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The number of extreme points of tropical polyhedra ☆

Xavier Allamigeon [a], Stéphane Gaubert [b], Ricardo D. Katz [c]

[a] Direction du Budget, 4ème sous-direction, Bureau des transports, 139 rue de Bercy, Télédoc 278, 75572 Paris Cedex 12, France
[b] INRIA and CMAP, École Polytechnique, 91128 Palaiseau Cedex, France
[c] CONICET, Instituto de Matemática "Beppo Levi", Universidad Nacional de Rosario, Avenida Pellegrini 250, 2000 Rosario, Argentina

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ABSTRACT

The celebrated upper bound theorem of McMullen determines the maximal number of extreme points of a polyhedron in terms of its dimension and the number of constraints which define it, showing that the maximum is attained by the polar of the cyclic polytope. We show that the same bound is valid in the tropical setting, up to a trivial modification. Then, we study the tropical analogues of the polars of a family of cyclic polytopes equipped with a sign pattern. We construct bijections between the extreme points of these polars and lattice paths depending on the sign pattern, from which we deduce explicit bounds for the number of extreme points, showing in particular that the upper bound is asymptotically tight as the dimension tends to infinity, keeping the number of constraints fixed. When transposed to the classical case, the previous constructions yield some lattice path generalizations of Gale's evenness criterion.

1. Introduction

A fundamental result in discrete convex geometry is McMullen's upper bound theorem, which settled a conjecture of Motzkin. We restate it for completeness.

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E-mail addresses: xavier.allamigeon@finances.gouv.fr, xavier.allamigeon@polytechnique.org (X. Allamigeon), stephane.gaubert@inria.fr (S. Gaubert), rkatz@fceia.unr.edu.ar (R.D. Katz).

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Theorem *(See [32].). Among all polytopes in $\mathbb{R}^d$ with p extreme points, the cyclic polytope maximizes the number of faces of each dimension.*

The reader is referred to [39,31] for more information. Recall that a cyclic polytope is the convex hull of p distinct points on the moment curve $\{(t, t^2, \dots, t^d) \mid t \in \mathbb{R}\}$.

In particular, the number of facets (faces of dimension $d-1$) is known to be at most

$$U(p,d) := \binom{p - \lfloor d/2 \rfloor}{\lfloor d/2 \rfloor} + \binom{p - \lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor - 1} \quad \text{for } d \text{ even, and}$$

$$U(p,d) := 2\binom{p - \lfloor d/2 \rfloor - 1}{\lfloor d/2 \rfloor} \quad \text{for } d \text{ odd.}$$

By duality, the same upper bound applies to the number of extreme points of a d-dimensional polytope defined as the intersection of p half-spaces.

In max-plus or tropical convexity, the addition and multiplication are replaced by the maximum and the addition, respectively. This unusual model of discrete convexity has been studied under different names by several authors. These include Zimmermann [40], Cohen, Gaubert and Quadrat [12, 13], with motivations from discrete event systems and optimal control [22,11], Kolokoltsov, Litvinov, Maslov and Shpiz [28,30], with motivations from variations calculus and quasi-classics asymptotics. The field attracted new attention after the work of Develin and Sturmfels [15], who connected it with current developments of tropical geometry. This has been the source of a number of works of the same authors and of Joswig and Yu, see in particular [25,16,26]. Tropical convexity can also be studied from the general perspective of abstract convexity, a point of view adopted by Singer, see [14, 35], and also by Briec and Horvath [8]. Some further works developing or applying tropical convexity include [10,17,27,23,1,21,29].

The notion of extreme point carries over to the tropical setting [9,18], and so, we may ask whether McMullen's theorem, or rather, its dual, concerning the number of extreme points, admits a tropical analogue.

Our first result, which we establish in Section 2, shows that a McMullen type bound is still valid in the tropical setting.

Theorem 1. *The number of extreme rays of a tropical cone in $(\mathbb{R} \cup \{-\infty\})^d$ defined as the intersection of p tropical half-spaces cannot exceed $U(p+d, d-1)$.*

The number $p+d$ instead of p for the number of constraints can be explained intuitively: in loose terms, in the tropical world, all the numbers are "positive", so the bound is the same as for a polyhedral cone of the same dimension in which d positivity constraints would have been added to the p explicit ones. The number $d-1$ instead of d for the dimension reflects the fact that we are dealing with cones, rather than with convex sets.

The most natural idea of proof of Theorem 1 would be to tropicalize the classical method, which relies on the f-vector theory. However, some pathological features of the notions of faces of tropical polytopes make somehow uneasy the development of a tropical analogue of this theory (see [16] for a discussion on faces). So, we choose here a different approach, and establish Theorem 1 as a corollary of the classical upper bound theorem, using a deformation argument in which the tropical polyhedron is seen as a degenerate limit of a sequence of classical polyhedra.

In the classical case, the polar of a cyclic polytope with p extreme points maximizes the number of extreme points among all the polytopes of dimension d defined by p inequalities. In the tropical case, the notion of polar can be defined as well [20,19]: the polar of a set of vectors consists of the tropically linear inequalities satisfied by these vectors, whereas the polar of a set of tropically linear inequalities consists of the vectors satisfying these inequalities. This leads us to define a family of tropical generalizations of the cyclic polytopes, in which a sign pattern is incorporated to encode inequalities. Our second result is the following.

Theorem 2. *The extreme rays of the polar of a signed cyclic polyhedral cone are in one to one correspondence with tropically allowed lattice paths.*

The definition of tropically allowed lattice paths is given in Section 3, in which this theorem is proved. We also give a characterization of the extreme rays of the classical (non-tropical) analogue of this polar (Theorem 6), showing that there are fewer extreme rays in the tropical case. The latter lattice path characterization is intimately related to Gale's evenness criterion, as shown in Theorem 7.

Recall that in the classical case, a point of a polyhedron is extreme if, and only if, the gradients of the constraints that it saturates form a family of full rank. The comparison between Theorem 2 and Theorem 6 reflects the fact that the same is not true in the tropical setting. Indeed, the proof of Theorem 2, which relies on a Cramer type result published under the collective name M. Plus [36] (Theorem 4 below, see [2] for a recent account and also [37] for an alternative approach due to Richter-Gebert, Sturmfels, and Theobald) and on the characterization of extreme points obtained by Allamigeon, Gaubert and Goubault in [3,4] (see Theorem 5 below), shows that in the tropical case some additional minimality conditions must be added to the classical rank condition. This fundamental discrepancy explains why there are fewer extreme points in the tropical case.

In Section 7, we estimate the maximal number of extreme rays of the polar of a signed cyclic polyhedral cone, for which we provide explicit lower and upper bounds. This is motivated by the question of knowing whether the upper bound à la McMullen of Theorem 1 is tight. As a consequence of Theorem 2, it follows that the upper bound is asymptotically tight for a fixed p, as d tends to infinity (see Remark 4, the upper bound is approached by the polar of a signed cyclic polyhedral cone). The analogy with classical convex geometry might suggest that the number of extreme rays of a tropical polyhedral cone defined by p inequalities in dimension d is maximized by the polar of a signed cyclic polyhedral cone. However, Example 4 below, building on the recent work [5], shows that this is not the case. Thus, it remains an open problem to find a "maximizing model", i.e. a family of tropical polyhedral cones reaching, for every (p,d), the maximal number of extreme rays for a tropical cone in dimension d defined as the intersection of p half-spaces.

Theorem 1 is actually a tropical analogue of the dual form of McMullen's theorem. The primal form shows that a polytope with p extreme points in dimension d has no more than $U(p,d)$ facets. In the tropical setting, the analogues of facets are only partially understood due to the previously mentioned pathological features of the notions of faces of tropical polytopes. However, instead of facets, we may count the numbers of extreme rays of the polar cone, as defined in [20]. These extreme rays determine a finite minimal family of inequalities from which any "valid inequality" (i.e. any inequality satisfied by the elements of the initial cone) can be obtained by taking tropical linear combinations, and this family is unique up to a scaling. A simple corollary of Theorem 1, Corollary 1 below, shows that the number of extreme rays of the polar of a tropical polyhedral cone with p extreme rays in dimension d is bounded by $2d + d(U(p+d, d-1) - (d-1))$. This gives a weak analogue of the primal form of McMullen's theorem (this bound is not expected to be tight).

Finally, we note that Theorem 2 may seem surprising (and perhaps even disappointing) in the light of the developments of enumerative tropical geometry, following the work of Mikhalkin [33]. A deep result there (Mikhalkin's correspondence theorem) is that certain classical enumerative invariants (the number of algebraic curves satisfying appropriate constraints) can be computed from their tropical analogues, taking into account certain multiplicities. The results of the present paper are limited to the linear case, but concern inequalities instead of equalities. Theorem 2 shows that the most natural enumerative object concerning inequalities, the number of extreme points, does not tropicalize. More precisely, its proof shows that when deforming a classical polyhedron to obtain a tropical polyhedron, some of the classical extreme points degenerate in points which are no longer extreme in the tropical sense.

2. Bounding the number of extreme points of a tropical polyhedron

The symbol $\mathbb{R}_{\max}$ will denote the max-plus semiring, which is the set $\mathbb{R} \cup \{-\infty\}$ equipped with the addition $(a,b) \mapsto a \oplus b := \max(a,b)$ and the multiplication $(a,b) \mapsto ab := a + b$. The zero and unit elements will be denoted by $\mathbb{0}$ and $\mathbb{1}$, respectively, so $\mathbb{0} = -\infty$ and $\mathbb{1} = 0$. If $A = (a_{ij})$ is a

$p \times d$ matrix with entries in $\mathbb{R}_{\max}$, the matrix vector product Ax is naturally defined for $x \in \mathbb{R}^d_{\max}$, $(Ax)_i := \bigoplus_{1 \leqslant j \leqslant d} a_{ij} x_j$, which can be rewritten as $\max_{1 \leqslant j \leqslant d} a_{ij} + x_j$ with the classical notation.

In the present section, we will apply some asymptotic arguments, mixing classical and max-plus algebra, and so we will mainly use the classical notation. However, in the next section, we shall make an intensive use of the "max-plus" notation, which will make clearer some analogies with the classical case. In all cases, the reader will easily avoid any ambiguity from the context.

A subset $\mathcal{C}$ of $\mathbb{R}^d_{\max}$ is a *tropical* (*convex*) *cone* if

$$u, v \in \mathcal{C}, \quad \lambda, \mu \in \mathbb{R}_{\max} \quad \Longrightarrow \quad \lambda u \oplus \mu v \in \mathcal{C}.$$

Here, we denote by $\oplus$ the tropical sum of vectors, which is nothing but the entrywise max, and we denote by λu the vector obtained by multiplying in the tropical sense (i.e., adding) the scalar λ by each entry of the vector u.

We say that a non-zero vector $u \in \mathcal{C}$ is an *extreme generator* of $\mathcal{C}$ if $u = v \oplus w$ with $v, w \in \mathcal{C}$ implies $u = v$ or $u = w$. The set of scalar multiples of an extreme generator of $\mathcal{C}$ is an *extreme ray* of $\mathcal{C}$. A subset $\mathcal{U}$ of a tropical cone $\mathcal{C}$ is said to be a *generating family* of $\mathcal{C}$ if any vector $x \in \mathcal{C}$ can be expressed as $x = \bigoplus_{1 \leqslant k \leqslant K} \lambda_k u_k$ for some $K \in \mathbb{N}$, where $\lambda_k \in \mathbb{R}_{\max}$ and $u_k \in \mathcal{U}$ for all $1 \leqslant k \leqslant K$. A tropical cone is *finitely generated* if it has a finite generating family.

Let us recall the following tropical analogue of the Minkowski theorem, established by Gaubert and Katz [17,18] and Butkovič, Schneider and Sergeev [9].

Theorem 3 *(Tropical Minkowski Theorem [9,18]). (See also [17].) A closed tropical cone is generated by its extreme rays.*

This applies in particular to finitely generated tropical cones, which are always closed [18]. Then, we get a refinement (with the added characterization in terms of extreme rays) of an observation made by several authors including Moller [34], Wagneur [38], and Develin and Sturmfels [15], showing that a finitely generated tropical cone has a "basis" (generating family with minimal cardinality) which is unique up to the multiplication of its vectors by possibly different scalars. Observe that every generating family of $\mathcal{C}$ must contain at least one vector in each extreme ray of $\mathcal{C}$.

To establish Theorem 1, we shall think of tropical convex cones as limits of classical convex cones along an exponential deformation. Let $\beta > 0$ denote a parameter, and let E_β denote the map from $\mathbb{R}^d_{\max}$ to $\mathbb{R}^d$ which sends the vector $x = (x_j)$ to the vector $(\exp(\beta x_j))$. We denote by L_β the inverse map of E_β.

We shall use repeatedly the following inequalities, which hold for any finite set of scalars $\{v_r\}_{r=1,\dots,n} \subset \mathbb{R}_{\max}$,

$$\max_{1 \leqslant r \leqslant n} v_r \leqslant \beta^{-1} \log\left(\sum_{1 \leqslant r \leqslant n} \exp(\beta v_r)\right) \leqslant \beta^{-1} \log n + \max_{1 \leqslant r \leqslant n} v_r. \tag{1}$$

Proof of Theorem 1. Consider the tropical cone $\mathcal{C}$ of $\mathbb{R}^d_{\max}$ defined as the intersection of the following p tropical half-spaces:

$$\max_{1 \leqslant j \leqslant d} a_{ij} + x_j \leqslant \max_{1 \leqslant j \leqslant d} b_{ij} + x_j, \quad 1 \leqslant i \leqslant p, \tag{2}$$

and let $\mathcal{C}(\beta) \subset \mathbb{R}^d$ denote the (ordinary) convex cone consisting of the vectors y satisfying the inequalities:

$$\begin{aligned} &y_i \geqslant 0, \quad 1 \leqslant i \leqslant d, \\ &\frac{1}{d} \sum_{1 \leqslant j \leqslant d} \exp(\beta a_{ij}) y_j \leqslant \sum_{1 \leqslant j \leqslant d} \exp(\beta b_{ij}) y_j, \quad 1 \leqslant i \leqslant p. \end{aligned} \tag{3}$$

If $x \in \mathcal{C}$, then

$$\frac{1}{d}\sum_{1\leqslant j\leqslant d}\exp\bigl(\beta(a_{ij}+x_j)\bigr)\leqslant \exp\Bigl(\beta\Bigl(\max_{1\leqslant j\leqslant d} a_{ij}+x_j\Bigr)\Bigr)$$
$$\leqslant \exp\Bigl(\beta\Bigl(\max_{1\leqslant j\leqslant d} b_{ij}+x_j\Bigr)\Bigr)$$
$$\leqslant \sum_{1\leqslant j\leqslant d}\exp\bigl(\beta(b_{ij}+x_j)\bigr),$$

which shows that $y := E_\beta(x)$ belongs to $\mathcal{C}(\beta)$.

