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Hemivariational inequalities for stationary Navier–Stokes equations ☆

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Abstract

In this paper we study a class of inequality problems for the stationary Navier–Stokes type operators related to the model of motion of a viscous incompressible fluid in a bounded domain. The equations are nonlinear Navier–Stokes ones for the velocity and pressure with nonstandard boundary conditions. We assume the nonslip boundary condition together with a Clarke subdifferential relation between the pressure and the normal components of the velocity. The existence and uniqueness of weak solutions to the model are proved by using a surjectivity result for pseudomonotone maps. We also establish a result on the dependence of the solution set with respect to a locally Lipschitz superpotential appearing in the boundary condition.

Keywords: Navier–Stokes equation; Hemivariational inequality; Subdifferential; Pseudomonotone; Nonconvex

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1. Introduction

In this paper we deal with a class of inequality problems for Navier–Stokes type operators related to the model of motion of viscous incompressible fluids. We study the stationary flow of inhomogeneous viscous fluid in a regular bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. The Navier–Stokes equations are the following:

$$-\nu \sum_{j=1}^{d} \frac{\partial^2 u_i}{\partial x_j^2} + \sum_{j=1}^{d} u_j \frac{\partial u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} = f_i, \quad i = 1, \ldots, d \text{ in } \Omega, \tag{1}$$

$$\sum_{j=1}^{d} \frac{\partial u_j}{\partial x_j} = 0 \quad \text{in } \Omega. \tag{2}$$

This system describes the flow of a viscous incompressible fluid which occupies the domain Ω, $u = \{u_i\}_{i=1}^d$ denotes the velocity of the fluid, p is the pressure, $f = \{f_i\}$ is the volume density of external forces and ν is a positive constant representing the coefficient of kinematic viscosity. Using the standard Lamb formulation, we rewrite (1)–(2) in an equivalent form (see (12)–(13) in Section 4):

$$-\nu \operatorname{rot} \operatorname{rot} u + \operatorname{rot} u \times u + \nabla h = f, \qquad \operatorname{div} u = 0 \quad \text{in } \Omega, \tag{3}$$

where a function $h = p + \frac{1}{2}|u|^2$ denotes the dynamic pressure. We consider this problem under the following boundary conditions:

$$h \in \partial j(x, u_N) \quad \text{and} \quad u_\tau = 0 \quad \text{on } \Gamma. \tag{4}$$

Here $\Gamma = \partial\Omega$, u_N and u_τ denote the normal and the tangential component of u on the boundary, $u_N = u \cdot n$, $u_\tau = u - u_N n$, n being the unit outward normal on Γ and ∂j is the Clarke subdifferential of a locally Lipschitz function $j(x, \cdot)$.

It should be noted that the subdifferential boundary condition in particular cases reduces to the classical boundary conditions. If the function $j(x, \cdot)$ is assumed to be convex the problem has been studied in papers by Chebotarev [6,7]. Next, still in a convex setting Chebotarev [8] considered the boundary conditions (4) for the Boussinesq equations and Konovalova [14] studied the evolution counterpart of (3)–(4). In all these papers the considered problems were formulated as variational inequalities involving maximal monotone operators (recall that the subdifferential of a convex function is a maximal monotone map, cf. e.g. [11,15,24]). In this paper, due to the absence of convexity of the superpotential j, the formulation of (3)–(4) is not longer a variational inequality and it leads to the expressions called hemivariational inequalities. The latter have been introduced and studied by P.D. Panagiotopoulos in the early eighties as variational formulations for several classes of mechanical problems with nonsmooth and nonconvex energy superpotentials. Since that time the notion of hemivariational inequality proved to be a useful and powerful tool for formulation and solving several problems coming from mechanics and engineering. In mechanics the hemivariational inequalities express the principles of virtual work or power, see, e.g., unilateral contact problems in nonlinear elasticity and viscoelasticity, problems describing frictional and adhesive effects, problem of delamination of plates, loading and unloading problems in engineering structures in Panagiotopoulos [20,22] and Naniewicz and Panagiotopoulos [19].

In a concrete situation the problem (3)–(4) describes a model in which it is desirable to regulate the boundary orifices in a tube (or channel): our aim is to reduce the pressure of the fluid on Γ when the normal velocity reaches a given value. The multivalued boundary condition can be used to model a control problem in which the pressure is regulated by a hydraulic control device. For other flow problems dealing with semipermeable walls and membranes, and the flow through porous media, we refer to Panagiotopoulos [21], Naniewicz and Panagiotopoulos [19, Chapter 5.5.3], Goeleven et al. [13, Chapter 2.11.9], Alekseev and Smishliaev [1], Migorski and Ochal [18] and Chebotarev [8,9] and the references therein.

The goal of the paper is to show the results on the existence and uniqueness of weak solutions to a hemivariational inequality corresponding to the problem (3)–(4). The existence will be proved by employing a surjectivity result for a pseudomonotone and coercive operator. Moreover, we study the sensitivity (stability) of the solution set of the problem with respect to perturbations in the boundary condition. We provide conditions under which such perturbations cause small perturbations of the solutions.

The paper is organized as follows. In Section 2 we recall some notation and present some auxiliary material. In Section 3 we consider abstract Navier–Stokes type operators and for inclusions involving such operators we present a surjectivity result. The formulation of the boundary value problem for the stationary Navier–Stokes equation with a subdifferential boundary condition as a hemivariational inequality is given in Section 4. In this section we deliver the results on the existence and uniqueness of the weak solution to the hemivariational inequality and present an example to which our results can be applied. Finally, in Section 5, we deal with the dependence of the solution with respect to changes of the boundary condition.

2. Preliminaries

In this section we introduce the notation and recall some definitions needed in the sequel.

Let V be a reflexive Banach space. We denote by $\langle \cdot , \cdot \rangle$ the pairing between V and its dual V^*.

Definition 1. An operator $T : V \to V^*$ is said to be pseudomonotone if

(i) it is bounded (i.e., it maps bounded subsets of V into bounded subsets of V^*);
(ii) $\langle Tu, u - v\rangle \leqslant \liminf \langle Tu_n, u_n - v\rangle$ for all $v \in V$ whenever the sequence $\{u_n\}$ converges weakly in V to u with $\limsup \langle Tu_n, u_n - u\rangle \leqslant 0$.

Remark 2. The condition (ii) of Definition 1 is equivalent (still under condition (i)) to the following one:

(ii)′ if $u_n \to u$ weakly in V and $\limsup \langle Tu_n, u_n - u\rangle \leqslant 0$, then $Tu_n \to Tu$ weakly in V^* and $\lim \langle Tu_n, u_n - u\rangle = 0$.

In fact, to show that (ii)′ implies (ii), it is enough to observe that for every $v \in V$ we have

$$\liminf \langle Tu_n, u_n - v\rangle \geqslant \liminf \langle Tu_n, u_n - u\rangle + \liminf \langle Tu_n, u - v\rangle = \langle Tu, u - v\rangle.$$

Conversely, putting $v = u$ in the condition in (ii), we have

$$0 \leqslant \liminf \langle Tu_n, u_n - u\rangle \leqslant \limsup \langle Tu_n, u_n - u\rangle \leqslant 0,$$

hence $\langle Tu_n, u_n - u\rangle \to 0$. Moreover, taking $v = u - \lambda w$, $\lambda \in \mathbb{R}$, $w \in V$, we get

$$\begin{aligned}\langle Tu, \lambda w\rangle &\leqslant \liminf \langle Tu_n, u_n - u + \lambda w\rangle \\ &= \lim \langle Tu_n, u_n - u\rangle + \liminf \langle Tu_n, \lambda w\rangle \leqslant \liminf \langle Tu_n, \lambda w\rangle.\end{aligned}$$

Since $\lambda \in \mathbb{R}$ is arbitrary, we obtain $\lim \langle Tu_n, w\rangle = \langle Tu, w\rangle$ for all $w \in V$.

Definition 3. A multivalued operator $T : V \to 2^{V^*}$ is said to be pseudomonotone if the following conditions hold:

(i) the set Tv is nonempty, bounded, closed and convex for all $v \in V$;
(ii) T is usc from each finite dimensional subspace of V into V^* endowed with the weak topology;
(iii) if $v_n \in V$, $v_n \to v$ weakly in V and $v_n^* \in Tv_n$ is such that $\limsup \langle v_n^*, v_n - v\rangle \leqslant 0$, then to each $y \in V$, there exists $v^*(y) \in Tv$ such that $\langle v^*(y), v - y\rangle \leqslant \liminf \langle v_n^*, v_n - y\rangle$.

