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Preprojective Modules over Artin Algebras

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In their study of the representation theory of finite-dimensional tensor algebras Dlab and Ringel [11] described certain modules which they called preprojective and preinjective modules. Platzeck in her work on the representation theory of artin algebras stably equivalent to hereditary artin algebras [14], which includes among other things the tensor, hereditary, and square radical zero artin algebras, found and studied modules which are clearly analogs of the preprojective and preinjective modules introduced by Dlab and Ringel. Our purpose in this paper is to develop a general theory of preprojective and preinjective modules over arbitrary artin algebras. As would be expected, the preprojective and preinjective modules described by the general theory coincide with the modules considered earlier by Dlab and Ringel and Platzeck in their respective situations.

In view of these remarks it is perhaps surprising that the original impetus for our work did not come from the theory of hereditary artin algebras or those stably equivalent to hereditary artin algebras. Rather it came from an effort to explain a much older result of Gabriel and Roiter [12, 15] concerning artin algebras of finite representation type in terms of the technics and ideas developed by Auslander and Reiten in connection with almost split sequences and irreducible morphisms [6, 7]. Our generalization of the Gabriel–Roiter result is as follows.

Let Λ be an artin algebra (for instance, a finite-dimensional algebra over a field) and ind Λ the full subcategory of the category of finitely generated Λ-modules consisting of the indecomposable Λ-modules. Then there is a unique collection of full subcategories $\{\mathbf{P}_i\}_{i\in N\cup\infty}$ of ind Λ where N is the non-negative integers having the following properties:

(a) If $A \in \mathbf{P}_i$ and $B \approx A$, then $B \in \mathbf{P}_i$ for all i.

(b) $\mathbf{P}_i \cap \mathbf{P}_j = \varnothing$ for $i \neq j$ and $\bigcup_{i\in N\cup\infty} \mathbf{P}_i = \operatorname{ind} \Lambda$.

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(c) $\mathbf{P}_i$ has only a finite number of nonisomorphic objects for each i in N.

(d) For each i in N, an indecomposable module A is in $\mathbf{P}_i$ if and only if every surjective morphism $f: B \to A$ is a splittable surjection whenever every indecomposable summand of B is in $\bigcup_{k \geqslant i} \mathbf{P}_k$.

This uniquely determined collection $\{\mathbf{P}_i\}_{i \in N \cup \infty}$ of subcategories of ind Λ is called the preprojective partition of ind Λ. In this terminology the Gabriel–Roiter theorem says that if Λ is of finite representation type, i.e., ind Λ has only a finite number of nonisomorphic modules, then ind Λ has a preprojective partition.

It is in terms of preprojective partitions that preprojective modules are defined; namely, a finitely generated Λ-module M is preprojective if each indecomposable summand of M is in $\mathbf{P}_i$ for some $i < \infty$. While preprojective modules are defined in terms of the preprojective partition, they can also be given descriptions which do not depend on the preprojective partition. For example, an indecomposable module M is preprojective if and only if there is a maximal submodule M' of M such that there are only a finite number of nonisomorphic indecomposable modules $B_1, \ldots, B_n$ having a morphism $f: B_i \to M$ with $\operatorname{Im} f$ not contained in M'. The definition and existence of preinjective partitions and preinjective modules is given by duality.

The proofs of the main existence theorem for preprojective and preinjective partitions are based on the following result concerning full subcategories $\mathbf{C}$ of ind Λ. Suppose $\mathbf{C}$ has the property that there are a finite number of indecomposable modules $B_1, \ldots, B_n$ such that if C is in ind Λ and not isomorphic to any B_i, then C is in $\mathbf{C}$. Then given an X in ind Λ we have:

(a) There is a morphism $f: C \to X$ where every indecomposable summand of C is in $\mathbf{C}$ such that for each morphism $g: L \to X$ with L in $\mathbf{C}$ there is an $h: L \to C$ satisfying $g = fh$.

(b) There is a morphism $f: X \to C'$ where every indecomposable summand of C' is in $\mathbf{C}$ such that for each morphism $g: X \to L$ with L in $\mathbf{C}$, there is an $h: C' \to L$ satisfying $hf = g$.

For technical as well as conceptual reasons, the initial discussion in this paper of preprojective and preinjective partitions as well as preprojective and preinjective modules is concerned with full subcategories of ind Λ rather than ind Λ itself. While this necessitates slightly more abstract and longer proofs, we feel this approach is justified on two counts.

First, some of the seemingly abstract notions have intrinsic value, especially the notion of mod Λ being contravariantly or covariantly finite over a full subcategory $\mathbf{C}$. As can be seen from the discussion here, the question of whether or not a subcategory $\mathbf{C}$ has left or right almost split morphisms is intimately connected with mod Λ being contravariantly or covariantly finite over $\mathbf{C}$. In another paper [8] we will show how these notions are connected with **C** having almost split sequences.

Second, the greater generality shows that many subcategories of ind Λ and not just ind Λ itself have preprojective and/or preinjective partitions: hence the greater applicability of the theory. This is particularly important in view of the work of Bautista and Martinez [9] showing that various classification problems, for instance, for representations of partially ordered sets, can be viewed as classification problems for specific subcategories in ind Λ for suitably chosen artin algebras Λ.

Roughly speaking the paper is divided into three parts. The first four sections deal with basic concepts and existence theorems. In Sections 5 through 7, the preprojective and preinjective partitions of subcategories of ind Λ are studied. The rest of the paper is devoted to applying these results to ind Λ as well as giving more detailed information concerning the structure of preprojective and preinjective modules.

A preliminary announcement of this work was given at the Antwerp Ring Theory Conference in the summer of 1978 [3].

After suitable reformulation, many of the notions and results of this paper are also applicable to lattices over orders over complete discrete valuation rings. In particular one obtains preprojective and preinjective partitions for lattices as well as preprojective and preinjective lattices. These matters will be discussed in another publication.

1. Minimal Morphisms

Throughout this section we assume that Λ is a left artin ring and all modules and morphisms are in mod Λ, the category of finitely generated left Λ-modules.

Suppose $f: B \to C$ and $g: X \to C$ are morphisms. We say that a morphism $h: X \to B$ is a *lifting of g to f* if $fh = g$. The morphism $g: X \to C$ is said to be *liftable to f* if there is a lifting $h: X \to B$ of g to f. Finally, we say that two morphisms $f_1: B_1 \to C$ and $f_2: B_2 \to C$ are *lifting equivalent* if a morphism $g: X \to C$ can be lifted to f_1 if and only if it can be lifted to f_2.

Clearly for a fixed C in mod Λ, the relation on the collection of all morphisms $X \to C$ given by $f_1: B_1 \to C$ is related to $f_2: B_2 \to C$ if f_1 and f_2 are lifting equivalent is an equivalence relation. Our purpose in this section is to describe how the morphisms in one equivalence class, called a lifting equivalence class, are related. Even though this material has been discussed previously (see [2, pp. 28–31]), we give here a self-contained account for the sake of completeness and ease of reference. We assume through out this discussion that C is a fixed module in mod Λ.

We begin with the following easily verified observations.

LEMMA 1.1. *Let* $f_1\colon B_1 \to C$ *and* $f_2\colon B_2 \to C$ *be morphisms in* mod Λ:

(a) f_1 *can be lifted to* f_2 *if and only if each* $g\colon X \to C$ *which can be lifted to* f_1 *can also be lifted to* f_2.

(b) f_1 *and* f_2 *are lifting equivalent if and only if* f_1 *can be lifted to* f_2 *and* f_2 *can be lifted to* f_1.

(c) *Suppose* f_1 *and* f_2 *are lifting equivalent and the diagram*

$$\begin{array}{ccc} B_1 & \xrightarrow{f_1} & C \\ \downarrow & & \| \\ B & \xrightarrow{f} & C \\ \downarrow & & \| \\ B_2 & \xrightarrow{f_2} & C \end{array}$$

commutes. Then f, f_1, *and* f_2 *are lifting equivalent.*

Proof. Left as an exercise to the reader.

Before giving the main result of this discussion, it is convenient to give the following definition.

A morphism $B \to^f C$ is said to be *right minimal* if every lifting of f to f is an isomorphism, i.e., $h\colon B \to B$ is an isomorphism if $fh = f$.

PROPOSITION 1.2. *Let* $\mathscr{E}$ *be a lifting equivalence class in the class of all morphisms* $X \to C$ *in* mod Λ.

(a) *If* $f\colon B \to C$ *in* $\mathscr{E}$ *has the property that* $l(B) \leqslant l(X)$ (*where* $l(Y)$ *means the length of* Y) *for all* X *such that there is* $g\colon X \to C$ *in* $\mathscr{E}$, *then* $f\colon B \to C$ *is right minimal. Thus there is an* $f\colon B \to C$ *in* $\mathscr{E}$ *which is right minimal.*

(b) *If* $f_1\colon B_1 \to C$ *and* $f_2\colon B_2 \to C$ *are two right minimal morphisms in* $\mathscr{E}$, *then every lifting of* f_1 *to* f_2 *is an isomorphism. Hence, there is an isomorphism* h: $B_1 \to B_2$ *such that* $f_1 = f_2 h$.

(c) *Let* $f\colon B \to C$ *be a morphism in* $\mathscr{E}$. *If* $B = B_1 \coprod B_2$ (*direct sum*) *such that* $f(B_2) = 0$, *then* $f \mid B_1 : B_1 \to C$ *in* $\mathscr{E}$.

(d) *Let* $f\colon B \to C$ *be in* $\mathscr{E}$. *Then there is a decomposition* $B = B_1 \coprod B_2$ *such that* $f(B_2) = 0$ *and* $f \mid B_1\colon B_1 \to C$ *is a right minimal morphism in* $\mathscr{E}$.

Proof (a) Let $h\colon B \to B$ be such that $fh = f$. Then we have the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \downarrow & & \| \\ h(B) & \xrightarrow{f|h(B)} & C \\ \downarrow & & \| \\ B & \xrightarrow{f} & C \end{array}$$

By Lemma 1.1(c), we know that $f \mid h(B)$: $h(B) \to C$ is in $\mathscr{E}$. Since $l(B) \leqslant l(X)$ for all g: $X \to C$ in $\mathscr{E}$, we have that $l(h(B)) \leqslant l(B) \leqslant l(h(B))$. Hence $h(B) = B$ and so h: $B \to B$ is surjective and thus an isomorphism since B has finite length. The rest of (a) is obvious.

(b) Suppose f_1: $B_1 \to C$ and f_2: $B_2 \to C$ are two right minimal morphisms in $\mathscr{E}$. Since they are both in $\mathscr{E}$ we know by Lemma 1.1(b) that there are morphisms h_1: $B_1 \to B_2$ and h_2: $B_2 \to B_1$ such that $f_2 h_1 = f_1$ and $f_2 = f_1 h_2$. Hence $f_1 h_2 h_1 = f_1$ and so $h_1 h_2$: $B_1 \to B_2$ is an isomorphism since f_1 is right minimal. Also $f_2 = f_2 h_1 h_2$ and so $h_1 h_2$: $B_2 \to B_1$ is an isomorphism since f_2 is right minimal. Hence h_1: $B_1 \to B_2$ is an isomorphism such that $f_2 h_1 = f_1$.

(c) Suppose f: $B \to C$ is in $\mathscr{E}$ and $B = B_1 \coprod B_2$ a decomposition such that $f(B_2) = 0$. Then we have the commutative diagram

$$\begin{array}{ccc} B_1 \coprod B_2 & \xrightarrow{f} & C \\ \downarrow p & & \| \\ B_1 & \xrightarrow{f|B_1} & C \\ \downarrow i & & \| \\ B_1 \coprod B_2 & \xrightarrow{f} & C \end{array}$$

where p: $B_1 \coprod B_2 \to B_1$ is the projection morphism and i: $B_1 \to B_1 \coprod B_2$ is the injection morphism. Therefore by Lemma 1.1(c) we have that $f \mid B_1$: $B_1 \to C$ is in $\mathscr{E}$.

(d) Let f: $B \to C$ be in $\mathscr{E}$. Then by part (a) we know there is a right minimal g: $B' \to C$ in $\mathscr{E}$. Thus there is a commutative diagram

$$\begin{array}{ccc} B' & \xrightarrow{g} & C \\ \downarrow h_1 & & \| \\ B & \xrightarrow{f} & C \\ \downarrow h_2 & & \| \\ B' & \xrightarrow{g} & C \end{array}$$

Since $g: B' \to C$ is right minimal, we know that $h_2 h_1: B' \to B'$ is an isomorphism. Therefore $B = \operatorname{Ker} h_2 \coprod \operatorname{Im} h$, and it is easy to see that $f(\operatorname{Ker} h_2) = 0$ and $f \mid \operatorname{Im} h_1$ is a right minimal morphism in $\mathscr{E}$ since $h_1: B' \to B$ induces an isomorphism $j: B' \to \operatorname{Im} h_1$ with the property $g = (f \mid \operatorname{Im} h_1) j$. Then $B = B_1 \coprod B_2$, where $B_1 = \operatorname{Im} h_1$ and $B_2 = \operatorname{Ker} h_2$ is our desired decomposition of B.

In order to simplify our terminology we give the following definition:

Let C be in mod Λ. Two morphisms $f_1: X_1 \to C$ and $f_2: X_2 \to C$ are said to be *isomorphic* if there is an isomorphism $h: X_1 \to X_2$ which is a lifting of f_1 to f_2. Obviously if a lifting $h: X_1 \to X_2$ of f_1 to f_2 is an isomorphism, then $h^{-1}: X_2 \to X_1$ is a lifting of f_2 to f_1.

Thus, in this terminology, Proposition 1.2(b) has the following form. Let $\mathscr{E}$ be a lifting equivalence class in the class of all morphisms $X \to C$ in mod Λ. Then any two right minimal morphisms in $\mathscr{E}$ are isomorphic.

In the rest of this paper, expecially Sections 2 and 3, we will often be interested in finding a solution to the following type of problem. Let C be in mod Λ and $\mathscr{S}$ a family of morphisms $Y \to C$. Is there a morphism $f: B \to C$ in $\mathscr{S}$ with the property that each morphism $g: X \to C$ in $\mathscr{S}$ is liftable to f? While our discussion so far gives no information concerning the existence of solutions to this problem, it does give considerable information concerning the structure of the solutions when they do exist, at least in the case $\mathscr{S}$ satisfies some mild additional conditions.

PROPOSITION 1.3. *Let C be in* mod Λ *and $\mathscr{S}$ a family of morphisms $X \to C$ satisfying the condition that if $f: X \to C$ is in $\mathscr{S}$ and $g: Y \to X$ is a splittable injection* (*i.e., there is a $t: X \to Y$ such that $tg = id_Y$*), *then the composition fg: $Y \to C$ is in $\mathscr{S}$. Suppose there is an $f: B \to C$ in $\mathscr{S}$ such that every g in $\mathscr{S}$ is liftable to f. Then*

(a) *There is a right minimal $f': B' \to C$ in $\mathscr{S}$ such that every g in $\mathscr{S}$ can be lifted to f.*

(b) *Any two right minimal morphisms $f': B' \to C$ in $\mathscr{S}$ such that every g in $\mathscr{S}$ can be lifted to f' are isomorphic.*

(c) *Suppose $f: B \to C$ in $\mathscr{S}$ has the property that every g in $\mathscr{S}$ can be lifted to f. Then there is a decomposition $B = B_1 \coprod B_2$ such that* (i) $f \mid B_1: B_1 \to C$ *is right minimal with the property every g in $\mathscr{S}$ can be lifted to $f \mid B_1$ and* (ii) $f(B_2) = 0$.

Proof. (a) and (c). Let $f: B \to C$ in $\mathscr{S}$ be such that every g in $\mathscr{S}$ can be lifted to f. We know by Proposition 1.2(c) that there is a decomposition $B = B_1 \coprod B_2$ with $f \mid B_1: B_1 \to C$ right minimal and $f(B_2) = 0$. Since the inclusion $i_1: B_1 \to B$ is a splittable injection, $f \mid B_1 = f i_1$ is in $\mathscr{S}$. Because $f(B_2) = 0$, we have that the projection $p_1: B \to B_1$ has the property $f = (f \mid B_1) p_1$. Thus every g in $\mathscr{S}$ can be lifted to $f \mid B_1$. This proves (a) and (c).

(b) Suppose $f_1: B_1 \to C$ and $f_2: B_2 \to C$ are two right minimal morphisms in $\mathcal{S}$ such that every g in $\mathcal{S}$ can be lifted to f_1 and f_2. Then there are $h_1: B_1 \to B_2$ and $h_2: B_2 \to B_1$ such that $f_1 = f_2 h_1$ and $f_2 = f_1 h_2$. From this it follows that h_1 and h_2 are isomorphisms since f_1 and f_2 are right minimal. This completes the proof of (b) and the entire proposition.

In order to fix definitions and for ease of reference, we now state the duals of the above concepts and results. No proofs are given since they follow easily from the above using dual arguments.

Suppose $f: B \to C$ and $g: B \to Y$ are morphisms. We say that a morphism $h: C \to Y$ is an *extension of* g *to* f if $hf = g$. The morphism $g: B \to Y$ is said to be *extendable to* $f: B \to C$ if there is an extension $h: C \to Y$ from f to g. Finally, we say that two morphisms $f_1: B_1 \to C_1$ and $f_2: B \to C_2$ are *extension equivalent* if a morphism $B \to Y$ can be extended to f_1 if and only if it can be extended to f_2.

Clearly for a fixed B in mod Λ, the relation on the collection of morphisms $B \to Y$ given by $f_1: B \to C_1$ is related to $f_2: B \to C_2$ if f_1 and f_2 are extension equivalent is an equivalence relation. The equivalence classes under this relation are called the *extension classes*.

While we leave the dual of Lemma 1.1 to the reader to state and prove, we will state the dual of Proposition 1.2 as soon as we define left minimal morphisms, which are the dual of right minimal morphisms.

A morphism $f: B \to C$ is *left minimal* if every extension of f to f is an isomorphism.

PROPOSITION 1.4. *Let B be in* mod Λ *and $\mathcal{F}$ an extension class of the class of all morphisms $B \to Y$ (B fixed) in* mod Λ.

(a) *If $f: B \to C$ has the property that $l(C) \leqslant l(Y)$ with $B \to Y$ in $\mathcal{F}$, then $f: B \to C$ is left minimal. Thus there is a left minimal $f: B \to C$ in $\mathcal{F}$.*

(b) *If $f_1: B \to C_1$ and $f_2: B \to C_2$ in $\mathcal{F}$ are left minimal, then every extension from f_1 to f_2 is an isomorphism. Hence there is an isomorphism $h: C_1 \to C_2$ such that $hf_1 = f_2$.*

(c) *Let $f: B \to C$ be in $\mathcal{F}$. If $C = C_1 \coprod C_2$ and the projections $p_i: C \to C_i$ have the property that $p_2 f: B \to C_2$ is zero, then $p_1 f: B \to C_1$ is in $\mathcal{F}$.*

(d) *Let $f: B \to C$ be in $\mathcal{F}$. Then there is a decomposition $C = C_1 \coprod C_2$ such that $p_2 f: B \to C_2$ is zero and $p_1 f: B \to C_1$ is a left minimal morphism in $\mathcal{F}$.*

Let B be in mod Λ. Two morphisms $f_1: B \to X_1$ and $f_2: B \to X_2$ are said to be *isomorphic* if there is an isomorphism $h: X_1 \to X_2$ which is an extension of f_2 to f_1.

With this definition in mind we give the dual of Proposition 1.3.

PROPOSITION 1.5. *Let B be in* mod Λ *and $\mathcal{T}$ a family of morphisms $B \to X$*

satisfying the condition that if $f: B \to X$ is in $\mathcal{T}$ and $g: X \to Y$ is a splittable surjection (i.e., there is an $s: Y \to X$ such that $gs = id_Y$), then the composition $gf: B \to Y$ is in $\mathcal{T}$. Suppose there is an $f: B \to C$ in $\mathcal{T}$ such that each g in $\mathcal{T}$ can be extended to f. Then

(a) *There is a left minimal $f': B \to C'$ in $\mathcal{T}$ such that each g in $\mathcal{T}$ can be extended to f'.*

(b) *Any two right minimal morphisms $f': B \to C'$ in $\mathcal{T}$ such that each g in $\mathcal{T}$ can be extended to f' are isomorphic.*

(c) *Suppose $f: B \to C$ in $\mathcal{T}$ has the property that each g in $\mathcal{T}$ can be extended to f. Then there is a decomposition $C = C_1 \coprod C_2$ such that the projections $p_i: C \to C_i$ have the property that $p_1 f$ is a left minimal morphism in $\mathcal{T}$ such that each g in $\mathcal{T}$ can be extended to $p_1 f$ and $p_2 f = 0$.*

2. Covers and Splitting Projectives

Throughout the rest of this paper we assume that R is a commutative artin ring and Λ an R-algebra which is a finitely generated R-module. It is obvious that the R-algebra structure on Λ induces an R-algebra structure on Λ^{op} in such a way that Λ^{op} is also a finitely generated R-module where Λ^{op} is the opposite ring of Λ. We denote the category of finitely generated left Λ-modules by mod Λ. Clearly mod Λ^{op} is the category of finitely generated right Λ-modules. Further, letting I be the injective envelope of $R/\text{rad}\, R$, the contravariant functor $\text{Hom}_R(\ , I)$: mod $\Lambda \to$ mod Λ^{op} given by $M \to \text{Hom}_R(M, I)$ is a duality with inverse $\text{Hom}_R(\ , I)$: mod $\Lambda^{op} \to$ mod Λ. We denote both of these dualities by D: mod $\Lambda \to$ mod Λ^{op} and D: mod $\Lambda^{op} \to$ mod Λ.

By a subcategory $\mathbf{C}$ of mod Λ we always mean a full subcategory having the property that if M in mod Λ is isomorphic to a summand (direct) of C in $\mathbf{C}$, then M is in $\mathbf{C}$. Associated with each subcategory $\mathbf{C}$ of mod Λ is the subcategory $D(\mathbf{C})$ of mod Λ^{op} consisting of all the Λ^{op}-modules isomorphic to $D(C)$ for some C in $\mathbf{C}$. The duality D: mod $\Lambda \to$ mod Λ^{op} clearly induces a duality D: $\mathbf{C} \to D(\mathbf{C})$ which gives an equivalence of $\mathbf{C}^{op}$, the opposite category of $\mathbf{C}$, with $D(\mathbf{C})$. We will usually identify the categories $\mathbf{C}^{op}$ and $D(\mathbf{C})$ by means of this equivalence.

In studying a subcategory $\mathbf{C}$ of mod Λ, it is often convenient to also consider the categories add $\mathbf{C}$ and ind $\mathbf{C}$ which we now describe.

The category add $\mathbf{C}$ is the subcategory of mod Λ consisting of all Λ-modules isomorphic to summands of finite sums of modules in $\mathbf{C}$. Thus add $\mathbf{C}$ is an additive subcategory of mod Λ in which idempotents split.

The category ind $\mathbf{C}$ is the subcategory of $\mathbf{C}$ consisting of all the indecomposable modules in $\mathbf{C}$. Clearly add ind $\mathbf{C}$ = add $\mathbf{C}$ and ind add $\mathbf{C}$ = ind $\mathbf{C}$. Also D(add $\mathbf{C}$) = add $D(\mathbf{C})$ and D(ind $\mathbf{C}$) = ind $D(\mathbf{C})$.

Of primary concern to us throughout this paper is the kinds of covers and cocovers a subcategory of mod Λ has; notions we now introduce.

A *cover* for a subcategory $\mathbf{C}$ of mod Λ is a subcategory $\mathbf{A}$ of ind $\mathbf{C}$ such that for each C in $\mathbf{C}$ there is a surjective morphism $f: A \to C$ with A in add $\mathbf{A}$. A cover $\mathbf{A}$ of $\mathbf{C}$ is said to be a *minimal cover* for $\mathbf{C}$ if no proper subcategory of $\mathbf{A}$ is a cover for $\mathbf{C}$. Clearly a subcategory $\mathbf{A}$ of $\mathbf{C}$ is a (minimal) cover for $\mathbf{C}$ if and only if $\mathbf{A}$ is a (minimal) cover for ind $\mathbf{C}$.

Dualizing, we obtain the dual notions of a cocover and minimal cocover for a subcategory of mod Λ.

A *cocover* for a subcategory $\mathbf{C}$ of mod Λ is a subcategory $\mathbf{B}$ of ind $\mathbf{C}$ such that for each C in $\mathbf{C}$ there is an injective morphism $f: C \to B$ with B in add $\mathbf{B}$. A cocover $\mathbf{B}$ of $\mathbf{C}$ is said to be a *minimal cocover* for $\mathbf{C}$ if no proper subcategory of $\mathbf{B}$ is a cocover for $\mathbf{C}$. Clearly a subcategory $\mathbf{B}$ of $\mathbf{C}$ is a (minimal) cocover for $\mathbf{C}$ if and only if $\mathbf{B}$ is a (minimal) cocover for ind $\mathbf{C}$. Finally, we observe that a subcategory $\mathbf{A}$ of $\mathbf{C}$ is a (minimal) cover for $\mathbf{C}$ if and only if $D(\mathbf{A})$ is a (minimal) cocover for $D(\mathbf{C})$. Thus the theory of covers of subcategories of mod Λ is the dual of the theory of cocovers of subcategories of mod Λ^{op}.

In order to motivate our discussion of covers and cocovers for subcategories $\mathbf{C}$ of mod Λ we consider the special case $\mathbf{C} = \text{mod}\,\Lambda$. Let $\mathbf{P}_0$ be the subcategory of ind mod Λ consisting of all the indecomposable projective modules in mod Λ. Then it is easily seen that $\mathbf{P}_0$ has the following properties.

(a) $\mathbf{P}_0$ is a cover for mod Λ.