Consider now the simplex

$$\Sigma := \Bigl\{ y\in\mathbb{R}^d \Bigm| y\geqslant 0,\ \sum_{1\leqslant j\leqslant d} y_j = 1\Bigr\}.$$

The extreme rays of the cone $\mathcal{C}(\beta)$ are in one to one correspondence with the extreme points of the convex set $\mathcal{C}(\beta)\cap\Sigma$. By eliminating the variable y_d, we identify the latter set with a convex subset of $\mathbb{R}^{d-1}$ defined by $p+d$ affine inequalities. It follows that the number $K(\beta)$ of extreme points of $\mathcal{C}(\beta)\cap\Sigma$ is such that

$$K(\beta)\leqslant U(p+d,d-1).$$

Let $\{u_k(\beta)\}_{k=1,\dots,K(\beta)}\subset\mathbb{R}^d$ denote a family obtained by ordering the extreme points of $\mathcal{C}(\beta)\cap\Sigma$ in an arbitrary way.

Since $u_k(\beta)\geqslant 0$, we can find a vector $v_k(\beta)\in\mathbb{R}^d_{\max}$ such that $u_k(\beta)=E_\beta(v_k(\beta))$.

Let us now fix a sequence β_m tending to infinity. Since $K(\beta)$ only takes a finite number of values, after replacing β_m by a subsequence, we may assume that $K := K(\beta_m)$ is independent of m.

Let us consider an arbitrary index k among $1,\dots,K$. Since $\sum_j (u_k(\beta))_j = 1$ and $(u_k(\beta))_j\geqslant 0$, we deduce that $\exp(\beta(v_k(\beta))_j)\leqslant 1$, and so, $v_k(\beta)$ belongs to the set $[-\infty,0]^d$. Since this set is compact, possibly after extracting K subsequences we may assume that for every index $1\leqslant k\leqslant K$, $v_k(\beta_m)$ tends to some vector $v_k\in[-\infty,0]^d$ as m tends to infinity.

By applying the map L_β to the relation $\sum_j\exp(\beta(v_k(\beta))_j)=\sum_j(u_k(\beta))_j=1$, we get thanks to Inequality (1),

$$\max_{1\leqslant j\leqslant d}\bigl(v_k(\beta)\bigr)_j\leqslant 0\leqslant \beta^{-1}\log d+\max_{1\leqslant j\leqslant d}\bigl(v_k(\beta)\bigr)_j,$$

and so

$$\max_{1\leqslant j\leqslant d}(v_k)_j = 0. \tag{4}$$

We claim that the family $\{v_k\}_{k=1,\dots,K}$ generates the tropical cone $\mathcal{C}$.

By setting $y=E_\beta(v_k(\beta))$ in Inequality (3), applying the order preserving map L_β to both sides of this expression, and using Inequality (1), we get

$$-\beta^{-1}\log d+\max_{1\leqslant j\leqslant d} a_{ij}+\bigl(v_k(\beta)\bigr)_j\leqslant \beta^{-1}\log d+\max_{1\leqslant j\leqslant d} b_{ij}+\bigl(v_k(\beta)\bigr)_j.$$

Taking $\beta := \beta_m$ and letting m tend to infinity, we deduce that

$$\max_{1\leqslant j\leqslant d} a_{ij}+(v_k)_j\leqslant \max_{1\leqslant j\leqslant d} b_{ij}+(v_k)_j,$$

which shows that $v_k\in\mathcal{C}$.

Consider now an arbitrary element $x\in\mathcal{C}$. Since $u_k(\beta)$ generates the convex cone $\mathcal{C}(\beta)$, we can express the vector $E_\beta(x)\in\mathcal{C}(\beta)$ as a linear combination

$$E_\beta(x)=\sum_{1\leqslant k\leqslant K}\delta_k u_k(\beta)=\sum_{1\leqslant k\leqslant K}\delta_k E_\beta\bigl(v_k(\beta)\bigr), \tag{5}$$

for some scalars $\delta_k\geqslant 0$, which can be written as $\delta_k=\exp(\beta\lambda_k(\beta))$ for some $\lambda_k(\beta)\in\mathbb{R}_{\max}$.

We deduce from (5) that

$$E_\beta(x) \geqslant \delta_k E_\beta\big(v_k(\beta)\big),$$

and so, for all j,

$$x_j \geqslant \lambda_k(\beta) + \big(v_k(\beta)\big)_j.$$

Choosing any index j such that $(v_k)_j = 0$, which exists by (4), we deduce that $\lambda_k(\beta_m)$ is bounded from above as m tends to infinity. Hence, after extracting a new subsequence, we may assume that $\lambda_k(\beta_m)$ converges to some scalar $\lambda_k \in \mathbb{R}_{\max}$.

Since by (5) for each j we have

$$\exp(\beta x_j) = \sum_{1\leqslant k\leqslant K} \exp\big(\beta\lambda_k(\beta) + \beta\big(v_k(\beta)\big)_j\big),$$

applying the map L_β to this equality and using Inequality (1), we obtain

$$\max_{1\leqslant k\leqslant K} \lambda_k(\beta) + \big(v_k(\beta)\big)_j \leqslant x_j \leqslant \beta^{-1}\log K + \max_{1\leqslant k\leqslant K} \lambda_k(\beta) + \big(v_k(\beta)\big)_j.$$

Then, as this holds for any j, letting $\beta = \beta_m$ tend to infinity we conclude that

$$x = \max_{1\leqslant k\leqslant K} \lambda_k + v_k.$$

This shows that the family of vectors $\{v_k\}_{k=1,\dots,K}$ generates the tropical cone $\mathcal{C}$. Since the number of extreme rays of $\mathcal{C}$ is bounded from above by the cardinality of any of its generating families, this concludes the proof of Theorem 1. $\square$

The polar [20,19] of a tropical polyhedral cone $\mathcal{C} \subset \mathbb{R}^d_{\max}$ consists of the pairs of vectors (a,b), where $a, b \in \mathbb{R}^d_{\max}$ are such that $ax \leqslant bx$ for all $x \in \mathcal{C}$. In other words, the polar represents the set of "valid inequalities" satisfied by the elements of $\mathcal{C}$. The extreme rays of the polar provide, in particular, a finite set of inequalities defining the original cone [20]. Assuming, without loss of generality, that for all $i \in \{1,\dots,d\}$ there exists $z \in \mathcal{C}$ such that $z_i \neq \mathbb{0}$, it is shown in [5], and it can readily be derived from [19, Theorem 5] or [6, Prop. 5.13], that any valid inequality corresponding to an extreme ray of the polar is either a multiple of the trivial inequality $x_i \leqslant x_i$, for some $i \in \{1,\dots,d\}$, or of the form $u_i x_i \leqslant \bigoplus_{1\leqslant j\leqslant d, j\neq i} u_j x_j$, again for some $i \in \{1,\dots,d\}$, where $u \in \mathbb{R}^d_{\max}$ is an extreme generator of the tropical cone

$$\mathcal{C}_i^\circ := \Big\{ v \in \mathbb{R}^d_{\max} \;\Big|\; v_i x_i \leqslant \bigoplus_{1\leqslant j\leqslant d,\ j\neq i} v_j x_j,\ \forall x \in \mathcal{C} \Big\}, \tag{6}$$

called the ith polar of $\mathcal{C}$. Observe that if $\mathcal{C}$ is a tropical polyhedral cone in $\mathbb{R}^d_{\max}$ with p extreme rays, then $\mathcal{C}_i^\circ$ is a tropical polyhedral cone which can be defined as the intersection of p tropical half-spaces (in (6), we can equivalently replace $\mathcal{C}$ by the set composed of one extreme generator for each extreme ray of $\mathcal{C}$). Therefore, by Theorem 1 the number of extreme rays of the ith polar of $\mathcal{C}$ is at most $U(p+d, d-1)$. However, since the trivial inequality $\mathbb{0} \leqslant x_j$ corresponds to an extreme ray of the ith polar of $\mathcal{C}$ for any $i \neq j$, the number of extreme rays of the polar of $\mathcal{C}$ is at most $2d + d(U(p+d, d-1) - (d-1))$, where among these extreme rays $2d$ correspond to the trivial inequalities $x_i \leqslant x_i$ and $\mathbb{0} \leqslant x_i$, for $i \in \{1,\dots,d\}$. In consequence, we obtain:

Corollary 1. *A tropical polyhedral cone in $(\mathbb{R} \cup \{-\infty\})^d$ with p extreme rays can be expressed as the intersection of at most $d(U(p+d, d-1) - (d-1))$ tropical half-spaces, and its polar has no more than $2d + d(U(p+d, d-1) - (d-1))$ extreme rays.*

3. The tropical signed cyclic polyhedral cone and its polar

We shall use the symmetrization of the max-plus semiring that M. Plus introduced in [36] to establish a max-plus analogue of the Cramer theorem. An intimately related Cramer theorem was established by Richter-Gebert, Sturmfels, and Theobald in [37]. In a nutshell, the result of [36] deals with max-plus linear systems in which signs are taken into account, whereas the result of [37] concerns systems of equations in the tropical sense: rather than requiring the maximum of "positive" terms of an expression to be equal to the maximum of its "negative" terms, it is only required that the maximum of the terms to be attained at least twice. The former Cramer theorem yields some information on amoebas (image by the valuation) of linear spaces over the field of real Puiseux series, whereas the latter Cramer theorem concerns amoebas over the field of complex Puiseux series. We refer the reader to the work by Akian, Gaubert and Guterman [2], which gives a unified view of these Cramer theorems, connecting them also with a further work of Izhakian [24]. In what follows, we need the version with signs, and use therefore the result of [36], referring the reader to [2] for more information.

The symmetrized max-plus semiring $\mathbb{S}_{\max}$ consists of three copies of $\mathbb{R}_{\max}$, glued by identifying the zero element. A number of $\mathbb{S}_{\max}$ is written formally either as a, $\ominus a$, or $a^\bullet$ for some $a \in \mathbb{R}_{\max}$. These three numbers are different, unless a is the zero element (i.e. $a = -\infty$). The *sign* $\operatorname{sgn} x$ of an element $x \in \mathbb{S}_{\max}$ is defined to be $+1$ if $x = a$ for some $a \in \mathbb{R}_{\max} \setminus \{-\infty\}$, -1 if $x = \ominus a$ for some $a \in \mathbb{R}_{\max} \setminus \{-\infty\}$, and 0 otherwise. The elements of the form a, $\ominus a$ and $a^\bullet$ are said to be *positive*, *negative* and *balanced*, respectively. The elements which are either positive or negative are said to be *signed*. We denote by $\mathbb{S}^\vee_{\max}$ the set of signed elements and by $\mathbb{S}^\bullet_{\max}$ the set of balanced elements, so that

$$\mathbb{S}_{\max} = \mathbb{S}^\vee_{\max} \cup \mathbb{S}^\bullet_{\max},$$

the intersection of the latter sets being reduced to the zero element. A vector is *signed* (resp. *balanced*) if each of its entries is signed (resp. balanced).

The *modulus* of $x \in \{a, \ominus a, a^\bullet\}$ is defined as $|x| := a$. The addition of two elements $x, y \in \mathbb{S}_{\max}$ is defined to be $\max(|x|, |y|)$ if the maximum is attained only by elements of positive sign, $\ominus \max(|x|, |y|)$ if it is attained only by elements of negative sign, and $\max(|x|, |y|)^\bullet$ otherwise. For instance, $(\ominus 3) \oplus (2 \oplus (\ominus 2)) = (\ominus 3) \oplus 2^\bullet = \ominus 3$. The multiplication is defined in such a way that the modulus and the sign are both morphisms. For instance, $(\ominus 3)(\ominus 4) = 7$, but $(\ominus 3)4^\bullet = 7^\bullet$. The semiring $\mathbb{S}_{\max}$ is equipped with an involution $x \mapsto \ominus x$, which sends the element a to $\ominus a$, and vice versa, and which fixes every balanced element $a^\bullet$. It is convenient to write, for $x, y \in \mathbb{S}_{\max}$, $x \ominus y := x \oplus (\ominus y)$. We shall identify an element $a \in \mathbb{R}_{\max}$ with the corresponding element of $\mathbb{S}_{\max}$, which yields an embedding of the semiring $\mathbb{R}_{\max}$ into $\mathbb{S}_{\max}$.

The additive and multiplicative rules of $\mathbb{S}_{\max}$ become intuitive if the element $a \in \mathbb{S}_{\max}$ is interpreted as the equivalence class of real functions of t belonging to $\Theta(t^a)$ as $t \to \infty$ (i.e., functions of t belonging to some interval $[Ct^a, C't^a]$ for some $C, C' > 0$), the element $\ominus a$ is interpreted as the equivalence class consisting of the opposites of the former functions, i.e. $-\Theta(t^a)$, whereas $a^\bullet$ represents the equivalence class $O(t^a)$. Then, the rule $(\ominus 3) \oplus (2 \ominus 2) = (\ominus 3) \oplus 2^\bullet = \ominus 3$ can be interpreted as the "classical" rule with asymptotic expansions: $-\Theta(t^3) + \Theta(t^2) - \Theta(t^2) = -\Theta(t^3) + O(t^2) = -\Theta(t^3)$.

Given p scalars $-\infty < t_1 < t_2 < \cdots < t_p$ in $\mathbb{R}_{\max}$, and a collection of signs $\epsilon_{ij} \in \{\oplus \mathbb{1}, \ominus \mathbb{1}\}$, $1 \leqslant i \leqslant p$, $1 \leqslant j \leqslant d$, we construct the $p \times d$ matrix $C := C(\epsilon, t)$ with entries in the symmetrized max-plus semiring

$$C_{ij} = \epsilon_{ij} t_i^{j-1}, \quad 1 \leqslant i \leqslant p, \ 1 \leqslant j \leqslant d.$$

Here, and in the rest of this section, the exponentiation should be understood in the tropical sense, as well as the adjective "non-zero". We denote by C_i the ith row of C, and we write $C_i = C_i^+ \ominus C_i^-$ where $C_i^+, C_i^- \in \mathbb{R}^d_{\max}$ are chosen in such a way that for all $1 \leqslant j \leqslant d$ exactly one of the jth entries of C_i^+ and C_i^- is non-zero.

