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The Spectral Abscissa of Partitioned Matrices*

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1. INTRODUCTION AND PRELIMINARIES

Let C^n denote the vector space of column n-tuples of complex numbers and let M_n denote the algebra of complex $n \times n$ matrices.

If $A \in M_n$, then the *spectral abscissa* of A, denoted $\alpha(A)$, is the largest real part of the eigenvalues of A, i.e.,

$$\alpha(A) = \max\{\text{Re}\, \lambda: \lambda \text{ is an eigenvalue of } A\}.$$

We denote by e_i $(i = 1,..., n)$ the vector in C^n whose components are $\delta_{i1},..., \delta_{in}$, where δ_{ij} is the Kronecker delta function. The identity matrix in M_n is denoted by I.

If $r_0, r_1,..., r_k$ are nonnegative integers such that

$$0 = r_0 < r_1 < \cdots < r_k = n, \tag{1}$$

then the direct-sum decomposition of C^n, given by

$$C^n = W_1 \oplus \cdots \oplus W_k,$$

where

$$W_j = \text{span}\{e_{r_{j-1}+1}, e_{r_{j-1}+2},..., e_{r_j}\},$$

will be called a *partition* of C^n. Clearly, a partition of C^n is completely determined by any finite collection of integers $\pi = \{r_0, r_1,..., r_k\}$ satisfying (1). By abuse of language, we will say that π *is* a partition of C^n. The projections associated with this partition are the k $n \times n$ matrices $P_1,..., P_k$, where

$$P_j = \sum_{q=r_{j-1}+1}^{r_j} e_q e_q^*.$$

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Let ϕ be a norm on C^n. The operator norm on M_n induced by ϕ (called also the matrix norm subordinate to ϕ [1]) will be denoted by $\| \ \|_\phi$. Thus, if $A \in M_n$, then

$$\| A \|_\phi = \max_{\substack{x \in C^n \\ x \neq 0}} (\phi(Ax)/\phi(x)).$$

It is known [1–3] that for $A = (a_{ij}) \in M_n$ and for the Hölder norms h_∞, h_1, h_2, we have, respectively,

$$\| A \|_{h_\infty} = \max_i \sum_{j=1}^{n} | a_{ij} |,$$

$$\| A \|_{h_1} = \max_j \sum_{i=1}^{n} | a_{ij} |,$$

$$\| A \|_{h_2} = \text{largest eigenvalue of } (A^*A)^{1/2}.$$

If $A, B \in M_n$, then we denote

$$g_\phi(A, B) = \lim_{h \downarrow 0} \frac{\| A + hB \|_\phi - \| A \|_\phi}{h}.$$

It is known [4–6] that this limit exists; it is called the right Gateaux derivative of the norm at A with respect to B. The number $g_\phi(I, B)$ is called the *logarithmic derivative* of B corresponding to the norm ϕ [7, 8]. The mapping $B \to g_\phi(I, B)$ $(B \in M_n)$ of M_n into the field of real numbers is called sometimes the logarithmic norm corresponding to ϕ, although it is not a norm [9]. The concept arises in stability problems of differential equations [5, 6]. It is known [5, 6] that for every $A \in M_n$ and for every norm ϕ on C^n, we have

$$\alpha(A) \leqslant g_\phi(I, A). \tag{2}$$

Thus, $g_\phi(I, A)$ gives an upper bound for the spectral abscissa $\alpha(A)$ of the matrix A.

It is also known [5, 6] that for $A = (a_{ij}) \in M_n$ and for the Hölder norms h_∞, h_1, h_2, we have, respectively,

$$g_{h_\infty}(I, A) = \max_i \left(\operatorname{Re} a_{ii} + \sum_{\substack{j=1 \\ j \neq 1}}^{n} | a_{ij} | \right),$$

$$g_{h_1}(I, A) = \max_j \left(\operatorname{Re} a_{jj} + \sum_{\substack{i=1 \\ i \neq j}}^{n} | a_{ij} | \right),$$

$$g_{h_2}(I, A) = \text{largest eigenvalue of } \tfrac{1}{2}(A + A^*).$$

Let π be a given partition of C^n, let $P_1, ..., P_k$ be the projections associated with π and let ϕ be a given norm on C^n. If $A \in M_n$, then we denote

$$A_{\pi\phi} = \begin{pmatrix} g_\phi(P_1, P_1AP_1) & \| P_1AP_2 \|_\phi & \cdots & \| P_1AP_k \|_\phi \\ \| P_2AP_1 \|_\phi & g_\phi(P_2, P_2AP_2) & \cdots & \| P_2AP_k \|_\phi \\ \cdots & \cdots & \cdots & \cdots \\ \| P_kAP_1 \|_\phi & \| P_kAP_2 \|_\phi & \cdots & g_\phi(P_k, P_kAP_k) \end{pmatrix}$$