(b) If $f: X \to P$ is surjection with P in $\mathbf{P}_0$, then f is a splittable surjection.

(c) If $\mathbf{A}$ is a cover for mod Λ, then $\mathbf{A}$ contains $\mathbf{P}_0$.

(d) $\mathbf{A}$ is a minimal cover for mod Λ if and only if $\mathbf{A} = \mathbf{P}_0$.

Since (a) and (b) are well known, it only remains to establish (c) and (d). But (c) follows readily from (b) (see Proposition 2.1) and (d) follows from (a) and (c). Thus $\mathbf{P}_0$ is the unique minimal cover for mod Λ which gives a complete characterization of the subcategory of mod Λ consisting of the indecomposable projective modules in terms of the notion of covers.

Obviously, a similar discussion can be carried out for cocovers of mod Λ, giving results dual to those given for covers. In particular, mod Λ has a unique minimal cocover which is the subcategory $\mathbf{I}_0$ of mod Λ consisting of the indecomposable injective modules in mod Λ.

This discussion suggests that for an arbitrary subcategory $\mathbf{C}$ of mod Λ there might be some connection between the covers (cocovers) of $\mathbf{C}$ and some sort of suitably defined notion of "projective" ("injective") objects in $\mathbf{C}$. The right notion of "projective" in $\mathbf{C}$ seems to be the one given in property (b) for the projectives.

Let $\mathbf{C}$ be a subcategory of mod Λ. We say that a C in $\mathbf{C}$ is a *splitting projective* in $\mathbf{C}$ if each surjective morphism $X \to C$ with X in add $\mathbf{C}$ is a splittable surjection.

Clearly C in $\mathbf{C}$ is a splitting projective in $\mathbf{C}$ if and only if each indecomposable summand of C is a splitting projective in $\mathbf{C}$. We denote by $\mathbf{P}_0(\mathbf{C})$ the subcategory of ind $\mathbf{C}$ consisting of the indecomposable splitting projectives in $\mathbf{C}$. It is clear that $\mathbf{P}_0(\mathbf{C}) = \mathbf{P}_0(\text{ind}\,\mathbf{C}) = \mathbf{P}_0(\text{add}\,\mathbf{C})$.

Dually, we say that C in $\mathbf{C}$ is a *splitting injective* in $\mathbf{C}$ if each injective morphism $C \to Y$ with Y in add $\mathbf{C}$ is a splittable injection. Obviously C in $\mathbf{C}$ is a splitting injective in $\mathbf{C}$ if and only if each indecomposable summand of C is a splitting injective in $\mathbf{C}$. We denote by $\mathbf{I}_0(\mathbf{C})$ the subcategory of ind $\mathbf{C}$ consisting of all the indecomposable splitting injectives in $\mathbf{C}$. It is clear that $\mathbf{I}_0(\mathbf{C}) = \mathbf{I}_0(\text{ind}\,\mathbf{C}) = \mathbf{I}_0(\text{add}\,\mathbf{C})$. Moreover it is easy to see that C in $\mathbf{C}$ is a splitting projective in $\mathbf{C}$ if and only if $D(C)$ is a splitting injective in $D(\mathbf{C})$. Hence $D(\mathbf{P}_0(\mathbf{C})) = \mathbf{I}_0(D(\mathbf{C}))$ and $D(\mathbf{I}_0(\mathbf{C})) = \mathbf{P}_0(D(\mathbf{C}))$.

While the notions of splitting projectives and splitting injectives are obviously intimately related to the usual notions of projectives and injectives there is the following important difference. Let $f: X \to Y$ be a surjective morphism in $\mathbf{C}$ and P a splitting projective in $\mathbf{C}$. Then it is *not* necessarily true that each g: $P \to Y$ can be lifted to f. A similar remark holds for splitting injectives in $\mathbf{C}$. On the other hand, if $\mathbf{C} = \text{mod}\,\Lambda$, then it is obvious that $\mathbf{P}_0(\mathbf{C})$ is the subcategory of indecomposable projective Λ-modules, while $\mathbf{I}_0(\mathbf{C})$ is the subcategory of indecomposable injective Λ-modules. It should also be noted, as we will show later, that there are subcategories $\mathbf{C}$ of mod Λ for which $\mathbf{P}_0(\mathbf{C})$ or $\mathbf{I}_0(\mathbf{C})$ is empty.

Let $\mathbf{A}$ be a cover for the subcategory $\mathbf{C}$ of mod Λ. In the case $\mathbf{C} = \text{mod}\,\Lambda$ we saw that $\mathbf{A}$ is a minimal cover for $\mathbf{C}$ if and only if $\mathbf{A} = \mathbf{P}_0(\mathbf{C})$. Our aim is to show that this result holds for arbitrary subcategories $\mathbf{C}$ of mod Λ. We begin with the following easily established preliminary result.

PROPOSITION 2.1. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ.

(a) *If* $\mathbf{A}$ *is a cover for* $\mathbf{C}$, *then* $\mathbf{A}$ *contains* $\mathbf{P}_0(\mathbf{C})$.

(b) *If* $\mathbf{B}$ *is a cocover for* $\mathbf{C}$, *then* $\mathbf{B}$ *contains* $\mathbf{I}_0(\mathbf{C})$.

Proof. (a) Let P be in $\mathbf{P}_0(\mathbf{C})$. Since $\mathbf{A}$ is a cover for $\mathbf{C}$, there is a surjective morphism $f: A \to P$ with A in add $\mathbf{A}$. Because P is a splitting projective in $\mathbf{C}$, the surjection $f: A \to P$ is a splitting surjection. Hence P is isomorphic to an indecomposable summand of A and thus must be in $\mathbf{A}$. Therefore $\mathbf{A} \supset \mathbf{P}_0(\mathbf{C})$.

(b) Dual of (a).

The rest of the proof that a cover $\mathbf{A}$ of $\mathbf{C}$ is minimal if and only if $\mathbf{A} = \mathbf{P}_0(\mathbf{C})$ is based on the following general definitions and results which are also of interest in their own right.

Let A and B be in mod Λ. The *trace of* B *in* A is the submodule $\tau_B(A)$ of A generated by all the homomorphic images of B in A. Since A is noetherian, it follows that there is a morphism $f: nB \to A$, where nB is a sum of n copies of B, such that $\text{Im} f = \tau_B(A)$. Also, viewing A as an End A-module, where End A

is the endomorphism ring of A, we have that $\tau_B(A)$ is an End A-submodule of A.
We now use these observations to prove the following general result.

PROPOSITION 2.2. *Let A and B be in* mod Λ *with A indecomposable.*

(a) *Suppose f_i: $A \to A$, $i = 1,..., n$ is a finite family of morphisms and f_0: $B \to A$ is a morphism such that the induced morphism f: $B \coprod nA \to A$ is a surjection which is not a splittable surjection. Then $\tau_B(A) = A$ and so there is a surjection $mB \to A$ of m copies of B(m finite) to A.*

(b) *Suppose f_i: $A \to A$, $i = 1,..., n$ is a finite family of morphisms and f_0: $A \to B$ a morphism such that the induced morphism f: $A \to nA \coprod B$ is an injection which is not a splittable injection. Then there is an injection $A \to mB$, where mB is a finite sum of copies of B.*

Proof. (a) Since the surjection f: $B \coprod nA \to A$ is not a splittable surjection, none of the f_i: $A \to A$ is an isomorphism. The fact that A is indecomposable implies that End A is local. Hence each of the f_i: $A \to A$ is in rad End A. Now f: $B \coprod nA \to A$ being surjective implies that the submodule $\sum_{i=1}^{n} f_i(A) + f_0(B)$ of A generated by the $f_i(A)$, $i = 1,..., n$ and $f_0(B)$ is all of A. Since each $f_i(A) \subset$ (rad End A) $\cdot$ A and $f_0(B) \subset \tau_B(A)$, we have that $A =$ (rad End A)$\cdot$ $A + \tau_B(A)$. Thus viewing A as an End A-module, we have by Nakayama's lemma that $\tau_B(A) = A$ since $\tau_B(A)$ is an End A-submodule of A and A is a finitely generated End A-module.

(b) Dual of (a).

We now prove our main result connecting minimal covers and cocovers of a subcategory **C** of mod Λ with $\mathbf{P}_0(\mathbf{C})$ and $\mathbf{I}_0(\mathbf{C})$.

THEOREM 2.3. *Let* **C** *be a subcategory of* mod Λ.

(a) *A cover* **A** *of* **C** *is a minimal cover of* **C** *if and only if* $\mathbf{A} = \mathbf{P}_0(\mathbf{C})$.

(b) *A cocover* **B** *of* **C** *is a minimal cocover of* **C** *is and only if* $\mathbf{B} = \mathbf{I}_0(\mathbf{C})$.

Proof. (a) We have already seen, Proposition 2.1, that every cover of **C** contains $\mathbf{P}_0(\mathbf{C})$. Hence if **A** is a cover of **C** and $\mathbf{A} = \mathbf{P}_0(\mathbf{C})$, then **A** is a minimal cover for **C**. We now finish the proof of (a) showing that if **A** is a cover of **C** and $\mathbf{A} \neq \mathbf{P}_0(\mathbf{C})$, then **A** is not minimal.

Suppose A in **A** is not in $\mathbf{P}_0(\mathbf{C})$. Then there is a surjective morphism f: $C \to A$ in **C** which is not a splittable surjection. Since **A** is a cover for **C**, we know there is a surjection g: $X \to C$ with X in **A**. Then the composition $h = gf$: $X \to A$ is a surjection which is not a splittable surjection since f is not a splittable surjection. We can write X as a sum $B \coprod nA$ where B has no summands isomorphic to A. Then by Proposition 2.2, we know there is a surjection $mB \to A$. Since B is in add($\mathbf{A} - \{A\}$), where $\{A\}$ is the subcategory of **A** of all modules isomorphic to A, it follows that $\mathbf{A} - \{A\}$ is a cover for **A** and hence for **C**. Hence, if there is an Λ in **A** which is not in $\mathbf{P}_0(\mathbf{C})$, then **A** is not a minimal cover for **C**. Therefore **A** being a minimal cover for **C** implies $\mathbf{P}_0(\mathbf{C}) = \mathbf{A}$, which finishes the proof of (a).

(b) Dual of (a).

As an immediate consequence of Theorem 2.3 we have the following.

COROLLARY 2.4. *Let* **C** *be a subcategory of* mod Λ.

(a) If $\mathbf{A}_1$, $\mathbf{A}_2$ *are two minimal covers of* **C**, *then* $\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{P}_0(\mathbf{C})$.

(b) *If* **A** *is a finite cover for* **C** (*i.e.,* **A** *has only a finite number of nonisomorphic objects*), *then* $\mathbf{P}_0(\mathbf{C})$ *is a finite minimal cover for* **C**.

(c) *If* $\mathbf{B}_1$ *and* $\mathbf{B}_2$ *are minimal cocovers for* **C**, *then* $\mathbf{B}_1 = \mathbf{B}_2 = \mathbf{I}_0(\mathbf{C})$.

(d) *If* **C** *has a finite cocover, then* $\mathbf{I}_0(\mathbf{C})$ *is a finite minimal cocover for* **C**.

Proof. (a) By Theorem 2.3 we know that $\mathbf{A}_1 = \mathbf{P}_0(\mathbf{C}) = \mathbf{A}_2$ and so $\mathbf{A}_1 = \mathbf{A}_2$.

(b) If **A** is a finite cover for **C**, then there is a subcategory $\mathbf{A}'$ of **A** with the smallest number of nonisomorphic objects such that $\mathbf{A}'$ is a cover for **C**. Then by Proposition 2.3, $\mathbf{A}' = \mathbf{P}_0(\mathbf{C})$ which shows that $\mathbf{P}_0(\mathbf{C})$ is a finite cover for **C**.

(c) and (d). Duals of (a) and (b), respectively.

Corollary 2.4 plays a critical role in the rest of this paper. We end this section with our first application of this result.

We recall that a module M of finite length over a principal ideal domain R has two standard representations: (a) $M \approx \coprod_{i=1}^{s} R/p_i^{n_i} R$, where each p_i is a prime, nonzero element in R and (b) $M \approx \coprod_{i=1}^{t} R/a_i R$, where $a_1 \mid a_2 \mid \cdots \mid a_t$. Moreover each of these representations is unique in the sense that the $p_i^{n_i}$ and $a_1, \ldots, a_t$ are unique for any such representations. Since the R-modules of the form $R/p^n R$ with a p prime element in R are precisely the indecomposable modules of finite length, representation (a) has a ready analog for modules in mod Λ, namely, the usual representation as a sum of indecomposable modules. We now show that each module M in mod Λ has two other decompositions each of which is the analog of the decomposition (b). Results similar to these were given by Roiter [15].

We say that a module L in mod Λ is *multiplicity free* if its representation $\coprod_{i=1}^{n} M_i$ as a sum of indecomposable modules M_i has the property $M_i \approx M_j$ implies $i = j$ for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant n$. A module L is said to be *covering indecomposable* if $\tau_K(L) \neq L$ for each proper summand K of L. Clearly if L is covering indecomposable, then L is multiplicity free.

By a *covering decomposition* of a module M we mean a representation of $M \approx \coprod_{i=1}^{t} L_i$ having the following properties: (a) each L_i is covering indecomposable and (b) $\tau_{L_i}(L_{i+1}) = L_{i+1}$ for all $i = 1, \ldots, t-1$.

Dually, a module L is said to be *cocovering indecomposable* if there is no injective morphism $L \to nK$ with K a proper summand of L. Clearly a cocovering indecomposable module is multiplicity free. By a *cocovering decomposition* of a module M we mean a representation of $M \approx \coprod_{i=1}^{s} L_i$ having the following properties: (a) each L_i is cocovering indecomposable and (b) for each $i = 1,..., s - 1$ there is an injection $L_{i+1} \to n_i L_i$ for some n_i.

Before showing that every module in mod Λ has both a covering and cocovering decomposition, we show that these representations are essentially unique. To this end it is convenient to introduce the following definition.

Let M be a module in mod Λ. We denote the subcategory of ind mod Λ consisting of those indecomposable modules which are isomorphic to summands of M by ind M. Clearly ind M is a finite category.

PROPOSITION 2.5. *Let M be a nonzero module in* mod Λ.

(a) *Let $M \approx \coprod_{i=1}^{n} L_i$ be a covering decomposition. Then for each $i = 1,..., n$ we have that* $\text{ind}\, L_i = \mathbf{P}_0(\text{ind}(M/\coprod_{j<i} L_i))$. *Thus if $M \approx \coprod_{j=1}^{m} L'_j$ is another covering decomposition, then $n = m$ and $L_i \approx L'_i$ for all $i = 1,..., n$.*

(b) *Let $M \approx \coprod_{j=1}^{s} J_j$ be a cocovering decomposition. Then for each $j = 1,..., s$ we have that* ind $J_j = \mathbf{I}_0(\text{ind}(M/\coprod_{i<j} J_i))$. *Thus if $M \approx \coprod_{j=1}^{t} J'_j$ is another cocovering decomposition, then $s = t$ and $J_j \approx J'_j$ for all $j = 1,..., s$.*

Proof. (a) Suppose $M \approx \coprod_{i=1}^{n} L_i$ is a covering decomposition. Since $M/\coprod_{j<i} L_i = \coprod_{k=i}^{n} L_k$, we have to show that $\mathbf{P}_0(\text{ind}(\coprod_{k=i}^{n} L_k)) = \text{ind}\, L_i$. Because $\text{ind}(\coprod_{k=i}^{n} L_k)$ is finite, we have by Corollary 2.4 that it suffices to show that $\text{ind}\, L_i$ is a minimal cover for $\text{ind}(\coprod_{k=i}^{n} L_k)$. Since $\tau_{L_k}(L_{k+1}) = L_{k+1}$ for all $k = 1,..., n - 1$, it follows that $\text{ind}(L_i)$ is a cover for $\text{ind}(\coprod_{i=k}^{n} L_i)$. But $\text{ind}(L_i)$ is a minimal cover for $\text{ind}(\coprod_{k=i}^{n} L_k)$. Suppose L'_1 is the sum of a complete set of nonisomorphic representatives of the objects of a proper subcategory of $\text{ind}(L_i)$. Because L_i is multiplicity free, it follows that L'_i is a proper summand of L_i. Therefore $\tau_{L'_i}(L_i) \neq L_i$ and so $\text{ind}(L'_i)$ is not a cover for $\text{ind}(\coprod_{k=i}^{n} L_k)$. Hence $\text{ind}(L_i)$ is a minimal cover for $\text{ind}(\coprod_{k=i}^{n} L_k)$ and so $\text{ind}(L_i) = \mathbf{P}_0(M/\coprod_{j<i} L_j)$ for all $i = 1,..., n$. This proves the first part of (a).

Since each L_i is multiplicity free, the fact that $\text{ind}(L_i) = \mathbf{P}_0\, \text{ind}(M/\coprod_{j<i} L_j)$ implies that L_i is isomorphic to the sum of any complete set of nonisomorphic representations of objects in $\mathbf{P}_0\, \text{ind}(M/\coprod_{j<i} L_j)$. Hence L_1 is isomorphic to the sum of any complete set of nonisomorphic representatives of objects in $\mathbf{P}_0$ $(\text{ind}(M))$. Thus if $M \approx \coprod_{j=1}^{m} L'_j$ is another covering decomposition for M, then $L_1 \approx L'_1$. Then $M/L_1 \approx M/L'_1$, and so $L_2 \amalg \cdots \amalg L_n$ and $L'_2 \amalg \cdots \amalg L'_m$ are covering decompositions of the same module M/L_1. Hence by induction on n, we have that $n = m$ and $L_i \approx L'_i$ for $i = 2,..., n$. Combining this with the fact that $L_1 \approx L_1$, we have finished the proof of (a).

(b) Dual of (a).

PROPOSITION 2.6. *Let M be a nonzero module in* mod Λ.

(a) *M has a covering decomposition $\coprod_{i=1}^{n} L_i$ which is unique in the sense that the L_i are uniquely determined up to isomorphism by M. Moreover* $\operatorname{ann}(L_i) \subset \operatorname{ann}(L_2) \subset \cdots \subset \operatorname{ann}(L_n)$.

(b) *M has a cocovering decomposition $\coprod_{j=1}^{m} J_j$ which is unique in the sense that the J_j are uniquely determined up to isomorphism by M. Moreover* $\operatorname{ann}(J_1) \subset \operatorname{ann}(J_2) \subset \cdots \subset \operatorname{ann}(J_m)$.

Proof. (a) Suppose $M = \coprod_{i=1}^{t} n_i M_i$ with each M_i indecomposable and $n_i \geqslant 1$. Proceed by induction on $\sum_{i=1}^{t} n_i$. If $\sum_{i=1}^{t} n_i = 1$, then M is indecomposable and $L_1 = M$ is our desired decomposition. Suppose (a) holds for all $M = \coprod_{i=1}^{t} n_i M_i$ with $1 \leqslant \sum_{i=1}^{t} n_i < k$ and suppose $\sum_{i=1}^{t} n_i = k$. Define L_1 to be the multiplicity-free module $\coprod N_j$ where N_j ranges over a complete set of nonisomorphic representatives of the objects in the finite subcategory $\mathbf{P}_0(\operatorname{ind} M)$. Since $\operatorname{ind}(M)$ is finite, we know by Corollary 2.4 that $\mathbf{P}_0(\operatorname{ind} M)$ is a finite minimal cover for ind M. In particular $\mathbf{P}_0(\operatorname{ind} M)$ is not empty so L_1 is not zero. Hence by the inductive hypothesis $M_2 = M/L_1$ has a unique covering decomposition $L_2 \coprod \cdots \coprod L_n$. We now claim that $L_1 \coprod L_2 \coprod \cdots \coprod L_n$ is a covering decomposition for M.

Since $\mathbf{P}_0(\operatorname{ind} M)$ is a minimal cover for ind M, it follows that L_1 has the following properties: (a) for each X in ind M there is a surjective morphism $nL_1 \to X$ for some n in N and (b) if L_1' is a proper summand of L_1, then there is an indecomposable X in ind M such that there is no surjection $nL_1' \to X$ for any n in N. Therefore there is certainly a surjective morphism $nL_1 \to L_2$. Moreover if there were a surjective morphism $nL_1' \to L_1$ with L_1' a proper summand of L_1, then $\operatorname{ind}(L_1')$ would be a cover for ind M since, by Proposition 2.5, $\operatorname{ind}(L_2)$ is a cover for $\operatorname{ind}(L_2 \coprod \cdots \coprod L_n)$. But the fact that L_1' is a proper summand of L_1 which is multiplicity free means that $\operatorname{ind} L_1'$ is a proper subcategory of $\operatorname{ind} L_1$. Therefore we would have that $\operatorname{ind} L_1$ is not a minimal cover for ind M. But by Corollary 2.4 we know that $\mathbf{P}_0(\operatorname{ind} M) = \operatorname{ind} L_1$ is a minimal cover for ind M. This contradiction shows that $\tau_{L_1'}(L_1) \neq L_1$ and so we have that $L_1 \coprod L_2 \coprod \cdots \coprod L_n$ is a covering decomposition of M.

The uniqueness of the L_i was shown in Proposition 2.5.

Since for each $i = 1,..., n-1$ we have that there is a surjection $n_i L_i \to L_{i+1}$, it follows that $\operatorname{ann}(L_i) \subset \operatorname{ann}(L_{i+1})$ for all $i = 1,..., n-1$. This completes the proof of (a).

(b) Dual of (a).

It is worth noting that in case Λ is a proper factor of a principal ideal domain and M is in mod Λ, then the covering and cocovering decompositions of M are the same, namely, the usual invariant factor decomposition of M. This is because modules over such a Λ are self-dual. However, for arbitrary Λ the covering and cocovering decomposition of M are usually not the same.

3. Preprojective Partitions

Let $\mathbf{C}$ be a subcategory of mod Λ. In this section we are primarily concerned with giving our basic existence theorem for when $\mathbf{C}$ has a preprojective partition or a preinjective partition; notions we introduce after the following notation.

Suppose $\mathbf{C}$ is a subcategory of mod Λ and $\mathbf{X}$ a subcategory of $\mathbf{C}$. We denote by $\mathbf{C}_{\mathbf{X}}$ the subcategory of $\mathbf{C}$ consisting of those objects in $\mathbf{C}$ with no summands in $\mathbf{X}$. Clearly $\mathbf{C}_{\mathbf{X}} = \mathbf{C}_{\text{ind}\mathbf{X}}$ and if $\mathbf{C}$ is a subcategory of ind Λ, then $\mathbf{C}_{\mathbf{X}} = \mathbf{C} - \mathbf{X}$.

Let $\mathbf{C}$ be a subcategory of mod Λ. We have already defined $\mathbf{P}_0(\mathbf{C})$ to be the subcategory of ind $\mathbf{C}$ consisting of the splitting projectives in ind $\mathbf{C}$. We define $\mathbf{P}_1(\mathbf{C}) = \mathbf{P}_0(\mathbf{C}_{\mathbf{P}_0(\mathbf{C})})$ and, by induction, $\mathbf{P}_k(\mathbf{C}) = \mathbf{P}_0(\mathbf{C}_{\mathbf{P}_0(\mathbf{C})\cup\cdots\cup\mathbf{P}_{k-1}(\mathbf{C})})$. Finally we denote the subcategory $\bigcup_{i<\infty}\mathbf{P}_i(\mathbf{C})$ of ind $\mathbf{C}$ by $\mathbf{P}(\mathbf{C})$ and ind$(\mathbf{C}_{\mathbf{P}(\mathbf{C})})$ by $\mathbf{P}_\infty(\mathbf{C})$. We now list some easily verified properties of these subcategories of $\mathbf{C}$.

(a) $\mathbf{P}_i(\mathbf{C}) \cap \mathbf{P}_j(\mathbf{C}) = \varnothing$ for all $i, j \leqslant \infty$ if $i \neq j$.

(b) If $\mathbf{P}_i(\mathbf{C}) = \varnothing$, then $\mathbf{P}_j(\mathbf{C}) = \varnothing$ for $i \leqslant j < \infty$.

We say that the collection $\{\mathbf{P}_i(\mathbf{C})\}_{i=0,\ldots,\infty}$ is a *preprojective partition of* $\mathbf{C}$ if $\mathbf{P}_i(\mathbf{C})$ is a finite cover for $\mathbf{C}_{\mathbf{P}_0(\mathbf{C})\cup\cdots\cup\mathbf{P}_{i-1}(\mathbf{C})}$ for each $i < \infty$. We say that $\mathbf{C}$ has a *preprojective partition* if $\{\mathbf{P}_i(\mathbf{C})\}_{i=0,\ldots,\infty}$ is a preprojective partition of $\mathbf{C}$. If $\mathbf{C}$ has a preprojective partition, then we say that C in $\mathbf{C}$ is *preprojective* if every indecomposable summand of C is in $\mathbf{P}(\mathbf{C}) = \bigcup_{i<\infty}\mathbf{P}_i(C)$.

Dually, we have already defined $\mathbf{I}_0(\mathbf{C})$ to be the subcategory of ind $\mathbf{C}$ consisting of the splitting injectives in ind $\mathbf{C}$. By induction we define $\mathbf{I}_k(\mathbf{C}) = \mathbf{I}_0(\mathbf{C}_{\mathbf{I}_0(\mathbf{C})\cup\cdots\cup\mathbf{I}_{k-1}(\mathbf{C})})$. Finally we denote the subcategory $\bigcup_{i<\infty}\mathbf{I}_i(\mathbf{C})$ of ind $\mathbf{C}$ by $\mathbf{I}(\mathbf{C})$ and ind$(\mathbf{C})_{\mathbf{I}(\mathbf{C})}$ by $\mathbf{I}_\infty(\mathbf{C})$. The following properties of these subcategories of $\mathbf{C}$ are easily verified.