Definition 1 *(Signed cyclic polyhedral cone).* The *signed cyclic polyhedral cone* with sign pattern (ϵ_{ij}) is the tropical cone of $(\mathbb{R}^d_{\max})^2$ generated by the elements (C_i^+, C_i^-), $1 \leqslant i \leqslant p$. The *polar* of this cone is the set $\mathcal{K}(\epsilon)$ of vectors $x \in \mathbb{R}^d_{\max}$ such that

$$C_i^- x \leqslant C_i^+ x, \quad \forall 1 \leqslant i \leqslant p.$$

The notion of tropical polar was introduced in [20], to which we refer the reader for more information. We note that a related cyclic polytope (without signs) was studied by Block and Yu in [10].

We shall often write $\mathcal{K}$ instead of $\mathcal{K}(\epsilon)$ for brevity.

Example 1. Let $d = 3$ and $p = 2$. Define $t_i = i - 1$ for $i = 1, 2$, and consider the sign pattern $(\epsilon_{ij}) = \left(\begin{smallmatrix} + & - & + \\ + & - & + \end{smallmatrix}\right)$. Then, the signed cyclic polyhedral cone with sign pattern (ϵ_{ij}) is the tropical cone of $(\mathbb{R}^3_{\max})^2$ generated by the vectors (C_i^+, C_i^-), $i = 1, 2$, where $C_1^+ = (0, -\infty, 0)$, $C_1^- = (-\infty, 0, -\infty)$, $C_2^+ = (0, -\infty, 2)$ and $C_2^- = (-\infty, 1, -\infty)$. Its polar is the set of vectors $(x_1, x_2, x_3) \in \mathbb{R}^3_{\max}$ such that:

$$\begin{pmatrix} -\infty & 0 & -\infty \\ -\infty & 1 & -\infty \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqslant \begin{pmatrix} 0 & -\infty & 0 \\ 0 & -\infty & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

This cone is represented on the left-hand side of Fig. 2 (see also Example 3) below.

We shall give a combinatorial construction of the extreme rays of $\mathcal{K}$. An inequality $ax \leqslant bx$ $(a, b \in \mathbb{R}^d_{\max})$ is said to be *saturated* by $y \in \mathbb{R}^d_{\max}$ if the equality $ay = by$ holds. By analogy with the classical case, we expect an extreme generator to be obtained by saturating k inequalities among $C_i^+ x \geqslant C_i^- x$, $1 \leqslant i \leqslant p$, and by setting $d - k - 1$ entries of x to zero. In this way, we get k equations for $k + 1$ degrees of freedom, and can hope the solution x to be unique up to a scalar multiple.

In order to implement this idea, given two sequences of indices $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_{k+1}\}$, where $k \leqslant d - 1$ and $i_1 < \cdots < i_k$, $j_1 < \cdots < j_{k+1}$, we consider the matrix $C(I, J)$ obtained by deleting the rows and columns of C whose indices do not belong to I and J, respectively. The matrices $C^+(I, J)$ and $C^-(I, J)$ are defined similarly.

We shall need to characterize the solutions z of the system $C^+(I, J)z = C^-(I, J)z$.

To this end, let us recall some basic consequences of the Cramer theorem of [36]. This result applies to systems of "balances". The balance relation in $\mathbb{S}_{\max}$ is defined by $x \nabla y$ if $x \ominus y \in \mathbb{S}^\bullet_{\max}$. It is a non-transitive relation, which allows one to make elimination arguments which are somehow similar to the case of rings, although the addition does not have an opposite law. In particular, if $x, y \in \mathbb{S}_{\max}$, $x = y$ implies that $x \ominus y \nabla \mathbb{0}$, and the converse holds if x and y are signed. The balance relation is extended to vectors of $\mathbb{S}^d_{\max}$, being understood entrywise.

Consider a linear system of the form $A'x \oplus b' = A''x \oplus b''$, where A', A'' are $n \times n$ matrices with entries in $\mathbb{R}_{\max}$, and $b', b'' \in \mathbb{R}^n_{\max}$. Let $A := A' \ominus A''$, which is a well defined matrix with entries in $\mathbb{S}_{\max}$. Similarly, let $b := b'' \ominus b'$. It follows from the previous discussion that if $A'x \oplus b' = A''x \oplus b''$, then, the balance relation $Ax \nabla b$ holds. Conversely, if x is a vector with positive entries, and if $Ax \nabla b$, then $A'x \oplus b' = A''x \oplus b''$.

The determinant of an $n \times n$ matrix $A = (a_{ij})$ with entries in $\mathbb{S}_{\max}$ is given by

$$\det A := \bigoplus_{\sigma \in S_n} \operatorname{sgn}(\sigma) a_{\sigma(1)1} \cdots a_{\sigma(n)n},$$

where $\operatorname{sgn}(\sigma) := \oplus \mathbb{1}$ if σ is even and $\operatorname{sgn}(\sigma) := \ominus \mathbb{1}$ if σ is odd. We denote by A^{adj} the transpose of the matrix of cofactors.

Theorem 4. *(See [36], see also [2, Th. 6.4].) Let A be an $n \times n$ matrix with entries in $\mathbb{S}_{\max}$ and $b \in \mathbb{S}^d_{\max}$. Then, every signed solution of the system of balances*

$$Ax \nabla b \tag{7}$$

satisfies

$$\det Ax \nabla A^{\mathrm{adj}} b.$$

Conversely, if $A^{\mathrm{adj}}b$ *is signed and if* $\det A$ *is invertible, then* $x = (\det A)^{-1} A^{\mathrm{adj}} b$ *is the unique signed solution of* (7).

By taking b to be the zero vector, it follows that the equation $Ax \nabla \mathbb{0}$ has a non-zero signed solution only if $\det A$ is balanced. The converse implication also holds [36], but we shall not need it here.

This max-plus analogue of Cramer theorem shows that the system of balances $Ax\nabla b$ can be solved by the usual Cramer rule, the determinants being interpreted as elements of $\mathbb{S}_{\max}$. In particular, it shows that if none of the Cramer determinants (the determinants appearing in the Cramer formula) is balanced, then the system $Ax\nabla b$ has a unique signed solution, given by the Cramer formulæ. Under the same circumstances, the original system $A'x \oplus b' = A''x \oplus b''$ has a solution in $\mathbb{R}_{\max}$ if and only if the solution of the system of balances is positive.

We now apply this result to the homogeneous system $C^+(I, J)z = C^-(I, J)z$, with k equations and $k+1$ unknowns.

Let us now consider the system of balances $C(I, J)z\nabla\mathbb{0}$, and let D_r denote the rth Cramer determinant of this system, which is the determinant of the matrix obtained from $C(I, J)$ by deleting column r, i.e. $C(I, J \setminus \{r\})$.

Lemma 1. *The Cramer determinants of the previous linear system are given by*

$$\begin{aligned}
D_{k+1} &= t_{i_1}^{j_1-1} t_{i_2}^{j_2-1} \cdots t_{i_k}^{j_k-1} \epsilon_{i_1 j_1} \epsilon_{i_2 j_2} \cdots \epsilon_{i_k j_k}, \\
D_1 &= t_{i_1}^{j_2-1} t_{i_2}^{j_3-1} \cdots t_{i_k}^{j_{k+1}-1} \epsilon_{i_1 j_2} \epsilon_{i_2 j_3} \cdots \epsilon_{i_k j_{k+1}}, \\
D_r &= t_{i_1}^{j_1-1} \cdots t_{i_{r-1}}^{j_{r-1}-1} t_{i_r}^{j_{r+1}-1} \cdots t_{i_k}^{j_{k+1}-1} \epsilon_{i_1 j_1} \cdots \epsilon_{i_{r-1} j_{r-1}} \epsilon_{i_r j_{r+1}} \cdots \epsilon_{i_k j_{k+1}}, \quad 2 \leqslant r \leqslant k.
\end{aligned}$$

Proof. When $A = C(I, J \setminus \{r\})$, we have

$$\det A = \bigoplus_{\sigma \in S_k} \operatorname{sgn}(\sigma) \epsilon_{i_{\sigma(1)} j_1} t_{i_{\sigma(1)}}^{j_1-1} \cdots \epsilon_{i_{\sigma(r-1)} j_{r-1}} t_{i_{\sigma(r-1)}}^{j_{r-1}-1} \epsilon_{i_{\sigma(r)} j_{r+1}} t_{i_{\sigma(r)}}^{j_{r+1}-1} \cdots \epsilon_{i_{\sigma(k)} j_{k+1}} t_{i_{\sigma(k)}}^{j_{k+1}-1}, \tag{8}$$

since

$$a_{\sigma(s)s} = \begin{cases} \epsilon_{i_{\sigma(s)} j_s} t_{i_{\sigma(s)}}^{j_s-1} & \text{if } s < r, \\ \epsilon_{i_{\sigma(s)} j_{s+1}} t_{i_{\sigma(s)}}^{j_{s+1}-1} & \text{if } s \geqslant r. \end{cases}$$

If we define $\bar\sigma$ by $\bar\sigma(s) = s$ for $s = 1, \dots, k$, it follows that

$$t_{i_{\sigma(1)}}^{j_1-1} \cdots t_{i_{\sigma(r-1)}}^{j_{r-1}-1} t_{i_{\sigma(r)}}^{j_{r+1}-1} \cdots t_{i_{\sigma(k)}}^{j_{k+1}-1} < t_{i_{\bar\sigma(1)}}^{j_1-1} \cdots t_{i_{\bar\sigma(r-1)}}^{j_{r-1}-1} t_{i_{\bar\sigma(r)}}^{j_{r+1}-1} \cdots t_{i_{\bar\sigma(k)}}^{j_{k+1}-1},$$

for all $\sigma \neq \bar\sigma$, because $t_i^{j-1} t_{i'}^{j'-1} < t_{i'}^{j-1} t_i^{j'-1}$ whenever $t_{i'} < t_i$ and $j < j'$. Therefore, the term corresponding to $\bar\sigma$ is the only one maximizing the modulus in (8), which implies that

$$D_r = \epsilon_{i_1 j_1} t_{i_1}^{j_1-1} \cdots \epsilon_{i_{r-1} j_{r-1}} t_{i_{r-1}}^{j_{r-1}-1} \epsilon_{i_r j_{r+1}} t_{i_r}^{j_{r+1}-1} \cdots \epsilon_{i_k j_{k+1}} t_{i_k}^{j_{k+1}-1},$$

and in particular D_r is signed. □

Corollary 2. *The system of balances* $C(I, J)z\nabla\mathbb{0}$ *has a signed non-zero solution* z*, which is unique up to a scalar multiple, and which is determined by the relations*

$$
\begin{aligned}
z_1 &= \ominus t_{i_1}^{j_2-j_1} \epsilon_{i_1 j_1} \epsilon_{i_1 j_2} z_2, \\
z_2 &= \ominus t_{i_2}^{j_3-j_2} \epsilon_{i_2 j_2} \epsilon_{i_2 j_3} z_3, \\
&\vdots \\
z_k &= \ominus t_{i_k}^{j_{k+1}-j_k} \epsilon_{i_k j_k} \epsilon_{i_k j_{k+1}} z_{k+1}.
\end{aligned}
\tag{9}
$$

Proof. Let A denote the matrix consisting of the first k columns of $C(I, J)$ and let b denote the opposite of the last column of $C(I, J)$. Define $\bar{z}$ to be the vector consisting of the first k coordinates of z. Then, we have $C(I, J)z \nabla \mathbb{0}$ if, and only if, $A\bar{z} \nabla b z_{k+1}$.

The Cramer theorem above implies that

$$D_{k+1} z_r \nabla (\ominus \mathbb{1})^{k-r+1} D_r z_{k+1}, \quad 1 \leqslant r \leqslant k.$$

Recall that when two elements of $\mathbb{S}_{\max}$ y and y' are both signed, $y \nabla y'$ implies $y = y'$. It follows that the relations (9) hold. The same theorem also shows that, conversely, setting $z_{k+1} = \mathbb{1}$, and defining z by (9), we obtain a solution of $C(I, J)z \nabla \mathbb{0}$. $\square$

We get as an immediate corollary.

Corollary 3. *The system $C^+(I, J)z = C^-(I, J)z$ has a non-zero solution in $\mathbb{R}^n_{\max}$ if and only if*

$$\epsilon_{i_1 j_1} \epsilon_{i_1 j_2} = \epsilon_{i_2 j_2} \epsilon_{i_2 j_3} = \cdots = \epsilon_{i_k j_k} \epsilon_{i_k j_{k+1}} = \ominus \mathbb{1}.$$

Then, this solution z is determined by (9)*, up to a scalar multiple.*

We shall denote by $z(I, J)$ the vector defined by (9) together with the normalization condition $z_{k+1} = \mathbb{1}$. The vector $z(I, J)$ is a candidate to be an extreme generator of $\mathcal{K}$. We shall see that only those subsets I, J meeting a special combinatorial condition that we express in terms of lattice paths actually yield an extreme generator.

We shall visualize a pair of integers (i, j), with $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant d$, as the position of the corresponding entry in a $p \times d$ matrix. So (p, d) is the position of the bottom right entry and $(1, 1)$ is the position of the top left entry.

We shall consider oriented lattice paths, which are sequences of positions starting from some top node $(1, i)$ and ending with some bottom node (p, j), in which at each step, the next position is either immediately at the right or immediately at the bottom of the current one. Thus, such a path consists of vertical segments (oriented downward) and of horizontal segments (oriented from left to right). An example of lattice path is given in Fig. 1, the initial (vertical) segment consists of the positions $(1, 2)$, $(2, 2)$, $(3, 2)$, the next (horizontal) segment consists of $(3, 2)$, $(3, 3)$, $(3, 4)$, $(3, 5)$, the next (vertical) segment consists of $(3, 5)$, $(4, 5)$, $(5, 5)$, $(6, 5)$, etc. Note that the initial and final segments may be restricted to a unique position.