Clearly, the partition π of C^n induces a partitioning $A = (A_{ij})_{i,j=1,...,k}$ of A and the $n \times n$ matrix P_iAP_j is nothing but the block A_{ij} bordered appropriately by zeros. The matrix $A_{\pi\phi}$ is an essentially nonnegative $k \times k$ matrix [10].

The purpose of this paper is to prove that for a given partition π of C^n and for a large class of norms ϕ on C^n we have

$$\alpha(A) \leqslant \alpha(A_{\pi\phi}) \tag{3}$$

for all $A \in M_n$. This inequality can give better upper bounds for the spectral abscissa of A than those given by (2). We will also see that inequality (2) is a special case of (3). Our result is similar to a result of Ostrowski [11] concerning the spectral radius of a square matrix.

2. Results

THEOREM 1. *Let A be a complex $n \times n$ matrix, let π be a partition of C^n with associated projections $P_1, ..., P_k$ and let ϕ be a norm on C^n such that $\| P_j \|_\phi = 1$ ($j = 1, ..., k$). Then $\alpha(A) \leqslant \alpha(A_{\pi\phi})$.*

Proof. Denote $\beta = \alpha(A_{\pi\phi})$. Since $A_{\pi\phi}$ is an essentially nonnegative matrix, β is an eigenvalue of $A_{\pi\phi}$.

First we will assume that $A_{\pi\phi}$ is irreducible. Then, there exists a positive eigenvector y of $(A_{\pi\phi})^T$ (T denotes transpose) corresponding to β, i.e., $y^TA_{\pi\phi} = \beta y^T$. Let $y^T = (\eta_i, ..., \eta_k)$. Denoting $q_{ii} = g_\phi(P_i, P_iAP_i)$, $q_{ij} = \| P_iAP_j \|_\phi$ ($i \neq j$; $i, j = 1, ..., k$), the equality $y^TA_{\pi\phi} = \beta y^T$ becomes

$$q_{1i}\eta_1 + q_{2i}\eta_2 + \cdots + q_{ki}\eta_k = \beta\eta_i \qquad (i = 1, ..., k). \tag{4}$$

Now, let λ be an arbitrary eigenvalue of A and let x be a corresponding eigenvector. It can be easily seen that the relation $Ax = \lambda x$ is equivalent to the following relations

$$(P_1AP_1)\, P_1x + (P_1AP_2)\, P_2x + \cdots + (P_1AP_k)\, P_kx = \lambda P_1x.$$
$$\cdots$$
$$\cdots$$

From these relations we obtain for all $h > 0$

$$(P_1 + hP_1AP_1)\,P_1x + h(P_1AP_2)\,P_2x + \cdots + h(P_1AP_k)\,P_kx = (1 + h\lambda)\,P_1x,$$
$$\cdots$$
$$\cdots$$

whence

$$| 1 + h\lambda |\,\phi(P_1x) \leqslant \| P_1 + hP_1AP_1 \|_\phi\,\phi(P_1x) + h\,\| P_1AP_2 \|_\phi\,\phi(P_2x) + \cdots + h\,\| P_1AP_k \|_\phi\,\phi(P_kx),$$
$$\cdots$$
$$\cdots$$

or, taking into account that

$$\| P_j \|_\phi = 1 \qquad (j = 1,..., k),$$

$$\frac{| 1 + h\lambda | - 1}{h}\,\phi(P_1x) \leqslant \frac{\| P_1 + hP_1AP_1 \|_\phi - \| P_1 \|_\phi}{h}\,\phi(P_1x) + \| P_1AP_2 \|_\phi\,\phi(P_2x) + \cdots + \| P_1AP_k \|_\phi\,\phi(P_kx).$$
$$\cdots$$
$$\cdots$$