(a) $\mathbf{I}_i(\mathbf{C}) \cap \mathbf{I}_j(\mathbf{C}) = \varnothing$ for $i, j \leqslant \infty$ with $i \neq j$.

(b) If $\mathbf{I}_i(\mathbf{C}) = \varnothing$, then $\mathbf{I}_j(\mathbf{C}) = \varnothing$ for $i \leqslant j < \infty$.

We say that the collection $\{\mathbf{I}_i(\mathbf{C})\}_{i=0,\ldots,\infty}$ is a *preinjective partition* of $\mathbf{C}$ if $\mathbf{I}_i(\mathbf{C})$ is a finite cocover for $\mathbf{C}_{\mathbf{I}_0(\mathbf{C})\cup\cdots\cup\mathbf{I}_{i-1}(\mathbf{C})}$ for each $i < \infty$. We say that $\mathbf{C}$ *has a preinjective partition* if $\{\mathbf{I}_i(\mathbf{C})\}_{i=0,\ldots,\infty}$ is a preinjective partition of $\mathbf{C}$. If $\mathbf{C}$ has a preinjective partition, then we say that a C in $\mathbf{C}$ is preinjective if every indecomposable summand of C is in $\mathbf{I}(\mathbf{C}) = \bigcup_{i<\infty}\mathbf{I}_i(\mathbf{C})$.

We have already seen that $D(\mathbf{P}_0(\mathbf{C})) = \mathbf{I}_0(D(\mathbf{C}))$ and $D(\mathbf{I}_0(\mathbf{C})) = \mathbf{P}_0(D(\mathbf{C}))$. Consequently it is easily seen that $D(\mathbf{P}_i(\mathbf{C})) = \mathbf{I}_i(D(\mathbf{C}))$ and $D(\mathbf{I}_i(\mathbf{C})) = \mathbf{P}_i(D(\mathbf{C}))$ for all $i \leqslant \infty$ and $D(\mathbf{P}(\mathbf{C})) = \mathbf{I}(D(\mathbf{C}))$ and $D(\mathbf{I}(\mathbf{C})) = \mathbf{P}(D(\mathbf{C}))$. From these remarks it follows that $\{\mathbf{P}_i(\mathbf{C})\}_{i=0,\ldots,\infty}$ is a preprojective partition of $\mathbf{C}$ if and only if $\{\mathbf{I}_i(D(\mathbf{C}))\}_{i=0,\ldots,\infty}$ is a preinjective partition of $D(\mathbf{C})$.

Before stating our general criterion for when a subcategory $\mathbf{C}$ of mod Λ has a preprojective or a preinjective partition, we give the following result due to

Gabriel and Roiter [12, 15] which was also the starting point of our own investigations.

THEOREM 3.1. *Let* **C** *be a subcategory of* mod Λ *of finite type* (*i.e.*, ind **C** *is finite*). *Then* **C** *has both preprojective and preinjective partitions.*

Proof. Since ind **C** is finite, ind **X** is finite for all subcategories **X** of **C**. Hence by Corollary 2.4, we know that $\mathbf{P}_0(\mathbf{X})$ is a finite cover for **X** for each subcategory **X** of **C**. Since by definition, $\mathbf{P}_i(\mathbf{C}) = \mathbf{P}_0(\mathbf{C}_{\mathbf{P}_0(\mathbf{C})\cup\cdots\cup\mathbf{P}_{i-1}(\mathbf{C})})$, it follows that each $\mathbf{P}_i(\mathbf{C})$ is a finite cover for $\mathbf{C}_{\mathbf{P}_0(\mathbf{C})\cup\cdots\cup\mathbf{P}_{i-1}(\mathbf{C})}$ for all $i < \infty$. Hence $\{\mathbf{P}_i(\mathbf{C})\}_{i=0,\ldots,\infty}$ is a preprojective partition of **C** which shows that **C** has a preprojective partition. That **C** also has a preinjective partition follows by duality.

In connection with Theorem 3.1 we point out the following relationship between the preprojective and preinjective partitions of subcategories of mod Λ of finite type and the covering and cocovering decompositions of modules introduced in Section 2.

PROPOSITION 3.2. *Let* **C** *be a subcategory of* mod Λ *of finite type. Let* $C_1, \ldots, C_t$ *be a complete set of nonisomorphic representatives of objects in* ind **C** *and let* $C = \coprod_{i=1}^{t} C_i$.

(a) *If* $C \approx \coprod_{i=1}^{r} L_i$ *is a covering decomposition for* C, *then* $\mathbf{P}_i(\mathbf{C}) =$ ind L_i *for* $i = 1,\ldots, r$ *and* $\mathbf{P}_j(\mathbf{C}) = \varnothing$ *for* $j > r$.

(b) *If* $C \approx \coprod_{i=1}^{s} J_i$ *is a cocovering decomposition of* C, *then* ind $J_j = \mathbf{I}_j(\mathbf{C})$ *for* $j = 1,\ldots, s$ *and* $\mathbf{I}_i(\mathbf{C}) = \varnothing$ *for* $i > s$.

Proof. Left as an exercise.

We now state our main criterion for when a subcategory **C** of mod Λ has a preprojective partition or a preinjective partition.

THEOREM 3.3. *Let* **C** *be a subcategory of* mod Λ.

(a) *Suppose for each* M *in* mod Λ *there is a morphism* $g\colon M \to C$ *with* C *in* add **C** *such that* $(g, X)\colon (C, X) \to (M, X)$ *is surjective for all* X *in* **C**, *where* $(C, X) = \operatorname{Hom}_\Lambda(C, X)$. *Then* **C** *has a preprojective partition.*

(b) *Suppose for each* M *in* mod Λ *there is a morphism* $f\colon C \to M$ *with* C *in* add **C** *such that* $(X, f)\colon (X, C) \to (X, M)$ *is surjective for all* X *in* **C**, *where* $(X, C) = \operatorname{Hom}_\Lambda(X, C)$. *Then* **C** *has a preinjective partition.*

The proof of Theorem 3.3 will occupy the rest of this section. We begin with the following lemmas which, when suitably specialized, give a criterion for when a subcategory **C** of mod Λ has a finite cocover. This criterion is especially well suited to proving Theorem 3.3.

LEMMA 3.4. *Let* $\mathbf{D} \supset \mathbf{C}$ *be subcategories of* mod Λ *and assume that* $\mathbf{I}_0(\mathbf{D})$ *is a* *finite cocover for* **D**. *Then a finite subcategory* **A** *of* ind **C** *is a cocover for* **C** *if for each* I *in* $\mathbf{I}_0(\mathbf{D})$ *there is a morphism* $f_I: A_I \to I$ *with* A_I *in* add **A** *such that* (X, f_I): $(X, A_I) \to (X, I)$ *is surjective for all* X *in* **C**. *Moreover if* $\mathbf{D} = \operatorname{mod} \Lambda$ *and* **A** *in* ind Λ *is a finite cocover for* **C**, *then there exists for each* I *in* $\mathbf{I}_0(\operatorname{mod} \Lambda)$ *a morphism* $f_I: A_I \to I$ *with* A_I *in* add A *such that* (X, f_I): $(X, A_I) \to (X, I)$ *is surjective for all* X *in* **C**.

Proof. Suppose that for each I in $\mathbf{I}_0(\mathbf{D})$ there exists a morphism $f_I: A_I \to I$ with A_I in add **A** such that (X, f_I): $(X, A_I) \to (X, I)$ is surjective for all X in **C**. Let X be in **C** and let $g: X \to I$ be an injective morphism with I in add $\mathbf{I}_0(\mathbf{D})$. Then there is an $h: X \to A_I$ such that $g = f_I h$. Therefore $h: X \to A_I$ is injective with A_I in add **A**. Since X is an arbitrary object in **C**, we have shown that **A** is a cocover for **C**. Now let $\mathbf{D} = \operatorname{mod} \Lambda$ and assume that **A** is a finite cocover for **C**. Let $A_1, \ldots, A_n$ be a complete set of nonisomorphic objects in **A**. Let I be in $\mathbf{I}_0(\operatorname{mod} \Lambda)$. Since (A_i, I) is a finitely generated End A^{op}-module, there is for each $i = 1, \ldots, n$ a morphism $f_i: n_i A_i \to I$ such that (A_i, f): $(A_i, n_i A_i) \to (A_i, I)$ is surjective. Therefore the induced morphism $f: \coprod n_i A_i \to I$ has the property that (A, f): $(A, \coprod n_i A_i) \to (A, I)$ is surjective for each A in **A**. Thus we have shown that for each indecomposable module I in $\mathbf{I}_0(\mathbf{D})$ there is a morphism $f_I: A_I \to I$ with A_I in add **A** such that (A, f_I): $(A, A_I) \to (A, I)$ is exact for each A in **A** and hence for each A in add **A**.

Let C be in **C**. Since we are assuming that **A** is a cocover for **C**, we know that there is a monomorphism: $g: C \to A$ with A in add **A**. Suppose $h: C \to I$ is a morphism with I in $\mathbf{I}_0$ (mod Λ). Then there is a $j: A \to I$ such that $h = jg$. Since A is in add **A**, we know by our previous remarks that there is $t: A \to A_I$ such that $j = f_I t$. Therefore $h = f_I t g$. Because tg is in (C, A_I) we have that h is in the image of (C, f_I): $(C, A_I) \to (C, I)$. Therefore (C, f_I): $(C, A_I) \to (C, I)$ is surjective for all C in **C**. This finishes the proof of the lemma.

Assume that **C** is a subcategory of mod Λ which has a finite cocover. As a consequence of Lemma 3.4 we have the following description of the minimal cocover $\mathbf{I}_0(\mathbf{C})$ of the subcategory **C**.

LEMMA 3.5. *Suppose* **C** *is a subcategory of* mod Λ *which has a finite cocover. Then the following hold for a finite cocover* **A** *of* **C**.

(a) *For each indecomposable injective* I *there is a unique* (*up to isomorphism*) *right minimal morphism* $f_I: A_I \to I$ *with* A_I *in* add **A** *such that* (X, f_I): $(X, A_I) \to (X, I)$ *is surjective for all* X *in* **C**.

(b) $\mathbf{I}_0(\mathbf{C}) = \bigcup_I$ ind A_I *as* I *runs through a complete set of nonisomorphic objects in* $\mathbf{I}_0$ (mod Λ).

Proof. (a) Since **A** is a finite cocover for **C**, we know by Lemma 3.4 that for each I in $\mathbf{I}_0$ (mod Λ) there is a morphism f': $A_I \to I$ with A_I in add **A** such that (X, f'): $(X, A_I) \to (X, I)$ is surjective for all X in **C**. Then by Proposition

1.3 we know there are unique (up to isomorphism) right minimal morphisms f_I: $A_I \to I$ with A_I isomorphic to a summand of A_I such that (X, f_1): $(X, A_I) \to (X, I)$ is surjective for all X in add $\mathbf{A}$. Since the A'_I are summands of the A_I, it follows that the A_I are in add $\mathbf{A}$ which finishes the proof of (a).

(b) Since the f_I: $A_I \to I$ have the property that (X, f_I): $(X, A_I) \to (X, I)$ is surjective for all X in $\mathbf{C}$, it follows from Lemma 3.4, that $\bigcup_I \operatorname{ind}(A_I)$, as I runs through a complete set of nonisomorphic objects in $\mathbf{I}_0$ (mod Λ), is a finite cocover for $\mathbf{C}$. Hence $\mathbf{I}_0(\mathbf{C}) \subset \bigcup_I \operatorname{ind}(A_I)$ by Proposition 2.1. We now show that $\mathbf{I}_0(\mathbf{C}) \supset \bigcup_I \operatorname{ind}(A_I)$ by showing that $\mathbf{I}_0(\mathbf{C}) \supset \operatorname{ind}(A_I)$ for each indecomposable injective I.

Since $\mathbf{C}$ has a finite cocover, we know by Corollary 2.4 that $\mathbf{I}_0(\mathbf{C})$ is the minimal cocover of $\mathbf{C}$. Therefore by Lemma 3.4 we know that given an I in $\mathbf{I}_0$ (mod Λ) there is a morphism g: $C_I \to I$ with C_I in $\mathbf{I}_0(\mathbf{C})$ such that (X, g) is surjective for all X in $\mathbf{C}$. In particular, there is an h: $A_I \to C_I$ such that $f_I = gh$. On the other hand there is an h': $C_I \to A_I$ such that $g = f_I h'$. Hence $f_I h' h = gh = f_I$ so that $h'h$: $A_I \to A_I$ is an isomorphism since f_I is right minimal. Thus A_I is isomorphic to a summand of C_I and so $\operatorname{ind} A_I \subset \mathbf{I}_0(\mathbf{C})$. Therefore we have our desired result $\mathbf{I}_0(\mathbf{C}) = \bigcup_I \operatorname{ind} A_I$ as I runs through a complete set of nonisomorphic objects in $\mathbf{I}_0(\text{mod } \Lambda)$.

Summarizing Lemmas 3.4 and 3.5 we have the following result.

PROPOSITION 3.6. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ. *Then the following statements are equivalent.*

(a) $\mathbf{C}$ *has a finite cocover.*

(b) *For each indecomposable injective module* I *there is a morphism* f_I: $A_I \to I$ *with* A_I *in* add $\mathbf{C}$ *such that* (C, f_I) *is surjective for all* C *in* $\mathbf{C}$.

(c) *For each indecomposable injective module* I *there is a unique (up to isomorphism) right minimal* f_I: $A_I \to I$ *with* A_I *in* add $\mathbf{C}$ *such that* (C, f_I) *is surjective for all* C *in* $\mathbf{C}$.

Moreover, if $\mathbf{C}$ *has a finite cocover, then* $\mathbf{I}_0(\mathbf{C})$, *its minimal cocover, can be described as follows. Let* $I_1, \ldots, I_t$ *be a complete set of nonisomorphic indecomposable injective modules and let* f_j: $A_j \to I_j$ *be the unique right minimal morphisms with* A_j *in* add $\mathbf{C}$ *such that* (C, f_j) *is surjective for all* C *in* $\mathbf{C}$. *Then* $\mathbf{I}_0(\mathbf{C}) = \bigcup_{j=1}^{t} \operatorname{ind} A_j$.

Dualizing Proposition 3.6 we obtain.

PROPOSITION 3.7. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ. *Then the following are equivalent.*

(a) $\mathbf{C}$ *has a finite cover.*

(b) *For each indecomposable projective module* P *there is a morphism* g_P: $P \to A_P$ *with* A_P *in* add $\mathbf{C}$ *such that* (g_P, C) *is surjective for all* C *in* $\mathbf{C}$.

(c) *For each indecomposable projective module P there is a unique (up to isomorphism) left minimal morphism $g_P\colon P \to A_P$ with A_P in* add $\mathbf{C}$ *such that* (g_P, C) *is surjective for all C in* $\mathbf{C}$.

Moreover, if $\mathbf{C}$ *has a finite cover, then* $\mathbf{P}_0(\mathbf{C})$, *its minimal cover, can be described as follows. Let $P_1, \ldots, P_t$ be a complete set of nonisomorphic indecomposable projective modules and let $g_j\colon P_j \to A_j$ be the unique left minimal morphisms with A_j in* add $\mathbf{C}$ *such that (g_j, C) is surjective for all C in* $\mathbf{C}$. *Then* $\mathbf{P}_0(\mathbf{C}) = \bigcup_{j=1}^{s} \operatorname{ind} A_j$.

Theorem 3.3, the result we want to prove, states: Suppose $\mathbf{C}$ is a subcategory of mod Λ with the property that for each M in mod Λ there is a C in $\mathbf{C}$ and a morphism $f\colon C \to M$ such that (X, f) is surjective for all X in $\mathbf{C}$. Then $\mathbf{C}$ has a preinjective partition. It obviously follows from Proposition 3.6 that $\mathbf{I}_0(\mathbf{C})$ is a finite cocover for any subcategory $\mathbf{C}$ of mod Λ satisfying the hypothesis of the theorem. Suppose we can show that $\mathbf{C}$ satisfying the hypothesis of Theorem 3.3 implies $\mathbf{C}_{\mathbf{I}_0(\mathbf{C})}$ also does. Then, proceeding by induction on n, it would follow trivially that $\mathbf{I}_n(\mathbf{C})$ is a finite cocover for $\mathbf{C}_{\mathbf{I}_0(\mathbf{C}) \cup \cdots \cup \mathbf{I}_{n-1}(\mathbf{C})}$ for all n, which is simply a restatement of Theorem 3.3. Hence to finish the proof of Theorem 3.3 it remains to show that if $\mathbf{C}$ is a subcategory of mod Λ satisfying the hypothesis of Theorem 3.3, then $\mathbf{C}_{\mathbf{I}_0(\mathbf{C})}$ also does. This will be a consequence of some of the following considerations.

We begin by recalling some of the basic facts concerning the radical of a category that we shall need in this section. Suppose $\mathbf{C}$ is a subcategory of mod Λ. For each A and B in $\mathbf{C}$ the radical of (A, B), denoted by $\mathrm{r}(A, B)$ consists of all $f\colon A \to B$ such that for each Z in ind C, every composition $Z \to A \to B \to Z$ is in rad End Z or, equivalently, is not an isomorphism. The radical of $\mathbf{C}$, denoted by $\mathrm{r}(\mathbf{C})$, consists of all the morphisms in $\mathrm{r}(A, B)$ for some A and B in $\mathbf{C}$. Further, for each integer $n \geqslant 0$, we define $\mathrm{r}^n(A, B)$ to consist of all morphisms $f\colon A \to B$ which can be written as a sum $\sum f_i$ where each f_i is a composition of at least n morphisms in $\mathrm{r}(\mathbf{C})$. We denote by $\mathrm{r}^n(\mathbf{C})$ the collection of all morphisms in $\mathrm{r}^n(A, B)$ for some A and B in $\mathbf{C}$. We now list some easily verified properties of $\mathrm{r}^n(A, B)$ for all integers $n \geqslant 0$ and A and B in $\mathbf{C}$.

(a) $\mathrm{r}^n(A, B)$ is a subgroup of (A, B).

(b) $\mathrm{r}^0(A, B) = (A, B)$, $\mathrm{r}^1(A, B) = \mathrm{r}(A, B)$, and $\mathrm{r}^n(A, B) \supset \mathrm{r}^{n+1}(A, B)$.

(c) The composition $U \to A \xrightarrow{f} B \to V$ is in $\mathrm{r}^n(U, V)$ if f is in $\mathrm{r}^n(A, B)$.

(d) $\mathrm{r}^n(A, B)$ is functorial in A and B so that $\mathrm{r}^n(\coprod_{i\in I} A_i, B) = \coprod_{i \in I} \mathrm{r}^n(A_i, B)$ and $\mathrm{r}^n(A, \coprod_{j\in J} B_j) = \coprod_{j\in J} \mathrm{r}^n(A, B_j)$ for all finite families of modules $\{A_i\}_{i\in I}$ and $\{B_j\}_{j\in J}$ in $\mathbf{C}$.

(e) $\mathrm{r}^n(A, B)$ is the same whether we view A and B as objects in $\mathbf{C}$ or add $\mathbf{C}$.

(f) $\mathrm{r}^n(A, A) \supset (\operatorname{rad} \operatorname{End} A)^n$.

(g) If f is in $r^i(A, B)$ and g is in $r^j(B, C)$, then gf is in $r^{i+j}(A, C)$.

(h) $r(A, B) = (A, B)$ if and only if no indecomposable summand of A is isomorphic to an indecomposable summand of B.

In connection with this last property we point out the following easily deduced facts.

LEMMA 3.8. *Suppose* $\mathbf{C}$ *is a subcategory of* mod Λ *and* B *is in* ind $\mathbf{C}$.

(a) *For a morphism* $f: A \to B$ *in* $\mathbf{C}$ *the following are equivalent*:

(i) f *is not a splittable surjection*,

(ii) f *is in* $r(A, B)$,

(iii) *for each* X *in* $\mathbf{C}$, *the induced morhism* $(X, f): (X, A) \to (X, B)$ *has* $\operatorname{Im}(X, f) \subset r(X, B)$.

(b) *For a morphism* $g: B \to C$ *in* $\mathbf{C}$, *the following are equivalent*:

(i) g *is not a splittable injection*,

(ii) g *is in* $r(B, C)$,

(iii) *for each* Y *in* $\mathbf{C}$, *the induced morphism* $(g, Y): (C, Y) \to (B, Y)$ *has* $\operatorname{Im}(g, Y) \subset r(B, Y)$.

Lemma 3.8 naturally raises the question: given a C in ind $\mathbf{C}$, is there a morphism $f: B \to C$ in add $\mathbf{C}$ such that $\operatorname{Im}((X, f): (X, B) \to (X, C)) = r(X, C)$ for all X in add $\mathbf{C}$ or, equivalently, for all X in ind $\mathbf{C}$? If $\mathbf{C} = \operatorname{mod} \Lambda$, it was shown in [6] that such an $f: B \to C$ always exists and it was called a right almost split morphism.

Similarly, one can ask: given an indecomposable B in ind $\mathbf{C}$, is there a g: $B \to C$ in add $\mathbf{C}$ such that $\operatorname{Im}((g, Y): (C, Y) \to (B, Y)) = r(B, Y)$ for all Y in add $\mathbf{C}$ or, equivalently, for all Y in ind $\mathbf{C}$? Again, if $\mathbf{C} = \operatorname{mod} \Lambda$, it was shown in [6] that such a $g: B \to C$ always exists and it was called a left almost split morphism.

These remarks suggest the following terminology.

Let $\mathbf{C}$ be a subcategory of mod Λ and let C be in ind $\mathbf{C}$. A morphism $B \to C$ in add $\mathbf{C}$ is said to be *right almost split* in $\mathbf{C}$ if for each X in add $\mathbf{C}$ we have that $\operatorname{Im}((X, f): (X, B) \to (X, C)) = r(X, C)$. Clearly if $f: B \to C$ is right almost split, then f is in $r(B, C)$. We say that $\mathbf{C}$ *has right almost split morphisms* if for each indecomposable C in ind $\mathbf{C}$ there is a right almost split morphism $f: B \to C$. It is easy to see that $\mathbf{C}$ has right almost split morphisms if and only if given any C in $\mathbf{C}$ (not necessarily indecomposable) there is a morphism $f: B \to C$ in add $\mathbf{C}$ such that $\operatorname{Im}(X, f) = r(X, C)$ for all X in add $\mathbf{C}$. Clearly such an f: $B \to C$ is in $r(B, C)$.

Similarly, if B is in ind $\mathbf{C}$ a morphism $g: B \to C$ in add $\mathbf{C}$ is said to be *left almost split* in $\mathbf{C}$ if for each Y in add $\mathbf{C}$ we have that $\operatorname{Im}((g, Y): (C, Y) \to (B, Y)) = r(B, Y)$. We say that $\mathbf{C}$ *has left almost split morphisms* if for each indecomposable B in ind $\mathbf{C}$ there is a left almost split morphism $f: B \to C$. It is easy to see that $\mathbf{C}$ has left almost split morphisms if and only if given any B in $\mathbf{C}$ (not necessarily indecomposable) there is a $g: B \to C$ in add $\mathbf{C}$ such that $\operatorname{Im}(g, Y) = r(B, Y)$ for all Y in add $\mathbf{C}$. Clearly such a $g: B \to C$ is in $r(B, C)$.

The fact that certain subcategories $\mathbf{C}$ of mod Λ have right and/or left almost split morphisms plays an important role in our proof of Theorem 3.3. While we do not have a complete description of such subcategories, we do have a sufficient condition which is based on the following notion.

Let $\mathbf{C} \supset \mathbf{D}$ be subcategories of mod Λ. We say that a module C in $\mathbf{C}$ is *contravariantly finite over* $\mathbf{D}$ if there is a morphism $f: D \to C$ with D in add $\mathbf{D}$ such that $(X, f): (X, D) \to (X, C)$ is surjective for all X in add $\mathbf{D}$. We say that C is *covariantly finite over* $\mathbf{D}$ if there is a morphism $g: C \to D$ with D in add $\mathbf{D}$ such that $(g, X): (D, X) \to (C, X)$ is surjective for all X in add $\mathbf{D}$. It is not hard to see that a C in $\mathbf{C}$ is contravariantly (covariantly) finite over $\mathbf{D}$ if and only if every indecomposable summand of C is contravariantly (covariantly) finite over $\mathbf{D}$.

The reason for this terminology is the following. Let C be in $\mathbf{C}$ and let $(\ , C) \mid \mathbf{D}$ denote the restriction of the representable functor $(\ , C)$ to $\mathbf{D}$. By definition, the functor $(\ , C) \mid \mathbf{D}$ is finitely generated if and only if there is an epimorphism $\alpha: (\ , D) \to (\ , C) \mid \mathbf{D}$ with D in add $\mathbf{D}$. But every morphism $\alpha: (\ , D) \to (\ , C) \mid \mathbf{D}$ is of the form $(\ , f): (\ , D) \to (\ , C) \mid \mathbf{D}$ for some morphism $f: D \to C$ since D is in add $\mathbf{D}$. Thus the condition that there exists a morphism $f: D \to C$ such that $(X, f): (X, D) \to (X, C)$ is surjective for all X in add $\mathbf{D}$ is equivalent to $(\ , C) \mid \mathbf{D}$ being finitely generated over $\mathbf{D}$. Hence the terminology C is contravariantly finite over $\mathbf{D}$. Similar remarks hold for C being covariantly finite over $\mathbf{D}$. Moreover it should be observed that C is contravariantly finite over $\mathbf{D}$ if and only if $D(C)$ in $D(\mathbf{C})$ is covariantly finite over $D(\mathbf{D})$.

Familiar arguments, based on Section 1, give the following result (see Propositions 1.2 and 1.3.)

PROPOSITION 3.9. *Let* $\mathbf{C} \supset \mathbf{D}$ *be subcategories of* mod Λ.