Definition 4. An operator $T : V \to 2^{V^*}$ is said to be generalized pseudomonotone if for every sequences $v_n \to v$ weakly in V, $v_n^* \to v^*$ weakly in V^*, $v_n^* \in Tv_n$ and $\limsup \langle v_n^*, v_n - v\rangle \leqslant 0$, we have $v^* \in Tv$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$.

The following result is well-known, cf. Browder and Hess [3] and Zeidler [24].

Proposition 5. *If $T : V \to 2^{V^*}$ is a generalized pseudomonotone operator which is bounded and has nonempty, closed and convex values, then T is pseudomonotone.*

We recall the definitions of the generalized directional derivative and the generalized gradient of Clarke for a locally Lipschitz function (see Clarke [10]).

Definition 6. Let $h : E \to \mathbb{R}$ be a locally Lipschitz function defined on a Banach space E. The generalized directional derivative of h at $x \in E$ in the direction $v \in E$, denoted by $h^0(x; v)$, is defined by

$$h^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{h(y + tv) - h(y)}{t}.$$

The generalized gradient of h at x, denoted by $\partial h(x)$, is a subset of a dual space E^* given by

$$\partial h(x) = \bigl\{\zeta \in E^*:\ h^0(x; v) \geqslant \langle \zeta, v\rangle_{E^* \times E} \text{ for all } v \in E\bigr\}.$$

The locally Lipschitz function h is called regular (in the sense of Clarke) at $x \in E$ if for all $v \in E$ the one-sided directional derivative $h'(x; v)$ exists and satisfies $h^0(x; v) = h'(x; v)$ for all $v \in E$.

Finally we state the chain rules for the generalized directional derivative and the generalized gradient which are needed in the sequel.

Proposition 7. *Let X and Y be Banach spaces, $L \in \mathcal{L}(Y, X)$ and let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a locally Lipschitz function. Then*

(i) $(f \circ L)^0(x; z) \leqslant f^0(Lx; Lz)$ *for* $x, z \in Y$,
(ii) $\partial(f \circ L)(x) \subseteq L^*\partial f(Lx)$ *for* $x \in Y$,

where $L^* \in \mathcal{L}(X^*, Y^*)$ denotes the adjoint operator to L. If in addition either f or $-f$ is regular, then in both (i) *and* (ii) *the equalities hold.*

For the proof of the proposition we refer to Theorem 2.3.10 of Clarke [10].

3. Abstract setting

In this section we deliver the main result of the paper on the existence of solutions to an abstract inclusion.

Let V be a reflexive separable Banach space and let V^* be its dual. We denote by H a Hilbert space such that $V \subset H$ with dense and compact embedding. Identifying H with its dual, we have an evolution triple of spaces $V \subset H \subset V^*$ (cf. Lions [15], Zeidler [24]). The norms in V, H and V^* are denoted by $\|\cdot\|_V$, $|\cdot|_H$ and $\|\cdot\|_{V^*}$, respectively. The pairing between V and V^* is denoted by $\langle\cdot,\cdot\rangle$.

Definition 8. An operator $N : V \to V^*$ is called a Navier–Stokes type operator if $Nv = Av + B[v]$, where

(1) $A : V \to V^*$ is a linear, continuous, symmetric operator such that
$$\langle Av, v\rangle \geqslant \alpha\|v\|_V^2 \quad \text{for } v \in V \text{ with } \alpha > 0;$$
(2) $B[v] = B(v, v)$, $B : V \times V \to V^*$ is a bilinear continuous operator satisfying the conditions:
  (2a) $\langle B(u, v), v\rangle = 0$ for $u, v \in V$,
  (2b) the map $B[\cdot] : V \to V^*$ is weakly continuous.

Lemma 9. *The Navier–Stokes type operator is coercive and pseudomonotone.*

Proof. The coerciveness of N is a consequence of the conditions (1) and (2a) of Definition 8, namely for every $v \in V$, we have
$$\langle Nv, v\rangle = \langle Av, v\rangle + \big\langle B(v, v), v\big\rangle \geqslant \alpha\|v\|_V^2. \tag{5}$$

The boundedness of N follows from the facts that A is linear, continuous and B is bilinear and continuous. Now we prove the condition (ii) of Definition 1. Let $u_n \to u$ weakly in V, $\limsup\langle Nu_n, u_n - u\rangle \leqslant 0$ and let $v \in V$. By the conditions (2a) and (2b) of Definition 8, we have

$$\big\langle B[u_n], u_n - v\big\rangle - \big\langle B[u], u - v\big\rangle = \big\langle B[u_n], u_n\big\rangle - \big\langle B[u_n], v\big\rangle - \big\langle B[u], u\big\rangle + \big\langle B[u], v\big\rangle$$
$$= \big\langle B[u], v\big\rangle - \big\langle B[u_n], v\big\rangle \to 0,$$

which implies

$$\lim\big\langle B[u_n], u_n - v\big\rangle = \big\langle B[u], u - v\big\rangle \quad \text{for all } v \in V. \tag{6}$$

Hence in particular we have $\lim\langle B[u_n], u_n - u\rangle = 0$. Thus

$$\limsup\langle Au_n, u_n - u\rangle = \limsup\langle Au_n, u_n - u\rangle + \lim\big\langle B[u_n], u_n - u\big\rangle$$
$$= \limsup\big\langle Au_n + B[u_n], u_n - u\big\rangle = \limsup\langle Nu_n, u_n - u\rangle \leqslant 0.$$

From the pseudomonotonicity of A, we obtain

$$\langle Au, u - v\rangle \leqslant \liminf\langle Au_n, u_n - v\rangle \quad \text{for all } v \in V,$$

which together with (6) yields

$$\langle Nu, u - v\rangle \leqslant \liminf\langle Nu_n, u_n - v\rangle \quad \text{for all } v \in V.$$

The proof is completed. □

In order to formulate the problem under consideration, we introduce a reflexive Banach space Z such that $V \subset Z \subset H \simeq H^* \subset Z^* \subset V^*$. We assume that the embeddings $V \subset Z \subset H$ are dense and compact. The pairing between Z and Z^* is denoted by $\langle\cdot,\cdot\rangle_{Z^*\times Z}$.

In what follows we also consider an operator $R : Z \to 2^{Z^*}$ which satisfies the hypothesis

$\underline{H(R)}$: $R : Z \to 2^{Z^*}$ is a multivalued map such that
- (i) R has nonempty, convex and weakly compact values;
- (ii) R has a graph closed in $Z \times (w\text{–}Z^*)$ topology;
- (iii) $\|Rz\|_{Z^*} \leqslant \bar{c}(1 + \|z\|_Z^\rho)$ for all $z \in Z$ with $\bar{c} > 0$ and $0 \leqslant \rho \leqslant 1$,

where w–Z^* denotes the space Z^* equipped with the weak topology.

The goal is now to establish certain properties of the operator $\mathcal{F} : V \to 2^{V^*}$ defined by

$$\mathcal{F}v = Nv + Rv \quad \text{for } v \in V.$$

Proposition 10. *Let N be the Navier–Stokes type operator and let R be an operator satisfying $H(R)$. Then*

- (a) *$\mathcal{F}$ is pseudomonotone*;
- (b) *if $0 \leqslant \rho < 1$, then $\mathcal{F}$ is coercive. If $\rho = 1$, then $\mathcal{F}$ is also coercive provided $\alpha - \bar{c}\beta^2 > 0$, where $\beta > 0$ is an embedding constant of $V \subset Z$.*

Proof. For the proof of (a) we apply Proposition 5. It is clear from $H(R)(i)$ that $\mathcal{F}$ has nonempty, convex and closed values. Moreover, from $H(R)$(iii) and Lemma 9, it follows that $\mathcal{F}$ is a bounded map. It remains to show that $\mathcal{F}$ is a generalized pseudomonotone. To this end, let $v_n \to v$ weakly in V, $v_n^* \to v^*$ weakly in V^*, $v_n^* \in \mathcal{F}v_n$ and $\limsup \langle v_n^*, v_n - v\rangle \leqslant 0$. We will show that $v^* \in \mathcal{F}v$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$. Since $v_n^* \in \mathcal{F}v_n$, we have $v_n^* = Nv_n + \zeta_n$ with $\zeta_n \in Rv_n$. From the continuity of the embedding $V \subset Z$, it follows that $\{v_n\}$ lies in a bounded subset of Z. Thus the boundedness of the map R allows to assume, by passing to a subsequence if necessary, that

$$\zeta_n \to \zeta \quad \text{weakly in } Z^* \quad \text{with } \zeta \in Z^*. \tag{7}$$

Since $V \subset Z$ compactly, we may also suppose that

$$v_n \to v \quad \text{in } Z. \tag{8}$$

By $H(R)$(ii) we deduce that $\zeta \in Rv$. Moreover, from the equality

$$\langle v_n^*, v_n - v\rangle = \langle Nv_n, v_n - v\rangle + \langle \zeta_n, v_n - v\rangle_{Z^*\times Z}$$

by using (7) and (8), we have

$$\limsup \langle Nv_n, v_n - v\rangle = \limsup \langle v_n^*, v_n - v\rangle \leqslant 0.$$

By virtue of the pseudomonotonicity of N (cf. Lemma 9), from Remark 2, we obtain

$$Nv_n \to Nv \quad \text{weakly in } V^* \tag{9}$$

and

$$\lim \langle Nv_n, v_n - v\rangle = 0. \tag{10}$$

Exploiting (7) and (9), and passing to the limit in the equality $v_n^* = Nv_n + \zeta_n$, we get $v^* = Nv + \zeta$ which together with $\zeta \in Rv$ implies that $v^* \in Nv + Rv = \mathcal{F}v$.