We shall say that such a lattice path is *tropically allowed* for the sign pattern (ϵ_{ij}) if the following conditions are valid:

(i) every sign occurring on the initial vertical segment, except possibly the sign at the bottom of the segment, is positive;
(ii) every sign occurring on the final vertical segment, except possibly the sign at the top of the segment, is positive;
(iii) every sign occurring in some other vertical segment, except possibly the signs at the top and bottom of this segment, is positive;
(iv) for every horizontal segment, the pair of signs consisting of the signs of the leftmost and rightmost positions of the segment is of the form $(+, -)$ or $(-, +)$;
(v) as soon as a pair $(-, +)$ occurs as the pair of extreme signs of some horizontal segment, the pairs of signs corresponding to all the horizontal segments below this one must also be equal to $(-, +)$.

$$
\begin{array}{c c}
 & \begin{array}{ccccccccccccccc} & j_1 & & & j_2 & j_3 & & & & j_4 & & j_5 & j_6 & & \end{array} \\
\begin{array}{c} \\ \\ i_1 \\ \\ \\ i_2 \\ \\ i_3 \\ \\ \\ \\ i_4 \\ \\ \\ \\ \\ i_5 \\ \\ \\ \\ \end{array} &
\begin{pmatrix}
\cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & + & \star & \star & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & \star & \star & \star & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & \star & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot
\end{pmatrix}
\end{array}
$$

Fig. 1. A tropically allowed lattice path.

The notion of (non-tropically) allowed lattice path is defined only by Conditions (i)–(iv). Hence, a tropically allowed path is allowed, but the converse is not true.

Fig. 1 gives an example of tropically allowed lattice path, the positions belonging to the path but the sign of which is irrelevant are indicated by the symbol "$\star$". The positions which do not belong to the path are indicated by the symbol "$\cdot$".

Example 2. Consider the 2×3 sign pattern of Example 1. Then, it can be checked that there are five tropically allowed lattice paths for this sign pattern which are: $\left(\begin{smallmatrix} + & \cdot & \cdot \\ + & \cdot & \cdot \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} \cdot & \cdot & + \\ \cdot & \cdot & + \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} + & \cdot & \cdot \\ + & - & \cdot \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} + & - & \cdot \\ \cdot & - & + \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} \cdot & - & + \\ \cdot & \cdot & + \end{smallmatrix}\right)$.

In order to prove Theorem 2, it is convenient to recall the notion of tangent cone introduced in [3], see also [4]. Given a cone $\mathcal{C}$ of $\mathbb{R}^d_{\max}$ defined as the intersection of a finite set of tropical half-spaces $A_r x \leqslant B_r x$, where A_r and B_r denote the rth rows of some matrices A and B, the *tangent cone* of $\mathcal{C}$ at $y \in \mathbb{R}^d_{\max}$ is defined as the tropical cone $\mathcal{T}(\mathcal{C}, y)$ of $\mathbb{R}^d_{\max}$ given by the system of inequalities

$$\max_{i \in \arg\max(A_r y)} x_i \leqslant \max_{j \in \arg\max(B_r y)} x_j \quad \text{for all } r \quad \text{such that} \quad A_r y = B_r y, \tag{10}$$

where $\arg\max(cy)$ is the argument of the maximum $cy = \max_{1 \leqslant i \leqslant d}(c_i + y_i)$ for any row vector c. The tangent cone of $\mathcal{C}$ at y provides a local description of $\mathcal{C}$ around y, leading to the following characterization of the extreme generators of a tropical polyhedral cone.

Theorem 5. *(See [3].) A vector $y \in \mathbb{R}^d_{\max}$ belongs to an extreme ray of a tropical polyhedral cone $\mathcal{C}$ if, and only if, there exists $s \in \{1, \dots, d\}$ such that*

$$\big(x \in \mathcal{T}(\mathcal{C}, y) \cap \{\mathbb{1}, \mathbb{0}\}^d \text{ and } x_s = \mathbb{1}\big) \implies (x_r = \mathbb{1} \text{ or } y_r = \mathbb{0}) \tag{11}$$

for all $r \in \{1, \dots, d\}$.

As a consequence, we obtain.

Corollary 4. *Let $\mathcal{C} := \{x \in \mathbb{R}^d_{\max} \mid A_r x \leqslant B_r x,\ 1 \leqslant r \leqslant p\}$ be a tropical polyhedral cone and let $y \in \mathbb{R}^d_{\max}$ be a vector in an extreme ray of $\mathcal{C}$. If t entries of y are zero, then y must saturate at least $d - t - 1$ inequalities among $A_r x \leqslant B_r x$, $1 \leqslant r \leqslant p$.*

Proof. Let s be an index satisfying the condition in Theorem 5. Among the inequalities that define $\mathcal{T}(\mathcal{C}, y)$, consider those with precisely one term on the right-hand side, i.e. those of the form

$$\bigoplus_{i \in I_h} x_i \leqslant x_h,$$

for some set of indices I_h. Let H be the set composed of such indices h. If y saturates strictly less than $d - t - 1$ inequalities, there exists $q \in \{1, \dots, d\}$ such that $q \notin \{s\} \cup H \cup \{j \mid y_j = \mathbb{0}\}$. Then, the vector $x \in \{\mathbb{1}, \mathbb{0}\}^d$ defined by $x_q := \mathbb{0}$ and $x_i := \mathbb{1}$ for all $i \neq q$ belongs to $\mathcal{T}(\mathcal{K}, y)$, which contradicts (11). □

Now we can restate Theorem 2 more precisely in the following way.

Theorem 2. *The extreme rays of the polar $\mathcal{K}$ of a signed cyclic polyhedral cone with sign pattern (ϵ_{ij}) are in one to one correspondence with the tropically allowed lattice paths for (ϵ_{ij}).*

Proof. Let $x \in \mathbb{R}^d_{\max}$ be a vector in an extreme ray of $\mathcal{K}$. Assume that $x_j \neq \mathbb{0}$ if, and only if, $j \in J = \{j_1, \dots, j_{k+1}\}$, where $k \leqslant d - 1$. Then, by Corollary 4 we know that x must saturate at least k inequalities among $C_i^- x \leqslant C_i^+ x$, $i = 1, \dots, p$. More precisely, we claim that x saturates exactly k inequalities. To see this, let $\bar{x} \in \mathbb{R}^{k+1}_{\max}$ be the vector obtained by deleting the entries of x which do not belong to J (or equivalently, are zero). Assume that x saturates the inequalities $C_i^- x \leqslant C_i^+ x$ for $i \in I$, where I has $k + 1$ elements. Then, we would have $C(I, J)\bar{x} \nabla \mathbb{0}$, where the determinant of the $(k + 1) \times (k + 1)$ matrix $C(I, J)$ is signed by Lemma 1, contradicting Cramer theorem above in the case of homogeneous systems of balances. This proves our claim.

Let $I = \{i_1, \dots, i_k\}$ be the set composed of the indices of the inequalities which x saturates. We assume $i_1 < \dots < i_k$ and $j_1 < \dots < j_{k+1}$. With this extreme ray, we associate the lattice path $\mathscr{P}$

$$(1, j_1), \dots, (i_1, j_1), \dots, (i_1, j_2), \dots, (i_2, j_2), \dots, (i_k, j_k), \dots, (i_k, j_{k+1}), \dots, (p, j_{k+1}). \tag{12}$$

In other words, the ordinates of the horizontal segments of this path are given by the indices of the inequalities which are saturated, and the abscissæ of the vertical segments are given by the indices j such that x_j is non-zero. Note that this path has k horizontal segments and that (i_r, j_r) and (i_r, j_{r+1}) are the leftmost and rightmost positions of the rth horizontal segment.

We claim that $\mathscr{P}$ is tropically allowed. In order to prove this, define $\bar{x} \in \mathbb{R}^{k+1}_{\max}$ as above. Since $C^+(I, J)\bar{x} = C^-(I, J)\bar{x}$, the "only if" part of Corollary 3 shows precisely that $\mathscr{P}$ satisfies Condition (iv). Hence, we may assume

$$\bar{x} = z(I, J) = \begin{pmatrix} t_{i_1}^{j_2 - j_1} t_{i_2}^{j_3 - j_2} \dots t_{i_k}^{j_{k+1} - j_k} \\ t_{i_2}^{j_3 - j_2} \dots t_{i_k}^{j_{k+1} - j_k} \\ \vdots \\ t_{i_k}^{j_{k+1} - j_k} \\ \mathbb{1} \end{pmatrix}, \tag{13}$$

which implies that

$$\left(C_i^+ \oplus C_i^-\right) x = \bigoplus_{1 \leqslant j \leqslant d} t_i^{j-1} x_j = \bigoplus_{1 \leqslant r \leqslant k+1} t_i^{j_r - 1} x_{j_r} = \bigoplus_{1 \leqslant r \leqslant k+1} t_i^{j_r - 1} t_{i_r}^{j_{r+1} - j_r} \cdots t_{i_k}^{j_{k+1} - j_k}, \tag{14}$$

for all $1 \leqslant i \leqslant p$. Then, if $i_s < i < i_{s+1}$, it follows that

$$t_i^{j_r - 1} t_{i_r}^{j_{r+1} - j_r} \cdots t_{i_k}^{j_{k+1} - j_k} < t_i^{j_{s+1} - 1} t_{i_{s+1}}^{j_{s+2} - j_{s+1}} \cdots t_{i_k}^{j_{k+1} - j_k},$$

for all $r \neq s+1$. Hence, the maximum in (14) is attained only for $j = j_{s+1}$. Since $C_i^- x \leqslant C_i^+ x$, the sign of $C_{ij_{s+1}}$ must be positive, implying that Condition (iii) is valid for $\mathscr{P}$. Analogously, when $i < i_1$

$$t_i^{j_r-1} t_{i_r}^{j_{r+1}-j_r} \cdots t_{i_k}^{j_{k+1}-j_k} < t_i^{j_1-1} t_{i_1}^{j_2-j_1} \cdots t_{i_k}^{j_{k+1}-j_k},$$

for all $r > 1$, so $C_i^- x \leqslant C_i^+ x$ implies that Condition (i) holds for $\mathscr{P}$. Finally, if $i > i_k$, we have

$$t_i^{j_r-1} t_{i_r}^{j_{r+1}-j_r} \cdots t_{i_k}^{j_{k+1}-j_k} < t_i^{j_{k+1}-1},$$

for all $r < k+1$, and the same argument as before shows that $\mathscr{P}$ satisfies Condition (ii).

When $i = i_s$ for some s, we have

$$t_i^{j_r-1} t_{i_r}^{j_{r+1}-j_r} \cdots t_{i_k}^{j_{k+1}-j_k} < t_i^{j_s-1} t_{i_s}^{j_{s+1}-j_s} \cdots t_{i_k}^{j_{k+1}-j_k} = t_i^{j_{s+1}-1} t_{i_{s+1}}^{j_{s+2}-j_{s+1}} \cdots t_{i_k}^{j_{k+1}-j_k},$$

for all $r \notin \{s, s+1\}$, which means that the tangent cone $\mathcal{T}(\mathcal{K}, x)$ of $\mathcal{K}$ at x is defined by the inequalities $x_{j_r} \geqslant x_{j_{r+1}}$ if $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (\oplus\mathbb{1}, \ominus\mathbb{1})$ and $x_{j_r} \leqslant x_{j_{r+1}}$ if $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (\ominus\mathbb{1}, \oplus\mathbb{1})$, for $r = 1, \ldots, k$. It is convenient to visualize the relations defining the tangent cone by constructing a digraph with nodes $j_1, \ldots, j_{k+1}$ and an arc from j_r to j_{r+1} (resp. from j_{r+1} to j_r) if the inequality $x_{j_r} \geqslant x_{j_{r+1}}$ (resp. $x_{j_{r+1}} \geqslant x_{j_r}$) belongs to these relations. For instance, the digraph associated with the relations $x_{j_1} \geqslant x_{j_2} \geqslant x_{j_3} \leqslant x_{j_4} \leqslant x_{j_5} \geqslant x_{j_6}$ is

$$j_1 \to j_2 \to j_3 \leftarrow j_4 \leftarrow j_5 \to j_6.$$

Theorem 5 requires the existence of a node j_s such that $x_{j_s} = \mathbb{1}$ implies $x_{j_r} = \mathbb{1}$ for all r. This can only occur if in the digraph associated with the tangent cone there is a directed path from any node to j_s. Since the digraph associated with $\mathcal{T}(\mathcal{K}, x)$ has a line structure, the only possibility for this to happen is that, when scanning the arcs of the digraph from left to right, the arcs must be directed to the right until node j_s, and then all the remaining arcs must be directed to the left. Since an arc directed to the right (resp. left) corresponds to a horizontal segment of the path whose pair of extreme signs is $(+,-)$ (resp. $(-,+)$), it follows that $\mathscr{P}$ must satisfy Condition (v). In consequence, $\mathscr{P}$ is tropically allowed.

Conversely, with a tropically allowed lattice path with k horizontal segments, we associate the sequences of indices $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_{k+1}\}$ obtained by taking the ordinates and abscissæ of its horizontal and vertical segments respectively, as illustrated for the tropically allowed path in Fig. 1. Note that the previous logic is reversible, meaning that if we define $x \in \mathbb{R}^d_{\max}$ by $x_{j_r} = z_r(I, J)$ for $r \in \{1, \ldots, k+1\}$ and $x_j = \mathbb{0}$ for $j \notin J$, then x is in an extreme ray of $\mathcal{K}$. More precisely, by Corollary 3, Condition (iv) implies that all the entries of $z(I, J)$ are positive so $x \in \mathbb{R}^d_{\max}$, Conditions (i), (ii) and (iii) imply that x belongs to $\mathcal{K}$, and finally Condition (v) and Theorem 5 show that x belongs to an extreme ray of $\mathcal{K}$. This concludes the proof of Theorem 2. $\square$

Example 3. Fig. 2 provides two examples of polars of signed cyclic polyhedral cones for $d = 3$. They are represented in barycentric coordinates: each element (x_1, x_2, x_3) of $\mathbb{R}^3_{\max}$ is represented as a barycenter with weights $(e^{x_1}, e^{x_2}, e^{x_3})$ of the three vertices of the outermost triangle. Then, two representatives of the same ray are represented by the same point. This barycentric representation is convenient to represent points with infinite coordinates, which are mapped to the boundary of the simplex.

The two cones are defined by $p = 2$ and $p = 5$ inequalities respectively, and, for all $1 \leqslant i \leqslant p$, $t_i = i - 1$ and $\epsilon_{ij} = \ominus\mathbb{1}$ if and only if $j = 2$. Note that the first cone is the one considered in Examples 1 and 2.

The extreme rays are depicted by blue points. For the first cone, a representative of each extreme ray is provided, and the corresponding tropically allowed path is given beside.

4. The number of extreme points of the classical polar of a signed cyclic polyhedral cone

We next give a characterization of the extreme rays of the polar of the classical analogue of the signed cyclic polyhedral cone, which shows that in the tropical case there exist fewer extreme rays. Therefore, in this section, all the operations should be understood in the usual algebra.