(a) *C in $\mathbf{C}$ is contravariantly finite over $\mathbf{D}$ if and only if there is a right minimal morphism $f: D \to C$ with D in* add $\mathbf{D}$ *having the property that (X, f): $(X, D) \to (X, C)$ is surjective for all X in* add D. *Moreover such an $f: D \to C$ is unique (up to isomorphism) and we denote by $f_C: C_{\mathbf{D}} \to C$ one particular such morphism chosen once and for all.*

(b) *C in $\mathbf{C}$ is covariantly finite over $\mathbf{D}$ if and only if there is a left minimal morphism $g: C \to D$ with D in* add $\mathbf{D}$ *which has the property that $(g, X): (D, X) \to (C, X)$ is surjective for all X in* add $\mathbf{D}$. *Moreover such a $g: C \to D$ is unique (up to isomorphism) and we denote by $g^C: C \to C^{\mathbf{D}}$ one particular such morphism chosen once and for all.*

Globalizing these remarks we have the following.

Let $\mathbf{C} \supset \mathbf{D}$ be subcategories of $\operatorname{mod} \Lambda$. We say that $\mathbf{C}$ is *contravariantly* (covariantly) *finite over* $\mathbf{D}$ if every C in $\mathbf{C}$ is contravariantly (covariantly) finite over $\mathbf{D}$. Finally we say that $\mathbf{C}$ is functorially finite over $\mathbf{D}$ if it is both contravariantly and covariantly finite over $\mathbf{D}$. The reader should observe that Theorem 3.3 can be restated as follows using this new terminology. Let $\mathbf{C}$ be a subcategory of $\operatorname{mod} \Lambda$. Then (a) $\mathbf{C}$ has a preprojective partition if $\operatorname{mod} \Lambda$ is covariantly finite over $\mathbf{C}$ and b) $\mathbf{C}$ has a preinjective partition if $\operatorname{mod} \Lambda$ is contravariantly finite over $\mathbf{C}$.

Returning to the question of when a subcategory $\mathbf{C}$ of $\operatorname{mod} \Lambda$ has right and/or left almost split morphisms, we have the following result.

PROPOSITION 3.10. *Suppose* $\mathbf{C} \supset \mathbf{D}$ *are subcategories of* $\operatorname{mod} \Lambda$.

(a) *If* $\mathbf{C}$ *has right almost split morphisms and is contravariantly finite over* $\mathbf{D}$, *then* $\mathbf{D}$ *has right almost split morphisms.*

(b) *If* $\mathbf{C}$ *has left almost split morphisms and is covariantly finite over* $\mathbf{D}$, *then* $\mathbf{D}$ *has left almost split morphisms.*

Proof. (a) Let D be an indecomposable object in $\mathbf{D}$ and $f: C \to D$ a right almost split morphism in $\mathbf{C}$, i.e., $(X, f): (X, C) \to (X, D)$ has image $\mathrm{r}(X, D)$ for all X in $\mathbf{C}$. Since C is in add $\mathbf{C}$ and $\mathbf{C}$ is contravariantly finite over $\mathbf{D}$, there is a morphism $f_C: \mathbf{C}_{\mathbf{D}} \to C$ with $C_{\mathbf{D}}$ in add $\mathbf{D}$ such that $\operatorname{Im}(X, f_C) = (X, C)$ for all X in $\mathbf{D}$. Consequently the composition $C_{\mathbf{D}} \to C \to D$ which we denote by $g: C_{\mathbf{D}} \to D$ has the property $\operatorname{Im}(X, g) = \mathrm{r}(X, D)$ for all X in add $\mathbf{D}$. Therefore $g: C_{\mathbf{D}} \to D$ is a right almost aplit morphism in $\mathbf{D}$. Since D was an arbitrary indecomposable in $\mathbf{D}$, this shows that $\mathbf{D}$ has right almost split morphisms.

(b) Dual of (a).

Letting $\mathbf{C} = \operatorname{mod} \Lambda$ in Proposition 3.10 and recalling Propositions 3.6 and 3.7 we obtain the following step toward proving Theorem 3.3.

LEMMA 3.11. *Let* $\mathbf{C}$ *be a subcategory of* $\operatorname{mod} \Lambda$.

(a) *Suppose* $\operatorname{mod} \Lambda$ *is contravariantly finite over* $\mathbf{C}$. *Then* $\mathbf{C}$ *has a finite cocover and right almost split morphisms.*

(b) *Suppose* $\operatorname{mod} \Lambda$ *is covariantly finite over* $\mathbf{C}$. *Then* $\mathbf{C}$ *has a finite cover and left almost split morphisms.*

In order to give our next result which, combined with Lemma 3.11, will finish the proof of Theorem 3.3, it is convenient to make the following definitions.

Let $\mathbf{C}$ be a subcategory of $\operatorname{mod} \Lambda$. A C in $\mathbf{C}$ is said to be *contravariantly nilpotent* if there is an integer n such that $\mathrm{r}^n(\ , C) = 0$, i.e., $\mathrm{r}^n(X, C) = 0$ for all X in $\mathbf{C}$. $\mathbf{C}$ is said to be *locally contravariantly nilpotent* if each C in $\mathbf{C}$ is contravariantly nilpotent. $\mathbf{C}$ is said to be *contravariantly nilpotent* if there is an n such that $\mathrm{r}^n(\ , C) = 0$ for all C in $\mathbf{C}$. Obviously if $\mathbf{C}$ is contravariantly nilpotent it is locally contravariantly nilpotent, but the converse is not necessarily true.

Dually, a C in $\mathbf{C}$ is said to be *covariantly nilpotent* if there is an n such that $\mathrm{r}^n(C, \ ) = 0$. $\mathbf{C}$ is said to be *locally covariantly nilpotent* if each C in $\mathbf{C}$ is covariantly nilpotent. $\mathbf{C}$ is said to be *covariantly nilpotent* if there is an integer n such that $\mathrm{r}^n(C, \ ) = 0$ for all C in $\mathbf{C}$. Obviously if $\mathbf{C}$ is covariantly nilpotent, it is locally covariantly nilpotent, but the converse is not necessarily true.

Finally, we say that $\mathbf{C}$ is nilpotent if it is both covariantly and contravariantly nilpotent. Obviously $\mathbf{C}$ is nilpotent if and only if there is an integer n such that $\mathrm{r}^n(C_1, C_2) = 0$ for all C_1, C_2 in $\mathbf{C}$. If $\mathbf{C}$ is of finite type, then the fact that the radical of the endomorphism ring of the sum of a complete set of nonisomorphic indecomposable modules in ind Λ is nilpotent implies that $\mathbf{C}$ is nilpotent. More generally, we have the following result of Harada and Sai [13].

LEMMA 3.12. *A subcategory* $\mathbf{C}$ *of* mod Λ *is nilpotent if it is bounded, i.e., there is an integer n such that the length of each indecomposable module in* $\mathbf{C}$ *is at most n.*

Returning to the proof of Theorem 3.3 we have the following result.

PROPOSITION 3.13. *Let* $\mathbf{C} \supset \mathbf{D}$ *be subcategories of* mod Λ.

(a) *Suppose* $\mathbf{C}$ *has right almost split morphisms and* $\mathbf{C}_\mathbf{D}$ *is locally contravariantly nilpotent. Then* $\mathbf{C}$ *is contravariantly finite over* $\mathbf{D}$.

(b) *Suppose* $\mathbf{C}$ *has left almost split morphisms and* $\mathbf{C}_\mathbf{D}$ *is locally covariantly nilpotent. Then* $\mathbf{C}$ *is covariantly finite over* $\mathbf{D}$.

(c) *If* $\mathbf{C}$ *has right and left almost split morphisms and* $\mathbf{C}_\mathbf{D}$ *is nilpotent, then* $\mathbf{C}$ *is functorially finite over* $\mathbf{D}$.

Proof. Since (c) is a trivial consequence of (a) and (b), and (b) is the dual of (a), it suffices to prove (a).

(a) Suppose we can show that $\mathbf{C}$ having right almost split morphisms implies that for each C in $\mathbf{C}_\mathbf{D}$ and each integer $n \geqslant 0$ there is a morphism $f: D \coprod Y \to C$ with Y in add $\mathbf{C}_\mathbf{D}$, D in add $\mathbf{D}$, and $f \mid Y$ in $\mathrm{r}^n(Y, C)$ such that $(X, f): (X, D \coprod Y) \to (X, C)$ is surjective for all X in add $\mathbf{D}$. Since we are assuming that C is contravariantly nilpotent, there is an n such that $\mathrm{r}^n(Y, C) = 0$ for all Y in $\mathbf{C}_\mathbf{D}$. Therefore for this value of n we have that there is an $f: D \to C$ with D in add $\mathbf{D}$ such that $(X, f): (X, D) \to (X, C)$ is surjective for all X in add $\mathbf{D}$. Hence we would have that each C in $\mathbf{C}_\mathbf{D}$ is covariantly finite over $\mathbf{D}$. Since each D in $\mathbf{D}$ is obviously covariantly finite over $\mathbf{D}$, we would have shown that $\mathbf{C}$ is covariantly finite over $\mathbf{D}$, under the hypothesis of the proposition. This would prove the proposition. Hence Proposition 3.13 follows from the following.

LEMMA 3.14. *Suppose* $\mathbf{C} \supset \mathbf{D}$ *are subcategories of* mod Λ.

(a) *Suppose* **C** *has right almost split morphisms. For each* C *in* $\mathbf{C_D}$ *and integer* $n \geqslant 0$, *there is a morphism* $f: D \coprod Y \to C$ *with* D *in* add **D** *and* Y *in* add $\mathbf{C_D}$ *such that* $f \mid Y$ *is in* $\mathrm{r}^n(Y, C)$ *and* $\mathrm{Im}(X, f) = (X, C)$ *for all* X *in* **D**.

(b) *Suppose* **C** *has left almost split morphisms. For each* C *in* $\mathbf{C_D}$ *and integer* $n \geqslant 0$, *there is a morphism* $g: C \to D \coprod Z$ *with* D *in* add **D** *and* Z *in* add $\mathbf{C_D}$ *such that* pg *is in* $\mathrm{r}^n(C, Z)$ *and* $\mathrm{Im}(g, X) = (C, X)$ *for all* X *in* **D**, *where* $p: D \coprod Z \to Z$ *is the projection morphism.*

Proof. Proceed by induction on n. For $n = 0, f = id_C$ works. Suppose $k > 0$ and the lemma is true for $0 \leqslant n < k$. Let C be in $\mathbf{C_D}$. By the induction hypothesis we know there is a morphism $f': D' \coprod Y' \to C$ with D' in add **D**, Y' in add $\mathbf{C_D}$ such that $f' \mid Y'$ is in $\mathrm{r}^{k-1}(Y', C)$ and $\mathrm{Im}(X, f') = (X, C)$ for all X in **D**. Since **C** has right almost split morphisms, we know that there is a morphism $g: Z \to Y'$ such that $\mathrm{Im}(X, g) = \mathrm{r}(X, Y')$ for all X in **C**. Since Y' is in add $\mathbf{C_D}$, no indecomposable summand of Y' is in **D**. Hence $\mathrm{r}(X, Y') = (X, Y')$ for all X in add **D** and so $\mathrm{Im}(X, g) = (X, Y')$ for all X in **D**. Thus the composition $D' \coprod Z \to^{(id_{D'}, g)} D' \coprod Y' \to^{f'} C$ which we denote by h has the property $\mathrm{Im}(X, h) = (X, C)$ for all X in add **D**.

Write Z as a sum $D'' \coprod Y''$ where D'' is in add **D** and Y'' is in add $\mathbf{C_D}$. Then $h \mid Y''$ is in $\mathrm{r}^k(Y'', C)$ since $h \mid Y''$ is the composition of $g \mid Y''$ which is in $\mathrm{r}(Y'', Y')$ and $f \mid Y'$ which is in $\mathrm{r}^{k-1}(Y', C)$. Therefore the morphism $h: D' \coprod D'' \coprod Y'' \to C$ is our desired morphism since $D' \coprod D''$ is in add **D**, Y'' is in add $\mathbf{C_D}$, $hh \mid Y''$ is in $\mathrm{r}^k(Y'', C)$, and $\mathrm{Im}(X, h) = (X, C)$ for all X in add **D**.

(b) Dual of (a).

We are now ready to prove Theorem 3.3. To do this we must show that if mod Λ is contravariantly finite over **C**, then **C** has a preinjective partition. Since mod Λ is contravariantly finite over **C**, we know by Lemma 3.11 that **C** has a finite cocover and right almost split morphisms. Therefore $\mathbf{I}_0(\mathbf{C})$ is the finite minimal cocover for **C**. Since **C** also has right almost split morphisms and $\mathbf{I}_0(\mathbf{C})$ is finite and hence nilpotent, it follows from Proposition 3.13 that **C** is contravariantly finite over $\mathbf{C}_{\mathbf{I}_0(\mathbf{C})}$. Therefore by Proposition 3.6, $\mathbf{C}_{\mathbf{I}_0(\mathbf{C})}$ has a finite cocover and so $\mathbf{I}_1(\mathbf{C}) = \mathbf{I}_0(\mathbf{C}_{\mathbf{I}_0(\mathbf{C})})$ is a finite cocover for $\mathbf{C}_{\mathbf{I}_0(\mathbf{C})}$. The fact that $\mathbf{I}_j(\mathbf{C})$ is a finite cocover for $\mathbf{C}_{\mathbf{I}_0(\mathbf{C}) \cup \cdots \cup \mathbf{I}_{j-1}(\mathbf{C})}$ for all $j \geqslant 0$ now follows easily by induction on j, proving part (b) of Theorem 3.3. The rest of Theorem 3.3 follows by duality.

4. Subcategories over Which mod Λ Is Functorially Finite

This section is devoted to giving various criteria for mod Λ to be covariantly or contravariantly finite over **C**. In view of Theorem 3.3 such subcategories automatically have preprojective or preinjective partitions depending on whether mod Λ is covariantly or contravariantly finite over **C**. Our main result is that if **C** has images (i.e., if $f: C_1 \to C_2$ is a morphism in **C**, then $\operatorname{Im} f$ is in **C**), then mod Λ is covariantly (contravariantly) finite over **C** if and only if **C** has a finite cover (cocover). Before taking up the proof of this result we point out the following results which are essentially recapitulations of results in Section 3.

We begin with the following, which is essentially Proposition 3.13 with **C** = mod Λ.

THEOREM 4.1. *Let* **C** *be a subcategory of* ind Λ *such that* ind $\Lambda_{\mathbf{C}}$ *is of bounded length. Then* mod Λ *is functorially finite over* **C** *and so* **C** *has the following properties.*

(a) **C** *has preprojective and preinjective partitions.*

(b) **C** *has right and left almost split morphisms.*

In particular, mod Λ *has preprojective and preinjective partitions.*

In some of our arguments in Section 3 we implicitely used the following.

PROPOSITION 4.2. *Let* **C** *be a subcategory of* mod Λ *of finite type. Then* mod Λ *is functorially finite over* **C**.

Proof. Let $C_1, \ldots, C_n$ be a complete set of nonisomorphic indecomposable objects in **C** and let $C = \coprod_{i=1}^{n} C_i$. Then (C, M) is a finitely generated End C^{op}-module for each M in mod Λ. Let $f_1, \ldots, f_t$ be a set of generators for (C, M) over End C^{op} and let $f: tC \to M$ be the morphism induced by the $f_i: C \to M$. Then $(C, f): (C, tC) \to (C, M)$ is surjective. Since each C_i is isomorphic to a summand of C, it follows that $(C_i, f): (C_i, tC) \to (C_i, M)$ is surjective for all $i = 1, \ldots, n$ and hence $(X, f): (X, tC) \to (X, M)$ is surjective for all X in **C**. Hence mod Λ is contravariantly finite over **C**.

The fact that mod Λ is also covariantly finite over **C** follows by duality. Therefore we have that mod Λ is functorially finite over **C** if **C** is of finite type.

Our next criterion for mod Λ being contravariantly or covariantly finite over a subcategory **C** concerns subcategories **C** which have the property that if $f: C_1 \to C_2$ is a morphism in **C**, then $\operatorname{Im} f$ is in **C**. Such subcategories **C** will be said to *have images*. It is easily seen that a subcategory **C** of mod Λ has images if and only $D(\mathbf{C})$ in mod Λ^{op} has images. This follows from the fact that if $f: C_1 \to C_2$ is a morphism in **C**, then $\operatorname{Im}(D(f): D(C_2) \to D(C_1)) = D(\operatorname{Im} f)$.

We now show that associated with each subcategory **C** of mod Λ is a smallest additive subcategory of mod Λ having images and containing **C**. For each subcategory **C** of mod Λ let Im **C** denote the collection of all M in mod Λ having the property that $M \approx \operatorname{Im}(f: C_1 \to C_2)$ for some C_1, C_2 in add **C**. Then we have the following.

PROPOSITION 4.3. *Let* $\mathbf{C}$ *be a subcategory of* $\operatorname{mod} \Lambda$. *Then* $\operatorname{Im} \mathbf{C}$ *has the following properties.*

(a) $\operatorname{Im} \mathbf{C}$ *is an additive subcategory of* $\operatorname{mod} \Lambda$ *having images.*

(b) $\operatorname{Add} \mathbf{C}$ *has images if and only if* $\operatorname{add} \mathbf{C} = \operatorname{Im} \mathbf{C}$.

(c) $\operatorname{Im} \mathbf{C} = \operatorname{Im} \operatorname{ind} \mathbf{C}$.

(d) $\operatorname{Im} D(\mathbf{C}) = D(\operatorname{Im} \mathbf{C})$.

(e) *A subcategory* $\mathbf{A}$ *of* $\operatorname{ind} \mathbf{C}$ *is a cover (cocover) for* $\mathbf{C}$ *if and only if it is a cover (cocover) for* $\operatorname{Im} \mathbf{C}$. *Hence* $\mathbf{C}$ *has a finite cover (cocover) if and only if* $\operatorname{Im} \mathbf{C}$ *has a finite cover (cocover).*

Proof. (a) Suppose $f: C_1 \to C_2$ is a morphism in $\operatorname{add} \mathbf{C}$ and $M = \operatorname{Im} f$. Suppose M' is a summand of M. Then there is a morphism $g: M \to M'$ such that the composition $M' \to^{\text{inc}} M \to^{g} M'$ is the identity where $M' \to^{\text{inc}} M$ is the inclusion morphism. Then the composition $C_1 \to^{f} M \to^{g} M' \to^{\text{inc}} C_2$ has M' as image. Thus M' is in $\operatorname{Im} \mathbf{C}$. So $\operatorname{Im} \mathbf{C}$ is a subcategory of $\operatorname{mod} \Lambda$. Since $\operatorname{Im} \mathbf{C}$ is obviously closed under finite sums, $\operatorname{Im} \mathbf{C}$ is an additive subcategory of $\operatorname{mod} \Lambda$. So it only remains to show that $\operatorname{Im} \mathbf{C}$ has images.

Suppose $g: M_1 \to M_2$ is a morphism where $M_i \approx \operatorname{Im}(f_i: C_i \to C_i')$ for $i = 1, 2$ with the C_i and C_i' in $\operatorname{add} \mathbf{C}$. Since the image of the composition $C_1 \to M_1 \to \operatorname{Im} g \to^{\text{inc}} C_2'$ is clearly $\operatorname{Im} g$ and C_1 and C_2' are in $\operatorname{add} \mathbf{C}$, it follows that $\operatorname{Im} g$ is in $\operatorname{Im} \mathbf{C}$. Thus $\operatorname{Im} \mathbf{C}$ has images. This completes the proof of (1).

(b)–(e) Trivial.

As our final preliminary remark concerning additive subcategories $\mathbf{C}$ of $\operatorname{mod} \Lambda$ with images we point out the following.

LEMMA 4.4. *Suppose* $\mathbf{C}$ *is an additive subcategory of* $\operatorname{mod} \Lambda$ *with images. Suppose* $\{f_i: X_i \to C\}_{i \in I}$ *is a family of morphisms in* $\mathbf{C}$ *and* $f: \coprod_{i \in I} X_i \to C$ *the induced morphism. Then* $\operatorname{Im} f$ *is in* $\mathbf{C}$.

Proof. Since C is a finitely generated module over an artin ring, C is noetherian, and so $\operatorname{Im} f$ is finitely generated. Hence there is a finite subset J of I such that $f(\coprod_{j \in J} X_j) = \operatorname{Im} f$. But $\coprod_{j \in J} X_j$ is in $\mathbf{C}$ since $\mathbf{C}$ is additive. Hence $\operatorname{Im} f = \operatorname{Im}(f \mid \coprod_{j \in J} X_j)$ is in $\mathbf{C}$ since $\mathbf{C}$ has images.

We now describe precisely when $\operatorname{mod} \Lambda$ is contravariantly or covariantly finite over an additive subcategory $\mathbf{C}$ with images.

THEOREM 4.5. *Suppose* $\mathbf{C}$ *is an additve subcategory of* $\operatorname{mod} \Lambda$ *with images.*

(a) $\operatorname{mod} \Lambda$ *is contravariantly finite over* $\mathbf{C}$ *if and only if* $\mathbf{C}$ *has a finite cocover.*

(b) $\operatorname{mod} \Lambda$ *is a covariantly finite over* $\mathbf{C}$ *if and only if* $\mathbf{C}$ *has a finite cover.*

(c) $\operatorname{mod} \Lambda$ *is functorially finite over* $\mathbf{C}$ *if and only if* $\mathbf{C}$ *has a finite cover and a finite cocover.*

Proof. (a) By Proposition 3.6 we know that **C** has a finite cocover if and only if I is contravariantly finite over **C** for all injective modules. Hence if mod Λ is contravariantly finite over **C**, then **C** has a finite cocover.

Suppose now that **C** has a finite cocover **A**. Then again by Proposition 3.6, we know that for each injective module I there is a morphism $f_I: A_I \to I$ with A_I in add **A** such that $(X, f): (X, A_I) \to (X, I)$ is surjective for all X in **C**. Let M be a Λ-module and inc: $M \to I(M)$ its injective envelope. Since $\operatorname{Im}(X, f_{I(M)}) = (X, I(M))$ for all X in **C**, it follows that the induced morphism $g: f^{-1}_{I(M)}(M) \to M$ has the property that $\operatorname{Im}(X, g) = (X, M)$ for all X in **C**. Letting $\{h_j: X_j \to A_{I(M)}\}_{j \in J}$ be the family of all morphisms in **C** with $\operatorname{Im} h_j \subset f^{-1}_{I(M)}(M)$, we know by Lemma 4.4 that the induced morphism $h: \coprod X_j \to A_{I(M)}$ has the property that $\operatorname{Im} h \subset f^{-1}_{I(M)}(M)$ is contained in **C** since **C** is an additive category with images. It is now easily seen that the induced morphism $f \mid \operatorname{Im} h: \operatorname{Im} h \to M$ has the property $\operatorname{Im}(X, f \mid \operatorname{Im} h) = (X, M)$ for all X in **C**. Therefore M is contravariantly finite over **C**. Because this is true for arbitrary Λ-modules M, it follows that mod Λ is contravariantly finite over **C** if **C** has a finite cocover.

(b) Dual of (a).

(c) Trivial consequence of (a) and (b).

We now give various examples of additive subcategories **C** closed under images with finite covers or cocovers.

We say that a subcategory **C** *has factor modules* if whenever there is a surjection $C \to C'$ with C in **C**, then C' is in **C**. Clearly if **C** has factor modules, then **C** has images. If **C** is an arbitrary subcategory of mod Λ, then the collection of modules M such that there is a surjective $C \to M$ with C in add **C** is easily seen to be an additive subcategory of mod Λ with factor modules which we denote by Fac **C**. Obviously **C** $\subset$ Fac **C** and **C** $=$ Fac **C** if and only if **C** is an additive subcategory of mod Λ with factor modules. In particular, if M is a module we will denote Fac add M more simply by Fac M.

PROPOSITION 4.6. *Suppose* **C** *is an additive subcategory of* mod Λ *with factor modules.*

(a) **C** *has a finite cocover so* mod Λ *is contravariantly finite over* **C**. *Hence* **C** *has a preinjective partition and right almost split morphisms.*

(b) **C** *has a finite cover if and only if* **C** $=$ Fac C *for some module C in* **C**.

(c) mod Λ *is functorially finite over* **C** *if and only if* **C** $=$ Fac C *for some C in* **C**.

Proof. (a) Let I be an indecomposable injective Λ-module. Let $\{f_i: C_i \to I\}_{i \in J}$ be the family of all morphisms with the C_i a complete set of nonisomorphic objects in **C**. Then by Lemma 4.4, there is a finite subset J' of J such that induced morphisms $f': \coprod_{j \in J'} C_j \to I$ and $f: \coprod_{i \in J} C_i \to I$ have the same images which denote by $\tau_{\mathbf{C}}(I)$. Then $\tau_{\mathbf{C}}(I)$ is in $\mathbf{C}$ since $\coprod_{j\in J} C_j$ is in $\mathbf{C}$ and $\mathbf{C}$ is closed under factor objects. Clearly $\tau_{\mathbf{C}}(I)$ has the property that if $g: C \to I$ is a morphism with C in $\mathbf{C}$, then $\operatorname{Im} g \subset \tau_{\mathbf{C}}(I)$.