Finally, from (7)–(10), we have

$$\begin{aligned}\lim \langle v_n^*, v_n\rangle &= \lim \langle Nv_n, v_n - v\rangle + \lim \langle Nv_n, v\rangle + \lim \langle \zeta_n, v_n\rangle_{Z^*\times Z}\\ &= \langle Nv, v\rangle + \langle \zeta, v\rangle_{Z^*\times Z} = \langle v^*, v\rangle,\end{aligned}$$

which completes the proof of (a).

For the proof of (b), we observe that by (5), we have

$$\langle \mathcal{F}v, v\rangle = \langle Nv, v\rangle + \langle \zeta, v\rangle_{Z^*\times Z} \geqslant \alpha \|v\|_V^2 + \langle \zeta, v\rangle_{Z^*\times Z} \quad \text{for all } v \in V$$

with $\zeta \in Rv$. From the hypothesis $H(R)$(iii) we deduce

$$\begin{aligned}\left|\langle \zeta, v\rangle_{Z^*\times Z}\right| &\leqslant \|\zeta\|_{Z^*}\|v\|_Z \leqslant \bar{c}\left(1 + \|v\|_Z^\rho\right)\|v\|_Z\\ &= \bar{c}\|v\|_Z + \bar{c}\|v\|_Z^{\rho+1} \leqslant \bar{c}\beta\|v\|_V + \bar{c}\beta^{\rho+1}\|v\|_V^{\rho+1},\end{aligned}$$

where $\beta > 0$ is such that $\|\cdot\|_Z \leqslant \beta\|\cdot\|_V$. Hence

$$\langle \zeta, v\rangle_{Z^*\times Z} \geqslant -\bar{c}\beta\|v\|_V - \bar{c}\beta^{\rho+1}\|v\|_V^{\rho+1}.$$

Therefore for $0 \leqslant \rho < 1$ the map $\mathcal{F}$ is coercive without assuming any additional conditions. If $\rho = 1$, then $\mathcal{F}$ is coercive provided $\alpha - \bar{c}\beta^2 > 0$. This finishes the proof of the proposition. □

The following follows from the fundamental surjectivity result of nonlinear analysis, cf. Zeidler [24, Section 32.4] or Denkowski et al. [11, Theorem 1.3.70].

Corollary 11. *Under the hypotheses of Proposition* 10, *the operator* $\mathcal{F}: V \to 2^{V^*}$ *is surjective, i.e., for every* $f \in V^*$ *there is* $u \in V$ *such that* $Nu + Ru \ni f$.

4. Application to hemivariational inequalities for Navier–Stokes equations

In this section we consider the boundary value problem for the stationary Navier–Stokes equation with a subdifferential boundary condition. We give a variational formulation of the problem and applying results of Section 3, we establish the existence of weak solutions. Finally, we comment on the uniqueness of solutions to this problem.

Let Ω be a bounded simply connected domain in $\mathbb{R}^d$, $d = 2, 3$, with connected boundary Γ of class C^2. We consider the following system of stationary Navier–Stokes equations:

$$-\nu \Delta u + (u \cdot \nabla)u + \nabla p = f, \qquad \nabla \cdot u = 0 \quad \text{in } \Omega. \tag{11}$$

The system describes the steady state flow of incompressible viscous fluid occupying the volume Ω subjected to given volume forces f. Here $u = \{u_i(x)\}_{i=1}^d$ is the velocity field, p the pressure, $\nu > 0$ the kinematic viscosity of the fluid ($\nu = 1/\mathrm{Re}$, where Re is the Reynolds number), $f = \{f_i(x)\}_{i=1}^d$ the density of external forces. The nonlinear term $(u \cdot \nabla)u$ in (11) (often called the convective term) is a symbolic notation for the vector $\{\sum_{j=1}^d u_j \frac{\partial u_i}{\partial x_j}\}_{i=1}^d$. The divergence free condition in (11) is the equation for law of mass conservation and it states that the motion is incompressible. Similarly as in the papers of Chebotarev [6–8], Konovalova [14] and Alekseev and Smishliaev [1], in order to give a variational formulation of (11) and make use of some results from those papers, it is desirable to use the standard Lamb formulation and rewrite the problem in the following way. By using the identities (see Girault and Raviart [12, Chapter I])

$$(u \cdot \nabla)u = \operatorname{rot} u \times u + \frac{1}{2}\nabla(u \cdot u), \qquad -\Delta u = \operatorname{rot}\operatorname{rot} u - \nabla \operatorname{div} u$$

and the incompressibility condition, we derive from (11) that

$$-\nu \operatorname{rot}\operatorname{rot} u + \operatorname{rot} u \times u + \nabla h = f \quad \text{in } \Omega, \tag{12}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega, \tag{13}$$

where the total head of the fluid, sometimes referred to as “total pressure” or “Bernoulli pressure,” is given by $h = p + \frac{1}{2}|u|^2$.

We suppose that on Γ the tangential components of the velocity vector are known and without loss of generality we put them equal to zero (the nonslip condition):

$$u_\tau = u - u_N n = 0 \quad \text{on } \Gamma, \tag{14}$$

where $n = \{n_i\}_{i=1}^d$ is the unit outward normal on the boundary Γ and $u_N = u \cdot n = \sum u_i n_i$ denotes the normal component of the vector u. Moreover, we assume the following subdifferential boundary condition:

$$h(x) \in \partial j\big(x, u_N(x)\big) \quad \text{for } x \in \Gamma. \tag{15}$$

Here $j:\Gamma \times \mathbb{R} \to \mathbb{R}$ is called a superpotential and denotes the function which is locally Lipschitz in the second variable and ∂j is the subdifferential of $j(x,\cdot)$ in the sense of Clarke (see Definition 6). We comment on a fluid flow control example which motivates the study of the problem (12)–(15). The condition (15) arises in the problem of motion of a fluid through a tube or channel: the fluid pumped into Ω can leave the tube at the boundary orifices while a device can change the sizes of the latter. In this problem we regulate the normal velocity of the fluid on the boundary to reduce the total pressure on Γ. For instance, we consider the boundary condition (15) with

$$\partial j(\xi) = \begin{cases} 0 & \text{if } \xi < 0, \\ [0,b] & \text{if } \xi = 0, \\ \frac{c-b}{d}\xi + b & \text{if } 0 < \xi < d, \\ [a,c] & \text{if } \xi = d, \\ \frac{a}{d}\xi & \text{if } \xi > d, \end{cases}$$

where $0 \leqslant a < b \leqslant c$ and $d > 0$. The condition $u_N > 0$ represents the outflow of the fluid through the boundary. If $u_N \in (0,d)$, the orifices on the boundary allow the fluid to infiltrate outside the tube; when the velocity increases so does the total pressure, say, linearly from the value b to the value c. If u_N reaches the value d, a mechanism opens the orifices wider and allows the fluid to pass through Γ. Therefore the pressure drops to a value a and we may assume that $h = c_1 u_N + c_2$ for $u_N > d$ with suitable constants c_1 and c_2. Moreover, in (15) we allow j to depend on the variable $x \in \Gamma$ which means that the subdifferential boundary condition can be of different character on different parts of Γ (see Example 18).