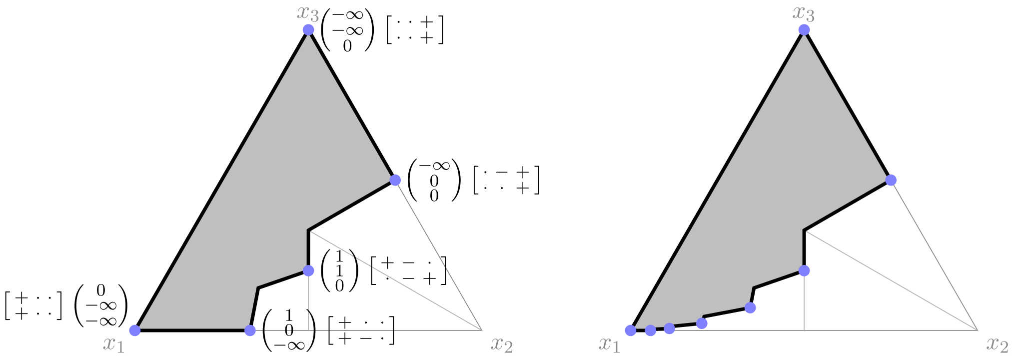

Fig. 2. The polars of two signed cyclic polyhedral cones in $\mathbb{R}^3_{\max}$.

Given p positive real numbers $t_1 < \cdots < t_p$ and a sign pattern (ϵ_{ij}), which now belongs to $\{+1, -1\}^{p\times d}$, we shall consider the usual polar of the signed cyclic polyhedral cone, which we still denote by $\mathcal{K}$,

$$\mathcal{K} := \{x \in \mathbb{R}^d \mid x \geqslant 0,\ Cx \geqslant 0\},$$

where $C_{ij} = \epsilon_{ij} t_i^{j-1}$ for $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant d$.

Like in the previous section, given two sequences of indices $I = \{i_1, \dots, i_k\}$ and $J = \{j_1, \dots, j_{k+1}\}$ where $k \leqslant d-1$, consider the matrix $C(I, J)$ obtained by keeping only the rows and columns of C whose indices belong to I and J, respectively.

Lemma 2. *The (ordinary) Cramer determinants D_r of the system $C(I, J)z = 0$ are given by*

$$D_r = \delta_r t_{i_1}^{j_1-1} \cdots t_{i_{r-1}}^{j_{r-1}-1} t_{i_r}^{j_{r+1}-1} \cdots t_{i_k}^{j_{k+1}-1} \epsilon_{i_1 j_1} \cdots \epsilon_{i_{r-1} j_{r-1}} \epsilon_{i_r j_{r+1}} \cdots \epsilon_{i_k j_{k+1}}, \tag{15}$$

for $1 \leqslant r \leqslant k+1$, where the scalars δ_r tend to 1 as the ratios $t_2/t_1, \dots, t_{p+1}/t_p$ tend to infinity.

Proof. For any permutation $\sigma \in S_k$ define

$$D_r(\sigma) := t_{i_{\sigma(1)}}^{j_1-1} \cdots t_{i_{\sigma(r-1)}}^{j_{r-1}-1} t_{i_{\sigma(r)}}^{j_{r+1}-1} \cdots t_{i_{\sigma(k)}}^{j_{k+1}-1},$$

so that

$$D_r = \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) D_r(\sigma) \epsilon_{i_{\sigma(1)} j_1} \cdots \epsilon_{i_{\sigma(r-1)} j_{r-1}} \epsilon_{i_{\sigma(r)} j_{r+1}} \cdots \epsilon_{i_{\sigma(k)} j_{k+1}}.$$

Let $\bar{\sigma}$ be defined by $\bar{\sigma}(s) := s$ for all s. We claim that for any $\sigma \neq \bar{\sigma}$, the quotient $D_r(\sigma)/D_r(\bar{\sigma})$ is a product of terms of the form t_{i_r}/t_{i_s}, where $i_r < i_s$. To see this, let $s = \max\{h \mid \sigma(h) \neq h\}$. Then, we have $\sigma(s) < s$ and there exists $q < s$ such that $s = \sigma(q)$. If we define σ' by $\sigma'(s) = s$, $\sigma'(q) = \sigma(s)$ and $\sigma'(h) = \sigma(h)$ for all $h \notin \{s, q\}$, it follows that

$$\frac{D_r(\sigma)}{D_r(\sigma')} = \frac{t_{i_{\sigma(s)}}^{j_{\hat{s}}-1} t_{i_s}^{j_{\hat{q}}-1}}{t_{i_s}^{j_{\hat{s}}-1} t_{i_{\sigma(s)}}^{j_{\hat{q}}-1}} = \left(\frac{t_{i_{\sigma(s)}}}{t_{i_s}}\right)^{j_{\hat{s}}-j_{\hat{q}}},$$

where $\hat{s} = s$ if $s < r$ and $\hat{s} = s+1$ otherwise, and the same applies to q. The claim follows by repeating this procedure till $\sigma' = \bar{\sigma}$.

Note that (15) is satisfied for

$$\delta_r := 1 + \sum_{\sigma \neq \bar{\sigma}} \frac{\operatorname{sgn}(\sigma) D_r(\sigma) \epsilon_{i_{\sigma(1)} j_1} \cdots \epsilon_{i_{\sigma(r-1)} j_{r-1}} \epsilon_{i_{\sigma(r)} j_{r+1}} \cdots \epsilon_{i_{\sigma(k)} j_{k+1}}}{D_r(\bar{\sigma}) \epsilon_{i_1 j_1} \cdots \epsilon_{i_{r-1} j_{r-1}} \epsilon_{i_r j_{r+1}} \cdots \epsilon_{i_k j_{k+1}}},$$

and from the discussion above it follows that δ_r tends to 1 as the ratios $t_2/t_1, \dots, t_{p+1}/t_p$ tend to infinity. □

As a consequence of the classical Cramer theorem we obtain.

Corollary 5. *Assume that the ratios $t_2/t_1, \dots, t_{p+1}/t_p$ are sufficiently large. Then, the system $C(I, J)z = 0$ has a non-zero solution, which is unique up to a scalar multiple, and which is determined by the relations*

$$\begin{aligned}
z_1 &= (-\gamma_1) t_{i_1}^{j_2 - j_1} \epsilon_{i_1 j_1} \epsilon_{i_1 j_2} z_2, \\
z_2 &= (-\gamma_2) t_{i_2}^{j_3 - j_2} \epsilon_{i_2 j_2} \epsilon_{i_2 j_3} z_3, \\
&\vdots \\
z_k &= (-\gamma_k) t_{i_k}^{j_{k+1} - j_k} \epsilon_{i_k j_k} \epsilon_{i_k j_{k+1}} z_{k+1},
\end{aligned} \tag{16}$$

where for $r = 1, \dots, k$ the scalars γ_r tend to 1 as the ratios $t_2/t_1, \dots, t_{p+1}/t_p$ tend to infinity.

Like in the previous section, we shall denote by $z(I, J)$ the vector defined by (16) together with the normalization condition $z_{k+1} = 1$, i.e.

$$z(I, J) := \begin{pmatrix} (-\gamma_1)\epsilon_{i_1 j_1}\epsilon_{i_1 j_2} t_{i_1}^{j_2 - j_1} t_{i_2}^{j_3 - j_2} \cdots t_{i_k}^{j_{k+1} - j_k} \\ (-\gamma_2)\epsilon_{i_2 j_2}\epsilon_{i_2 j_3} t_{i_2}^{j_3 - j_2} \cdots t_{i_k}^{j_{k+1} - j_k} \\ \vdots \\ (-\gamma_k)\epsilon_{i_k j_k}\epsilon_{i_k j_{k+1}} t_{i_k}^{j_{k+1} - j_k} \\ 1 \end{pmatrix}. \tag{17}$$

Theorem 6. *If the ratios $t_2/t_1, \dots, t_{p+1}/t_p$ are sufficiently large, the extreme rays of $\mathcal{K}$ are in one to one correspondence with the (non-tropically) allowed lattice paths for the sign pattern (ϵ_{ij}).*

Proof. Let $x \in \mathbb{R}^d$ be in an extreme ray of $\mathcal{K}$. Assume that $\{j \mid x_j \neq 0\} = \{j_1, \dots, j_{k+1}\}$, where $k \leqslant d-1$. Then, x must saturate at least k inequalities among $C_i x \geqslant 0$, $i = 1, \dots, p$. Indeed, like in the tropical case, x saturates precisely k inequalities, because otherwise, by Lemma 2 and Cramer theorem, it would be equal to the null vector.

Let $\{i_1, \dots, i_k\}$ be the indices of the inequalities which x saturates. We assume $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{k+1}$. With x we associate the lattice path $\mathscr{P}$ defined by (12). We next show that $\mathscr{P}$ is allowed.

Let $\bar{x} \in \mathbb{R}^{k+1}$ be the vector obtained from x by deleting its null entries. Since $\bar{x}$ satisfies $C(I, J)\bar{x} = 0$ and the entries of $\bar{x}$ are positive, by Corollary 5 it follows that the signs on the extreme positions of every horizontal segment of $\mathscr{P}$, i.e. $\epsilon_{i_r j_r}$ and $\epsilon_{i_r j_{r+1}}$, must be opposite. In other words, $\mathscr{P}$ satisfies Condition (iv).

Since we may assume $\bar{x} = z(I, J)$, it follows that

$$C_i x = \sum_{1 \leqslant j \leqslant d} \epsilon_{ij} t_i^{j-1} x_j = \sum_{1 \leqslant r \leqslant k+1} \epsilon_{i j_r} t_i^{j_r - 1} x_{j_r} = \sum_{1 \leqslant r \leqslant k+1} \epsilon_{i j_r} t_i^{j_r - 1} \gamma_r t_{i_r}^{j_{r+1} - j_r} \cdots t_{i_k}^{j_{k+1} - j_k}, \tag{18}$$

for all $1 \leqslant i \leqslant p$, where we define $\gamma_{k+1} := 1$.

If we take $i_s < i < i_{s+1}$, note that

$$C_i x = \epsilon_{i j_{s+1}} t_i^{j_{s+1} - 1} \gamma_{s+1} t_{i_{s+1}}^{j_{s+2} - j_{s+1}} \cdots t_{i_k}^{j_{k+1} - j_k} (1 + \kappa_i)$$

where

$$\kappa_i = \sum_{1\leqslant r\leqslant s} \frac{\gamma_r}{\gamma_{s+1}} \left(\frac{t_{i_r}}{t_i}\right)^{j_{r+1}-j_r} \cdots \left(\frac{t_{i_s}}{t_i}\right)^{j_{s+1}-j_s} + \sum_{s+2\leqslant r\leqslant k+1} \frac{\gamma_r}{\gamma_{s+1}} \left(\frac{t_i}{t_{i_{s+2}}}\right)^{j_{s+3}-j_{s+2}} \cdots \left(\frac{t_i}{t_{i_r}}\right)^{j_{r+1}-j_r}.$$

Since κ_i tend to 0 as the ratios $t_2/t_1, \ldots, t_{p+1}/t_p$ tend to infinity, it follows that $\epsilon_{ij_{s+1}} = +1$ must be satisfied in order to have $C_i x \geqslant 0$. This means that Condition (iii) is valid for $\mathscr{P}$. A similar argument shows that Conditions (i) and (ii) also hold and thus $\mathscr{P}$ is allowed.

Conversely, like in the tropical case, note that the previous logic is reversible. With an allowed lattice path with k horizontal segments, we associate the sequences of indices $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_{k+1}\}$ obtained by taking the ordinates and abscissæ of its horizontal and vertical segments respectively. If we define $x \in \mathbb{R}^d$ by $x_{j_r} = z_r(I, J)$ for $r \in \{1, \ldots, k+1\}$ and $x_j = 0$ for $j \notin J$, then x is in an extreme ray of $\mathcal{K}$ for sufficiently large ratios $t_2/t_1, \ldots, t_{p+1}/t_p$. More precisely, by Corollary 5, Condition (iv) implies that all the entries of $z(I, J)$ are positive, so $x_j \geqslant 0$ for all j. This fact together with Conditions (i), (ii) and (iii) imply that x belongs to $\mathcal{K}$. Finally, note that Lemma 2 shows that the gradients of the inequalities that x saturates, i.e. $C_{i_r} x \geqslant 0$ for $r \in I$ and $x_j \geqslant 0$ for $j \notin J$, form a family of full rank. Therefore, x belongs to an extreme ray of $\mathcal{K}$. This concludes the proof. $\square$

The following theorem shows that the bound $U(p+d, d-1)$ is attained by the polar of the classical analogue of the signed cyclic polyhedral cone. Its proof also shows that the lattice path characterization of Theorem 6 may be thought of as a generalization of Gale's evenness criterion, since the latter is recovered by considering the special case in which the sign pattern is $\epsilon_{ij} = (-1)^j$.

Theorem 7. *The number of extreme rays of the classical polar $\mathcal{K}$ of the signed cyclic polyhedral cone with sign pattern $\epsilon_{ij} := (-1)^{j-1}$ is exactly $U(p+d, d-1)$.*

Proof. Given the set $\{1, \ldots, n\}$, we shall say that a subset Q of $\{1, \ldots, n\}$ satisfies Gale's evenness condition, if for any $i, j \in \{1, \ldots, n\} \setminus Q$ the number of elements in Q between i and j is even. It is known (see [31]) that the number of subsets Q of $\{1, \ldots, n\}$ with k elements satisfying the evenness condition is $U(n, k)$. We shall show that the number of extreme rays of $\mathcal{K}$ is $U(p+d, d-1)$ by constructing a bijective correspondence between allowed lattice paths for the sign pattern (ϵ_{ij}) and subsets of $\{1, \ldots, p+q\}$ with $d-1$ elements which satisfy Gale's evenness condition.