Let $I_1, \ldots, I_n$ be a complete set of nonisomorphic indecomposable injective Λ-modules. We claim that the family $C_1 = \tau_{\mathbf{C}}(I), \ldots, C_n = \tau_{\mathbf{C}}(I_n)$ of objects in $\mathbf{C}$ has the property that if C is in $\mathbf{C}$, then there is an injective morphism $C \to \coprod_{i=1}^n n_i C_i$. For there is an injection $f: C \to \coprod_{i=1}^n n_i I_i$ and for each projection $p_j: \coprod_{i=1}^n n_i I_i \to I_j$ we have that $\operatorname{Im} p_j f \subset \tau_{\mathbf{C}}(I_j) = C_j$. Therefore $\operatorname{Im} f \subset \coprod_{i=1}^n n_i C_i$. This shows that the subcategory of ind $\mathbf{C}$ consisting of the modules isomorphic to the C_j is a finite cocover for $\mathbf{C}$. Since $\mathbf{C}$ has images and a finite cocover we have by Theorem 4.5 that mod Λ is contravariantly finite over $\mathbf{C}$. Hence the proof of (a) is complete.

(b) and (c) Trivial.

The dual to a subcategory $\mathbf{C}$ having factor modules is that $\mathbf{C}$ have submodules. We say that $\mathbf{C}$ *has submodules* if whenever there is an injection $C' \to C$ with C in $\mathbf{C}$, then C' is in $\mathbf{C}$. Clearly if $\mathbf{C}$ has submodules, then $\mathbf{C}$ has images. Also it is obvious that $\mathbf{C}$ has submodules if and only if $D(\mathbf{C})$ has factor modules.

If $\mathbf{C}$ is an arbitrary subcategory of mod Λ, then the collection of modules M such that there is an injection $M \to C$ with C in add $\mathbf{C}$ is easily seen to be an additive subcategory of mod Λ with submodules which we denote by Sub $\mathbf{C}$. Obviously $\mathbf{C} \subset$ Sub $\mathbf{C}$ and $\mathbf{C} =$ Sub $\mathbf{C}$ if and only if $\mathbf{C}$ is an additive subcategory of mod Λ with submodules. In particular, if M is a module, we will denote Sub add M more simply by Sub M.

We now state without proof the dual of Proposition 4.6.

PROPOSITION 4.7. *Suppose* $\mathbf{C}$ *is an additive subcategory of* mod Λ *with submodules.*

(a) $\mathbf{C}$ *has a finite cover so* mod Λ *is covariantly finite over* $\mathbf{C}$. *Hence* $\mathbf{C}$ *has a preprojective partition and left almost split morphisms.*

(b) $\mathbf{C}$ *has a finite cocover if and only if* $\mathbf{C} =$ sub C *for some* C *in* $\mathbf{C}$.

(c) mod Λ *is functorially finite over* $\mathbf{C}$ *if and only if* $\mathbf{C} =$ Sub C *for some* C *in* $\mathbf{C}$.

In the course of proving Propositions 4.6 *and* 4.7 *we implicitly proved the following which we now state without proof.*

PROPOSITION 4.8. *Let* $I_1, \ldots, I_n$ *and* $P_1, \ldots, P_n$ *be complete sets of nonisomorphic indecomposable injective and projective modules respectively and let* $\mathbf{C}$ *be an additive subcategory of* mod Λ.

(a) *Suppose* $\mathbf{C}$ *has factor modules.*

(i) *Each* Λ*-module* M *has a unique maximal submodule* $\tau_{\mathbf{C}}(M)$ *in* $\mathbf{C}$.

(ii) *For each Λ-module M. the inclusion $\tau_{\mathbf{C}}(M) \to M$ induces an isomorphism $(C, \tau_{\mathbf{C}}(M)) \to (C, M)$ for all C in* **C**.

(iii) $\mathbf{I}_0(\mathbf{C})$ *consists of the nonzero modules isomorphic to $\tau_{\mathbf{C}}(I_j)$ for $j = 1,..., n$.*

(iv) *If $C_1 \to C_2$ is an injection in* **C**, *then $(C_2, V) \to (C_1, V)$ is surjective for all V in* add $\mathbf{I}_0(\mathbf{C})$.

(b) *Suppose* **C** *has submodules.*

(i) *Each Λ-module M has a unique submodule $M_{\mathbf{C}}$ minimal with respect to $M/M_{\mathbf{C}}$ being in* **C**.

(ii) *The canonical morphism $M \to M/M_{\mathbf{C}}$ has the property that $(M/M_{\mathbf{C}}, C) \to (M, C)$ is an isomorphism for all C in* **C**.

(iii) $\mathbf{P}_0(\mathbf{C})$ *consists of the nonzero modules isomorphic to $P_i/(P_i)_{\mathbf{C}}$ for some $i = 1,..., n$.*

(iv) *If $C_1 \to C_2$ is a surjection in* **C**, *then $(U, C_1) \to (U, C_2)$ is surjective for all U in* add $\mathbf{P}_0(\mathbf{C})$.

As a corollary to this proposition we want to point out the following.

COROLLARY 4.9. *Let* **C** *be an additive subcategory of* mod Λ.

(1) *Assume* **C** *has factor modules. Then the functor $M \to \tau_{\mathbf{C}}(M)$ is the right adjoint of the inclusion from* **C** *to* mod Λ.

(2) *Assume* **C** *has submodules. Then the functor $M \to M/M_{\mathbf{C}}$ is the left adjoint of the inclusion from* **C** *to* mod Λ.

Finally we point out the following connection between arbitrary categories closed under images and those closed under submodules and factor modules.

PROPOSITION 4.10. *Let* **C** *be an additive subcategory of* mod Λ. *Then*

(i) **C** *has images if and only if* **C** = Sub **C** ∩ Fac **C**.

(ii) **C** *has images and a finite cover if and only if* **C** = Sub **C** ∩ Fac M *for some M in* mod Λ.

(iii) **C** *has images and a finite cocover if and only if* **C** = Fac **C** ∩ Sub M *for some M in* mod Λ.

(iv) **C** *has images and a finite cover and cocover if and only if* **C** = Sub M ∩ Fac N *for some M and N in* mod Λ.

Proof. The proof of this is straightforward and left to the reader.

5. PREPROJECTIVE AND PREINJECTIVE MODULES

Let, as usual, mod Λ denote the category of finitely generated modules over the artin algebra Λ. Let **C** be a subcategory of mod Λ. We recall from Section 3 that $\mathbf{C}$ is said to have a preprojective partition if there exists a partition $\mathbf{P}_i$, $i = 0, 1, \ldots, \infty$, of ind $\mathbf{C}$, the category of indecomposable modules in $\mathbf{C}$, satisfying the following properties.

(i) $\bigcup_{i=0}^{\infty} \mathbf{P}_i = \operatorname{ind} \mathbf{C}$.

(ii) $\mathbf{P}_i \cap \mathbf{P}_j = \varnothing$ when $i \neq j$.

(iii) $\mathbf{P}_i$ is finite when $i < \infty$ and is a minimal cover for $\bigcup_{j=i}^{\infty} \mathbf{P}_j$.

An indecomposable module A in $\mathbf{C}$ was defined to be preprojective in $\mathbf{C}$ if $A \in \bigcup_{i<\infty} \mathbf{P}_i$ and a module M in $\mathbf{C}$ was defined to be preprojective in $\mathbf{C}$ if $M \in \operatorname{add} \bigcup_{i<\infty} \mathbf{P}_i$. We denote by $\mathbf{P}$ the subcategory of ind $\mathbf{C}$ consisting of the indecomposable preprojective modules.

Dually, $\mathbf{C}$ is said to have a preinjective partition if there exists a partition $\mathbf{I}_i$, $i = 0, 1, 2, \ldots, \infty$, of ind $\mathbf{C}$ satisfying the following properties.

(i) $\bigcup_{i=0}^{\infty} \mathbf{I}_i = \operatorname{ind} \mathbf{C}$.

(ii) $\mathbf{I}_i \cap \mathbf{I}_j = \varnothing$ *when* $i \neq j$.

(iii) $\mathbf{I}_i$ is finite when $i < \infty$ and is a minimal cocover for $\bigcup_{j=i}^{\infty} \mathbf{I}_j$.

A module M in $\mathbf{C}$ is defined to be preinjective if M is in add $\bigcup_{i<\infty} \mathbf{I}_i$. We denote by $\mathbf{I}$ the subcategory of ind $\mathbf{C}$ consisting of the indecomposable preinjective modules.

Our main purpose in this section is to give various characterizations of the preprojective and preinjective modules as well as some structure theorems for these types of modules.

Before going on we recall the definition of the trace of a subcategory $\mathbf{C}$ in mod Λ in a module M. Let A and B be two finitely generated Λ-modules, then $\tau_B(A)$ is defined to be the submodule of A generated by $\{\operatorname{Im} f \mid f \in \operatorname{Hom}(B, A)\}$. Similarly, if A is a finitely generated Λ-module and $\mathbf{C}$ is a subcategory of mod Λ, then $\tau_{\mathbf{C}}(A)$ is defined to be the submodule of A generated by $\{\operatorname{Im} f \mid f \in \operatorname{Hom}(B, A), B \in \mathbf{C}\}$. Observe also that $\tau_{\mathbf{C}}(\ )$ is a subfunctor of the identity functor.

We now start out with the following characterization of the preprojective modules.

THEOREM 5.1. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ *having a preprojective partition denoted by* $\mathbf{P}_i$, $i = 0, 1, 2, \ldots, \infty$. *The following are equivalent for an indecomposable module* A *in* $\mathbf{C}$.

(i) *A is preprojective, i.e.,* $A \in \mathbf{P} = \bigcup_{i<\infty} \mathbf{P}_i$.

(ii) *There exists an* $n < \infty$ *such that* $A \in \mathbf{P}_n$.

(iii) *There exists an* $n < \infty$ *such that if there is a nonsplittable surjective morphism* $B \to A$ *in* $\mathbf{C}$, *then* B *has a summand from* $\bigcup_{i=0}^{n-1} \mathbf{P}_i$.

(vi) *There exists a finite subcategory* $\mathbf{A}$ *of* ind $\mathbf{C}$ *such that if there is a nonsplittable surjective morphism* $B \to A$ *in* $\mathbf{C}$, *then* B *contains a summand isomorphic to a module in* $\mathbf{A}$.

(v) *There exists a simple module S and a nonzero morphism $f: A \to S$ such that $\tau_M(A) \subset \operatorname{Ker} f$ for all but a finite number of modules M in* ind **C**.

(vi) $\bigcup_{i<\infty} \tau_{\mathbf{P}_i}(A) \neq A$ [$\tau_{\varnothing}(A) = 0$ *by definition*].

(vii) $\tau_{\mathbf{P}_\infty}(A) \neq A$.

Proof. We want to prove the theorem by proving (i) $\Rightarrow$ (ii), (ii) $\Rightarrow$ (iii), (iii) $\Rightarrow$ (iv), (iv) $\Rightarrow$ (v), (v) $\Rightarrow$ (vi), (vi) $\Rightarrow$ (vii), and (vii) $\Rightarrow$ (i).

(i) $\Rightarrow$ (ii). This is just the definition that $\mathbf{P} = \bigcup_{i<\infty} \mathbf{P}_i$.

(ii) $\Rightarrow$ (iii). Let $B \to A$ be a nonsplittable surjective morphism in **C**. By (ii) there exists an n such that $A \in \mathbf{P}_n$. But then A is an indecomposable splitting projective module in $\bigcup_{i=n}^{\infty} \mathbf{P}_i$. Therefore, since $B \to A$ is a nonsplittable surjective morphism B is not in add $\bigcup_{i=n}^{\infty} \mathbf{P}_i$ and hence has to contain a summand isomorphic to a module in $\bigcup_{i=0}^{n-1} \mathbf{P}_i$.

(iii) $\Rightarrow$ (iv). Let $B \to A$ be a nonsplittable surjective morphism in **C**. Then by (iii), B contains a summand of $\bigcup_{i=0}^{n-1} \mathbf{P}_i$ for some $n < \infty$ depending only on A. Each $\mathbf{P}_i$, $i < \infty$, is finite so $\bigcup_{i=0}^{n-1} \mathbf{P}_i$ is finite and hence (iv) follows.

(iv) $\Rightarrow$ (v). To prove this, assume **A** is a finite subcategory of ind **C** such that for each nonsplittable surjective morphism $B \to A$ in **C**, B contains a summand isomorphic to a module in **A**. Let $A' = \tau_{\mathbf{C}'}(A)$ where $\mathbf{C}' = \operatorname{ind} \mathbf{C} - (\{A\} \cup \mathbf{A})$. We now claim that $A' \neq A$. Assume for a moment that this was proven. Let A'' be a maximal submodule of A containing A'. Then $\tau_M(A) \subseteq A' \subseteq A''$ for all M in ind **C** which are not in $\mathbf{A} \cup \{A\}$ and $\mathbf{A} \cup \{A\}$ is a finite subcategory of ind **C**. Hence $\tau_M(A) \subset \operatorname{Ker} f$ for all but a finite number of modules M in ind **C** where $f: A \to A/A'' = S$ is the natural surjective morphism.

This shows that if we are able to prove $A' \neq A$ we have proven (vi) $\Rightarrow$ (v). Assume to the contrary that $A' = A$. Then there exists a B in add $\mathbf{C}'$ and a surjective morphism $B \to A$ which is not a splitting surjective morphism since A is not in $\mathbf{C}'$. Hence B has to contain a summand isomorphic to module in **A**. This is a contradiction since $\mathbf{A} \cap \mathbf{C}' = \varnothing$.

(v) $\Rightarrow$ (vi) Assume there exists a simple module S and a nonzero morphism $f: A \to S$ such that $\tau_M(A) \subset \operatorname{Ker} f$ for all but a finite number of modules M in ind **C**. Under this hypothesis we want to prove $\bigcup_{i<\infty} \tau_{\mathbf{P}_i}(A) \neq A$.

Assume to the contrary that $\bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A) = A$. Then $\tau_{\mathbf{P}_i}(A) \not\subset \operatorname{Ker} f$ for any $i < \infty$. Hence, for each i, there exists an indecomposable module B_i in $\mathbf{P}_i$ such that $\tau_{B_i}(A) \not\subset \operatorname{Ker} f$. Since the $\mathbf{P}_i$ are disjoint, the B_i are all nonisomorphic. This is contradiction and proves (v) $\Rightarrow$ (vi).

(vi) $\Rightarrow$ (vii). By assumption $\bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A) \neq A$ but then $\tau_{\mathbf{P}_\infty}(A) \neq A$ since $\tau_{\mathbf{P}_\infty}(A) \subset \tau_{\mathbf{P}_i}(A)$ for all $i < \infty$.

(vii) $\Rightarrow$ (i). This is trivial.

Before we state the dual result we want to introduce the dual of the trace, called the cotrace, which is a quotient functor of the identity functor.

Let A and B be in $\operatorname{mod} \Lambda$. Then the cotrace $\operatorname{Co} \tau_B(A)$ of B in A is A/A' where $A' = \bigcap \{\operatorname{Ker} f \mid f \in (A, B)\}$. Now this quotient is determined by A' and $A' = \bigcap \{\operatorname{Ker} f \mid f \in (A, B)\}$ is denoted by $\operatorname{Rej}_B(A)$, the reject of B in A. Similarly if A is in $\operatorname{mod} \Lambda$ and $\mathbf{B}$ is a subcategory of $\operatorname{mod} \Lambda$, $\operatorname{Rej}_{\mathbf{B}}(A)$ is defined to be $\bigcap \{\operatorname{Ker} f \mid f \in (A, B), B \in \mathbf{B}\}$ and $\operatorname{Co} \tau_{\mathbf{B}}(A) = A/\operatorname{Rej}_{\mathbf{B}}(A)$. The $\operatorname{Rej}_{\mathbf{B}}(\ )$ is a subfunctor of the identity functor on $\operatorname{mod} \Lambda$. The reject is not the dual of the trace, but rather the kernel of the map from the identity onto the cotrace and hence all statements about the cotrace may be transformed into statements about the reject. This justifies that we are using the trace and reject in dual statements even though the reject is not the formal dual of the trace. By definition, $\operatorname{Rej}_{\varnothing}(A) = A$.

We now state the dual of Theorem 5.1.

THEOREM 5.2. *Let* $\mathbf{C}$ *be a subcategory of* $\operatorname{mod} \Lambda$ *having a preinjective partition denoted by* $\mathbf{I}_i$, $i = 0, 1, 2,..., \infty$. *The following are equivalent for an indecomposable module* A *in* $\mathbf{C}$.

(i) A *is preinjective, i.e.,* $A \in \mathbf{I} = \bigcup_{i<\infty} \mathbf{I}_i$.

(ii) *There exists an* $n < \infty$ *such that* $A \in \mathbf{I}_n$.

(iii) *There exists an* $n < \infty$ *such that if there is a nonsplittable injective morphism* $A \to B$ *in* $\mathbf{C}$, *then* B *contains a summand from* $\bigcup_{i=0}^{n-1} \mathbf{I}_i$.

(iv) *There exists a finite subcategory of indecomposable modules* $\mathbf{A}$ *of* $\mathbf{C}$ *such that if there is a nonsplittable injective morphism* $A \to B$ *in* $\mathbf{C}$, *then* B *contains a summand isomorphic to a module in* $\mathbf{A}$.

(v) *There exists a simple module* S *and a nonzero morphism* $f: S \to A$ *such that* $f(S) \subset \operatorname{Rej}_B(A)$ *for all but a finite number of indecomposable modules* B *in* $\mathbf{C}$.

(vi) *The submodule* A' *of* A *generated by* $\operatorname{Rej}_{\mathbf{I}_i}(A)$, $i < \infty$ *is different from zero.*

(vii) $\operatorname{Rej}_{\mathbf{I}_\infty}(A) \neq 0$.

Proof. The proof of this theorem is dual to the proof of Theorem 5.1 and is left to the reader.

Let $\mathbf{C}$ be a subcategory of $\operatorname{mod} \Lambda$ having a preprojective partition. We have seen in Theorem 5.1 that the submodules $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$ and $A_0' = \tau_{\mathbf{P}_\infty}(A)$ determine whether an indecomposable module A in $\mathbf{C}$ is preprojective or not. Since $\tau_{\mathbf{A}}(\ )$ is a subfunctor of the identity functor on $\operatorname{mod} \Lambda$ for any subcategory $\mathbf{A}$ we know that $\tau_{\mathbf{A}}(M \coprod M') = \tau_{\mathbf{A}}(M) \coprod \tau_{\mathbf{A}}(M')$. From this observation with a similar remark for the reject the following consequence of Theorems 5.1 and 5.2 telling when an arbitrary module M has a preprojective or preinjective summand respectively.

COROLLARY 5.3. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ.

(a) *Assume* $\mathbf{C}$ *has a preprojective partition and let* M *be a module in* $\mathbf{C}$. *Then the following are equivalent.*

(i) *M has a preprojective summand.*

(ii) $M_0 = \bigcap_{i<0} \tau_{\mathbf{P}_i}(M) \neq M$.

(iii) $M_0' = \tau_{\mathbf{P}_\infty}(M) \neq M$.

(b) *Assume* $\mathbf{C}$ *has a preinjective partition and let* M *be a module in* $\mathbf{C}$. *Then the following are equivalent.*

(i) *M has a preinjective summand.*

(ii) *The submodule M' of M generated by* $\mathrm{Rej}_{\mathbf{I}_i}(M)$ $i < \infty$ *is different from zero.*

(iii) $\mathrm{Rej}_{\mathbf{I}_\infty}(M) \neq 0$.

Proof. This is a direct consequence of Theorem 5.1 and Theorem 5.2, respectively.

We are now going to study the submodules $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$ and $A_0' = \tau_{\mathbf{P}_\infty}(A)$ of an indecomposable preprojective module A in a subcategory $\mathbf{C}$ of mod Λ having a preprojective partition and give different characterizations of them.

LEMMA 5.4. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ *having a preprojective partition and let A be an indecomposable preprojective module in* $\mathbf{C}$. *Then* $A_0' = \tau_{\mathbf{P}_\infty}(A) \subset A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$.

Proof. Observe that for each module M in $\mathbf{P}_\infty$ and $i < \infty$ there exists a module M_i in add $\mathbf{P}_i$ and a surjective morphism $f_i\colon M_i \to M$. Therefore $\tau_{\mathbf{P}_\infty}(A) \subset \tau_{\mathbf{P}_i}(A)$ for all $i < \infty$ which completes the proof of the lemma.

PROPOSITION 5.5. *Let* $\mathbf{C}$ *be subcategory of* mod Λ *having a preprojective partition and let M be a module in* $\mathbf{C}$. *Then* $M_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(M)$ *is characterized by each of the following properties.*

(i) *M_0 is the unique minimal submodule of M containing $\tau_B(M)$ for all but a finite number of indecomposable modules B in* $\mathbf{C}$.

(ii) *M_0 is the unique minimal submodule of M such that for all but a finite number of indecomposable modules B in* $\mathbf{C}$ *there exists no nonzero morphism to M/M_0 factoring through the natural morphism $M \to M/M_0$.*

Proof. (i) First we observe that there is a unique minimal submodule M' such $\tau_B(M) \subset M'$ for all but a finite number of indecomposable modules in $\mathbf{C}$ since the set of submodules of M with this property is closed under intersections.

Now let $N_i = \tau_{\mathbf{P}_i}(M)$. Then $N_0 \supset N_1 \supset N_2 \supset \cdots$ is a descending chain of submodules of M, hence stops say at k $< \infty$, since M is artinian. Then $\tau_B(M) \subset N_k = M_0$ for all B in $\bigcup_{i=k}^{\infty} \mathbf{P}_i$. Hence $M_0 \supset M'$. We now want to prove that this is an equality. Assume to the contrary that this inclusion is proper. Then $\tau_{\mathbf{P}_i}(M) \not\subset M'$ for any $i < \infty$, hence there exist infinitely many B in ind $\mathbf{C}$ with $\tau_B(M) \not\subset M'$ since $\mathbf{P}_i \cap \mathbf{P}_j = \varnothing$ when $i \neq j$. This is a contradiction and hence $M' = M_0$.

(ii) This is just a reformulation from submodules to kernels of morphisms.

If we assume that the subcategory $\mathbf{C}$ of mod Λ with a preprojective partition is also closed under images, there are some other characterizations of the submodule $M_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(M)$ for a module M in $\mathbf{C}$.

Proposition 5.6. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ *closed under images and having a preprojective partition. Let M be a module in C. Then the following are true for* $M_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(M)$.

(i) M_0 *is in* $\mathbf{C}$ *and is the unique minimal submodule of M such that* $\tau_B(M) \subset$ *for all but a finite number of modules in* ind $\mathbf{C}$.

(ii) $M_0 = \tau_{\mathbf{P}_\infty}(M)$.

(iii) M_0 *is the unique submodule of M maximal with respect to the properties that it is in* $\mathbf{C}$ *and does not contain any preprojective summands.*

Proof. (i) Since $\mathbf{C}$ is closed under images we have that M_0 is in $\mathbf{C}$. The rest is just Proposition 5.5.(i)

(ii) Since $\bigcap_{i<\infty} \tau_{\mathbf{P}_i}(M_0) = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(M) = M_0$ and $M_0 \in \mathbf{C}$, we have by Proposition 5.3 that M_0 does not contain any preprojective summands. Hence $M_0 \supset \tau_{\mathbf{P}_\infty}(M)$. The other inclusion is clear from Lemma 5.4.

(iii) Since add $\mathbf{P}_\infty$ is closed under images it is clear that there exists a unique submodule M' of M maximal with respect to the property that M' is in $\mathbf{C}$ and without preprojective summands. Now the claim that this submodule is M_0 follows trivially from (ii).

Corollary 5.7. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ *having a preprojective partition, let A be a preprojective module in* $\mathbf{C}$, *and let* $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$.

(i) *Let M be a module in* $\mathbf{C}$. *Now, if there exists an* $f: M \to A$ *such that* Im $f \not\subset A_0$, *then M contains a preprojective summand.*

(ii) $A_0 = 0$ *if and only if there is only a finite number of nonisomorphic indecomposable modules B in* $\mathbf{C}$ *such that* Hom$(B, A) \neq 0$.

(iii) *If* $\mathbf{C}$ *is closed under submodules, then any submodule of A not contained in* A_0 *contains a preprojective summand. In particular, if* $x \notin A_0$, Λx, *the submodule generated by x, has a preprojective summand.*

Proof. The proof is left to the reader.

We now want to collect the dual statements of Proposition 5.5, Proposition 5.6, and Corollary 5.7 in one proposition for a subcategory $\mathbf{C}$ of mod Λ having a preinjective partition.

PROPOSITION 5.8. *Let $\mathbf{C}$ be a subcategory of* mod Λ *having a preinjective partition and let M be a module in $\mathbf{C}$. Let $M^0 = \sum_{i<\infty} \mathrm{Rej}_{\mathbf{I}_i}(M)$ be the submodule of M generated by all* $\mathrm{Rej}_{\mathbf{I}_i}(M)$, $i < \infty$.

(a) *Then each of the following properties characterizes M^0.*

(i) *M^0 is the unique maximal submodule of M contained in* $\mathrm{Rej}_B(M)$ *for all but a finite number of nonisomorphic indecomposable modules B in $\mathbf{C}$.*

(ii) *M^0 is the unique maximal submodule of M such that for all but a finite number of nonisomorphic indecomposable modules B in $\mathbf{C}$ there exists no nonzero morphism M^0 to B factoring through the inclusion $M^0 \to M$.*

(b) *If $\mathbf{C}$ in addition to having a preinjective partition is closed under images we have that*

(i) *$M^0 = \mathrm{Rej}_{\mathbf{I}_\infty}(M)$ and M/M^0 is in $\mathbf{C}$.*

(ii) *M^0 is the submodule of M minimal with respect to the property M/M^0 is in $\mathbf{C}$ and does not contain any preinjective summands.*

(c) *Let A be a preinjective module in $\mathbf{C}$.*

(i) *If B is a module in $\mathbf{C}$ and there exists a morphism $f: A \to B$ with $A^0 \not\subset$* Ker f, *then B contains a preinjective summand.*

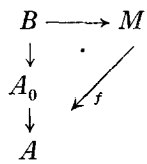
(ii) *$A^0 = A$ if and only if there are only a finite number of nonisomorphic indecomposable modules B in $\mathbf{C}$ such that* $\mathrm{Hom}(A, B) \neq 0$.