Letting $h \to 0$, we obtain

$$(\operatorname{Re} \lambda)\,\phi(P_1x) \leqslant q_{11}\phi(P_1x) + q_{12}\phi(P_2x) + \cdots + q_{1k}\phi(P_kx).$$
$$\cdots$$
$$\cdots$$

Multiplying these relations by $\eta_1 ,..., \eta_k$, respectively, and adding them, we obtain, after making use of (4),

$$(\operatorname{Re} \lambda)\, c \leqslant \beta c,$$

where

$$c = \eta_1\phi(P_1x) + \cdots + \eta_k\phi(P_kx).$$

Since $\eta_j > 0$ for all $j = 1,..., k$ and $x = P_1x + \cdots + P_kx \neq 0$, we have $c > 0$. Thus $\operatorname{Re} \lambda \leqslant \beta = \alpha(A_{\pi\phi})$. Since this inequality holds for every eigenvalue of A, we obtain $\alpha(A) \leqslant \alpha(A_{\pi\phi})$.

Now, let us assume that $A_{\pi\phi}$ is reducible. Then, without loss of generality, we may assume that $A_{\pi\phi}$ can be partitioned as

$$A_{\pi\phi} = \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1s} \\ 0 & Q_{22} & \cdots & Q_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & Q_{ss} \end{pmatrix},$$

where Q_{jj} $(j = 1,..., s)$ is either an irreducible square matrix or a 1×1 zero matrix. But, whenever an off-diagonal element $\| P_iAP_j \|_\phi$ of $A_{\pi\phi}$ is equal to zero, then $P_iAP_j = 0$ and so the partitioning

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{pmatrix},$$

induced by the partition π of C^n, can be partitioned further as

$$A = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1s} \\ 0 & B_{22} & \cdots & B_{2s} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & B_{ss} \end{pmatrix}.$$

Clearly, $Q_{jj} = (B_{jj})_{\pi_j \phi_j}$ ($j = 1,..., s$) where π_j and ϕ_j are the restrictions of π and ϕ, respectively, to an appropriate subspace of C^n.

Let λ be an eigenvalue of A. Then λ is an eigenvalue of B_{jj} for some $j \in \{1,..., s\}$. Since Q_{jj} is irreducible (for our purposes, a 1×1 zero matrix can be viewed as an irreducible matrix since it admits a positive eigenvector), we have from the first part of the proof

$$\operatorname{Re} \lambda \leqslant \alpha((B_{jj})_{\pi_j \phi_j}) = \alpha(Q_{jj}) \leqslant \max_j \alpha(Q_{jj}) = \alpha(A_{\pi\phi}).$$

Since this is true for every eigenvalue λ of A, we have $\alpha(A) \leqslant \alpha(A_{\pi\phi})$. This completes the proof.

COROLLARY 1. *Let $A = (a_{ij})$ be a complex $n \times n$ matrix and denote*

$$\tilde{A} = \begin{pmatrix} \operatorname{Re} a_{11} & |a_{12}| & \cdots & |a_{1n}| \\ |a_{21}| & \operatorname{Re} a_{22} & \cdots & |a_{2n}| \\ \cdots & \cdots & \cdots & \cdots \\ |a_{n1}| & |a_{n2}| & \cdots & \operatorname{Re} a_{nn} \end{pmatrix}.$$

Then $\alpha(A) \leqslant \alpha(\tilde{A})$.

Proof. In Theorem 1, taking π to be the finest partition of C^n (i.e., $\pi = \{0, 1, 2,..., n\}$ and $\phi = h_\infty$, for example, we obtain $A_{\pi\phi} = \tilde{A}$ and so $\alpha(A) \leqslant \alpha(\tilde{A})$.

Remark 1. A direct proof of Corollary 1 can be found in [12].

Remark 2. If in Theorem 1 we take π to be the coarsest partition of C^n, (i.e., $\pi = \{0, n\}$), then we reobtain inequality (2).