Given an allowed lattice path $\mathscr{P}$ for the sign pattern (ϵ_{ij}), let $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_{k+1}\}$ be the sets of ordinates and abscissæ of its horizontal and vertical segments respectively. With $\mathscr{P}$ we associate the subset Q of $\{1, \ldots, p+q\}$ defined by

$$Q := \{i + d \mid i \in I\} \cup \{d - j + 1 \mid j \notin J\}.$$

The set Q may be visualized by scanning first the columns of the matrix ϵ from right to left, keeping only the columns not in J, and scanning then the rows of ϵ from top to bottom, keeping now the rows in I. The following illustrates the definition of Q for a special lattice path, the elements of Q are listed by numbers on the top and left borders of the matrix so that $Q = \{2, 3, 7, 8, 10, 11, 13, 14\}$:

$$\begin{array}{cc}
 & \begin{array}{ccccccccc} \cdot & 8 & 7 & \cdot & \cdot & \cdot & 3 & 2 & \cdot \end{array} \\
\begin{array}{c} 10 \\ 11 \\ \cdot \\ 13 \\ 14 \\ \cdot \\ \cdot \end{array} &
\begin{pmatrix}
+ & - & + & - & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & - & + & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & + & - & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +
\end{pmatrix}
\end{array}.$$

We next show that Q satisfies the evenness condition. With this aim, firstly it is convenient to note that for the considered sign pattern, in any allowed lattice path, the pairs of signs on the extreme positions of the horizontal segments alternate between $(+,-)$ and $(-,+)$.

We start by showing that for any $i', i'' \in \{1,\dots,p\} \setminus I$ the number of indices in I between i' and i'' is even. To see this, assume that $\{i_s,\dots,i_q\}$ is a maximal sequence of consecutive indices in I between i' and i''. By consecutive sequence of indices we mean that $i_{r+1} = i_r + 1$ for $s \leqslant r \leqslant q-1$. Then, by Conditions (i) and (iii) we must have $(\epsilon_{i_s j_s} \epsilon_{i_s j_{s+1}}) = (+,-)$ because $i_s - 1 \notin I$. In the same way, since $i_q + 1 \notin I$, from Conditions (ii) and (iii) it follows that $(\epsilon_{i_q j_q} \epsilon_{i_q j_{q+1}}) = (-,+)$. This implies that the number of elements in $\{i_s,\dots,i_q\}$ is even because the pairs of signs on the extreme positions of the horizontal segments of $\mathscr{P}$ alternate between $(+,-)$ and $(-,+)$. This means that the number of elements in Q between $i'+d$ and $i''+d$ can be expressed as a sum of even numbers, and therefore it is also even.

Analogously, for any $j_r, j_s \in J$, there is an even number of elements in $\{1,\dots,d\} \setminus J$ between j_r and j_s. Indeed, note that if $\epsilon_{i_r j_r} = +1$ (resp. $\epsilon_{i_r j_r} = -1$), then by Condition (iv) we have $\epsilon_{i_r j_{r+1}} = -1$ (resp. $\epsilon_{i_r j_{r+1}} = +1$), which means that the number of elements in $\{1,\dots,d\} \setminus J$ between j_r and j_{r+1} is even, because in the considered sign pattern the signs alternate between $+1$ and -1 on each row. Therefore, the number of elements in Q between $d - j_r + 1$ and $d - j_s + 1$ can be expressed as a sum of even numbers, and thus it is also even.

Finally, consider the case $i = i' + d$ and $j = d - j_s + 1$ for some $j_s \in J$ and $i' \notin I$. We claim that the number of elements in Q between i and j is even. Note that if $1 \notin I$, thanks to the previous results, it suffices to show that j_1 is odd, but this follows from Condition (i) because we must have $(-1)^{j_1 - 1} = +1$. On the other hand, if $1 \in I$, let $\{i_1,\dots,i_s\}$ be the maximal sequence of consecutive indices in I containing $i_1 = 1$. Due to the results above, note that to prove our claim, it is enough to show that $i_s + j_1 - 1$ is even. To prove this, we consider two cases. Assume first that $(\epsilon_{i_1 j_1}, \epsilon_{i_1 j_2}) = (+,-)$. Then, j_1 is odd because $(-1)^{j_1 - 1} = +1$. Since $i_s + 1 \notin I$, from Conditions (ii) and (iii) it follows that the pair of signs $(\epsilon_{i_s j_s}, \epsilon_{i_s j_{s+1}})$ can never be equal to $(+,-)$. Therefore, we conclude that i_s is even, which means that $i_s + j_1 - 1$ is also even. Assume now that $(\epsilon_{i_1 j_1}, \epsilon_{i_1 j_2}) = (-,+)$. Then, j_1 is even and, like in the previous case, by Conditions (ii) and (iii) the pair of signs $(\epsilon_{i_s j_s}, \epsilon_{i_s j_{s+1}})$ can never be equal to $(+,-)$. Therefore, we conclude that i_s is odd, which means that $i_s + j_1 - 1$ is even.

In consequence, Q satisfies Gale's evenness condition.

Conversely, let Q be a subset of $\{1,\dots,p+d\}$ with $d-1$ elements which satisfies Gale's evenness condition. Define the sets $I := \{i - d \mid i \in Q, i > d\}$ and $J := \{d - j + 1 \mid j \notin Q, j \leqslant d\}$. Since Q has $d-1$ elements, we can write $I = \{i_1,\dots,i_k\}$ and $J = \{j_1,\dots,j_{k+1}\}$ for some $k \leqslant d-1$, where we assume that $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_{k+1}$. Then, with Q we associate the lattice path $\mathscr{P}$ defined by (12). We next show that $\mathscr{P}$ is allowed.

Firstly, note that applying the evenness condition to $d - j_r + 1$ and $d - j_{r+1} + 1$, for $r \leqslant k$, it follows that the columns j_r and j_{r+1} of (ϵ_{ij}) always have opposite signs. This means that $\mathscr{P}$ satisfies Condition (iv), so it remains to show that Conditions (i), (ii) and (iii) are also satisfied. We divide the proof into two cases.

Assume first that $1 \notin I$. Then, by considering $i = d+1$ and $j = d - j_1 + 1$ in the evenness condition, we conclude that j_1 must be odd. Therefore, the initial vertical segment of $\mathscr{P}$ is contained in a column with only $+$ signs, so Condition (i) holds. If $k \geqslant 1$, since columns j_1 and j_2 have opposite signs, we know that $(\epsilon_{i_1 j_1}, \epsilon_{i_1 j_2}) = (+,-)$. Let $\{i_1,\dots,i_s\}$ be the maximal sequence of consecutive indices in I containing i_1. Since the indices in $\{i_1,\dots,i_s\}$ are consecutive, Condition (iii) holds for the vertical segments of $\mathscr{P}$ contained in columns $j_1,\dots,j_s$. Moreover, using the fact that columns j_r and j_{r+1} always have opposite signs, for $r \leqslant s$ we have $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (+,-)$ if r is odd and $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (-,+)$ if r is even. If $i_s < p$, by the evenness condition there is an even number of elements in Q between $i_1 + d - 1$ and $i_s + d + 1$, so $i_s - i_1$ must be odd. Therefore, we have $(\epsilon_{i_s j_s}, \epsilon_{i_s j_{s+1}}) = (-,+)$ and thus the $(s+1)$th vertical segment of $\mathscr{P}$ is contained in a column with only $+$ signs. If this is not the final vertical segment of $\mathscr{P}$, i.e. if $s \neq k$, we conclude that Condition (iii) is satisfied for this segment and that $(\epsilon_{i_{s+1} j_{s+1}}, \epsilon_{i_{s+1} j_{s+2}}) = (+,-)$. Repeating this argument, by considering successively sequences of consecutive indices in I, it follows that Condition (iii) holds for $\mathscr{P}$. Moreover, after a finite number of steps, we either conclude that $i_k = p$ or that $(\epsilon_{i_k j_k}, \epsilon_{i_k j_{k+1}}) = (-,+)$, and in both cases Condition (ii) is satisfied. This proves that $\mathscr{P}$ is allowed.

Now assume that $1 \in I$, which implies that Condition (i) holds. Let $\{i_1, \dots, i_s\}$ be the maximal sequence of consecutive indices in I containing $i_1 = 1$. Considering the evenness condition with $i = d + i_s + 1$ and $j = d - j_1 + 1$, we conclude that $i_s + j_1$ must be odd. If j_1 is odd and i_s is even, for $r \leqslant s$ we have $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (+,-)$ for r odd and $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (-,+)$ for r even, so in particular $(\epsilon_{i_s j_s}, \epsilon_{i_s j_{s+1}}) = (-,+)$. Otherwise, i.e. if j_1 is even and i_s is odd, it follows that for $r \leqslant s$, $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (-,+)$ if r is odd and $(\epsilon_{i_r j_r}, \epsilon_{i_r j_{r+1}}) = (+,-)$ if r is even. Therefore, again we have $(\epsilon_{i_s j_s}, \epsilon_{i_s j_{s+1}}) = (-,+)$. In both cases, we can now apply the same argument used in the case $1 \notin I$ to show that $\mathscr{P}$ satisfies Conditions (ii) and (iii). This concludes the proof of the theorem. $\square$

We denote by $N^{\text{tpath}}(\epsilon)$ (resp. $N^{\text{path}}(\epsilon)$) the number of tropically (resp. non-tropically) allowed lattice paths for the sign pattern ϵ. We also denote by $N^{\text{trop}}(p,d)$ the maximal number of extreme rays of a tropical cone in dimension d defined as the intersection of p half-spaces. We have shown that

$$\max_{\epsilon \in \{\oplus \mathbb{1}, \ominus \mathbb{1}\}^{p \times d}} N^{\text{tpath}}(\epsilon) \leqslant N^{\text{trop}}(p,d) \leqslant U(p+d, d-1) = \max_{\epsilon \in \{\pm 1\}^{p \times d}} N^{\text{path}}(\epsilon). \tag{19}$$

The analogy with classical convex geometry may suggest that the two leftmost quantities in the latter expression coincide, meaning that the maximal number of extreme rays is attained, for every (p,d), by the polar of some signed cyclic polyhedral cone. However, the following counter-example, that we include for the convenience of the reader, shows that this is not the case. This counter-example is actually a byproduct of the study of the external representation of tropical polyhedra which is carried out in [5].

Example 4. Take $d = 5$, $p = 7$, and consider the matrices

$$A = \begin{pmatrix} 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \\ 0 & -\infty & -\infty & -\infty & -\infty \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -\infty & 1 & 5 & 7 & 3 \\ -\infty & 2 & 7 & 5 & 2 \\ -\infty & 3 & 6 & 6 & 1 \\ -\infty & 4 & 3 & 2 & 7 \\ -\infty & 5 & 4 & 1 & 6 \\ -\infty & 6 & 1 & 4 & 5 \\ -\infty & 7 & 2 & 3 & 4 \end{pmatrix}.$$

Then, $\mathcal{C} = \{x \in \mathbb{R}^5_{\max} \mid Ax \leqslant Bx\}$ is a tropical polyhedral cone in dimension 5 defined by the intersection of 7 tropical half-spaces, and it has 47 extreme rays, whereas the left term in (19) is only 44. The number of extreme rays of this tropical cone was obtained using the algorithm of [4] and its OCaml TPLib implementation [7]. The value 44 for the left term in (19) was also obtained by a computer program (evaluating $N^{\text{tpath}}(\epsilon)$ for all the possible 7×5 sign patterns, using the method of Section 6 below).

Hence, finding a family of tropical polyhedral cones reaching the bound $N^{\text{trop}}(p,d)$ for every (p,d) remains an open problem, and the previous counter-example, as well as the characterization of the extreme points of the polar in terms of "weighted" hypergraph transversals [5], actually suggests that the maximizing polyhedral cones may be of a different nature than in the classical case. However, as a contribution to this problem, we next show that the inequalities defining these cones may still be assumed to be in "general position" (in a tropical sense).

5. Tropical half-spaces in general position

Recall that a $k \times k$ matrix M with entries in $\mathbb{R}_{\max}$ is tropically non-singular if the tropical permanent (some authors call this notion tropical determinant)

$$\operatorname{tper} M := \max_{\sigma \in S_k} \sum_{1 \leqslant i \leqslant k} M_{i\sigma(i)}$$

is finite and is attained by exactly one permutation σ.

Consider a tropical polyhedral cone defined by the system of inequalities $Ax \leqslant Bx$, where A, B are $p \times d$ matrices with entries in $\mathbb{R}_{\max}$ which we may require to satisfy $A_{ij} B_{ij} = \mathbb{0}$ for all $1 \leqslant i \leqslant p$ and $1 \leqslant j \leqslant d$. Let $C := A \oplus B$. We say that the latter inequalities are *in general position* if any $k \times k$ submatrix of C is tropically non-singular.

An elementary part of the proof of McMullen's upper bound theorem is to show that the number of facets of a polytope with p vertices in dimension d is maximized by a simplicial polytope. The following theorem may be thought of as a tropical version of the dual of this result.

Theorem 8. *The maximal number of extreme rays of a tropical cone defined as the intersection of p tropical half-spaces in dimension d is attained when these half-spaces are in general position.*

Proof. The proof is similar in its spirit to the one of Theorem 1. We can choose a sequence of perturbed matrices

$$A(m) \leqslant A \quad \text{and} \quad B(m) \geqslant B, \quad m \in \mathbb{N}, \tag{20}$$

in such a way that for all m, every square submatrix of the matrix $C(m) := A(m) \oplus B(m)$ is tropically non-singular and $A(m) \to A$, $B(m) \to B$ as m tends to infinity. For instance, if $B_{ij} > \mathbb{0}$, we may require that $(B(m))_{ij} > B_{ij}$ and $(A(m))_{ij} = \mathbb{0}$, whereas if $A_{ij} > \mathbb{0}$, we may require that $(A(m))_{ij} < A_{ij}$ and $(B(m))_{ij} = \mathbb{0}$. The matrices $A(m)$ and $B(m)$ may be chosen arbitrarily close to A and B, respectively, and if their entries are rationally independent, every submatrix of $C(m)$ must be tropically non-singular. Let $\mathcal{C} := \{x \in \mathbb{R}^d_{\max} \mid Ax \leqslant Bx\}$ and $\mathcal{C}(m) := \{x \in \mathbb{R}^d_{\max} \mid A(m)x \leqslant B(m)x\}$. Due to Property (20), we have $\mathcal{C}(m) \supset \mathcal{C}$.