(d) *If $\mathbf{C}$ is closed under factors and A is a preinjective module in $\mathbf{C}$ and x an element of A such that $A^0 \not\subset \Lambda x$, the submodule of A generated by x, then $A/\Lambda x$ has a preinjective summand.*

Assume now that $\mathbf{C}$ is a subcategory of mod Λ which is closed under images and having a preprojective partition. We then know by Theorem 4.5(b) that mod Λ is covariantly finite over $\mathbf{C}$ and hence $\mathbf{C}$ has left almost split morphisms. We are now able to characterize the indecomposable modules in $\mathbf{C}$ which are summands of $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$ for some preprojective module A in $\mathbf{C}$ in terms of these minimal left almost split morphisms.

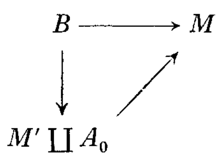
PROPOSITION 5.9. *Let $\mathbf{C}$ be a subcategory of* mod Λ *closed under images and having a preprojective partition. Let B be a module in $\mathbf{P}_\infty$ and let $B \to M$ be a minimal left almost split morphism. Then B is a summand of $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$ for some preprojective module A if and only if M contains a preprojective summand.*

Proof. Observe that we may assume that A is an indecomposable preprojective module. Suppose that B in $\mathbf{P}_\infty$ is a summand of A_0 for an indecomposable preprojective module A. Then the composed morphism $B \to A_0 \to A$ is not a splittable injective morphism and can therefore be factored through the minimal left almost split morphism $B \to M$. We then get the diagram



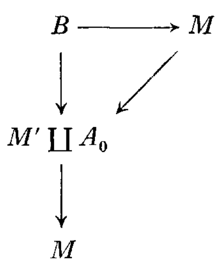
If $\operatorname{Im} f \subset A_0$, $B \to M$ will be a splittable injective morphism since $B \to A_0$ is a splittable injective morphism. This is a contradiction, hence $\operatorname{Im} f \not\subset A_0$ and therefore by Corollary 5.7(i), M contains a preprojective summand.

For the other implication, assume B is a module in $\mathbf{P}_\infty$ and M contains a preprojective summand where $B \to M$ is a minimal left almost split morphism. Decompose M as $M' \coprod A$ where A is an indecomposable preprojective summand of M. Then the image of the composed morphism $B \to M \to A$ is in A_0. Hence, the minimal left almost split morphism $B \to M$ can be factored to give the commutative diagram



Since the morphism $B \to M$ is not a splittable injective morphism, the induced morphism $B \to M \to M'$ is not a splittable injective morphism either.

Now look at the induced morphism $B \to M' \coprod A_0$. If this morphism is not a splittable injective morphism we get a commutative diagram



But $B \to M$ a left minimal morphism; hence, this is a contradiction since $A_0 \neq A$. This shows that $B \to M' \coprod A_0$ is a splittable monomorphism. Since B is indecomposable, the induced morphism $B \to M' \coprod A_0 \to A_0$ has to be a splittable monomorphism, since the induced morphism $B \to M' \coprod A_0 \to M'$ is not. This completes the proof of the proposition.

The dual statement is as follows.

PROPOSITION 5.10. *Let* $\mathbf{C}$ *be a subcategory of* mod Λ *closed under images and having a preinjective partition. Let B be a module in* $\mathbf{I}_\infty$ *and let* $M \to B$ *be a minimal right almost split morphism in* $\mathbf{C}$. *Then B is a summand of* A/A^0 *for a preinjective module A if and only if M contains a preinjective summand.*

As the final general result in this section we have the following.

PROPOSITION 5.11. (i) *Assume* $\mathbf{C}$ *is a subcategory of* mod Λ *having a preprojective partition. If* ind $\mathbf{C}$ *is infinite, then there is no bound on the length of the preprojective modules in* ind $\mathbf{C}$.

(ii) *Assume C is a subcategory of* mod Λ *having a preinjective partition. If* ind $\mathbf{C}$ *is infinite, then there is no bound on the length of the preinjective modules n* ind $\mathbf{C}$.

Proof. We will prove (i). (ii) follows by duality.

(i) Let $\mathbf{P}_i$, $i = 1, 2,..., \infty$, denote the preprojective partition of $\mathbf{C}$. Since for all n, $\mathbf{P}_n$ is a cover for $\bigcup_{i=n}^{\infty} \mathbf{P}_i$, we can get for all $n < \infty$ a chain of epimorphisms $P_0 \to P_1 \to \cdots \to P_n$ with P_i in add $\mathbf{P}_i$. This gives that for each n there exists chains of maps $P_0' \to P_1' \to P_2' \to \cdots \to P_n'$ with each $P_i' \in \mathbf{P}_i$ $i = 0, 1,..., n$ and the composition of the maps nonzero. Since $\mathbf{P}_i \cap \mathbf{P}_j = \varnothing$ if $i \neq j$, the maps $P_i' \to P_{i+1}'$ are not isomorphisms. Then using the following Lemma of Harada and Sai [13] we get our desired result.

LEMMA 5.12. *Let Λ be any ring and* $\{M_i\}_{i=1,2,...}$ *a collection of indecomposable modules of length less than or equal to m and let* $f_i: M_i \to M_{i+1}$ *be nonisomorphisms between these modules. Then there exists an m' such that* $f_{m'} \circ \cdots \circ f_2 \circ f_1 = 0$.

6. SUBCATEGORIES OF FINITE TYPE

Throughout this section $\mathbf{C}$ will denote a subcategory of mod Λ closed under images and having both a preprojective and a preinjective partition denoted by $\mathbf{P}_i$, $i = 0, 1,..., \infty$, and $\mathbf{I}_i$, $i = 0, 1,..., \infty$, respectively.

The main purpose of this section is to use the results of Section 5 to give characterizations of when $\mathbf{C}$ is of finite type in terms of the preprojective and preinjective partitions of $\mathbf{C}$. The main result is the following.

THEOREM 6.1. *The following are equivalent for* $\mathbf{C}$.

(i) $\mathbf{C}$ *is of finite type.*

(ii) $\mathbf{I}_\infty = \varnothing$.

(ii′) $\mathbf{P}_\infty = \varnothing$.

(iii) *All modules in* $\mathbf{C}$ *are preinjective in* $\mathbf{C}$.

(iii′) *All modules in* $\mathbf{C}$ *are preprojective in* $\mathbf{C}$.

(iv) *All preprojective modules in* $\mathbf{C}$ *are preinjective in* $\mathbf{C}$.

(iv′) *All preinjective modules in* $\mathbf{C}$ *are preprojective in* $\mathbf{C}$.

(v) *All preprojective modules in* $\mathbf{C}$ *which are quotients of some module in* $\mathbf{P}_0$ *are preinjective in* $\mathbf{C}$.

(v′) *All preinjective modules in* $\mathbf{C}$ *which are submodules of some module in* $\mathbf{I}_0$ *are preprojective in* $\mathbf{C}$.

Proof. The proof of the theorem will go as follows: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) $\Rightarrow$ (i). The proof of the rest is then the dual cycle and is left to the reader.

(i) $\Rightarrow$ (ii) is trivial since $\mathbf{C}$ is of finite type.

(ii) $\Rightarrow$ (iii). This is just the definition of $\mathbf{I}_\infty$.

(iii) $\Rightarrow$ (iv). This follows directly from the fact that if every indecomposable module in $\mathbf{C}$ is preinjective, then every preprojective module in $\mathbf{C}$ is preinjective.

(iv) $\Rightarrow$ (v) is also a triviality.

This leaves only (v) $\Rightarrow$ (i) which will be proven after some intermediate results occupying the rest of this section.

Associated with $\mathbf{C}$ we have for each $n = 0, 1, 2,..., n < \infty$ the subcategories $\mathbf{P}^n = \bigcup_{i<n} \mathbf{P}_i$ and $\mathbf{I}^n = \bigcup_{i<n} \mathbf{I}_i$, and for $n = \infty$ let $\mathbf{P}^\infty = \mathbf{P} = \bigcup_{i<\infty} \mathbf{P}_i$ and $\mathbf{I}^\infty = \mathbf{I} = \bigcup_{i<\infty} \mathbf{I}_i$. Now $\mathbf{C}_{\mathbf{P}^n}$ and $\mathbf{C}_{\mathbf{I}^n}$ denote as usual the subcategories of $\mathbf{C}$ consisting of modules without summands from $\mathbf{P}^n = \bigcup_{i<n} \mathbf{P}_i$ and $\mathbf{I}^n = \bigcup_{i<n} \mathbf{I}_i$, respectively. Since $\mathbf{P}^n = \bigcup_{i<n} \mathbf{P}_i$ and $\mathbf{I}^n = \bigcup_{i<n} \mathbf{I}_i$ are finite for all $n < \infty$, $\mathbf{C}_{\mathbf{P}^n}$ and $\mathbf{C}_{\mathbf{I}^n}$ have their own preprojective and preinjective partitions which we will denote by $\mathbf{P}_i(\mathbf{C}_{\mathbf{P}^n})$, $\mathbf{I}_i(\mathbf{C}_{\mathbf{P}^n})$, $\mathbf{P}_i(\mathbf{C}_{\mathbf{I}^n})$, and $\mathbf{I}_i(\mathbf{C}_{\mathbf{I}^n})$, $i = 0, 1, 2,..., \infty$, respectively. In order not to make the statements too complicated we will only deal with the subcategories $\mathbf{C}_{\mathbf{I}^n}$, $n < \infty$ since the results for $\mathbf{C}_{\mathbf{P}^n}$ will follow by duality.

PROPOSITION 6.2. *Let* $\mathbf{C}$ *be as before.*

(i) *Let* A *be any module in* $\mathbf{C}$, *let* B *be a module in* $\mathbf{C}_{\mathbf{I}^n}$ *for some* $n = 0, 1,..., \infty$, *and let* $f: A \to B$ *be a morphism. Then* $\operatorname{Im} f \in \mathbf{C}_{\mathbf{I}^n}$.

(ii) $\mathbf{C}_{\mathbf{I}^n}$ *has a preprojective partition for* $n = 0, 1,..., \infty$ *and also a preinjective partition if* $n < \infty$.

Remark. In the next section we will show that $\mathbf{C}_{\mathbf{I}^\infty}$ does not have any preinjective partition if $\mathbf{C}_{\mathbf{I}^\infty}$ is nonempty.

Proof. (i) Let $f: A \to B$ be a morphism with A in $\mathbf{C}$ and B in $\mathbf{C}_{\mathbf{I}^n}$. Then $\operatorname{Im} f$ is in $\mathbf{C}$ since $\mathbf{C}$ is closed with respect to images. Assume for a moment that $\operatorname{Im} f$ contains a summand M' from $\mathbf{I}^n$. The composed morphism $M' \to \operatorname{Im} f \to B$ is a nonsplittable injective morphism since B does not contain any summand from $\mathbf{I}^n$, but then we know by Theorem 5.2(iii) that B has to contain a summand from $\mathbf{I}^n$, which gives us the desired contradiction.

(ii) If $n < \infty$, then $\mathbf{I}^n$ is a finite category and hence by Lemma 3.12, $\mathbf{I}^n$ is nilpotent. Proposition 3.13(iii) then states that $\mathbf{C}$ is functorially finite over $\mathbf{C}_{\mathbf{I}^n}$ and hence $\mathbf{C}_{\mathbf{I}^n}$ has both a preprojective and a preinjective partition.

To prove the rest of (ii) we will use Theorem 4.5(b) which says that if a subcategory of mod Λ is closed under images and has a finite cover, then mod Λ is covariantly finite over $\mathbf{C}$ and hence $\mathbf{C}$ has a preprojective partition. From part (i) of the proposition we have that $\mathbf{C}_\mathbf{I}$ is not only closed under images but also contains $\operatorname{Im} f$ if $f: A \to B$ is a morphism in mod Λ with $A \in \mathbf{C}$ and B in $\mathbf{C}_\mathbf{I}$. Now let P_0 be the sum of a complete set of nonisomorphic modules in $\mathbf{P}_0$. It is not hard to see that $\operatorname{add}(P_0/\operatorname{Rej}_{\mathbf{I}_\infty} P_0)$ is a finite cover for $\mathbf{C}_\mathbf{I} = \operatorname{add} \mathbf{I}_\infty$. This then together with Theorem 4.5(b) completes the proof of the proposition.

We now study the minimal covers for $\mathbf{C}_{\mathbf{I}^n}$ as n varies and find some relations between them. Before we start we make the following definition.

Definition. Let $\mathbf{A}$ be a finite subcategory of ind Λ. Define $l(\mathbf{A})$ to be $\sum l(A)$ where the sum is taken over a complete set of nonisomorphic objects A in $\mathbf{A}$ and $l(A)$ denotes the length of the module A.

Proposition 6.3. *Let* $\mathbf{C}$ *be as before. Then* $\mathbf{P}_0$ *is a cover for* $\mathbf{C}_{\mathbf{I}_0}$ *if and only if* $\mathbf{P}_0 \cap \mathbf{I}_0 = \varnothing$. *Moreover, if* $\mathbf{P}_0 \cap \mathbf{I}_0 \neq \varnothing$, *then* $l\mathbf{P}_0 > l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}_0}))$ *and the modules in* $\mathbf{P}_0(\mathbf{C}_{\mathbf{I}_0})$ *are quotients of modules in* $\mathbf{P}_0$.

Proof. The proof of the first assertion in the proposition is trivial.

To prove the second one, assume $\mathbf{P}_0 \cap \mathbf{I}_0 \neq \varnothing$. Then we know that P_0, the sum of a complete set of nonisomorphic modules in $\mathbf{P}_0$, contains a summand from $\mathbf{I}_0$. Hence $A = \operatorname{Rej}_{\bigcup_{i=1}^{\infty} \mathbf{I}_i} P_0 \neq 0$. Further, by Proposition 6.2(i) we know that $P_0/A \in \mathbf{C}_{\mathbf{I}^0}$ since P_0/A is the image of morphism $f: P_0 \to M$ with M in $\mathbf{C}_{\mathbf{I}^0}$. Since all modules in $\mathbf{C}_{\mathbf{I}^0}$ are quotients of a direct sum of copies of P_0 and $A \subset \operatorname{Ker} g$ for all morphism $g: P_0 \to C$ with C in $\mathbf{C}_{\mathbf{I}^0}$, we have that all modules in $\mathbf{C}_{\mathbf{I}^0}$ are quotients of sums of copies of P_0/A. Therefore, $\operatorname{add}(P_0/A)$ is a finite cover for $\mathbf{C}_{\mathbf{I}^0}$ and the first claim follows from the inequalities $l(\mathbf{P}_0) = l(P_0) > l(P_0/A) \geqslant l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^0}))$.

Since the $\operatorname{Rej}_{\mathbf{C}_{\mathbf{I}^0}}$ is a subfunctor of the identity functor, we know that $A = \operatorname{Rej}_{\mathbf{C}_{\mathbf{I}^0}}(P_0)$ is mapped to itself by all morphisms $f: P_0 \to P_0$. Therefore every indecomposable summand of P_0/A is a quotient of a module in $\mathbf{P}_0$. This finishes the proof of the proposition.

As a corollary to this proposition we have the following.

COROLLARY 6.4. *Let* $\mathbf{C}$ *be as before. Then*

(i) $l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n})) \geqslant l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^{n+1}}))$ *with equality if and only if* $\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n}) = P_0(\mathbf{C}_{\mathbf{I}^{n+1}})$.

(ii) *For* $n < \infty, l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n})) > l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^{n+1}}))$ *if and only if* $\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n}) \cap \mathbf{I}_n \neq \varnothing$.

(iii) *For* $n < \infty, \mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n})$ *is a minimal cover for* $\mathbf{C}_{\mathbf{I}^k}$ *for all* $k \geqslant n$ *if and only if* $P_0(\mathbf{C}_{\mathbf{I}^n}) \cap I_k = \varnothing$ *for all* $n \leqslant k < \infty$.

Proof. (i), (ii), and (iii) are all trivial consequences of the proposition.

We are now in position to complete the proof of Theorem 6.1 by proving (v) implies (i), the last unproven implication. To do this, we assume all preprojective modules in $\mathbf{C}$ which are quotients of some module in $\mathbf{P}_0$ are preinjective and show that $\mathbf{C}$ is of finite type.

By Proposition 6.3 we have that the modules in $\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^k})$ are all quotients of modules in $\mathbf{P}_0$ for all $k < \infty$. Further all these modules are preprojective in $\mathbf{C}$ by Theorem 5.1(iii) since $\mathbf{I}^k$ is finite when $k < \infty$ and therefore by the assumption also preinjective. This is then the same as saying that $\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n}) \cap (\bigcup_{n<j<\infty} \mathbf{I}_i) \neq \varnothing$ for all $n < \infty$, which by Corollary 6.4(iii) is the same as saying that there exists no $n < \infty$ such that $\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n})$ is a minimal cover for $\mathbf{C}_{\mathbf{I}^k}$ for all $k \geqslant n$.

This shows by Corollary 6.4(i) that there is no lower bound greater than zero for $l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n}))$. This then implies that $l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n})) = 0$ for some $n < \infty$ and hence $\operatorname{ind} \mathbf{C} = \mathbf{I}^n$ for some $n < \infty$ which proves that $\mathbf{C}$ is of finite type. This completes the proof of the theorem.

7. THE CATEGORIES $\mathbf{P}_\infty$ AND I_∞

In Section 6 we were studying subcategories of mod Λ closed under images with preprojective and a preinjective partition and characterized those of finite type. We will also in this section assume that $\mathbf{C}$ is a subcategory of mod Λ closed under images and with a preprojective and a preinjective partition denoted by $\mathbf{P}_i$, $i = 0, 1,..., \infty$, and $\mathbf{I}_i$, $i = 0, 1,..., \infty$, respectively. Contrary to Section 6 we will now be mainly interested in the case where $\mathbf{C}$ is of infinite type. We will be especially interested in properties of the subcategories $\mathbf{P}_\infty$ and $\mathbf{I}_\infty$ of $\mathbf{C}$.

In Proposition 6.2(ii) we showed that $\mathbf{C}_\mathbf{I} = \operatorname{add} I_\infty$ has a preprojective partition and, by duality, that $\mathbf{C}_\mathbf{P} = \operatorname{add} \mathbf{P}_\infty$ has a preinjective partition. We will now show that $\mathbf{C}_\mathbf{I}$ does not have a preinjective partition and $\mathbf{C}_\mathbf{P}$ does not have a preprojective partition.

PROPOSITION 7.1. *Let* $\mathbf{C}$ *be as before and assume* $\mathbf{C}$ *is of infinite type. Then*

(i) $\mathbf{C_P}$ *does not contain any splitting projective modules and hence no monimal cever.*

(ii) $\mathbf{C_I}$ *does not contain any splitting injective modules and hence no minimal cocever.*

Proof. We will prove (i). (ii) follows then by duality. Since $\mathbf{C}$ is closed under images and has a preprojective and preinjective partition, mod Λ is functorially finite over $\mathbf{C}$ by Theorem 4.5 and hence $\mathbf{C}$ has both left and right almost split morphism. $\mathbf{P}_\infty$ is nonempty by Theorem 6.1 so let A be a module in $\mathbf{P}_\infty$ and look at the right almost split morphism $M \to A$ in $\mathbf{C}$. Since A is in $\mathbf{P}_\infty$ we know that for each $i = 0, 1, 2,...$ there exists a nonsplittable surjective morphism f_i: $P_i \to A$ with P_i in add($\mathbf{P}_i$). These morphisms factor through the right almost split morphism $M \to A$. Let $M_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(M)$ which is in $\mathbf{C_P}$ by Proposition 5.6(i) since $\mathbf{C}$ is closed under images. This implies that there exists a $k < \infty$ such that f_i: $P_j \to A$ factors through the composed morphism $M_0 \to M \to A$ for all $j \geqslant k$. From this we get that $M_0 \to A$ is a surjective morphism in $\mathbf{C_P}$. Since the morphism $M \to A$ is not a splittable surjective morphism, it follows that $M_0 \to A$ is also not a splittable surjective morphism. This shows that A is not a splitting projective in $\mathbf{C_P}$. Since A was an arbitrary module in $\mathbf{P}_\infty =$ ind $\mathbf{C_P}$ we have shown that $\mathbf{C_P}$ does not contain any splitting projective modules. This also shows that $\mathbf{C_P}$ does not have a minimal cover since all modules in a minimal cover are splitting projectives by Theorem 3.4(a). The first part of the proposition is now proven. The second part follows by duality.

As a consequence of Proposition 7.1 we have:

COROLLARY 7.2. *Let* $\mathbf{C}$ *be as before. Then the following are equivalent.*

(i) $\mathbf{C}$ *is of infinite type.*

(ii) $\mathbf{P}_\infty \neq \varnothing$.

(ii′) $\mathbf{I}_\infty \neq \varnothing$.

(iii) $\mathbf{P}_\infty$ *is of infinite type.*

(iii′) $\mathbf{I}_\infty$ *is of infinite type.*

Let $\mathbf{C}$ be as before and of infinite type. From Proposition 6.2 we know that $\mathbf{C_P}$ has a preinjective partition and by duality $\mathbf{C_I}$ has a preprojective partition. Denote these by $\mathbf{I}_i(\mathbf{P}_\infty)$, $i = 0, 1,..., \infty$, and $\mathbf{P}_i(\mathbf{I}_\infty)$, $i = 0, 1,..., \infty$, respectively. From Proposition 7.1 we know that $\mathbf{P}_\infty$ does not contain any splitting projective modules and hence has no preprojective partition and $\mathbf{I}_\infty$ does not contain any splitting injective modules and hence has no preinjective partition. Some natural questions in this situation are

(i) If $\mathbf{I}_\infty(\mathbf{P}_\infty)$ is different from zero, does it contain any splitting projective or splitting injective modules?

(i′) If $\mathbf{P}_\infty(\mathbf{I}_\infty)$ is different from zero, does it contain any splitting injective or splitting projective modules?

(ii) What is the relation between $\mathbf{I}_\infty(\mathbf{P}_\infty)$, $\mathbf{P}_\infty(\mathbf{I}_\infty)$ and $\mathbf{P}_\infty \cap \mathbf{I}_\infty$?

We know that all preinjective modules M in $\mathbf{C}$ which are in $\mathbf{P}_\infty$ are preinjective in $\mathbf{P}_\infty$. The problem is, can there be any other preinjective modules in $\mathbf{P}_\infty$ except those which are already preinjective in $\mathbf{C}$? The next proposition deals with this problem and gives a necessary condition for this to happen.

PROPOSITION 7.3. *Let* $\mathbf{C}$ *be as before and of infinite type.*

(i) *If there exists a module* M *in* $\mathbf{P}_\infty$ *which is preinjective in* $\mathbf{P}_\infty$ *but is not preinjective in* $\mathbf{C}$, *then there exist infinitely many indecomposable nonisomorphic modules* A *in* $\mathbf{C}$ *which are both preprojective and preinjective in* $\mathbf{C}$.

(ii) *If there exists a module* N *in* $\mathbf{I}_\infty$ *which is preprojective in* $\mathbf{I}_\infty$ *but is not preprojective in* $\mathbf{C}$, *then there exist infinitely many indecomposable nonisomorphic modules* B *in* $\mathbf{C}$ *which are both preprojective and preinjective in* $\mathbf{C}$.

Proof. (i) Let M be a preinjective module in $\mathbf{P}_\infty$ which is not preinjective in $\mathbf{C}$. Then for each $i = 0, 1, 2,...$ there exists a module B_i in add $\mathbf{I}_i$ and an injective morphism $f: M \to B_i$. B_i and B_j have no common summands if $i \neq j$, therefore only a finite number of the B_i are in $\mathbf{C}_\mathbf{P} = \text{add}\,\mathbf{P}_\infty$, i.e., all but a finite number of the B_i contains a preprojective direct summand which then is both a preprojective and a preinjective module in $\mathbf{C}$.

(ii) Dual to (i).

THEOREM 7.4. *Let* $\mathbf{C}$ *be as before of infinite type satisfying one of the following conditions.*

(i) *If* A *is a preprojective module in* ind $\mathbf{C}$ *and* $A \to B$ *is a minimal left almost split morphism in* $\mathbf{C}$, *then all summands of* B *are preprojective.*

(ii) *If* C *is a preinjective module in* ind $\mathbf{C}$ *and* $B \to C$ *is a minimal right almost split morphism, then all summands of* B *are preinjective.*

Under these hypothesis $\mathbf{P}_\infty\mathbf{I}_\infty = \mathbf{I}_\infty\mathbf{P}_\infty = \mathbf{P}_\infty \cap \mathbf{I}_\infty$.

Proof. Assume $\mathbf{C}$ satisfies (ii).

If now $\mathbf{P}_\infty\mathbf{I}_\infty \neq \mathbf{P}_\infty \cap \mathbf{I}_\infty$ we know by Proposition 7.3 that there are infinitely many preprojective modules which are preinjective. Therefore by the assumption each minimal cover of mod $\Lambda_{\mathbf{I}^n}$ contain a preinjective module. But then by Corollary 6.4 we know that $l(\mathbf{P}_0(\mathbf{C}_{\mathbf{I}^n})) = 0$ for some $n < \infty$ and hence $\mathbf{C}$ is of finite type. This is a contradiction and therefore $\mathbf{P}_\infty\mathbf{I}_\infty = \mathbf{P}_\infty \cap \mathbf{I}_\infty$ which finish the first part of the proposition.

If now $\mathbf{I}_\infty\mathbf{P}_\infty \neq \mathbf{P}_\infty \cap \mathbf{I}_\infty$ we know again by Proposition 7.3 that there are infinitely many preprojective modules which are preinjective, hence by the same argument as above $\mathbf{C}$ has to be of finite type. This is again a contradiction and hence $\mathbf{I}_\infty \mathbf{P}_\infty = \mathbf{P}_\infty \cap \mathbf{I}_\infty$.