3. EXAMPLES

EXAMPLE 1. Let

$$A = \left(\begin{array}{ccc|c} -6+i & 0 & 0 & 0 \\ 0 & -4 & 0 & 1 \\ 0 & 0 & -4 & 1 \\ \hline 1 & 2 & 6 & -4+i \end{array}\right)$$

and consider the partition $\pi = \{0, 3, 4\}$ of C^4 which induces the indicated partitioning of A. Taking the norm h_∞ on C^4, we have

$$A_{\pi h_\infty} = \begin{pmatrix} -4 & 1 \\ 9 & -4 \end{pmatrix}.$$

The eigenvalue of $A_{\pi h_\infty}$ are -1, and -7. Thus $\alpha(A) \leqslant \alpha(A_{\pi h_\infty}) = -1$. In particular, A is stable. Note that from the inequality (2), in the case of the most easily computable upper bounds, we obtain only

$$\alpha(A) \leqslant g_{h_\infty}(I, A) = 5, \qquad \alpha(A) \leqslant g_{h_1}(I, A) = 2.$$

The actual value of $\alpha(A)$ is $\frac{1}{2}(31)^{1/2} - 4 \approx -1.216$.

EXAMPLE 2. Let

$$A = \left(\begin{array}{c|ccc} 5 & 2 & 5 & 1 \\ \hline 1 & -5 & 0 & 0 \\ 1 & 0 & -5 & 0 \\ 0 & 0 & 0 & -7 \end{array}\right)$$

and consider the partition $\pi = \{0, 1, 4\}$ of C^4 which induces the indicated partitioning of A. Taking the norm h_∞ on C^4, we have

$$A_{\pi h_\infty} = \begin{pmatrix} 5 & 8 \\ 1 & -5 \end{pmatrix}.$$

The eiegenvalue of $A_{\pi h_\infty}$ are $\pm(33)^{1/2}$. Thus

$$\alpha(A) \leqslant \alpha(A_{\pi h_\infty}) = (33)^{1/2} \approx 5.745.$$

Actually we have $\alpha(A) = (32)^{1/2} \approx 5.657$. Inequality (2), for $\phi = h_\infty$, h_1, gives, respectively,

$$\alpha(A) \leqslant g_{h_\infty}(I, A) = 13, \qquad \alpha(A) \leqslant g_{h_1}(I, A) = 7.$$

The upper bound $g_{h_2}(I, A)$ is more difficult to compute. It is the largest eigenvalue of the self-adjoint matrix

$$\tfrac{1}{2}(A + A^*) = \begin{pmatrix} 5 & 1.5 & 3 & 0.5 \\ 1.5 & -5 & 0 & 0 \\ 3 & 0 & -5 & 0 \\ 0.5 & 0 & 0 & -7 \end{pmatrix}.$$

We obtain

$$\alpha(A) \leqslant g_{h_2}(I, A) = \tfrac{1}{2}((171)^{1/2} - 1) \approx 6.039.$$

It is interesting to note that we can obtain a better upper bound than the last one without even computing the spectral abscissa of $A_{\pi h_\infty}$. Indeed, applying inequality (2) to $A_{\pi h_\infty}$, with $\phi = h_1$, we obtain

$$\alpha(A) \leqslant \alpha(A_{\pi h_\infty}) \leqslant g_{h_1}(I, A_{\pi h_\infty}) = 6.$$

Remark 3. Examples 1 and 2 show that for a given $A \in M_n$, a given partition π of C^n and a given norm ϕ on C^n (satisfying the assumptions of Theorem 1), the number $\alpha(A_{\pi\phi})$ may be a better upper bound for $\alpha(A)$ than the most easily computable upper bounds of $\alpha(A)$ given by inequality (2). If $\alpha(A) < \alpha(A_{\pi\phi})$, then $\alpha(A_{\pi\phi})$ cannot be smaller than $g_\psi(I, A)$ for every norm ψ on C^n, since $\inf g_\psi(I, A) = \alpha(A)$, where the infimum is taken over all norms ψ on C^n [7, 9]. However, it may happen that $\alpha(A) = \alpha(A_{\pi\phi})$ but $\alpha(A) < g_\psi(I, A)$ for every norm ψ on C^n. We illustrate this by a very simple example.

EXAMPLE 3. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then $\alpha(A) = 1$. Since 1 is a double root of the minimal polynomial of A, we have $\alpha(A) < g_\psi(I, A)$ for every norm ψ on C^3 [8, 9]. Consider the partition $\pi = \{0, 2, 3\}$ of C^3. *Then*

$$A_{\pi h} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$

and we have $\alpha(A_{\pi h_\infty}) = 1$, i.e., $\alpha(A) = \alpha(A_{\pi h_\infty})$.

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