Let $K(p, d)$ denote the maximal number of extreme rays of (tropical) polyhedral cones in dimension d defined by systems of p inequalities in general position. For all $m \in \mathbb{N}$, let $\{u_k(m)\}_{k=1,\dots,K(m)}$ denote a generating family of $\mathcal{C}(m)$, which by Theorem 3 can be obtained by selecting precisely one element in each extreme ray of $\mathcal{C}(m)$, so that $K(m) \leqslant K(p, d)$. Possibly after extracting a subsequence, we may assume that $K := K(m)$ is independent of m. Every vector $u_k(m)$ can be chosen to be normalized (e.g. to have the maximum of its entries equal to $\mathbb{1}$) and so, perhaps after extracting again a subsequence, we may assume that $u_k(m)$ has a limit $u_k \in \mathbb{R}^d_{\max}$ different from the zero vector $\mathbb{0}$ as m tends to infinity. Since $\mathcal{C} \subset \mathcal{C}(m)$, we deduce that for all vectors $v \in \mathcal{C}$, and for all $m \in \mathbb{N}$, we can find some scalars $\lambda_k(m)$ such that

$$v = \bigoplus_{1 \leqslant k \leqslant K} \lambda_k(m) u_k(m).$$

Since every $u_k(m)$ has some entry i (depending on k and m) equal to $\mathbb{1}$, we deduce that $\lambda_k(m) \leqslant v_i$, and so $\lambda_k(m) \leqslant \max_j v_j$ for all k and m. Hence, $\lambda_k(m)$, which is bounded as m tends to infinity, must have an accumulation point $\lambda_k \in \mathbb{R}_{\max}$, and we deduce that

$$v = \bigoplus_{1 \leqslant k \leqslant K} \lambda_k u_k.$$

Moreover, by passing to the limit in $A(m)u_k(m) \leqslant B(m)u_k(m)$, we deduce that $Au_k \leqslant Bu_k$, showing that $u_k \in \mathcal{C}$. It follows that $\{u_k\}_{1 \leqslant k \leqslant K}$ is a generating family of $\mathcal{C}$. Since the number of extreme rays of a polyhedral cone is bounded by the cardinality of any of its generating families, we deduce that the number of extreme rays of $\mathcal{C}$ is bounded by $K(p, d)$. $\square$

6. Computing the number of tropically allowed paths

We next give an inductive formula allowing one to compute the number $N^{\text{tpath}}(\epsilon)$ of tropically allowed lattice paths for the sign pattern $\epsilon = (\epsilon_{ij})$ in a time which is linear in the size of the pattern.

First, we write the signs ϵ_{ij}, $1 \leqslant i \leqslant p, 1 \leqslant j \leqslant d$ in a $p \times d$ table, that we complete by adding one dummy row at the top numbered 0 and one dummy row at the bottom numbered $p + 1$.

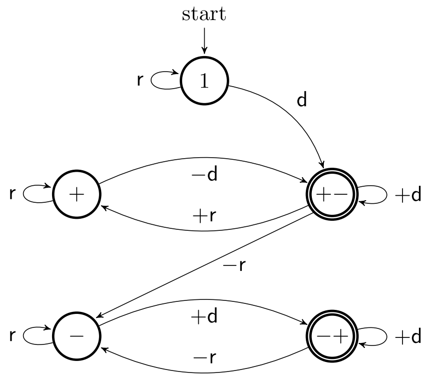

Fig. 3. An automaton recognizing tropically allowed paths.

We shall consider paths starting from the position $(0, 1)$ (row 0, column 1) and ending at some position $(p + 1, j)$ (row $p + 1$, column j). Such paths are said to be tropically allowed if the subpath lying in rows $1, \dots, p$ is tropically allowed.

We represent every lattice path by a word in the alphabet $\{\mathsf{d}, \mathsf{r}\}$. The letter d represents a downward move, whereas the letter r represents a move to the right. (The letter d should not be confused with the symbol d for the dimension.) For instance, if $p = 1$ and $d = 2$, the word drd corresponds to the path

$$\begin{array}{lll} (0,1) & & \\ \mathsf{d}\downarrow & & \\ (1,1) & \xrightarrow{\mathsf{r}} & (1,2) \\ & & \mathsf{d}\downarrow \\ & & (2,2) \end{array}$$

Consider now the automaton represented in Fig. 3, in which the state denoted by 1 (with an incoming arrow) is initial and the states denoted by the symbols $+-$ and $-+$ (with double circles) are final.

The arcs are labeled by letters, and sometimes by signs. We next introduce an acceptance condition which slightly differs from the classical one in automata theory, in order to take into account the sign pattern.

A word is said to be *accepted* by the automaton if the following holds. We read the letters of the word from left to right, performing at the same time the corresponding moves (downward, or to the right) in the table and in the automaton (following arcs). A move is accepted only if the sign ϵ_{ij} of the current position of the table is the same as the sign of the corresponding arc originating from the current state on the automaton (if there is no sign on this arc, ϵ_{ij} can be arbitrary). The word is accepted if, when starting from position $(0, 1)$ in the table and from the initial node in the automaton, every successive move is accepted, leading to a state of the automaton which is final, and if the final position in the table is at some point $(p + 1, j)$ with $1 \leqslant j \leqslant d$, which means that the word contains precisely $p + 1$ occurrence of the letter d and at most $d - 1$ occurrences of the letter r.

For instance, if $p = 1$ and $d = 2$, and if the sign pattern is $[+, -]$, the word dd is accepted since it corresponds to the following path in the automaton:

$$1 \xrightarrow{\mathsf{d}} +- \xrightarrow{+\mathsf{d}} +-$$

Similarly, the word drd is accepted, since it corresponds to the path:

$$1 \xrightarrow{\mathsf{d}} +- \xrightarrow{+\mathsf{r}} + \xrightarrow{-\mathsf{d}} +-$$

The introduction of the previous automaton is motivated by the following result.

Proposition 1. *The tropically allowed lattice paths are in one to one correspondence with the words that are accepted by the automaton, and each of these words corresponds precisely to one path in the automaton.*

Proof. Imagine a pen, drawing the path starting from the top left position and making only moves downward or to the right. We shall see that the states of the automaton are used to record the information necessary to determine how the pen can be moved to draw a tropically allowed lattice path.

First, the pen is at position $(0,1)$ (on the dummy top row), and the current state of the automaton is the initial state, 1. Then, the pen may either stay on the dummy top row, moving to the right, or leave the dummy row, moving down, corresponding to the two arcs $1 \xrightarrow{\mathsf{r}} 1$ and $1 \xrightarrow{\mathsf{d}} +-$. Assume that the latter arc has been chosen after a sequence of moves to the right (which cannot exceed $d-1$ due to the final acceptance condition), so that the pen is now at some position $(1, j)$ with $1 \leqslant j \leqslant d$. Then, the pen always may move to the right, beginning a horizontal segment. If $\epsilon_{1j} = \oplus\mathbb{1}$, in accordance with Condition (i), the pen may also move down. These moves correspond to the three arcs leaving node $+-$ in the automaton: we use the state $+$ (resp. $-$) to record that the horizontal segment which has been opened starts with a $+$ (resp. $-$ sign).

Consider now the situation in which $\epsilon_{1j} = \oplus\mathbb{1}$ and a move to the right has been selected, so that the current state in the automaton is $+$ and the position of the pen is now $(1, j+1)$. Since the sign of every position of a horizontal segment which is not extreme does not matter in the definition of tropically allowed path, the move to the right can always be selected. By Condition (iv), a downward move can be accepted only if the sign at the current position is $-$, since a horizontal segment which began with a $+$ must end by a $-$, and since the downward move ends the current horizontal segment. The latter move corresponds to the arc $+ \xrightarrow{-\mathsf{d}} +-$ in the automaton.

Similarly, the state $-$ indicates that the pen is now drawing a horizontal segment starting from a $-$ sign, and the state $-+$ indicates that such a segment has been closed. Observe that there is an arc from state $+-$ to state $-$, but no arc from state $-+$ to state $+$, because, by Condition (v), the pair $(-,+)$ may always appear after a pair $(+,-)$ as the signs of the extreme positions of a horizontal segment, whereas the opposite is not allowed.

With this interpretation in mind, it is readily seen that every accepted word bijectively corresponds to a tropically allowed lattice path.

An inspection of the automaton also shows that it is unambiguous, meaning that there is precisely one path in the automaton for each accepted word. Indeed, the unambiguity stems from the fact that at each state, there is at most one leaving arc with a given letter and sign. □

The inductive formula to compute $N^{\mathrm{tpath}}(\epsilon)$ is next obtained by some elementary bookkeeping.

We denote by $\chi^+_{ij}(\epsilon)$ the number which is 1 if $\epsilon_{ij} = \oplus\mathbb{1}$ and 0 otherwise. Similarly, $\chi^-_{ij}(\epsilon)$ is 1 if $\epsilon_{ij} = \ominus\mathbb{1}$ and 0 otherwise. For $0 \leqslant i \leqslant p+1, 1 \leqslant j \leqslant d+1$, define the numbers $N^+_{ij}(\epsilon)$, $N^-_{ij}(\epsilon)$, $N^{+-}_{ij}(\epsilon)$, $N^{-+}_{ij}(\epsilon)$, and $N^1_{0j}(\epsilon)$ by the following inductive formulæ

$$
\begin{aligned}
&N^1_{0j}(\epsilon) = N^1_{0j+1}(\epsilon) + N^{+-}_{1j}(\epsilon), \quad 1 \leqslant j \leqslant d,\\
&N^+_{ij}(\epsilon) = N^+_{ij+1}(\epsilon) + \chi^-_{ij}(\epsilon) N^{+-}_{i+1j}(\epsilon), \quad 0 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant d,\\
&N^-_{ij}(\epsilon) = N^-_{ij+1}(\epsilon) + \chi^+_{ij}(\epsilon) N^{-+}_{i+1j}(\epsilon), \quad 0 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant d,\\
&N^{+-}_{ij}(\epsilon) = \chi^+_{ij}(\epsilon) N^{+-}_{i+1j}(\epsilon) + \chi^+_{ij}(\epsilon) N^+_{ij+1}(\epsilon) + \chi^-_{ij}(\epsilon) N^-_{ij+1}(\epsilon), \quad 0 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant d,\\
&N^{-+}_{ij}(\epsilon) = \chi^-_{ij}(\epsilon) N^-_{ij+1}(\epsilon) + \chi^+_{ij}(\epsilon) N^{-+}_{i+1j}(\epsilon), \quad 0 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant d,
\end{aligned}
$$

together with the boundary conditions

$$
\begin{aligned}
&N^s_{id+1}(\epsilon) = 0, \quad 0 \leqslant i \leqslant p+1,\ s \in \{+,-,+-,-+\},\\
&N^1_{0d+1}(\epsilon) = 0,\\
&N^s_{p+1j}(\epsilon) = 1, \quad 1 \leqslant j \leqslant d,\ s \in \{+-,-+\},
\end{aligned}
$$

$$N^s_{p+1\,j}(\epsilon) = 0, \quad 1 \leqslant j \leqslant d,\ s \in \{+,-\}.$$

Corollary 6 *(Computing the number of tropically allowed paths). For all sign patterns ϵ, we have*

$$N^{\mathrm{tpath}}(\epsilon) = N^1_{00}(\epsilon).$$

Proof. We claim that for each state s of the automaton, and for all $0 \leqslant i \leqslant p$, $1 \leqslant j \leqslant d$, $N^s_{ij}(\epsilon)$ represents the number of possible sequences of remaining moves of a pen drawing a tropically allowed path, given that the current position of the pen is (i,j) and that the previous moves of the pen led to this position and to state s.

We observe that the equations above, except for the two ones which determine the boundary values $N^s_{p+1\,j}(\epsilon)$, are readily obtained from the automaton. For instance, the formula for $N^{+-}_{ij}(\epsilon)$ as a sum of three terms corresponds to the three options: move down if the sign ϵ_{ij} is positive; open a horizontal segment with initial sign $+$ under the same condition; or open a horizontal segment with initial sign $-$ if ϵ_{ij} is negative. The other formulæ are obtained in a similar way. Note that the boundary conditions which determine $N^s_{p+1\,j}(\epsilon)$ force the final state to be either $+-$ or $-+$, meaning that every horizontal path which has been opened must have been closed. Using these considerations, one readily shows the claim by a backward induction on (i,j), initialized when $i = p+1$ or $j = d+1$. $\square$

Remark 1. To compute the number of (non-tropically) allowed paths, it suffices to add an arc $-+ \xrightarrow{+\mathrm{r}} +$ in the automaton. Then, we must add a third term $\chi^+_{ij}(\epsilon)N^+_{i\,j+1}(\epsilon)$ in the expression of $N^{-+}_{ij}(\epsilon)$, and one can check that the number $N^1_{00}(\epsilon)$ now determines the number of allowed paths. One can also check that $N^{-+}_{ij}(\epsilon) = N^{+-}_{ij}(\epsilon)$, meaning that the automaton is no longer minimal (the states $+-$ and $-+$ can be identified).

7. Upper and lower estimates for the number of extreme rays of the polar of signed cyclic polyhedral cones

We showed that $N^{\mathrm{trop}}(p,d)$, the maximal number of extreme rays of a tropical polyhedral cone defined by p inequalities in dimension d is bounded from above by its classical analogue, $U(p+d, d-1)$, and bounded from below by $N^{\mathrm{tpath}}(p,d)$, the maximal number of tropically allowed lattice paths for a $p \times d$ signed pattern, see (19). The asymptotic behavior of $U(p,d)$ is easily determined. In this section, we provide explicit estimates for the lower bound $N^{\mathrm{tpath}}(p,d)$ and derive its asymptotic behavior as p or d tends to infinity.

We shall say that a tropically allowed path is of $-+$ type if the pair of signs consisting of the signs of the leftmost and rightmost positions of each of its horizontal segments is $(-,+)$. Tropically allowed paths of $+-$ type are defined in a symmetric way. Recall that a tropically allowed path consists of a path of $+-$ type followed by a path of $-+$ type, one of these being possibly empty.

Let $N^{-+}(p,d)$ (resp. $N^{+-}(p,d)$) denote the maximal number of tropically allowed paths of $-+$ type (resp. $+-$ type) in a $p \times d$ sign pattern. We shall also need $N^{+-}_{\ell}(p,d)$, which denotes the maximal number of tropically allowed paths of $+-$ type using the last column of a $p \times d$ sign pattern. We make the following observation:

$$N^{-+}(p,d) = N^{+-}(p,d). \tag{21}$$

Indeed, if we read a tropically allowed path of $-+$ type for a $p \times d$ sign pattern in a reverse way (starting from the end), it becomes a tropically allowed path of $+-$ type in the reversed sign pattern (in which the bottom right corner becomes the top left corner), and vice versa.

We first bound $N^{\mathrm{tpath}}(p,d)$ from above.