We have now proven half of the theorem. The other half follows by duality.

The rest of this section is devoted to applying some of our results to the situation that $\Lambda = R$, a commutative local artin ring and $\mathbf{C} = \operatorname{mod} \Lambda$. In particular, we develop several criteria for R to be Gorenstein, i.e., self-injective.

To do this, we need a result from [10] about the indecomposability of quotient modules of a sum of copies of the same module M. For the convenience of the reader, we state this result here.

THEOREM 7.5. *Let R be any ring with identity, A and B be two unitary modules and Φ and ψ be two homomorphisms from A to B satisfying the following*:

(i) $\operatorname{Ext}^1(B, \phi)\colon \operatorname{Ext}^1(B, A) \to \operatorname{Ext}^1(B, B)$ *is a monomorphism.*

(ii) *If h is an idempotent endomorphism of A with $\Phi h = t\Phi$ for some endomorphism t of B, then $h = 0$ or $h = 1$.*

(iii) *If t is an endomorphism of B with $t\phi = 0$, then t is in the radical of* $\operatorname{End}(B)$.

(iv) *For any $f, g \in \operatorname{End}(B)$ and $h \in \operatorname{End}(A)$ if $f\psi = g\phi + \phi h$, then $f\psi = 0$.*

(v) *If $f \in \operatorname{End}(B)$ and $f\psi = 0$, then $f\phi = 0$.*

Let $g_n\colon A \coprod A \coprod \cdots \coprod A \to B \coprod B \coprod B \coprod \cdots \coprod B$ *(n copies of each) be given by the matrix*

$$\begin{pmatrix} \phi & 0 & 0 & \cdots & 0 & 0 \\ \psi & \phi & 0 & \cdots & 0 & 0 \\ 0 & \psi & \phi & \cdots & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & \cdots & \phi & 0 \\ 0 & 0 & 0 & \cdots & \psi & \phi \end{pmatrix}$$

Then $\operatorname{coker} g_n$ *is indecomposable for all $n \in N$.*

Now if R is a commutative local artinian ring and $l(\operatorname{Soc} R) \geqslant 2$, *we know that there are two different simple ideals in R. Let ϕ and ψ be embeddings of* $R/\mathrm{r} = S$ (r *is the radical of R) into R such that $\phi(S) \cap \psi(S) = 0$. Then it is clear that ϕ and ψ satisfy the conditions in Theorem 7.5 and we can use this to construct infinitely many indecomposable modules.*

THEOREM 7.6. *Let R be a commutative local ring. Then the following statements are equivalent.*

(i) *R is self-injective.*

(ii) *R is preinjective as an R-module.*

(iii) *I, the indecomposable injective R-module, is preprojective.*

Proof. If R is a commutative artinian ring, the duality D is a duality from $\operatorname{mod} R$ to $\operatorname{mod} R$. This induces a duality between the preprojective and the preinjective R-modules. Hence (ii) is equivalent to (iii).

(i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (i). Assume R is preinjective in $\operatorname{mod} R$. To show that R is self-injective it suffices to show that R has simple socle. Assume to the contrary that R does not have a simple socle. Then we can find morphisms ϕ and ψ from $R/\mathrm{r} = S$ to R satisfying the conditions of Theorem 7.5. So let g_n: $S \coprod S \coprod \cdots \coprod S \to \Lambda \coprod \Lambda \coprod \cdots \coprod \Lambda$ (n copies of each) be given by the matrix

$$\begin{pmatrix} \phi & 0 & \cdots & 0 & 0 \\ \psi & \phi & \cdots & 0 & 0 \\ & & & & \\ 0 & 0 & \cdots & \phi & 0 \\ 0 & 0 & \cdots & \psi & \phi \end{pmatrix}$$

Then by Theorem 7.5 $\operatorname{coker} g_n$ is indecomposable for all $n \in N$. Note also that the composed morphism

$$R \xrightarrow{(1,0,\ldots,0)} R \coprod R \coprod \cdots \coprod R \to \operatorname{coker} g_n$$

is a monomorphism. Hence R can be embedded in infinitely many nonisomorphic indecomposable modules which gives the desired contradiction since R was assumed to be preinjective. Therefore the socle of R is simple. But then $R \subset I$ (I the indecomposable injective R-module) and the length of them are the same, so $R = I$.

As a consequence of Theorem 7.6 we have:

COROLLARY 7.7. *Suppose R is a commutative local self-injective artinian ring. Then*

(i) r, *the radical of R, is preprojective if and only if R is Nakayama.*

(ii) $R/\operatorname{Soc} R$ *is preinjective if and only if R is Nakayama.*

Proof. (i) Suppose r, the radical of R, is preprojective. Then it is also a preprojective $R/\operatorname{Soc} R$-module since $\operatorname{Soc} R \cdot \mathrm{r} = 0$. But since r is an injective $R/\operatorname{Soc} R$-module we have by Theorem 7.6 that $R/\operatorname{Soc} R$ is self-injective. It is well-known result that this is equivalent to the ring being Nakayama.

The other implication is trivial since all modules in ind R are preprojective if R is of finite representation type and R is Nakayama if and only if it is of finite representation type.

(ii) This is just the dual of (i) and left to the reader.

THEOREM 7.8. *Suppose R is a commutative local artin ring and let* $\mathfrak{a}_i = \operatorname{Rej}_{\mathbf{I}_i} R$. *Then*

(i) $\mathfrak{a}_n = 0$ *for all n if R is not self-injective.*

(ii) $\mathfrak{a}_n = \operatorname{Soc} R$ *for all $n \geqslant 1$ if R is a self-injective ring and not a Nakayama ring.*

(iii) $\mathfrak{a}_n = \operatorname{Soc}^n R$ *if R is a Nakayama ring where* $\operatorname{Soc}^n R = p_{n-1}^{-1} \operatorname{Soc}(R/\operatorname{Soc}^{n-1} R)$ *with p_{n-1} the natural morphism* $R \to R/\operatorname{Soc}^{n-1} R$.

Proof. (i) Assume R is not self-injective. Then by Theorem 7.6, R as an R-module is not preinjective, and hence $\mathfrak{a}_n = \operatorname{Rej}_{\mathbf{I}_i}(R) = 0$ for all n.

(ii) Suppose R is a self-injective ring which is not Nakayama. Then by Corollary 7.7(ii), $R/\operatorname{Soc} R$ is not preinjective and therefore $\mathfrak{a}_n = \operatorname{Soc} R$ for all n.

(iii) This follows since $R/\operatorname{Soc}^n R$ is injective as $R/\operatorname{Soc}^n R$-module when R is Nakayama.

It would be nice to know if a similar result holds for arbitrary indecomposable self-injective rings.

8. THE PREPROJECTIVE AND PREINJECTIVE IRREDUCIBLE MAPPING PROPERTY

In the rest of this paper we will deal with the preprojective and preinjective partitions of mod Λ which we know exist by Theorem 3.3. We will denote the preprojective partition by $\mathbf{P}_i$, $i = 0, 1,..., \infty$, and the preinjective partition by $\mathbf{I}_i$, $i = 0, 1,..., \infty$, throughout the rest of the paper. Some of the earlier results will also be restated for mod Λ. Before we start we recall some constructions and notion from [6, 7].

If Λ is an artin algebra over the commutative artin ring R, we will let D denote the functor $\operatorname{Hom}(\ , I_0(R/\operatorname{rad} R))$ from mod Λ to mod Λ^{op}, where Λ^{op} is the opposite ring of Λ, I_0 stands for the injective envelope, and rad R is the radical of R. D is a duality. Another way to obtain a module in mod Λ^{op} from a module in mod Λ is by means of the following construction. Let M be a module in mod Λ and let $P_1 \xrightarrow{f} P_0 \to M \to 0$ be a minimal projective presentation of M. We now apply the functor $\operatorname{Hom}_\Lambda(\ , \Lambda)$ and we denote by $\operatorname{Tr} M$ the Cokernel of $\operatorname{Hom}_\Lambda(f, \Lambda)$. This gives a duality Tr: $\underline{\text{mod}}\, \Lambda \to \underline{\text{mod}}\, \Lambda^{\text{op}}$, the transpose, where $\underline{\text{mod}}\, \Lambda$ is the category mod Λ modulo projectives.

DEFINITION. An exact sequence $0 \to A \to B \to C \to 0$ in mod Λ is called an almost split sequence if

(i) A and C are indecomposable,

(ii) the morphism $B \to C$ is a minimal right almost split morphism.

(iii) The morphism $A \to B$ is a minimal left almost split morphism.

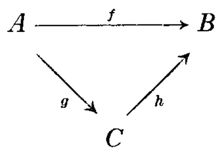
From [6] we have the following existence and uniqueness theorem for almost split sequences.

THEOREM 8.1. (a) *Let C be a finitely generated nonprojective indecomposable Λ-module. Then there exists a unique up to isomorphism, almost split sequence $0 \to A \to B \to C \to 0$. Moreover $A \simeq D \operatorname{Tr} C$.*

(b) *Let A' be a finitely generated noninjective indecomposable Λ-module. Then there exists a unique up to isomorphism almost split sequence $0 \to A' \to B' \to C' \to 0$. Moreover $C' \simeq \operatorname{Tr} DA'$.*

Another notion taken from [7] which is closely connected to minimal left and right almost split morphism is that of irreducible morphism.

DEFINITION. Let A and B be in mod Λ. A morphism $f: A \to B$ is called an irreducible morphism if f is neither a splittable epimorphism nor a splittable monomorphism but whenever the diagram



commutes either g is a splittable monomorphism or h is a splittable epimorphism.

The connection between irreducible morphisms and minimal left and right almost split morphisms is given in the next proposition taken from [7].

PROPOSITION 8.2. *Let A be an indecomposable module in* mod Λ. *A morphism $f: A \to C$ is irreducible if and only if there exists a morphism $f': A \to C'$ such that the induced morphism $A \to C \coprod C'$ is a minimal left almost split morphism. A morphism $g: B \to A$ is irreducible if and only if there exists a morphisms $g': B' \to A$ such that the induced morphism $B \coprod B' \to A$ is a minimal right almost split morphism.*

By using the notion of irreducible morphism we are now able to give a necessary condition on an indecomposable module A in mod Λ to be preprojective.

PROPOSITION 8.3. *Let A be an indecomposable preprojective module in* mod Λ. *Then there exist indecomposable modules $A = M_1, M_2, \ldots, M_k$ and irreducible morphisms $M_{i+1} \to M_i$, $i = 1, 2, \ldots, k-1$, with each M_i, $i = 2, \ldots, k-1$, preprojective and M_k projective.*

Note. This proposition can be strengthened to the result that the composition of the irreducible morphisms is nonzero, but we leave this out here since the proof is based on some other technics.

Proof. The proof goes by induction on i where $A \in \mathbf{P}_i$. If $A \in \mathbf{P}_0$, A is projective and the result holds with $k = 1$ and $M_1 = A$. Assume now that if $B \in \mathbf{P}_i$, $i < n$, then there exists an integer k and indecomposable modules $B = M_1, ..., M_k$ with irreducible morphisms $M_{j+1} \to M_j$, $j = 1, ..., k - 1$, M_j, $j = 2, ..., k - 1$, preprojective and M_k projective. Let A be a module in $\mathbf{P}_{n+1}$. A is not projective so the right almost split morphism $C \to A$ in mod Λ is surjective and it is not a splittable surjection, hence C contains a summand from $\mathbf{P}^{n-1} = \bigcup_{i=0}^{n-1} \mathbf{P}_i$ since A is a splitting projective module in $(\text{mod}\, \Lambda)_{\mathbf{P}^{n-1}}$. Let B be such a preprojective module, then there exists an irreducible morphism $B \to A$ and by induction hypothesis there exist indecomposable modules $M_1', ..., M_k'$ and irreducible morphisms $M_{j+1}' \to M_j'$, $j = 1, ..., k - 1$, with $M_1' = B$, M_j' preprojective for $j = 2, ..., k - 1$, and M_k' projective. Then $M_1, ..., M_{k+1}$, where $M_1 = A$ and $M_j = M_{j-1}'$ for $j = k + 1, ..., 2$, have the right properties. This completes the proof of the proposition.

We know that in many cases the converse of Proposition 8.3 also holds. However, this is not true in general. Alperin [1] has shown that this is not the case for the group algebra $F[A_5]$ where F is an algebraically closed field of characteristic 2. It would be nice to know a classification of the artin algebras where Proposition 8.1 completely describes the preprojective modules; i.e., all artin algebras with the property that if A is an indecomposable preprojective module, B is an indecomposable module, and there exists an irreducible morphism $A \to B$, then B is preprojective. We will say that an artin algebra with this property satisfies the *irreducible mapping property* for *preprojective modules.*

In their work on the representation theory for hereditary tensor algebras, Dlab and Ringel [11] defined an indecomposable module A to be preprojective if $A = \text{Tr}\, D^n P$ for some $n \in N$ (the natural numbers) and some indecomposable projective module P. In [5] Auslander and Platzeck proved that these modules could also be described as indecomposable modules A such that $\text{Hom}(B, A) \neq 0$ for only a finite number of indecomposable modules B. They also showed that this was equivalent to the existence of indecomposable modules M_i, $i = 1, ..., k$, with M_k projective, $M_1 = A$, and irreducible morphisms $f_i\colon M_{i+1} \to M_i$.

By now combining the result in Theorem 6.1(iv) specialized to mod Λ and the result of Auslander and Platzeck, we obtain that the modules we have defined to be preprojective in mod Λ coincide with Dlab and Ringel's definition of preprojective modules for hereditary artin algebras. Moreover, the second description of these modules by Auslander and Platzeck shows that the hereditary artin algebras satisfy the irreducible mapping property for preprojective modules.

We now state the dual result of Proposition 8.3, giving a necessary condition, on an indecomposable module A to be preinjective in mod Λ.

PROPOSITION 8.4. *Let A be an indecomposable preinjective module in* mod Λ. *Then there exists indecomposable modules $A = M_1, M_2, ..., M_k$ and irreducible* *morphisms* $M_i \to M_{i+1}$, $i = 1, 2, \ldots, k-1$, *with each* M_i, $i = 1, 2, \ldots, k-1$, *preinjective and* M_k *injective.*

Proof. The proof is dual to the proof of Proposition 8.3 and is left to the reader.

Again, the converse of Proposition 8.4 is not valid in general but holds in many cases. We will say that an artin algebra Λ satisfies the *irreducible mapping property* for *preinjective modules* if whenever $f: A \to B$ is an irreducible morphism between indecomposable modules A and B and B is preinjective, A is also preinjective. By the duality this is the same as Λ^{op} (the opposite ring of Λ) satisfies the irreducible mapping property for preprojective modules.

PROPOSITION 8.5. *Let A be an indecomposable preprojective module in* mod Λ *and let B be an indecomposable module with an irreducible morphism $A \to B$. Then B or D* Tr *B is preprojective. Moreover, if $A \in \mathbf{P}_n$ and B is not preprojective, then D* Tr *$B \in \mathbf{P}_j$ for some $j \leqslant n-1$.*

Proof. If B is preprojective, there is nothing to prove, so we may assume that $\mathbf{B} \in \mathbf{P}_\infty$, hence B, is not projective. Let A be in $\mathbf{P}_n$ and look at the almost split sequence

$$0 \to D\,\mathrm{Tr}\,B \to C \to B \to 0.$$

Since we have an irreducible morphism $A \to B$, A is a summand of C. Since mod Λ is closed under images and has both a preprojective and a preinjective partition, we know by Proposition 7.1 that add $\mathbf{P}_\infty = \mathbf{C_P}$ does not contain any splitting projective modules. Therefore there exists a nonsplittable surjective morphism $F \to B$ in $\mathbf{C_P}$. Since $C \to B$ is minimal right almost split, the surjective morphism $F \to B$ can be lifted to C. Hence we have a surjective morphism $D\,\mathrm{Tr}\,B \coprod F \to C$. Composing this morphism with the splittable surjective morphism $C \to A$ which represents A as a summand of C, we obtain a nonsplittable surjective morphism $D\,\mathrm{Tr}\,B \coprod F \to A$. Since A is a splitting projective module in mod $\Lambda_{\mathbf{P}^n}$, $D\,\mathrm{Tr}\,B \coprod F$ has to contain a summand from $\mathbf{P}^{n-1}$. But all summands in F are in $\mathbf{P}_\infty$; hence D Tr B is in $\mathbf{P}^{n-1}$ which completes the proof of the proposition.

As a corollary to this proposition we have the following characterization of the case when Λ satisfies the irreducible mapping property for preprojective modules.

COROLLARY 8.6. *For an artin algebra Λ the following are equivalent.*

(i) *Λ satisfies the irreducible mapping property for preprojective modules.*

(ii) Tr *DB is preprojective for all noninjective indecomposable preprojective modules B.*

Proof. (i) $\Rightarrow$ (ii). Assume B is an indecomposable preprojective noninjective module in mod Λ and that Λ satisfies the irreducible mapping property for preprojective modules. If $0 \to B \to C \to \operatorname{Tr} DB \to 0$ is an almost split sequence, we get by choosing any indecomposable summand C' of C a chain of irreducible morphisms $B \to C' \to \operatorname{Tr} DB$ with B', C', and $\operatorname{Tr} DB$ indecomposable. Hence C', and therefore also $\operatorname{Tr} DB$, is preprojective.

(ii) $\Rightarrow$ (i) Now assume $\operatorname{Tr} DB$ is preprojective for all indecomposable noninjective preprojective modules B and let $f: A \to C$ be an irreducible morphism with A preprojective and C indecomposable. If C is projective, there is nothing to prove, so we may assume C is not projective. Then by Proposition 8.5, C or $D \operatorname{Tr} C$ is preprojective. In particular by the hypothesis, $C = \operatorname{Tr} D(D \operatorname{Tr} C)$ is preprojective since $D \operatorname{Tr} C$ is not injective.

We now state the dual results for preinjective modules in mod Λ without proofs.

PROPOSITION 8.7. *Let A be a preinjective module and assume there is an irreducible morphism $B \to A$ with B indecomposable. Then B or* $\operatorname{Tr} DB$ *is preinjective. Moreover if $A \in \mathbf{I}_n$ and B is not preinjective, then* $\operatorname{Tr} DB \in \mathbf{I}_j$ *for some* $j \leqslant n - 1$.

COROLLARY 8.8. *For an artin algebra Λ the following are equivalent:*

(i) *Λ satisfies the irreducible mapping property for preinjective modules.*

(ii) *$D \operatorname{Tr} B$ is preinjective for all indecomposable nonprojective preinjective modules B.*

9. THE SUBMODULE A_0 OF A

In this section we continue our study of the preprojective and preinjective partitions of mod Λ.

In Theorem 5.1 we gave several characterizations of the preprojective modules for an arbitrary subcategory of mod Λ which, of course, holds for the preprojective modules in mod Λ itself. Keeping the same notation as in Section 5 we know that the submodule $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i} A$ for an indecomposable module A plays an important role in determinating whether A is preprojective or not, and this submodule together with its dual will be studied in this section. At the end of this section we give some conditions on the preprojective modules which are equivalent to A_0 being 0.

We start out by recalling the different characterizations of the submodule A_0 of a preprojective module A in mod Λ.

THEOREM 9.1. *Let A be a preprojective module in* mod Λ *and let* $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$. *Then*

(i) $A_0 = \tau_{\mathbf{P}_\infty}(A)$ ($\tau_\varnothing(A) = 0$ *by definition*).

(ii) A_0 *is the unique submodule of A maximal among submodules A' of A without preprojective summands.*

(iii) A_0 *is the unique submodule of A minimal among submodule A' of A such that $\tau_M(A) \subset A'$ for all but a finite number of indecomposable modules M in* mod Λ.

(iv) A_0 *is the unique submodule of A minimal among submodules A' of A such that for all but a finite number of modules B in* ind Λ *there is no nonzero morphism from B to A/A' factoring through the natural morphism $A \to A/A'$.*

From [4] we have the following way of calculating the length of a functor from mod Λ to Ab, the category of abelian groups.

If F is a functor from mod Λ to abelian groups, then $l(F) = \sum lF(A)$, where the sum is taken over all nonisomorphic indecomposable modules A in mod Λ with $F(A) \neq 0$ and $lF(A)$ is the length of $F(A)$ as End(A)-module. In particular, if F is a factor of a representable functor, then $l(F) < \infty$ if and only if there are only a finite number of nonisomorphic modules A in ind Λ such that $F(A) \neq 0$. These observations enable us to reformulate parts of Proposition 9.1 as follows.

PROPOSITION 9.2. *Let A be a preprojective module in* mod Λ *and let A_0 be as usual. Then the canonical surjection $p\colon A \to A/A_0$ has the following properties*:

(i) $l(\text{Im}(\ ,p)) < \infty$.

(ii) *If $f\colon A \to B$ has $l(\text{Im}(\ ,f)) < \infty$, then there is a unique morphism $g\colon A/A_0 \to B$ such that $f = gp$.*

Proof. The proof of this is straightforward and left to the reader.

We can now write down some consequences of these propositions.

COROLLARY 9.3. *Let A be a preprojective module in* mod Λ *and let* $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$.

(i) *Let B be a submodule of A such that $B \not\subset A_0$. Then B contains a preprojective summand.*

(ii) *If $x \in A$ and $x \notin A_0$, then the submodule of A generated by x has a preprojective summand.*

(iii) $A_0 = 0$ *if and only if* Hom$(B, A) \neq 0$ *for only a finite number of modules B in* ind Λ.

(iv) $A_0 = 0$ *if and only if all submodules of A are preprojective.*

(v) $A_0 = 0$ *if and only if* $l((\ ,A)) < \infty$.

Proof. All these statements follows easily from the proposition.

Using Proposition 8.7 we give an analog of Proposition 5.9, characterizing the summands of the submodules A_0 of the preprojective modules A in mod Λ.

PROPOSITION 9.4. *Let B be in $\mathbf{P}_\infty$. Then B is a summand of A_0 for some preprojective module A if and only if* Tr DB *is preprojective.*

Proof. Let B be a noninjective module in $\mathbf{P}_\infty$ and let $0 \to B \to M \to \operatorname{Tr} DB \to 0$ be an almost split sequence. Then we know by Proposition 5.9 that B is summand of A_0 for a preprojective module A if and only if M contains a preprojective summand. But this happens if and only if there exist a preprojective module C and an irreducible morphism $C \to \operatorname{Tr} DB$. By Proposition 8.7 this happens if and only if Tr DB is preprojective.

We have been studying the submodule A_0 of a preprojective module A. By dualizing these results we get information about $B^0 = \sum_{i<\infty} \operatorname{Rej}_{\mathbf{I}_i}(B)$, the submodule of B generated by $\operatorname{Rej}_{\mathbf{I}_i}(B)$, $i < \infty$, for preinjective modules B.

PROPOSITION 9.5. *Let B be a preinjective module in* mod Λ *and let* $B^0 = \sum_{i<\infty} \operatorname{Rej}_{\mathbf{I}_i}(B)$. *Then*

(i) $B^0 = \operatorname{Rej}_{\mathbf{I}_\infty}(B)$ ($\operatorname{Rej}_\varnothing(B) = B$ *by definition*).

(ii) *B^0 is the unique submodule of B minimal along submodules B' of B such that B/B' does not contain any preinjective summands.*

(iii) *B^0 is the unique submodule of B maximal among submodules B' of B such that $B' \subset \operatorname{Rej}_M(B)$ for all but a finite number of indecomposable modules M in* mod Λ.

(iv) *B^0 is the unique submodule of B maximal among submodules B' of B such that for all but a finite number of modules M in* ind Λ *there are no nonzero morphisms from B' to M factoring through the natural inclusion $B' \to B$.*

PROPOSITION 9.6. *Let B be a preinjective module in* mod Λ *and let B^0 be as usual. Then the canonical inclusion $i: B^0 \to B$ has the following properties.*

(i) $l(\operatorname{Im}(i, \,)) < \infty$.

(ii) *If $f: C \to B$ is such that $l(\operatorname{Im}(f, \,)) < \infty$, then there exists a morphism $g: C \to B^0$ such that $f = ig$.*

COROLLARY 9.7. *Let B be a preinjective module in* mod Λ *and let* $B^0 = \sum_{i<\infty} \operatorname{Rej}_{\mathbf{I}_i}(B)$.

(i) *If B' is a submodule of B such that $B^0 \not\subset B'$, then B/B' contains a preinjective summand.*

(ii) *If $x \in B$ such that $B^0 \not\subset (x)$, then $B/(x)$ contains a preinjective summand where (x) denotes the submodule of B generated by x.*

(iii) $B^0 = B$ *if and only if* $\mathrm{Hom}(B, M) \neq 0$ *for only a finite number of modules* M *in* ind Λ.

(iv) $B^0 = B$ *if and only if* $l((B, \ )) < \infty$.

The dual of Proposition 9.4 states:

PROPOSITION 9.8. *Let* A *be in* $\mathbf{I}_\infty$. *Then* A *is a summand of* B/B^0 *where* $B^0 = \sum_{i<\infty} \mathrm{Rej}_{\mathbf{I}_i} B$ *for a preinjective module* B *if and only if* $D\,\mathrm{Tr}\,A$ *is preinjective.*

We have been giving descriptions of the summands of A_0 for the preprojective modules A and the summands of B/B^0 for the preinjective modules B. We will now take a look at the summands of A/A_0 for the preprojective modules A and the summands of B^0 for the preinjective modules B.