In order to give the weak formulation of the problem (12)–(15), we introduce the following notation:

$$\mathcal{W} = \left\{ w \in C^\infty\left(\Omega;\mathbb{R}^d\right)\colon \operatorname{div} w = 0 \text{ in } \Omega,\ w_\tau = 0 \text{ on } \Gamma \right\}.$$

We denote by V and H the closure of $\mathcal{W}$ in the norms of $H^1(\Omega;\mathbb{R}^d)$ and $L^2(\Omega;\mathbb{R}^d)$, respectively. We define $A:V \to V^*$ and $B[\cdot]:V \to V^*$ by

$$\langle Au, v\rangle = \nu \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v \, dx,$$

$$\left\langle B(u,v), w\right\rangle = \int_\Omega (\operatorname{rot} u \times v)\cdot w \, dx, \qquad B[v] = B(v,v)$$

for $u, v, w \in V$. It is known from Bykhovski and Smirnov [4] that in the case of simply connected domain Ω, the bilinear form

$$((u,v))_V = \int_\Omega \operatorname{rot} u \cdot \operatorname{rot} v \, dx$$

generates a norm in V, $\|u\|_V = ((u,u))_V^{1/2}$, which is equivalent to the $H^1(\Omega;\mathbb{R}^d)$-norm. It is clear that the operator A is coercive with a constant $\alpha > 0$. Multiplying the equation of motion (12) by $v \in V$ and applying the Green formula, we obtain

$$\langle Au + B[u], v\rangle + \int_\Gamma h v_N \, d\sigma(x) = \langle F, v\rangle,$$

where $\langle F, v\rangle = \int_\Omega f \cdot v \, dx$. From the relation (15), by using the definition of the Clarke subdifferential, we have

$$\int_\Gamma h v_N \, d\sigma(x) \leqslant \int_\Gamma j^0\big(x, u_N(x); v_N(x)\big) \, d\sigma(x),$$

where $j^0(x,\xi;\eta)$ denotes the directional derivative of $j(x,\cdot)$ at the point $\xi \in \mathbb{R}$ in the direction $\eta \in \mathbb{R}$. The two last relations yield the following weak formulation which is called a hemivariational inequality:

$$\begin{cases} \text{find } u \in V \text{ such that} \\ \langle Au + B[u], v\rangle + \int_\Gamma j^0(x, u_N(x); v_N(x)) \, d\sigma(x) \geqslant \langle F, v\rangle \text{ for every } v \in V. \end{cases} \tag{16}$$

We have shown that the hemivariational inequality (16) is derived from (12)–(15). The following remark shows that in some sense the converse statement also holds.

Remark 12. If $u \in V$ is a solution to the hemivariational inequality (16) and u is sufficiently smooth, then there exists a distribution h such that the conditions (12)–(15) hold. Indeed, since $u \in V$ from the definition of V we have $\operatorname{div} u = 0$ in Ω and $u_\tau = 0$ on Γ. Let us now take $v = \pm w$, $w \in V \cap C_0^\infty(\Omega;\mathbb{R}^d)$ in (16). Since w is arbitrary and $j^0(x, u_N; 0) = 0$, we obtain $\langle Au + B[u], w\rangle = \langle F, w\rangle$. From Proposition 1.1 in Chapter I of Temam [23] it follows that $Au + B[u] + \nabla h = F$ which implies (12). Next let $v \in V$. Multiplying the last equation by v and integrating by parts over Ω, we get

$$\langle Au + B[u], v\rangle + \int_\Gamma h v_N \, d\sigma(x) = \langle F, v\rangle.$$

Comparing this equality with (16) entails $\int_\Gamma [j^0(x, u_N(x); v_N(x)) - h v_N] \, d\sigma(x) \geqslant 0$ for every $v \in V$. Arguing as in Proposition 3.3.1 of Panagiotopoulos [20], we deduce $j^0(x, u_N(x); v_N(x)) \geqslant h v_N$ on Γ. This shows the subdifferential condition (15).

In what follows we will prove the existence of solutions to (16). In order to show that the operator $N : V \to V^*$ given by $Nv = Av + B[v]$ for $v \in V$, which appears in (16), is a Navier–Stokes type operator, it is enough to prove that B satisfies the condition (2) of Definition 8. To this end we introduce the trilinear form $b : [H^1(\Omega;\mathbb{R}^d)]^3 \to \mathbb{R}$ defined by

$$b(u,v,w) = \big\langle B(u,v), w\big\rangle \quad \text{for } u,v,w \in H^1\big(\Omega;\mathbb{R}^d\big).$$

Analogously as in Lemmata 1.1, 1.3 and 1.5 in Chapter II of Temam [23], we can show that b is continuous, $b(u,v,w) = -b(u,w,v)$, $b(u,v,v) = 0$ for $u,v,w \in H^1(\Omega;\mathbb{R}^d)$ and that if $u_n \to u$ weakly in V, then

$$b(u_n,u_n,v) \to b(u,u,v) \quad \text{for all } v \in V.$$

This means that the bilinear operator $B : V \times V \to V^*$ satisfies the condition (2) of Definition 8.

Concerning the superpotential j, we admit the following hypothesis:

$\underline{H(j)}$: $j : \Gamma \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) $j(\cdot, \xi)$ is measurable on Γ for each $\xi \in \mathbb{R}$ and $j(\cdot, 0) \in L^1(\Gamma)$;
(ii) $j(x, \cdot)$ is locally Lipschitz on $\mathbb{R}$ for each $x \in \Gamma$;
(iii) $|\eta| \leqslant c_1(1 + |\xi|^\rho)$ for all $\eta \in \partial j(x, \xi)$, $(x, \xi) \in \Gamma \times \mathbb{R}$ with $c_1 > 0$ and $0 \leqslant \rho \leqslant 1$.

We define now the functional $J : L^2(\Gamma; \mathbb{R}^d) \to \mathbb{R}$ by

$$J(v) = \int_\Gamma j\big(x, v_N(x)\big)\, d\sigma(x) \quad \text{for } v \in L^2\big(\Gamma; \mathbb{R}^d\big). \tag{17}$$

Lemma 13. *Assume that the integrand $j : \Gamma \times \mathbb{R} \to \mathbb{R}$ verifies $H(j)$. Then the functional J defined by* (17) *satisfies*

$\underline{H(J)}$: *$J : L^2(\Gamma; \mathbb{R}^d) \to \mathbb{R}$ is a functional such that*

(i) *J is well-defined and Lipschitz on bounded subsets of $L^2(\Gamma; \mathbb{R}^d)$;*
(ii) *$\|\zeta\|_{L^2(\Gamma;\mathbb{R}^d)} \leqslant \tilde{c}(1 + \|v\|^\rho_{L^2(\Gamma;\mathbb{R}^d)})$ for all $\zeta \in \partial J(v)$, $v \in L^2(\Gamma; \mathbb{R}^d)$ with $\tilde{c} > 0$;*
(iii) *for all $u, v \in L^2(\Gamma; \mathbb{R}^d)$, we have*

$$J^0(u; v) \leqslant \int_\Gamma j^0\big(x, u_N(x); v_N(x)\big)\, d\sigma(x), \tag{18}$$

where $J^0(u; v)$ denotes the directional derivative of J at a point $u \in L^2(\Gamma; \mathbb{R}^d)$ in the direction $v \in L^2(\Gamma; \mathbb{R}^d)$.

Moreover, if additionally either j or $-j$ is regular in the sense of Clarke, then J or $-J$ is regular, respectively and the inequality (18) *becomes equality.*

Proof. First we study the properties of the integrand j. We define $j_1 : \Gamma \times \mathbb{R}^d \to \mathbb{R}$ by $j_1(x, \xi) = j(x, \xi_N)$ for $(x, \xi) \in \Gamma \times \mathbb{R}^d$. We observe that $j_1(x, \xi) = j(x, L\xi)$, where $L \in \mathcal{L}(\mathbb{R}^d, \mathbb{R})$, $L\xi = \xi_N = \xi \cdot n$ and that $L^* \in \mathcal{L}(\mathbb{R}, \mathbb{R}^d)$ is given by $L^*r = rn$ for $r \in \mathbb{R}$. From the hypotheses $H(j)$(i) and (ii), and the fact that L is linear continuous operator, we have

$$\begin{cases} j_1(\cdot, \xi) \text{ is measurable for all } \xi \in \mathbb{R}^d, \ j_1(\cdot, 0) \in L^1(\Gamma), \\ j_1(x, \cdot) \text{ is locally Lipschitz for } x \in \Gamma. \end{cases}$$

Using these properties, from Proposition 7(ii), we obtain

$$\partial j_1(x, \xi) = \partial\big(j(x, L\xi)\big) \subset L^*\partial j(x, L\xi) = \partial j(x, \xi_N)n, \tag{19}$$

where all subdifferentials are taken with respect to the second variable. We show the following estimate:

$$|\eta|_{\mathbb{R}^d} \leqslant c_1\big(1 + |\xi|^\rho_{\mathbb{R}^d}\big) \quad \text{for all } \eta \in \partial j_1(x, \xi), (x, \xi) \in \Gamma \times \mathbb{R}^d. \tag{20}$$