Proposition 2. *For every p, d,*

$$N^{\mathrm{tpath}}(p,d) \leqslant \big(p(d-1)+1\big)2^{d-1}. \tag{22}$$

Proof. In the first place, we claim that

$$N^{\text{tpath}}(p,d) \leqslant \left(\sum_{r=1}^{p} \sum_{m=1}^{d-1} N_{\ell}^{+-}(r-1,m) N^{-+}(p-r,d-m) \right) + N^{+-}(p,d). \tag{23}$$

Indeed, in this expression (r,m) represents the leftmost position of the first $-+$ segment, if any, of a tropically allowed path for a given sign pattern. Then, the part of the path before this segment must be of $+-$ type in the $(r-1)\times m$ upper left submatrix of which it uses the last column, accounting for the term $N_{\ell}^{+-}(r-1,m)$, whereas the part of the path after this segment must be of $-+$ type in the $(p-r)\times(d-m)$ bottom right submatrix. The term outside the parenthesis represents the paths which are purely of $+-$ type. The case $m=d$ is excluded because (r,m) is supposed to be the leftmost position of a $-+$ segment, so it cannot belong to the last column.

We claim that, for every p,

$$N^{-+}(p,d) \leqslant 2^{d}-1. \tag{24}$$

To see this, let $j_1,\dots,j_{k+1}$ denote the columns used by a tropically allowed path of $-+$ type in a $p\times d$ sign pattern (ϵ_{ij}). This path is uniquely determined by $j_1,\dots,j_{k+1}$ because due to Conditions (i) and (iii), the vertical ordinates $i_1,\dots,i_k$ of its horizontal segments are given recursively by $i_1=\min\{i \mid \epsilon_{ij_1}=\ominus\mathbb{1}\}$ and $i_r=\min\{i>i_{r-1} \mid \epsilon_{ij_r}=\ominus\mathbb{1}\}$, for $r=2,\dots,k$. Since $\{j_1,\dots,j_{k+1}\}$ can be any non-empty subset of $\{1,\dots,d\}$, the bound (24) follows. A similar argument shows that

$$N_{\ell}^{+-}(p,d) \leqslant 2^{d-1}$$

because in this case d always belongs to $\{j_1,\dots,j_{k+1}\}$, so $j_{k+1}=d$ and $\{j_1,\dots,j_k\}$ can be any subset of $\{1,\dots,d-1\}$.

Collecting the previous bounds and using the fact that $N^{-+}(p,d)=N^{+-}(p,d)$, from (23) we obtain

$$N^{\text{tpath}}(p,d) \leqslant p \sum_{1\leqslant m\leqslant d-1} 2^{m-1}\bigl(2^{d-m}-1\bigr)+2^{d}-1,$$

which implies (22). □

The following propositions provide lower bounds for the maximal number of tropically allowed paths in a $p\times d$ sign pattern.

Proposition 3. *For $p \geqslant 2d$, we have*

$$N^{\text{tpath}}(p,d) \geqslant (p-2d+7)\bigl(2^{d-2}-2\bigr). \tag{25}$$

Proof. We shall give a $p\times d$ sign pattern which has at least $(p-2d+7)(2^{d-2}-2)$ tropically allowed paths. Consider the $p\times d$ sign pattern (ϵ_{ij}), with a natural symbol shape (♮), defined as follows:

$$\epsilon_{ij}=\ominus\mathbb{1} \quad\Longleftrightarrow\quad \begin{cases} i=2 \quad \text{and} \quad j\geqslant d-3,\\ i=d-1 \quad \text{and} \quad j\leqslant p-d+4,\\ 3\leqslant i\leqslant d-2 \quad \text{and} \quad j\leqslant i-2,\\ 3\leqslant i\leqslant d-2 \quad \text{and} \quad j\geqslant i+p-d+2.\end{cases}$$

An example for $p=14$ and $d=7$ is given on the left-hand side of Fig. 4.

Let $\{j_1,\dots,j_k\}$ be any non-empty subset of $\{3,\dots,d-2\}$ and $i\in\{d-3,\dots,p-d+3\}$. Then, it can be checked that the following lattice paths

$$(1,2),\dots,(i_1,2),\dots,(i_1,j_1),\dots,(i_k,j_k),\dots,(i,j_k),\dots,(i,d-1),(i+1,d-1),$$
$$(i+1,d),\dots,(p,d)$$
$$(1,1),\dots,(i,1),(i,2),(i+1,2),\dots,(i+1,j_1),\dots,(h_1,j_1),\dots,(h_k,j_k),\dots,$$
$$(h_k,d-1),\dots,(p,d-1)$$

$$
\begin{pmatrix}
+ & + & - & - & - & - & + \\
+ & + & + & - & - & - & + \\
+ & + & + & + & - & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & + & + & + & - & + \\
+ & - & - & + & + & + & + \\
+ & - & - & - & + & + & + \\
+ & - & - & - & - & + & +
\end{pmatrix}
\quad
\begin{array}{c}
\\ i_1 \\ i_2 \\ \\ \\ i \\ \\ \\ \\ \\ \\ \\ \\ \\ \\
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & j_1 & j_2 & & & \end{array} \\
\begin{pmatrix}
\cdot & + & - & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & + & - & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & + & + & - & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & - & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +
\end{pmatrix}
\end{array}
\quad
\begin{array}{c}
\\ \\ \\ \\ \\ \\ \\ \\ i \\ \\ \\ \\ h_1 \\ \\ h_2
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} & & j_1 & & j_2 & & \end{array} \\
\begin{pmatrix}
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
+ & - & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & - & + & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & - & + & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & - & + & +
\end{pmatrix}
\end{array}
$$

Fig. 4. Sign pattern with a natural symbol shape and two tropically allowed paths.

$$(1,1),\dots,(i_1,1),\dots,(i_1,j_1),\dots,(i_k,j_k),\dots,(i,j_k),\dots,(i,d-1),(i+1,d-1),$$
$$(i+1,d),\dots,(p,d)$$
$$(1,1),\dots,(i,1),(i,2),(i+1,2),\dots,(i+1,j_1),\dots,(h_1,j_1),\dots,(h_k,j_k),\dots,(h_k,d),$$
$$\dots,(p,d)$$

where $i_r = j_r - 2$ and $h_r = j_r + p - d + 2$ for $r = 1,\dots,k$, are tropically allowed. Examples of the first and last cases are given in Fig. 4 for $k = 2$. Indeed, note that in the last two cases $\{j_1,\dots,j_k\}$ can also be empty, in which case these paths reduce to

$$(1,1),\dots,(i,1),\dots,(i,d-1),(i+1,d-1),(i+1,d),\dots,(p,d) \quad \text{and}$$
$$(1,1),\dots,(i,1),(i,2),(i+1,2),\dots,(i+1,d),\dots,(p,d)$$

respectively. Therefore, since all these paths are different, for this sign pattern we have at least $2(p-2d+7)(2^{d-4}-1)+2(p-2d+7)2^{d-4} = (p-2d+7)(2^{d-2}-2)$ tropically allowed paths. □

Proposition 4. *For $d \geqslant 2p+1$, we have*

$$N^{\text{tpath}}(p,d) \geqslant U(d,d-p-1). \tag{26}$$

Proof. Consider the $p \times d$ sign pattern (ϵ_{ij}) defined by $\epsilon_{ij} := \ominus\mathbb{1}$ if and only if $i + j$ is odd. We shall show that for this sign pattern, when $d \geqslant 2p+1$, there exist at least $U(d,d-p-1)$ tropically allowed lattice paths.

Let Q be any subset of $\{1,\dots,d\}$ with $d-p-1$ elements which satisfies Gale's evenness condition, i.e. such that for any $j', j'' \in \{1,\dots,d\} \setminus Q$ the number of elements in Q between j' and j'' is even. Assume that $\{1,\dots,d\} \setminus Q = \{j_1,\dots,j_{p+1}\}$, where $j_1 < \cdots < j_{p+1}$. Then, the lattice path

$$(1,j_1),(1,j_2),(2,j_2),(2,j_3),\dots,(p,j_p),(p,j_{p+1})$$

is tropically allowed. Indeed, by Gale's evenness condition applied to $j' = j_r$ and $j'' = j_{r+1}$, the signs in the positions (r,j_r) and (r,j_{r+1}) must be opposite. Since the signs in the positions (r,j_{r+1}) and $(r+1,j_{r+1})$ are also opposite, we conclude that $(\epsilon_{rj_r},\epsilon_{rj_{r+1}}) = (+,-)$ or $(\epsilon_{rj_r},\epsilon_{rj_{r+1}}) = (-,+)$ for all $1 \leqslant r \leqslant p$, depending on whether j_1 is odd or not. Therefore, the path above is tropically allowed.

Since there are $U(d,d-p-1)$ subsets of $\{1,\dots,d\}$ with $d-p-1$ elements which satisfy the evenness condition, the proposition follows. □

The following proposition points out cases in which the upper bound is attained.

Proposition 5. *The upper bound $U(p+d, d-1)$ for $N^{\text{trop}}(p,d)$ is attained for $p \leqslant 3$, for $d \leqslant 4$, and for $p = 4$ and d even.*

Proof. We shall only give the sign patterns for which the polar of the signed cyclic polyhedral cone attains the bound, leaving the details to the reader.

For $d \leqslant 4$ it is enough to define $\epsilon_{ij} = \ominus\mathbb{1}$ if and only if $j = 2$.

When $p = 1$ the maximizing sign pattern is given by $\epsilon_{1j} = \ominus\mathbb{1}$ if and only if j is even.

For $p = 2$ we have to define $\epsilon_{ij} = \ominus\mathbb{1}$ if and only if $i + j$ is odd, but $\epsilon_{pd} = \oplus\mathbb{1}$ when d is odd even if $p + d$ is odd.

The case $p = 3$ needs to be divided. If d is even the maximizing sign pattern is given by $\epsilon_{ij} = \ominus\mathbb{1}$ if and only if $i + j$ is odd, but $\epsilon_{pd} = \oplus\mathbb{1}$ even if $p + d$ is odd. When d is odd, $\epsilon_{ij} = \ominus\mathbb{1}$ if and only if $i + j$ is even, but $\epsilon_{11} = \oplus\mathbb{1}$ and $\epsilon_{pd} = \oplus\mathbb{1}$ even if $p + d$ is even.

Finally, when $p = 4$ and d is even, the maximizing sign pattern is given by $\epsilon_{ij} = \ominus\mathbb{1}$ if and only if $i + j$ is even, except for ϵ_{11} and ϵ_{pd} which must be equal to $\oplus\mathbb{1}$. $\square$

Remark 2. The bound $U(p+d, d-1)$ can be written as

$$\binom{p+k}{k-1} + \binom{p+k-1}{k-2} \quad \text{when } d = 2k-1, \text{ and}$$

$$2\binom{p+k}{k-1} \qquad \text{when } d = 2k.$$

Remark 3. An interesting situation arises when the dimension d is kept fixed, whereas the number of constraints p tends to infinity. Then, it follows readily from the previous formula that

$$U(p+d, d-1) = \Theta\big(p^{\lfloor \frac{d-1}{2} \rfloor}\big) \quad \text{as } p \to \infty,$$

whereas it follows from Propositions 2 and 3 that

$$N^{\text{tpath}}(p,d) = \Theta(p) \quad \text{as } p \to \infty$$

(these asymptotic expansions of course are not uniform in d). Hence, when the dimension d is fixed, and assuming that $d \geqslant 5$, the maximal number of extreme points of the polar of a signed cyclic polyhedral cone grows much more slowly in the tropical case than in the classical case.

Remark 4. When the number of constraints p is kept fixed, whereas d tends to infinity, it is easily seen that the upper bound $U(p+d, d-1)$ for the number of extreme rays $N^{\text{trop}}(p,d)$ is equivalent to the lower bound $U(d, d-p-1)$ of Proposition 4. It follows that

$$N^{\text{trop}}(p,d) \sim U(p+d, d-1) \quad \text{as } d \to \infty.$$

In other words, the inequalities in (19) are asymptotically tight when $d \to \infty$.

We illustrate the previous results by displaying, in Table 1, for each value of (p,d) the best bounds for $N^{\text{trop}}(p,d)$ which follow from the present study. Each entry of the table is an interval containing $N^{\text{trop}}(p,d)$. When the upper and lower bounds coincide, we write a number instead of the interval reduced to this number. The upper bounds come from Theorem 1. To get lower bounds, we use Theorem 2, which implies that $N^{\text{trop}}(p,d) \geqslant N^{\text{tpath}}(\epsilon)$ for all sign patterns. Then, we consider explicit sign patterns ϵ, which come either from Proposition 5, or from computer experiments. Indeed, for all the values of p, d such that $pd \leqslant 30$, we computed $N^{\text{tpath}}(\epsilon)$ for the 2^{pd} sign patterns ϵ, so that the lower bound actually gives $N^{\text{tpath}}(p,d)$. From these "low dimensional" cases, we derived some plausible values for the patterns ϵ maximizing or approaching $N^{\text{tpath}}(\epsilon)$ for higher values of (p,d), in particular variations on the "natural" pattern introduced in the proof of Proposition 3. Experiments actually indicate that there is no simple universal maximizing sign pattern. Finding the optimal patterns (and so, computing $N^{\text{tpath}}(p,d)$) seems to be an interesting combinatorial problem, which is beyond the scope of the present paper. Of course, the most interesting open problem is to determine $N^{\text{trop}}(p,d)$.

Table 1
Lower and upper bounds for $N^{\text{trop}}(p,d)$, the maximal number of extreme rays of a tropical polyhedral cone defined by p inequalities in dimension d.

$d \backslash p$	1	2	3	4	5	6	7	8	9	10	11
3	4	5	6	7	8	9	10	11	12	13	14
4	6	8	10	12	14	16	18	20	22	24	26
5	9	14	20	[26, 27]	[32, 35]	[38, 44]	[44, 54]	[50, 65]	[56, 77]	[62, 90]	[68, 104]
6	12	20	30	42	[55, 56]	[68, 72]	[82, 90]	[96, 110]	[110, 132]	[124, 156]	[138, 182]
7	16	30	50	[71, 77]	[96, 112]	[124, 156]	[152, 210]	[180, 275]	[208, 352]	[236, 442]	[264, 546]
8	20	40	70	[112, 112]	[159, 168]	[216, 240]	[280, 330]	[340, 440]	[401, 572]	[452, 728]	[508, 910]
9	25	55	105	[172, 182]	[250, 294]	[321, 450]	[436, 660]	[613, 935]	[751, 1287]	[869, 1729]	[981, 2275]
10	30	70	140	[252, 252]	[370, 420]	[538, 660]	[668, 990]	[898, 1430]	[1320, 2002]	[1642, 2730]	[1902, 3640]
11	36	91	196	[363, 378]	[584, 672]	[805, 1122]	[1122, 1782]	[1357, 2717]	[1799, 4004]	[2771, 5733]	[3528, 8008]

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