \times Z)^* = Y^* \times Z^*$ such that

$$0 < y^*(y) + z^*(z) \quad \text{for all } (y,z) \in M. \tag{19}$$

Let $(y,z) \in M$. Then there exists $y' \in X_0$ such that $y - F(x,y') \in \operatorname{int} H$, $g(y') - z \in \operatorname{int} K$. Hence, for every $c \in \operatorname{int} H$, $k \in \operatorname{int} K$, $t > 0$, $t' > 0$, we have $(y + tc, z) \in M$, and $(y, z - t'k) \in M$, this implies that $y^* \in H^*$ and $z^* \in -K^*$. In a way similar to the proof of Theorem 3.1, we have $y^* \neq 0$. It is from $C \setminus \{0\} \subset \operatorname{int} H$ and $0 \neq y^* \in H^*$, we can see that $y^* \in C^\sharp$. We have that

$$\big(F(x,y) + c, g(y) - k\big) \in M \quad \text{for all } y \in X_0,\ c \in \operatorname{int} H,\ k \in \operatorname{int} K.$$

By (19), we can obtain that

$$0 \leqslant y^*\big(F(x,y)\big) + z^*\big(g(y)\big) \quad \text{for all } y \in X_0. \tag{20}$$

It is clear that

$$\big(F(x,x)+tc,\, g(x)-tk\big)\in M \quad \text{for all } c\in \operatorname{int} H,\ k\in \operatorname{int} K,\ t>0.$$

By (19) and assumption (A), we have

$$0<y^*\big(F(x,x)+tc\big)+z^*\big(g(x)-tk\big)=ty^*(c)+z^*\big(g(x)\big)-tz^*(k).$$

Letting $t\to 0$, we obtain $0\leqslant z^*(g(x))$. Noting that $x\in A$, and $z^*\in -K^*$, we have $z^*(g(x))\leqslant 0$. Thus,

$$z^*\big(g(x)\big)=0. \tag{21}$$

From $F(x,x)=0$, (20), and (21), we get

$$y^*\big(F(x,x)\big)+z^*\big(g(x)\big)=\min_{y\in X_0} y^*\big(F(x,y)\big)+z^*\big(g(y)\big). \tag{22}$$

Conversely, let $x\in A$, and suppose that there exist $y^*\in C^\sharp$, $z^*\in -K^*$ such that

$$z^*\big(g(x)\big)=0$$

and

$$y^*\big(F(x,x)\big)+z^*\big(g(x)\big)=\min_{y\in X_0}\big\{y^*\big(F(x,y)\big)+z^*\big(g(y)\big)\big\}. \tag{23}$$

We will show that x is a globally efficient solution to the VEPC, that is, there exists a pointed convex cone H such that $C\setminus\{0\}\subset \operatorname{int} H$ and

$$F(x,A)\cap\big((-H)\setminus\{0\}\big)=\emptyset. \tag{24}$$

Suppose to the contrary that for any pointed convex cone H with $C\setminus\{0\}\subset\operatorname{int} H$, we have

$$F(x,A)\cap\big((-H)\setminus\{0\}\big)\neq\emptyset. \tag{25}$$

By $y^*\in C^\sharp$, we set

$$H_0=\big\{y\in Y:\ y^*(y)>0\big\}\cup\{0\}. \tag{26}$$

We have $C\setminus\{0\}\subset\operatorname{int} H_0$, and H_0 is a pointed convex cone. By (25), there exists $y_{H_0}\in A$ such that

$$F(x,y_{H_0})\in\big(F(x,A)\cap\big((-H_0)\setminus\{0\}\big)\big).$$

By the definition of H_0, we have that

$$y^*\big(F(x,y_{H_0})\big)<0. \tag{27}$$

Notice $y_{H_0}\in A$, $g(y_{H_0})\in K$, we have

$$z^*\big(g(y_{H_0})\big)\leqslant 0. \tag{28}$$

From (27) and (28), we obtain that

$$y^*\big(F(x,y_{H_0})\big)+z^*\big(g(y_{H_0})\big)<0.$$

But by (23), we have

$$0=y^*\big(F(x,x)\big)+z^*\big(g(x)\big)=\min_{y\in X_0}\big\{y^*\big(F(x,y)\big)+z^*\big(g(y)\big)\big\}.$$

This is a contradiction. Hence, x is a globally efficient solution to the VEPC. $\square$

4. Application

In this section, we use the results of Section 3 to get the optimality conditions for weakly efficient solution, Henig efficient solution, globally efficient solution, and superefficient solution to the vector variational inequalities and vector optimization problems, respectively.