In fact, from (19) we know that for $\eta \in \partial j_1(x,\xi)$ we have $\eta = an$, $a \in \partial j(x,\xi_N)$. Hence by $H(j)$(iii) we obtain $|\eta|_{\mathbb{R}^d} = |a| \leqslant c_1(1+|\xi_N|^\rho) \leqslant c_1(1+|\xi|^\rho_{\mathbb{R}^d})$ which implies (20). Next, we observe that $H(J)$(i) follows from Theorem 2.7.5 of Clarke [10]. The estimate in $H(J)$(ii) is a consequence of (20). By the Fatou lemma, we also have

$$J^0(u;v) \leqslant \int_\Gamma j_1^0\big(x,u(x);v(x)\big)\,d\sigma(x) \quad \text{for } u,v \in L^2\big(\Gamma;\mathbb{R}^d\big). \tag{21}$$

This inequality together with the following one

$$j_1^0(x,\xi;\eta) \leqslant j^0(x,L\xi;L\eta) = j^0(x,\xi_N;\eta_N) \quad \text{for } \xi,\eta \in \mathbb{R}^d$$

(cf. Proposition 7(i)) implies $H(J)$(iii). Furthermore, if either j or $-j$ is regular, we know (cf. Clarke [10, Theorem 2.7.2]) that (21) becomes equality. Also by using Proposition 7(i), we have $j_1^0(x,\xi;\eta) = j^0(x,\xi_N;\eta_N)$ for $\xi,\eta \in \mathbb{R}^d$. Hence we deduce

$$\begin{aligned} J^0(u;v) &= \int_\Gamma j_1^0\big(x,u(x);v(x)\big)\,d\sigma(x) \\ &= \int_\Gamma j^0\big(x,u_N(x);v_N(x)\big)\,d\sigma(x), \quad u,v \in L^2\big(\Gamma;\mathbb{R}^d\big). \end{aligned}$$

This completes the proof of the lemma. □

We continue the formulation of the problem in the form of an operator inclusion. We need to introduce the operator of a subdifferential type. To this end we define the space Z to be the closure of $\mathcal{W}$ in the norm of $H^\delta(\Omega;\mathbb{R}^d)$ with $\delta \in (\frac{1}{2},1)$. We have

$$V \subset Z \subset H \simeq H^* \subset Z^* \subset V^*$$

with all embeddings being dense and compact. Denoting by $i: V \to Z$ the embedding injection and by $\gamma: Z \to L^2(\Gamma;\mathbb{R}^d)$ and $\gamma_0: H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ the trace operators, for all $v \in V$ we get $\gamma_0 v = \gamma(iv)$. For simplicity we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for $v \in V$.

We define the operator $R: Z \to 2^{Z^*}$ by

$$Rz = \gamma^*\big(\partial J(\gamma z)\big) \quad \text{for } z \in Z, \tag{22}$$

where $\gamma^*: L^2(\Gamma;\mathbb{R}^d) \to Z^*$ is the adjoint operator to γ.

The reason we have introduced the operator R of the form (22) is explained in the remark below.

We consider the following inclusion:

$$\text{find} \quad u \in V \quad \text{such that} \quad Au + B[u] + \gamma^*\big(\partial J(\gamma u)\big) \ni F. \tag{23}$$

Definition 14. An element $u \in V$ is a solution to (23) if and only if there exists $\eta \in Z^*$ such that $Au + B[u] + \eta = F$ and $\eta \in \gamma^*(\partial J(\gamma u))$.

Remark 15. If the functional J is of the form (17) and $H(j)$ holds, then every solution to (23) is also a solution to the inequality (16). Moreover, if either j or $-j$ is regular, then the converse is also true. Indeed, if $u \in V$ solves (23), then for every $v \in V$, we have $\langle Au + B[u], v\rangle + \langle \eta, v\rangle_{Z^*\times Z} = \langle F, v\rangle$ with $\eta = \gamma^*\zeta$ and $\zeta \in \partial J(\gamma u)$. From the definition of the subdifferential we obtain $\langle \zeta, z\rangle_{L^2(\Gamma;\mathbb{R}^d)} \leqslant J^0(\gamma u; z)$ for all $z \in L^2(\Gamma;\mathbb{R}^d)$ and therefore by using $H(J)$(iii) we get

$$\langle \eta, v\rangle_{Z^*\times Z} = \langle \gamma^*\zeta, v\rangle_{Z^*\times Z} = \langle \zeta, \gamma v\rangle_{L^2(\Gamma;\mathbb{R}^d)} \leqslant J^0(\gamma u; \gamma v)$$
$$\leqslant \int_\Gamma j^0\big(x, u_N(x); v_N(x)\big)\, d\sigma(x)$$

for every $v \in V$. Hence u is also a solution to (16).

Now we will show that under regularity of j or $-j$ every solution to (16) solves also (23). From Lemma 13 we have

$$\big\langle F - Au - B[u], v\big\rangle \leqslant \int_\Gamma j^0\big(x, u_N(x); v_N(x)\big)\, d\sigma(x) = J^0(\gamma u; \gamma v).$$

By the chain rule (see Proposition 7(ii)), we get $\partial(J\circ\gamma)(v) = \gamma^* \circ \partial J(\gamma v)$ so

$$F - Au - B[u] \in \partial(J\circ\gamma)(u) = \gamma^*\big(\partial J(\gamma u)\big),$$

which implies (23).

In view of Remark 15, we will establish the existence of solutions to (23).

Lemma 16. *If the functional J verifies $H(J)$, then the operator R given by* (22) *satisfies $H(R)$.*

Proof. The values of R are nonempty and convex which immediately follows from the analogous properties of the Clarke subdifferential.

To show that the values of R are weakly compact, let $z \in Z$ and $\{z_n^*\} \subset Rz$. Thus $z_n^* = \gamma^* w_n$ with $w_n \in \partial J(\gamma z)$. Since $\partial J(\gamma z)$ is a weakly compact subset of $L^2(\Gamma;\mathbb{R}^d)$, we can find a subsequence $\{w_{n_k}\}$ of $\{w_n\}$ such that $w_{n_k} \to w_0$ weakly in $L^2(\Gamma;\mathbb{R}^d)$ with $w_0 \in \partial J(\gamma z)$. From the fact that γ^* is linear continuous, we have $z_{n_k}^* = \gamma^* w_{n_k} \to \gamma^* w_0 =: z^*$ weakly in Z^*. So $z^* = \gamma^* w_0$ and $w_0 \in \partial J(\gamma z)$ imply $z^* \in Rz$, which shows that the values of R are weakly compact in Z^*.

Next, we prove that R satisfies $H(R)$(ii). Let $\{z_n\} \subset Z$, $\{z_n^*\} \subset Z^*$ be such that $z_n^* \in Rz_n$, $z_n \to z$ in Z and $z_n^* \to z^*$ weakly in Z^*. We will show that $z^* \in Rz$. By assumption we have $z_n^* = \gamma^* w_n$ and $w_n \in \partial J(\gamma z_n)$. Using the fact that $\partial J : L^2(\Gamma;\mathbb{R}^d) \to 2^{L^2(\Gamma;\mathbb{R}^d)}$ is a bounded map (cf. $H(J)$(ii)), we may assume that $w_n \to w_0$ weakly in $L^2(\Gamma;\mathbb{R}^d)$. Hence $z_n^* = \gamma^* w_n \to \gamma^* w_0 = z^*$ weakly in Z^*. From the closedness of the graph of ∂J in $L^2(\Gamma;\mathbb{R}^d) \times (w\text{–}L^2(\Gamma;\mathbb{R}^d))$ topology (cf. [10]), passing to the limit in the relation $w_n \in \partial J(\gamma z_n)$, we obtain $w_0 \in \partial J(\gamma z)$. This together with $z^* = \gamma^* w_0$ implies $z^* \in \gamma^*(\partial J(\gamma z)) = Rz$ and proves the closedness of the graph of R in $Z \times (w\text{–}Z^*)$ topology.

Finally, by using $H(J)$(ii), for $z \in Z$, we have

$$
\begin{aligned}
\|Rz\|_{Z^*} &\leqslant \|\gamma^*\| \big\|\partial J(\gamma z)\big\|_{L^2(\Gamma;\mathbb{R}^d)} \leqslant \|\gamma^*\|\tilde{c}\big(1+\|\gamma z\|^{\rho}_{L^2(\Gamma;\mathbb{R}^d)}\big) \\
&\leqslant \tilde{c}\|\gamma^*\|\big(1+\|\gamma\|^{\rho}\|z\|^{\rho}_Z\big) \leqslant \hat{c}\big(1+\|z\|^{\rho}_Z\big) \qquad (24)
\end{aligned}
$$

with a positive constant $\hat{c}>0$, where $\|\gamma\| = \|\gamma^*\| = \|\gamma\|_{\mathcal{L}(Z;L^2(\Gamma;\mathbb{R}^d))}$. This shows that $H(R)$(iii) holds and completes the proof of the lemma. □

Now we are in a position to deduce from Remark 15, Lemma 16 and Corollary 11 the main result of this section.

Theorem 17. *Let hypothesis $H(j)$ hold and $f \in V^*$. If $0 \leqslant \rho < 1$, then the hemivariational inequality* (16) *corresponding to the Navier–Stokes system* (12)–(15) *admits a solution. The same conclusion holds for $\rho = 1$, provided $\alpha - \hat{c}\beta^2 > 0$, where α is a coercivity constant of A, $\hat{c}$ is a constant in* (24) *and β is the embedding constant of $V \subset Z$.*

Example 18. Let us assume that the boundary Γ of Ω consists of two disjoint parts such that $\Gamma = \Gamma_1 \cup \Gamma_2$. Given real numbers $\lambda_1 < \lambda_2 \leqslant \lambda_3 < \lambda_4$ and $h_0 < 0 < h_1$, we consider the function $j : \Gamma \times \mathbb{R} \to \mathbb{R}$ such that

$$
j(x,\lambda) = \begin{cases}
\frac{h_1}{2(\lambda_2-\lambda_1)}(\lambda-\lambda_1)^2 & \text{if } x \in \Gamma_1,\ \lambda < \lambda_2, \\
\frac{h_1}{2}(\lambda_2-\lambda_1) & \text{if } x \in \Gamma_1,\ \lambda \geqslant \lambda_2, \\
0 & \text{if } x \in \Gamma_2,\ \lambda \leqslant \lambda_3, \\
\frac{h_0}{2(\lambda_3-\lambda_4)}(\lambda-\lambda_3)(\lambda+\lambda_3-2\lambda_4) & \text{if } x \in \Gamma_2,\ \lambda > \lambda_3.
\end{cases} \qquad (25)
$$

Then for $x \in \Gamma_1$ we have

$$
\partial j(x,\lambda) = \begin{cases}
\frac{h_1}{\lambda_2-\lambda_1}(\lambda-\lambda_1) & \text{if } \lambda < \lambda_2, \\
[0,h_1] & \text{if } \lambda = \lambda_2, \\
0 & \text{if } \lambda > \lambda_2,
\end{cases}
$$

while for $x \in \Gamma_2$ we have

$$
\partial j(x,\lambda) = \begin{cases}
0 & \text{if } \lambda < \lambda_3, \\
[h_0,0] & \text{if } \lambda = \lambda_3, \\
\frac{h_0}{\lambda_3-\lambda_4}(\lambda-\lambda_4) & \text{if } \lambda > \lambda_3.
\end{cases}
$$

It is clear that for $(x,\lambda) \in \Gamma \times \mathbb{R}$, we have $|\eta| \leqslant c_1(1+|\lambda|)$ for all $\eta \in \partial j(x,\lambda)$ with $c_1 = \max\big\{-h_0, h_1, \frac{h_1}{\lambda_2-\lambda_1}, \frac{h_0}{\lambda_3-\lambda_4}\big\}$. Since n denotes the unit outward normal on Γ, the condition $u_N \geqslant 0$ ($u_N \leqslant 0$, respectively) represents the outflow (inflow, respectively) of the fluid through the boundary. The boundary condition $u_N = 0$ means that there is no flow across the boundary. In particular, if $\lambda_2 = \lambda_3 = 0$, the function (25) describes the following boundary conditions for velocity and the total head:

$$
\text{on } \Gamma_1: \quad \begin{cases}
\text{if } u_N < 0, \text{ then } h = h_1\lambda_1^{-1}(\lambda_1 - u_N), \\
\text{if } u_N = 0, \text{ then } 0 \leqslant h \leqslant h_1, \\
\text{if } u_N > 0, \text{ then } h = 0,
\end{cases}
$$

and

$$\text{on } \Gamma_2: \quad \begin{cases} \text{if } u_N < 0, \ \text{then } h = 0, \\ \text{if } u_N = 0, \ \text{then } h_0 \leqslant h \leqslant 0, \\ \text{if } u_N > 0, \ \text{then } h = h_0 \lambda_4^{-1}(\lambda_4 - u_N). \end{cases}$$

We now address the question of uniqueness of solutions to the inclusion (23). To this end we need an additional hypothesis on the functional J.

$\underline{H(J)_1}$: $J : L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ satisfies $H(J)$ and the following relaxed monotonicity condition:

$$\langle z_1 - z_2, w_1 - w_2\rangle_{L^2(\Gamma;\mathbb{R}^d)} \geqslant -m\|w_1 - w_2\|^2_{L^2(\Gamma;\mathbb{R}^d)} \tag{26}$$

for all $z_i \in \partial J(w_i)$, $w_i \in L^2(\Gamma;\mathbb{R}^d)$, $i = 1, 2$, with $m > 0$.

Proposition 19. *Let the operators A and B satisfy conditions* (1) *and* (2) *of Definition* 8, *let $H(J)$ hold, $f \in V^*$ and let $u \in V$ be a solution to* (23). *If $0 \leqslant \rho < 1$, then there exists a constant $C > 0$ such that*

$$\|u\|_V \leqslant C. \tag{27}$$

If $\rho = 1$ and $\alpha - \tilde{c}\beta^2\|\gamma\|^2 > 0$, then (27) *holds with $C := \frac{\|f\|_{V^*} + \tilde{c}\beta\|\gamma\|}{\alpha - \tilde{c}\beta^2\|\gamma\|^2}$. If $0 \leqslant \rho \leqslant 1$, the condition* (26) *holds and $\alpha - m\beta^2\,\|\gamma\|^2 - c_b C > 0$, where $c_b > 0$ is the continuity constant of the form b associated to the operator B, then the solution to problem* (23) *is unique.*

Proof. We start with the proof of a priori estimate (27). Since $u \in V$ solves (23), we have

$$\langle Au, u\rangle + \big\langle B[u], u\big\rangle + \langle \eta, u\rangle_{Z^* \times Z} = \langle F, u\rangle$$

with $\eta = \gamma^* z$ and $z \in \partial J(\gamma u)$. By $H(J)$(ii), we get $\|\eta\|_{Z^*} \leqslant \|\gamma^*\|\|z\|_{L^2(\Gamma;\mathbb{R}^d)} \leqslant \tilde{c}\|\gamma\|(1 + \|\gamma\|^\rho\|u\|_Z^\rho)$, which implies

$$\big|\langle \eta, u\rangle_{Z^* \times Z}\big| \leqslant \tilde{c}\|\gamma\|\beta\big(1 + \|\gamma\|^\rho\beta^\rho\|u\|_V^\rho\big)\|u\|_V,$$

where $\beta > 0$ is such that $\|\cdot\|_Z \leqslant \beta\|\cdot\|_V$. Hence and from the properties (1), (2) of Definition 8, we deduce

$$\alpha\|u\|_V^2 - \tilde{c}\beta^{\rho+1}\|\gamma\|^{\rho+1}\|u\|_V^{\rho+1} - \tilde{c}\beta\|\gamma\|\|u\|_V \leqslant \|f\|_{V^*}\|u\|_V.$$

Then

$$\alpha\|u\|_V \leqslant \tilde{c}\big(\beta\|\gamma\|\big)^{\rho+1}\|u\|_V^\rho + \tilde{c}\beta\|\gamma\| + \|f\|_{V^*}.$$

For $\rho < 1$ the bound (27) follows. If $\rho = 1$, then $(\alpha - \tilde{c}\beta^2\|\gamma\|^2)\|u\|_V \leqslant \tilde{c}\beta\|\gamma\| + \|f\|_{V^*}$, so (27) also holds with the suitable positive constant C.