Let X_0 be a nonempty convex subset of X, $g : X_0 \to Z$ be a mapping. Let

$$A = \{x \in X_0\colon\ g(x) \in K\}.$$

Theorem 4.1. *Assume that $g : X_0 \to Z$ is a K-concave mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, $\operatorname{int} C \neq \emptyset$, and that $T : A \to L(X, Y)$ is a mapping. Then $x \in A$ is a weakly efficient solution to the VVIC if and only if there exist $y^* \in C^* \setminus \{0\}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big((Tx, x - x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big((Tx, y - x)\big) + z^*\big(g(y)\big)\big\}.$$

Proof. Let $F(x, y) = (Tx, y - x)$, $x, y \in A$. It is clear that for each $x \in X_0$, $F(x, x) = 0$, and $F(x, y)$ is C-convex in y. By assumption, we can see that the conditions of Theorem 3.1 are satisfied. Combined with Definition 2.4, we have that $x \in A$ is a weakly efficient solution to the VVIC if and only if there exist $y^* \in C^* \setminus \{0\}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and

$$y^*\big((Tx, x - x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big((Tx, y - x)\big) + z^*\big(g(y)\big)\big\}.$$

The proof is completed. □

Similar as in the proof of Theorem 4.1, by Theorem 3.2, Corollary 3.1, and Theorem 3.3, we can get the following theorems.

Theorem 4.2. *Assume that $g : X_0 \to Z$ is a K-concave mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, C has a base B, and that $T : A \to L(X, Y)$ is a mapping. Then $x \in A$ is a Henig efficient solution to the VVIC if and only if there exist $y^* \in C^{\Delta}(B)$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big((Tx, x - x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big((Tx, y - x)\big) + z^*\big(g(y)\big)\big\}.$$

Corollary 4.1. *Assume that $g : X_0 \to Z$ is a K-concave mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, C has a bounded closed base B, and that $T : A \to L(X, Y)$ is a mapping. Then $x \in A$ is a superefficient solution to the VVIC if and only if there exist $y^* \in \operatorname{int} C^*$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big((Tx, x - x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big((Tx, y - x)\big) + z^*\big(g(y)\big)\big\},$$

where $\operatorname{int} C^*$ is the interior of C^* with respect to $\beta(Y^*, Y)$.

Theorem 4.3. *Assume that $g : X_0 \to Z$ is a K-concave mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, C has a base B, and that $T : A \to L(X, Y)$ is a mapping. Then $x \in A$ is a globally efficient solution to the VVIC if and only if there exist $y^* \in C^{\sharp}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big((Tx, x - x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big((Tx, y - x)\big) + z^*\big(g(y)\big)\big\}.$$

Theorem 4.4. *Assume that $g : X_0 \to Z$ is a K-concave mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, $\operatorname{int} C \neq \emptyset$, and $f : A \to Y$ is a C-convex mapping. Then $x \in A$ is a weakly efficient solution to the VOPC if and only if there exist $y^* \in C^* \setminus \{0\}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0}\big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}.$$

Proof. Let $F(x,y) = f(y) - f(x)$, $x, y \in A$. It is clear that for each $x \in X_0$, $F(x,x) = 0$. Since f is a C-convex mapping, $F(x,y)$ is C-convex in y. By assumption, the conditions of Theorem 3.1 are satisfied. Combined with Definition 2.5, we have that $x \in A$ is a weakly efficient solution to the VOPC if and only if there exist $y^* \in C^* \setminus \{0\}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}.$$

The proof is completed. □

Similar as in the proof of Theorem 4.4, by Theorem 3.2, Corollary 3.1, Theorem 3.3, and Definition 2.5, we can get the following theorems.

Theorem 4.5. *Assume that $g : X_0 \to Z$ is a K-concave mapping, $f : A \to Y$ is a C-convex mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, and C has a base B. Then $x \in A$ is a Henig efficient solution to the VOPC if and only if there exist $y^* \in C^{\Delta}(B)$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}.$$

Corollary 4.2. *Assume that $g : X_0 \to Z$ is a K-concave mapping, $f : A \to Y$ is a C-convex mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, and that C has a bounded closed base B. Then $x \in A$ is a superefficient solution to the VOPC if and only if there exist $y^* \in \operatorname{int} C^*$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\},$$

where $\operatorname{int} C^*$ *is the interior of C^* with respect to $\beta(Y^*, Y)$.*