Next we assume $\rho \in [0, 1]$ and $\alpha - m\beta^2\|\gamma\|^2 - c_b C > 0$, and let u_1, $u_2 \in V$ be two solutions of (23). We have

$$A(u_1 - u_2) + B[u_1] - B[u_2] + \eta_1 - \eta_2 = 0$$

with $\eta_k = \gamma^* z_k$ and $z_k \in \partial J(\gamma u_k)$ for $k = 1, 2$. By hypothesis $H(J)_1$, we have

$$\begin{aligned}\langle \eta_1-\eta_2, u_1-u_2\rangle_{Z^*\times Z} &= \langle z_1-z_2, \gamma u_1-\gamma u_2\rangle_{L^2(\Gamma;\mathbb{R}^d)}\\ &\geqslant -m\|\gamma(u_1-u_2)\|^2_{L^2(\Gamma;\mathbb{R}^d)}\\ &\geqslant -m\|\gamma\|^2\|u_1-u_2\|^2_Z \geqslant -m\beta^2\|\gamma\|^2\|u_1-u_2\|^2_V.\end{aligned}$$

Hence and from the inequality $\langle B[u_1]-B[u_2], u_1-u_2\rangle = b(u_1-u_2, u_2, u_1-u_2) \leqslant c_b\|u_2\|_V\|u_1-u_2\|^2_V$, we obtain

$$\alpha\|u_1-u_2\|^2_V - m\|\gamma\|^2\beta^2\|u_1-u_2\|^2_V \leqslant c_b\|u_2\|_V\|u_1-u_2\|^2_V.$$

So $(\alpha - m\beta^2\|\gamma\|^2 - c_bC)\|u_1-u_2\|^2_V \leqslant 0$ which implies $u_1 = u_2$ and completes the proof. □

We remark that when $J \equiv 0$ (so $m = 0$ and $\tilde{c} = 0$), the uniqueness of solutions was obtained by Temam [23] in Theorem 1.3, p. 167. In this case the condition of Proposition 19 under which we proved uniqueness reduces to $\alpha^2 - c_b\|f\|_{V^*} > 0$.

We close this section with an example of the functional which satisfies hypothesis $H(J)_1$.

Example 20. Let us consider the functional $J : L^2(\Gamma;\mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(v) = \int_\Gamma \left(\int_0^{v_N(x)} \varphi(s)\,ds \right) d\sigma(x) \quad \text{for all } v \in L^2\big(\Gamma;\mathbb{R}^d\big)$$

(for simplicity we drop the x-dependence in the integrand of J), where the function φ satisfies the following hypothesis:

$\underline{H(\varphi)}$: $\varphi \in L^\infty_{\mathrm{loc}}(\mathbb{R})$ verifies the growth condition $|\varphi(s)| \leqslant c_0(1+|s|)$ for $s \in \mathbb{R}$ with $c_0 > 0$ and

$$\operatorname*{ess\,inf}_{\xi_1\neq\xi_2} \frac{\varphi(\xi_1)-\varphi(\xi_2)}{\xi_1-\xi_2} \geqslant -m \quad \text{with some } m > 0. \tag{28}$$

We associate with φ a multivalued map $\hat{\varphi} : \mathbb{R} \to 2^{\mathbb{R}}$ defined by $\hat{\varphi}(\xi) = [\underline{\varphi}(\xi), \overline{\varphi}(\xi)]$, where

$$\underline{\varphi}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|t-\xi|\leqslant\delta} \varphi(t), \qquad \overline{\varphi}(\xi) = \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|t-\xi|\leqslant\delta} \varphi(t)$$

and $[\cdot\,,\cdot]$ denotes the interval. Roughly speaking, $\hat{\varphi}$ results from φ by "filling in the gaps" procedure. As a consequence of Theorem 1.2.20 of Chang [5], J is Lipschitz continuous on bounded sets in $L^2(\Gamma;\mathbb{R}^d)$ and there is a locally Lipschitz function $j : \mathbb{R} \to \mathbb{R}$, determined up to an additive constant by the relation $j(s) = \int_0^s \varphi(\tau)\,d\tau$ and $\partial j(s) = \hat{\varphi}(s)$ for $s \in \mathbb{R}$. Thus we have $J(v) = \int_\Gamma j(v_N(x))\,d\sigma(x) = \int_\Gamma j_1(v(x))\,d\sigma(x)$ for $v \in L^2(\Gamma;\mathbb{R}^d)$, where $j_1 : \mathbb{R}^d \to \mathbb{R}$ is given by $j_1(\xi) = j(\xi_N)$ for $\xi \in \mathbb{R}^d$. Since j_1 is locally Lipschitz and $\partial j_1(\xi) = \partial(j(\xi_N)) \subset \partial j(\xi_N)n = \hat{\varphi}(\xi_N)n$, we have for $\eta \in \partial j_1(\xi)$

$$|\eta|_{\mathbb{R}^d} \leqslant c_0\big(1+|\xi_N|\big) \leqslant c_0\big(1+|\xi|_{\mathbb{R}^d}\big) \quad \text{for all } \xi \in \mathbb{R}^d$$

(cf. the proof of Lemma 13). We will show the relaxed monotonicity condition (26). Let $w_1, w_2, z_1, z_2 \in L^2(\Gamma;\mathbb{R}^d)$, $z_1 \in \partial J(w_1)$ and $z_2 \in \partial J(w_2)$. By Theorem 2.7.5

of Clarke [10] we have $\partial J(v) \subset \int_\Gamma \partial j_1(v(x))d\sigma(x)$ for all $v \in L^2(\Gamma;\mathbb{R}^d)$. Therefore $z_k(x) \in \partial j_1(w_k(x)) \subset \hat{\varphi}(w_{kN})n$ for a.e. $x \in \Gamma$ and $k = 1, 2$. Hence

$$z_k(x) = a_k(x)n, \qquad a_k(x) \in \hat{\varphi}(w_{kN}) \quad \text{a.e. on } \Gamma \text{ for } k = 1, 2. \tag{29}$$

On the other hand, from (28), we get

$$\operatorname*{ess\,inf}_{\xi_1 > \xi_2} \frac{\underline{\varphi}(\xi_1) - \overline{\varphi}(\xi_2)}{\xi_1 - \xi_2} \geqslant -m. \tag{30}$$

Let $\Gamma^1 = \{x \in \Gamma\colon\ w_{1N}(x) > w_{2N}(x)\}$ and $\Gamma^2 = \{x \in \Gamma\colon\ w_{2N}(x) > w_{1N}(x)\}$. Using (29) and (30), we obtain

$$\begin{aligned}
&\langle z_1 - z_2, w_1 - w_2\rangle_{L^2(\Gamma;\mathbb{R}^d)} \\
&\quad = \int_\Gamma \big(a_1(x)n - a_2(x)n\big) \cdot \big(w_1(x) - w_2(x)\big)\, d\sigma(x) \\
&\quad = \int_\Gamma \big(a_1(x) - a_2(x)\big)\big(w_{1N}(x) - w_{2N}(x)\big)\, d\sigma(x) \\
&\quad = \int_{\Gamma^1} \big(a_1(x) - a_2(x)\big)\big(w_{1N}(x) - w_{2N}(x)\big)\, d\sigma(x) \\
&\qquad + \int_{\Gamma^2} \big(a_1(x) - a_2(x)\big)\big(w_{1N}(x) - w_{2N}(x)\big)\, d\sigma(x) \\
&\quad \geqslant \int_{\Gamma^1} \big(\underline{\varphi}\big(w_{1N}(x)\big) - \overline{\varphi}\big(w_{2N}(x)\big)\big)\big(w_{1N}(x) - w_{2N}(x)\big)\, d\sigma(x) \\
&\qquad + \int_{\Gamma^2} \big(\underline{\varphi}\big(w_{2N}(x)\big) - \overline{\varphi}\big(w_{1N}(x)\big)\big)\big(w_{2N}(x) - w_{1N}(x)\big)\, d\sigma(x) \\
&\quad \geqslant -m \int_{\Gamma^1} \big|w_{1N}(x) - w_{2N}(x)\big|^2\, d\sigma(x) - m \int_{\Gamma^2} \big|w_{2N}(x) - w_{1N}(x)\big|^2\, d\sigma(x) \\
&\quad = -m \int_\Gamma \big|w_{1N}(x) - w_{2N}(x)\big|^2\, d\sigma(x) \geqslant -m \int_\Gamma \big\|w_1(x) - w_2(x)\big\|^2\, d\sigma(x) \\
&\quad = -m\|w_1 - w_2\|^2_{L^2(\Gamma;\mathbb{R}^d)},
\end{aligned}$$

which proves the relaxed monotonicity condition (26).

We remark that the growth condition (28) appearing in $H(\varphi)$ was earlier considered by Miettinen [16], cf. also Migórski [17].

5. Dependence result

In this section we study the dependence of solutions of hemivariational inequality (23) with respect to the superpotential J given by (17). We consider a sequence of functions $j^k : \Gamma \times \mathbb{R} \to \mathbb{R}$ for $k \in \mathbb{N} \cup \{\infty\}$ and define $J^k : L^2(\Gamma; \mathbb{R}^d) \to \mathbb{R}$ by $J^k(v) = \int_\Gamma j^k(x, v_N(x))\, d\sigma(x)$ for $v \in L^2(\Gamma; \mathbb{R}^d)$. We admit the following hypothesis:

$\underline{H(j)_1}$: $j^k : \Gamma \times \mathbb{R} \to \mathbb{R}$, $k \in \mathbb{N} \cup \{\infty\}$, are such that

(i) $j^k(\cdot, \xi)$ are measurable on Γ for all $\xi \in \mathbb{R}$ and $j^k(\cdot, 0) \in L^1(\Gamma)$;
(ii) $j^k(x, \cdot)$ are locally Lipschitz on $\mathbb{R}$ for all $x \in \Gamma$;
(iii) $|\eta^k| \leqslant c_1(1 + |\xi|^\rho)$ for all $\eta^k \in \partial j^k(x, \xi)$, $(x, \xi) \in \Gamma \times \mathbb{R}$ with $c_1 > 0$ and $0 \leqslant \rho \leqslant 1$ independent of k;
(iv) $j^\infty(x, \cdot)$ is regular in the sense of Clarke;
(v) $\limsup_{k\to\infty} \operatorname{Gr} \partial j^k(x, \cdot) \subset \operatorname{Gr} \partial j^\infty(x, \cdot)$ for all $x \in \Gamma$, where the upper limit is taken in the sense of Kuratowski (cf. [2,11]).

The main result of this section is the following.

Theorem 21. *Assume that $H(j)_1$ holds and $f \in V^*$. Let $\{u^k\}_{k\in\mathbb{N}}$ denote a sequence of solutions of the problem* (23)*, when J is replaced by J^k. Then there exists a subsequence of $\{u^k\}$* (*denoted by the same symbol*) *such that $u^k \to u^\infty$ weakly in V, where $u^\infty \in V$ is a solution to* (23) *corresponding to J^∞.*

This result is important in fluid mechanics applications, since it shows what kind of tolerances is admissible in the mathematical model. It demonstrates that perturbations of the superpotential j of type $H(j)_1$ (and therefore, of the boundary conditions) cause small perturbations of the solutions.

Proof. The existence of solutions $\{u^k\}$, for every fixed $k \in \mathbb{N}$, follows from Theorem 17. By Proposition 19 and $H(j)_1$ (where the bounds hold uniformly in k), we know that $\{u^k\}$ remains in a bounded subset of V. Thus, for a subsequence, we have

$$u^k \to u^\infty \quad \text{weakly in } V \quad \text{with } u^\infty \in V. \tag{31}$$

By the compactness of the trace of V into $L^2(\Gamma; \mathbb{R}^d)$, it follows that $u^k \to u^\infty$ in $L^2(\Gamma; \mathbb{R}^d)$. This implies $u_N^k = u^k \cdot n \to u^\infty \cdot n = u_N^\infty$ in $L^2(\Gamma)$ and for a next subsequence

$$u_N^k(x) \to u_N^\infty(x) \quad \text{for a.e. } x \in \Gamma. \tag{32}$$

Since u^k is a solution to (23), we know that $Au^k + B[u^k] + \eta^k = F$, where $\eta^k = \gamma^* w^k$ and $w^k \in \partial J^k(\gamma u^k)$. We conclude by $H(j)_1$ and Lemma 13 that $\{w^k\}$ lies in a bounded subset of $L^2(\Gamma; \mathbb{R}^d)$. So, up to a subsequence, we have

$$w^k \to w^\infty \quad \text{weakly in } L^2(\Gamma; \mathbb{R}^d) \quad \text{with } w^\infty \in L^2(\Gamma; \mathbb{R}^d). \tag{33}$$

Consequently

$$\eta^k = \gamma^* w^k \to \gamma^* w^\infty =: \eta^\infty \quad \text{weakly in } Z^*. \tag{34}$$

Because $A(\cdot) + B[\cdot]: V \to V^*$ is a Navier–Stokes type operator, from (31), we get $Au^k + B[u^k] \to Au^\infty + B[u^\infty]$ weakly in V^*. Hence and from (34) it follows that $Au^\infty + B[u^\infty] + \eta^\infty = F$. To conclude the proof, it remains to show that $w^\infty \in \partial J^\infty(\gamma u^\infty)$.

Since the integrands j^k for $k \in \mathbb{N}$ satisfy $H(j)_1$(i)–(iii), we apply Theorem 2.7.5 of Clarke [10] to the functionals J^k and we obtain

$$\partial J^k(v) \subset \int_\Gamma \partial j_1^k\big(x, v(x)\big)\, d\sigma(x) \quad \text{for all } v \in L^2\big(\Gamma; \mathbb{R}^d\big) \text{ and } k \in \mathbb{N},$$

where $j_1^k(x,\xi) = j^k(x,\xi_N)$ for $(x,\xi) \in \Gamma \times \mathbb{R}^d$. Proposition 7 gives $\partial j_1^k(x,\xi) \subset \partial j^k(x,\xi_N)n$ for all $k \in \mathbb{N}$ (compare also (19)). Therefore $w^k \in \int_\Gamma \partial j^k(x, u_N^k(x))n\, d\sigma(x)$. This means (see [10, Section 2.7]) that there exists a sequence $\{z^k\} \subset L^2(\Gamma;\mathbb{R}^d)$ satisfying

$$z^k(x) \in \partial j^k\big(x, u_N^k(x)\big)n \quad \text{a.e. } x \in \Gamma \tag{35}$$

and such that

$$\big\langle w^k, \psi\big\rangle_{L^2(\Gamma;\mathbb{R}^d)} = \int_\Gamma z^k(x)\cdot\psi(x)\, d\sigma(x) \quad \text{for all } \psi \in L^2\big(\Gamma;\mathbb{R}^d\big). \tag{36}$$

From (35), it is clear that

$$z^k(x) = a^k(x)n \quad \text{with } a^k \in L^2(\Gamma)$$

and

$$a^k(x) \in \partial j^k\big(x, u_N^k(x)\big) \quad \text{a.e. } x \in \Gamma. \tag{37}$$

It follows from hypothesis $H(j)_1$(iii) that $\{a^k\}$ remains in a bounded subset in $L^2(\Gamma)$. Thus

$$\begin{aligned} &a^k \to a^\infty \quad \text{weakly in } L^2(\Gamma) \quad \text{with } a^\infty \in L^2(\Gamma),\\ &z^k \to z^\infty \quad \text{weakly in } L^2(\Gamma;\mathbb{R}^d) \quad \text{with } z^\infty \in L^2\big(\Gamma;\mathbb{R}^d\big). \end{aligned} \tag{38}$$

Hence we have $z^\infty(x) = a^\infty(x)n$. Applying Theorem 7.2.1 of Aubin and Frankowska [2], from (32), (37) and (38), we deduce

$$a^\infty(x) \in \overline{\text{conv}}\Big(\limsup_{z\to u_N^\infty(x),\, k\to\infty} \partial j^k(x,z)\Big) \subset \partial j^\infty\big(x, u_N^\infty(x)\big) \quad \text{for a.e. } x \in \Gamma.$$

The latter follows from $H(j)_1$(v) since $\partial j^\infty(x,\cdot)$ has closed and convex values. Passing to the limit in (36), by (38) and (33), we obtain

$$\big\langle w^\infty, \psi\big\rangle_{L^2(\Gamma;\mathbb{R}^d)} = \int_\Gamma z^\infty(x)\cdot\psi(x)\, d\sigma(x) \quad \text{for all } \psi \in L^2\big(\Gamma;\mathbb{R}^d\big).$$

Hence and from $z^\infty(x) = a^\infty(x)n \in \partial j^\infty(x, u_N^\infty(x))n$, we get

$$w^\infty \in \int_\Gamma \partial j^\infty\big(x, u_N^\infty(x)\big) n \, d\sigma(x).$$

Exploiting the regularity of $j^\infty(x, \cdot)$, by Proposition 7(ii) it follows that $\partial j_1^\infty(x, u^\infty(x)) = \partial j^\infty(x, u_N^\infty(x))n$, where $j_1^\infty(x, \xi) = j^\infty(x, \xi_N)$ for $(x, \xi) \in \Gamma \times \mathbb{R}^d$. This together with [10, Theorem 2.7.5] shows that

$$w^\infty \in \int_\Gamma \partial j_1^\infty\big(x, u^\infty(x)\big) n \, d\sigma(x) = \partial J^\infty\big(\gamma u^\infty\big),$$

which concludes the proof of the theorem. □

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