Theorem 4.6. *Assume that $g : X_0 \to Z$ is a K-concave mapping, $f : A \to Y$ is a C-convex mapping, there exists $x_0 \in X_0$ such that $g(x_0) \in \operatorname{int} K$, and that C has a base B. Then $x \in A$ is a globally efficient solution to the VOPC if and only if there exist $y^* \in C^{\sharp}$, $z^* \in -K^*$ such that $z^*(g(x)) = 0$ and*

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in X_0} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}.$$

Example 4.1. Let $X = R$, $X_0 = [-1, 1]$, $Y = Z = R^2$, and let

$$C = K = R_+^2 = \big\{x = (x_1, x_2)\colon\ x_1 \geqslant 0,\ x_2 \geqslant 0\big\}.$$

We define the mappings $f, g : [-1, 1] \to R^2$ by

$$f(x) = \big(x, x^2\big), \quad x \in [-1, 1],$$
$$g(x) = (-x, -x), \quad x \in [-1, 1],$$

respectively, and let

$$A = \big\{x \in [-1, 1]\colon\ g(x) \in R_+^2\big\}.$$

It is clear that f is C-convex on X_0, g is K-concave on X_0, $A = [-1, 0]$, and $g(-1/2) \in \operatorname{int} K$. We can see that $V_W = [-1, 0]$.

If $x = 0$, we pick $y^* = (0, 1) \in (R_+^2)^* \setminus \{0\}$ and $z^* = (0, 0) \in -R_+^2$. Then $z^*(g(0)) = 0$ and

$$y^*\big(f(0)\big) + z^*\big(g(0)\big) = \min_{y \in [-1,1]} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\},$$

that is,

$$0 = \big\langle (0, 1), (0, 0)\big\rangle = \min_{y \in [-1,1]} \big\langle (0, 1), \big(y, y^2\big)\big\rangle = \min_{y \in [-1,1]} y^2.$$

If $-1 \leqslant x < 0$, we pick $y^* = (\eta_1, \eta_2) = (1, -1/2x) \in (R^2_+)^* \setminus \{0\} = R^2_+ \setminus \{0\}$ and $z^* = (0,0) \in -R^2_+$. We have $z^*(g(x)) = 0$. We show that

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in [-1,1]} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}$$

holds. Define the function h as the following:

$$h(y) = y + \eta_2 y^2, \quad y \in [-1, 1].$$

The first and second derivatives of $h(y)$ are

$$h'(y) = 1 + 2\eta_2 y, \qquad h''(y) = 2\eta_2.$$

Let $h'(y) = 0$. We have

$$1 + 2\eta_2 y = 0. \tag{29}$$

Thus, $y = -(1/2\eta_2) = -(1/(-2/2x)) = x$ is a unique solution of (29). We can see that $h(y)$ attains its infimum at x, that is,

$$h(x) = \min_{y \in [-1,1]} h(y).$$

That is,

$$x + \eta_2 x^2 = \min_{y \in [-1,1]} \big\{y + \eta_2 y^2\big\}. \tag{30}$$

By (30), we have that

$$\begin{aligned} y^*\big(f(x)\big) + z^*\big(g(x)\big) &= \big\langle(1, -1/2x), \big(x, x^2\big)\big\rangle + 0 = x/2 = x + \eta_2 x^2 \\ &= \min_{y \in [-1,1]} \big\{y + \eta_2 y^2\big\} = \min_{y \in [-1,1]} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}. \end{aligned}$$

Example 4.2. Let X, X_0, Y, Z, C, K, f, g, and A be as in Example 4.1. It is clear that R^2_+ has a bounded closed base B. Since $C^* = (R^2_+)^* = R^2_+$, and by Lemma 2.1, we can see that

$$\operatorname{int} C^* = C^{\Delta}(B) = C^{\sharp} = \operatorname{int} R^2_+.$$

Since f is R^2_+-convex, by Theorem 2.1 of [24], we can see that $V_H = V_G = V_S = [-1, 0)$. As in the proof of Example 4.1, we can see that for any $x \in [-1, 0)$, there exists $y^* = (1, -1/2x) \in \operatorname{int} R^2_+$, and $z^* = (0,0) \in -(R^2_+)^*$ such that $z^*(g(x)) = 0$ and

$$y^*\big(f(x)\big) + z^*\big(g(x)\big) = \min_{y \in [-1,1]} \big\{y^*\big(f(y)\big) + z^*\big(g(y)\big)\big\}.$$

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