Let A be a preprojective module and let $P \xrightarrow{f} A/A_0$ be a projective cover for A/A_0. We have then that all nonzero morphisms from any module M in ind Λ which factors through f factors through the natural epimorphism $A \to A/A_0$. If we denote by $P(M, A/A_0)$ the subgroup of $\mathrm{Hom}_\Lambda(M, A/A_0)$ consisting of morphisms factoring through a projective module, we get that all $f \in P(M, A/A_0)$ factor through the natural epimorphism $A \to A/A_0$. But from Proposition 9.1(iv) we have that there are only a finite number of indecomposable modules M with a nonzero morphism $M \to A/A_0$ which factor through the natural morphism $A \to A/A_0$. From this discussion we have the following proposition.

PROPOSITION 9.9. *Let* A *be a preprojective module,* A_0 *as usual, and* $P(M, N)$ *be the subgroup of* $\mathrm{Hom}_\Lambda(M, N)$ *consisting of morphisms factoring through a projective module. Then* $P(M, A/A_0)$ *is different from zero for only a finite number of modules* M *in* ind Λ. *Moreover if* $P(M, A/A_0) \neq 0$, *then* M *is preprojective.*

We do not know a complete characterization of the indecomposable modules occurring as summands of A/A_0 for the preprojective modules A in mod Λ, but we have a complete description of all indecomposable modules B such that $P(M, B)$ is different from zero for only a finite number of indecomposable modules M in mod Λ.

PROPOSITION 9.10. *Let* M *be a module in* mod Λ. *Then* $P(N, M)$ *is different from zero for only a finite number of modules* N *in* ind Λ *if and only if* $\Lambda_0 \cdot M = 0$, *where* $\Lambda_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(\Lambda)$, i.e., $l(P(\ , M)) < \infty$ *if and only if* $\Lambda_0 \cdot M = 0$.

Proof. Assume first that $\Lambda_0 \cdot M = 0$ and let $P \xrightarrow{f} M$ be a projective cover for M. Then we know that $\Lambda_0 \cdot P = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(P) \subset \mathrm{Ker}\, f$. Hence, there is only a finite number of indecomposable modules N such that $\tau_N(P)$ is not in $\mathrm{Ker}\, f$; i.e., $P(N, M)$ is different from zero for only a finite number of modules N in ind Λ. This finishes one of the implications.

For the other implication, assume $P(N, M)$ is different from zero for only a finite number of modules N in ind Λ and assume to the contrary that $\Lambda_0 \cdot M \neq 0$. This is the same as saying that there exists a morphism $f\colon \Lambda \to M$ such that $\Lambda_0 \not\subset \operatorname{Ker} f$. But then for each $i = 0, 1,\ldots,$ there exists a module N in $\mathbf{P}_i$ and a nonzero morphism $g\colon N \to M$ factoring through f, i.e., $P(N, M)$ is different from zero for infinitely many modules N in ind Λ. This is a contradiction; hence $\Lambda_0 \cdot M = 0$ which completes the proof of the proposition.

COROLLARY 9.11. *$\Lambda_0 = 0$ if and only if $l(P(\ , M)) < \infty$ for all modules M in* mod Λ.

For an hereditary artin algebra Λ, $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A) = 0$ for all preprojective modules A in mod Λ. This shows that ind(mod(Λ/Λ_0) is by no means a complete characterization of the indecomposable modules occurring as summands of A/A_0 for preprojective modules A in mod Λ. It would be nice to know if the summands of A/A_0 for the preprojective modules A can be completely described. For instance, is it possible that these modules are the preprojective modules over Λ/Λ_0?

We now discuss when $\Lambda_0 = 0$.

Let $\underline{\mathrm{mod}}\,\Lambda$ be the category given by the following. The objects are the same as in mod Λ and if $A, B \in \underline{\mathrm{mod}}\,\Lambda$, then $\underline{\mathrm{Hom}}(A, B) = \mathrm{Hom}(A, B)/P(A, B)$.

Similarly, let $\overline{\mathrm{mod}}\,\Lambda$ be the category given by the following: The objects are the same as in mod Λ and if $A, B \in \overline{\mathrm{mod}}\,\Lambda$, then $\overline{\mathrm{Hom}}(A, B) = \mathrm{Hom}(A, B)/I(A, B)$, where $I(A, B)$ is the subgroup of $\mathrm{Hom}(A, B)$ consisting of morphism factoring through an injective module.

From [6] we have the following.

THEOREM 9.12. $D\,\mathrm{Tr}\colon \underline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Lambda$ *is an equivalence with inverse* $\mathrm{Tr}\,D\colon \overline{\mathrm{mod}}\,\Lambda \to \underline{\mathrm{mod}}\,\Lambda$.

In what is coming up we will need the following lemma.

LEMMA 9.13. *Let $0 \to D\,\mathrm{Tr}\,C \to B \to C \to 0$ be an exact sequence and assume $l(P(\ , C)) < \infty$ and $l((\ , D\,\mathrm{Tr}\,C)) < \infty$. Then $l((\ , C))$ and $l((\ , B))$ are also finite.*

Proof. By Theorem 9.12, $D\,\mathrm{Tr}\colon \underline{\mathrm{mod}}\,\Lambda \to \overline{\mathrm{mod}}\,\Lambda$ is an equivalence and we have therefore that $D\,\mathrm{Tr}\colon \underline{\mathrm{Hom}}(X, C) \simeq \overline{\mathrm{Hom}}(D\,\mathrm{Tr}\,X, D\,\mathrm{Tr}\,C)$ for all X in ind Λ. Since $l((\ , D\,\mathrm{Tr}\,C)) < \infty$, we have that $l(\overline{\mathrm{Hom}}(\ , D\,\mathrm{Tr}\,C)) < \infty$ and therefore also $l(\underline{\mathrm{Hom}}(\ , C)) < \infty$. But, $0 \to P(\ , C) \to (\ , C) \to \underline{\mathrm{Hom}}(\ , C) \to 0$ is exact so $l((\ , C)) < \infty$ since $l(P(\ , C))$ and $l(\underline{\mathrm{Hom}}(\ , C))$ are finite. Finally, the exact sequence $0 \to (\ , D\,\mathrm{Tr}\,C) \to (\ , B) \to (\ , C)$ gives that $l((\ , B)) < \infty$.

PROPOSITION 9.14. *Let Λ be an artin algebra such that $\Lambda_0 = 0$.*

(a) *The following are equivalent for a module* A *in* $\operatorname{ind} \Lambda$.

(i) *A is preprojective.*

(ii) $l((\ ,A)) < \infty$ *and if* $f: B \to A$ *is an irreducible morphism then* $l((\ ,B)) < \infty$.

(iii) *There exist an* n *and modules* M_i, $i = 1,\ldots, n$, *with* M_1 *projective,* $M_n \approx A$, *and irreducible morphisms* $f_i: M_i \to M_{i+1}$, $i = 1,\ldots, n-1$.

(b) *Λ satisfies the irreducible mapping property for preprojective modules.*

Proof. (a) That (ii) implies (i) follows directly from Theorem 5.1(iv) and that (i) implies (iii) is just Proposition 8.3. The proof of (iii) implies (ii) will be by induction on n where n is the number of modules M_i in $\operatorname{ind} \Lambda$ needed to connect $M_n \approx A$ to a projective module $P \approx M_1$ by irreducible morphisms $M_i \to M_{i+1}$. If $n = 1$, then A is a projective module; hence by assumption $l((\ ,A)) < \infty$. Further if $f: B \to A$ is irreducible, B is a summand of $\mathrm{r}A$; hence $(\ ,B) \subset (\ ,A)$ and therefore $l((\ ,B)) \leqslant l((\ ,A)) < \infty$. Assume now the claim is true for $n = k$. We want to prove it for $n = k+1$. By assumption there exists a module M_k in $\operatorname{ind} \Lambda$ and an irreducible morphism $M_k \xrightarrow{f} A$ such that $l((\ ,M_k)) < \infty$ and whenever there is an irreducible morphism $g: B \to M_k$, then $l((\ ,B)) < \infty$. We may assume that A is not projective, so look at the almost split sequence $0 \to D\operatorname{Tr} A \to C \to A \to 0$. Since there is an irreducible morphism $f: M_k \to A$, we have that M_k is a summand in C and there exists an irreducible morphism $h: D\operatorname{Tr} A \to M_k$. Therefore by induction hypothesis $l((\ ,D\operatorname{Tr} A)) < \infty$. By combining Corollary 9.11 and Lemma 9.13 we get that $l((\ ,A)) < \infty$ and $l((\ ,C)) < \infty$. If now $h': B' \to A$ is an irreducible morphism, then B' is a summand of C and hence $l((\ ,B')) < \infty$ which completes the proof that (iii) implies (ii).

(b) is a trivial consequence of (a).

THEOREM 9.15. *For an artin algebra* Λ *the following are equivalent*:

(i) $\Lambda_0 = 0$.

(ii) $l((\ ,\Lambda)) < \infty$.

(ii) $l(P(\ ,M)) < \infty$ *for all modules* M *in* $\operatorname{mod} \Lambda$.

(iv) *A module* A *in* $\operatorname{ind} \Lambda$ *is preprojective if and only if* $l((\ ,A)) < \infty$.

(v) *A module* A *in* $\operatorname{ind} \Lambda$ *is preprojective if and only if* $A_0 = 0$.

Proof. The equivalence of (i), (ii), and (iii) is trivial by Corollary 9.11. Similarly, the equivalence of (iv) and (v) is just Corollary 9.3(v). Since in particular Λ is preprojective, (iv) implies (ii) and finally, by Proposition 9.14, we have that (ii) $\Rightarrow$ (iv).

From [2] we have the following relation on $\operatorname{ind} \Lambda$. Let A and B be in $\operatorname{ind} \Lambda$, then $A \sim B$ if $A \approx B$ or there exists an irreducible morphism $f: A \to B$ or an irreducible morphism $g: B \to A$. This relation generates an equivalence relation on ind Λ. Let $[A]$ denote the equivalence class of A with respect to this relation.

With this notation in mind we have the following result.

PROPOSITION 9.16. *Let Λ be an artin algebra. Then the following are equivalent.*

(i) $\Lambda_0 = 0$.

(ii) *A module A in* ind Λ *is preprojective if and only if* $[A] = [P]$ *for some projective module P in* ind Λ.

Proof. Observe first that the "only if" part of (ii) is always satisfied by Proposition 8.3. Therefore we only have to prove the equivalence of (i) $\Lambda_0 = 0$ and (ii′) .A module A in ind Λ is preprojective if $[A] = [P]$ for some projective module P in ind Λ.

We want to prove that (i) implies (ii′). So assume $\Lambda_0 = 0$ and let A be a module in ind Λ such that $[A] = [P]$ for a projective module P in ind Λ. If there are modules M_i, $i = 1,..., n$, and irreducible morphisms $M_{i+1} \to M_i$ with M_n projective and $M_1 \approx A$, Proposition 9.14 gives that A is preprojective. Also, if B is preprojective and $C \to B$ is an irreducible morphism, it follows that C is preprojective since $l((\ , B)) < \infty$ and all modules X in ind Λ with $\operatorname{Hom}(X, B) \neq 0$ is preprojective. Now by an easy induction argument we get that A is preprojective which proves that (i) implies (ii).

For the other implication, assume $\Lambda_0 \neq 0$. Then there exists a projective module P in ind Λ with $P_0 \neq 0$. Let X be an indecomposable summand of P_0. Then by Proposition 9.4 we know that Tr DX is preprojective, but then $[X] = [\operatorname{Tr} DX] = [P']$ for some projective module P' in ind Λ. By (ii), this implies that X is preprojective which is a contradiction since all indecomposable summands of P_0 are in $\mathbf{P}_\infty$ by Theorem 9.1. This finishes the proof of the proposition.

For the sake of completeness we now state the dual of the results from Proposition 9.9 to Proposition 9.16.

PROPOSITION 9.17. *Let B be a preinjective module, $B^0 = \sum_{i<\infty} \operatorname{Rej}_{\mathbf{I}_i} B$ and $I(M, N)$ be the subgroup of* $\operatorname{Hom}_\Lambda(M, N)$ *consisting of morphisms factoring through an injective module. Then $I(B^0, M)$ is different from zero for only a finite number of modules M in* ind Λ *which, in addition, are all preinjective.*

PROPOSITION 9.18. *Let M be a module in* ind Λ. *Then $I(M, X)$ is different from zero for only a finite number of modules X in* ind Λ *if and only if $\tau_M(I) \subset I^0$ for all injective modules I in* mod Λ, *i.e., $l(I(M,\ )) < \infty$ if and only if $\tau_M(I) \subset I^0$ for all injective modules in* ind Λ.

COROLLARY 9.19. *$I^0 = I$ for all injective module I in* ind Λ *if and only if $l(I(M, -)) < \infty$ for all modules M in* mod Λ.

LEMMA 9.20. *Let* $0 \to A \to B \to \operatorname{Tr} DA \to 0$ *be an exact sequence and assume* $l(I(A,)) < \infty$ *and* $l(\operatorname{Tr} DA,)) < \infty$. *Then* $l((A,)) < \infty$ *and* $l((B,)) < \infty$.

THEOREM 9.21. *For an artin algebra* Λ, *the following are equivalent*:

(i) $I^0 = I$ *for all injective* Λ-*modules* I.

(ii) $l((I,)) < \infty$ *for all injective* Λ-*modules* I.

(iii) $l(I(M,)) < \infty$ *for all modules* M *in* mod Λ.

(iv) *A module* B *is preinjective in* ind Λ *if and only if* $l((B,)) < \infty$.

(v) *A module* B *is preinjective in* ind Λ *if and only if* $B^0 = B$.

PROPOSITION 9.22. *Let* Λ *be an artin algebra such that* $I^0 = I$ *for all injective modules in* mod Λ. *Then*

(i) Λ *satisfies the irreducible mapping property for preinjective modules.*

(ii) B *in* ind Λ *is preinjective if and only if* $[B] = [I]$ *for some injective module* I *in* ind Λ.

The obvious examples where $l((, \Lambda)) < \infty$ are the rings of finite type and the hereditary artin algebras. However, these rings are not the only artin algebras Λ satisfying the property that $l((, \Lambda)) < \infty$. Green has, in private correspondence, given a method of constructing many such algebras. Hereditary and artin algebras of finite type also have $l((I,)) < \infty$ for all injective I. Dualizing Green's construction it is possible to obtain artin algebras satisfying this condition. It would be interesting to have a description of the artin algebras Λ such that $l((, \Lambda)) < \infty$ and also a description of the artin algebras satisfying both $l((, \Lambda)) < \infty$ and $l((I,)) < \infty$ for I injective.

10. STRONGLY PREPROJECTIVE MODULES

We now return to the study of the submodule $A_0 = \bigcap_{i<\infty} \tau_{\mathbf{P}_i}(A)$ for a preprojective module A in ind Λ. For hereditary artin algebras and more generally for all artin algebras with $l((, \Lambda)) < \infty$ we know that $A_0 = 0$ for all preprojective modules A in ind Λ, but in general we only know that $A_0 \neq A$. The result we are going to prove is that if $A_0 \subset \mathrm{r}A$ for all preprojective modules A in ind Λ where r is the radical of Λ, then Λ satisfies the irreducible mapping property for preprojective modules. The dual result holds for preinjective modules and is as follows: Λ satisfies the irreducible mapping property for preinjective modules if $\operatorname{Soc} B \subset B^0 = \sum_{i<\infty} \operatorname{Rej}_{\mathbf{I}_i} B$ for all preinjective modules B in ind Λ.

DEFINITION. A module A is called strongly preprojective if $A_0 \subset \mathrm{r}A$ where r is the radical of Λ.

Note. All strongly preprojective modules are preprojective.

DEFINITION. A module B is called strongly preinjective if $B^0 \supset \operatorname{Soc} B$ is the socle of B.

Note. All strongly preinjective modules are preinjective.
We may reformulate these definitions as follows.

PROPOSITION 10.1. (a) *Let A be in* $\operatorname{ind} \Lambda$. *Then the following are equivalent*:

(i) *A is strongly preprojective.*

(ii) *For any simple module S in* $\operatorname{ind} \Lambda$ *there exist only a finite number of modules M in* $\operatorname{ind} \Lambda$ *with a nonzero morphism $f: M \to S$ factoring through A.*

(iii) *For all simple modules S in* $\operatorname{ind} \Lambda$ *and morphisms $g: A \to S$,* $l(\operatorname{Im}(\,, g)) < \infty$

(b) *Let B be in* $\operatorname{ind} \Lambda$. *Then the following are equivalent.*

(i) *B is strongly preinjective.*

(ii) *For any simple module S in* $\operatorname{ind} \Lambda$ *there exist only a finite number of modules M in* $\operatorname{ind} \Lambda$ *with a nonzero morphism $S \to M$ factoring through B.*

(iii) *For all simple modules S in* $\operatorname{ind} \Lambda$ *and morphisms g: $S \to B$,* $l(\operatorname{Im}(g, \,)) < \infty$.

Proof. The proof of this is straightforward and left to the reader.

Let $\underline{\operatorname{Hom}}(M, N)$ be the quotient group $\operatorname{Hom}(M, N)/P(M, N)$ where $P(M, N)$ is the subgroup of $\operatorname{Hom}(M, N)$ consisting of morphisms factoring through a projective module. From Corollary 9.3(iii) we have that $A_0 = 0$ for a preprojective module A in $\operatorname{mod} \Lambda$ if and only if $\operatorname{Hom}(N, A) \neq 0$ for only a finite number of modules N in $\operatorname{ind} \Lambda$. By using the quotient group $\underline{\operatorname{Hom}}(M, A)$ of $\operatorname{Hom}(M, A)$ for a preprojective module A, we give a condition for A_0 to be in $\mathrm{r}A$.

PROPOSITION 10.2. *Let A be a module in* $\operatorname{ind} \Lambda$. *If* $\underline{\operatorname{Hom}}(M, A) \neq 0$ *for only a finite number of modules M in* $\operatorname{ind} \Lambda$, *then $A_0 \subset \mathrm{r}A$ and hence A is strongly preprojective.*

Proof. Assume $\underline{\operatorname{Hom}}(M, A) \neq 0$ for only a finite number of modules M in $\operatorname{ind} \Lambda$. Then there is an n, $0 < n < \infty$, such that every $f: B \to A$ with $B \in \mathbf{P}_m$, $n \leqslant m < \infty$, factors through a projective module, i.e., every $f: B \to A$ is equal to some hg where $g: B \to P$ and $h: P \to A$ with P projective. Therefore $\operatorname{Im} f = \operatorname{Im} hg$, $\operatorname{Im} f = \operatorname{Im} hg \subset h(\mathrm{r}P) \subset \mathrm{r}A$ since B is not projective. This shows that $\tau_{\mathbf{P}_m}(A) \subset \mathrm{r}A$ for $n \leqslant m < \infty$ and hence $A_0 \subset \mathrm{r}A$.

Another condition on a module A in $\operatorname{ind} \Lambda$ which implies that A is strongly preprojective is given in the next proposition.

PROPOSITION 10.3. *Let A be in* $\operatorname{ind} \Lambda$. *Then* $\underline{\operatorname{Hom}}(M, A) \neq 0$ *for only a* *finite number of modules M in* ind Λ *if and only if* $\operatorname{Ext}_\Lambda^1(A, N) \neq 0$ *for only a finite number of modules N in* ind Λ. *Hence, A is strongly projective if* $\operatorname{Ext}_\Lambda^1(A, N) \neq 0$ *for only a finite number of modules N in* ind Λ.

Proof. The proof is easily obtained by using the isomorphism

$$\operatorname{Hom}_R(\underline{\operatorname{Hom}}_\Lambda(M, A), I_0(R/\operatorname{rad} R)) \approx \operatorname{Ext}_\Lambda^1(A, D \operatorname{Tr} M),$$

where R is the center of Λ and $I_0(R/\operatorname{rad} R)$ stands for the injective envelope of R modulo the radical of R. See [2, p. 19.]

From Platzeck's work on the representation theory of an artin algebra Λ stably equivalent to a hereditary artin algebra [14] we know that the following are equivalent for a module A in ind Λ:

(i) $\underline{\operatorname{Hom}}(M, A) \neq 0$ for only a finite number of modules M in ind Λ.

(ii) There exist modules M_i, $i = 0, 1,..., n$, in mod Λ with M_0 projective, $M_n \approx A$, and irreducible morphisms $M_i \to M_{i+1}$.

This result together with Proposition 8.3 and Proposition 10.2 then gives that $\underline{\operatorname{Hom}}(M, A) \neq 0$ for only a finite number of modules M in ind Λ is a complete description of the preprojective modules in mod Λ when Λ is stably equivalent to a hereditary artin algebra.

Let $\overline{\operatorname{Hom}}(M, N) = \operatorname{Hom}(M, N)/I(M, N)$ where $I(M N)$ is the subgroup of $\operatorname{Hom}(M, N)$ consisting of morphisms $f: M \to N$ that factors through an injective module. The duals of Propositions 10.2 and 10.3 now state:

Proposition 10.4. *Let B be in* ind Λ.

(i) *If* $\overline{\operatorname{Hom}}(B, N) \neq 0$ *for only a finite number of modules N in* ind Λ, $\operatorname{Soc} B \subset B^0 = \sum_{i<\infty} \operatorname{Rej}_{\mathbf{I}_i} B$ *and hence B is strongly preinjective.*

(ii) $\overline{\operatorname{Hom}}(B, N) \neq 0$ *for only a finite number of modules N in* ind Λ *if and only if* $\operatorname{Ext}^1(M, B) \neq 0$ *for only a finite number of modules M in* ind Λ.

For an artin algebra Λ stably equivalent to a hereditary artin algebra we get by dualizing the result of Platzeck that the preinjective modules B in ind Λ *are completely described by the property that* $\overline{\operatorname{Hom}}(B, M) \neq 0$ *for only a finite number of indecomposable modules M in* ind Λ. *Therefore, all preinjective modules are strongly preinjective in* ind Λ *when Λ is stably equivalent to an hereditary artin algebra.*

Before proving that Λ satisfies the irreducible mapping property for preprojective modules if all preprojective modules are strongly preprojective, we give a couple of general results.

Proposition 10.5. *Let A be in* ind Λ. *Then*

(i) $A \in \mathbf{P}_0$ *if and only if A is projective in* mod Λ.

(ii) *If A is not projective, then $A \in \mathbf{P}_1$ if and only if there exists an irreducible morphism $C \to A$ with C projective.*

(iii) *$A \in \mathbf{P}_1$ if and only if $A = \operatorname{Tr} DB$ where B is a summand of* r *as a left module.*

Proof. (i) This is just the definition of $\mathbf{P}_0$.

(ii) By Proposition 4.1 in [7] we know for a nonprojective module A in ind Λ that the following are equivalent. (a) $D \operatorname{Tr} A$ is a summand of r as a left module. (b) There exists an irreducible morphism $C \to A$ with C projective.

This shows that the statements in (ii) and (iii) are equivalent, so we need only prove (iii).

Assume first that $A \in \mathbf{P}_1$. We then know that there is an irreducible morphism $C \to A$ with C in $\mathbf{P}_0$. Since A is not projective we know that $A = \operatorname{Tr} D(D \operatorname{Tr} A)$ and there exists an irreducible morphism $D \operatorname{Tr} A \to C$ with C projective. But then we know that $D \operatorname{Tr} A$ is isomorphic to a summand in rC; i.e., $D \operatorname{Tr} A$ is isomorphic to a summand in r.

To prove the other implication assume $A = \operatorname{Tr} DB$ where B is an indecomposable summand of r. Then A is not projective and there exists a projective module P in ind Λ and an irreducible morphism $P \to A$. Now let $A' \to A$ be an epimorphism with A' in mod $\Lambda_{\mathbf{P}_0}$. Since P is projective, we know that the irreducible morphism $P \to A$ factors through the epimorphism $A' \to A$; i.e., we have obtained a commuting diagram

$$\begin{array}{ccc} P & \longrightarrow & A \\ \downarrow & \nearrow & \\ A' & & \end{array}$$

Since A' is in mod $\Lambda_{\mathbf{P}_0}$, $P \to A'$ is not a splittable injective morphism; hence $A' \to A$ is a splittable surjective morphism. This shows that A is a splitting projective module in mod $\Lambda_{\mathbf{P}_0}$, hence A is in $\mathbf{P}_1$. This completes the proof of the proposition.

In general, very little is known about which modules appear in $\mathbf{P}_n$, $n \geqslant 2$. The only case where one has a good description is in the hereditary case where Todorov [16] has given the following description of the modules in $\mathbf{P}_n$: $A \in \mathbf{P}_n$ if and only if the following two conditions hold. (a) For every irreducible morphism $f: M \to A$ with M in ind Λ, M is in $\mathbf{P}_{n-1} \cup \mathbf{P}_n$. (b) There is an M_0 in $\mathbf{P}_{n-1}$ and an irreducible morphism $f: M_0 \to A$.

Dualizing Proposition 10.5 we obtain the following.

PROPOSITION 10.6. *Let B be in* ind Λ.

(i) *$B \in \mathbf{I}_0$ if and only if B is injective.*

(ii) $B \in \mathbf{I}_1$ *if and only if* B *is not injective but there exists an irreducible morphism* $f: B \to N$ *with* N *injective.*

(iii) $B \in \mathbf{I}_1$ *if and only if* B *is not injective but* $\operatorname{Tr} DB$ *is a summand of* $N/\operatorname{Soc} N$ *for some injective module* N *where as usual* $\operatorname{Soc} N$, *the socle of* N, *is the maximal semisimple submodule of* N.

The main result in this section is a consequence of these observations and the following proposition.

PROPOSITION 10.7. *Let* A *be a nonprojective module in* $\operatorname{ind} \Lambda$. *If the middle term* B *of the almost split sequence*

$$0 \to D \operatorname{Tr} A \to B \to A \to 0$$

is strongly preprojective, then A *is strongly preprojective and* $D \operatorname{Tr} A$ *is also strongly preprojective if* $D \operatorname{Tr} A$ *is not simple.*

Proof. Let A be a nonprojective module in $\operatorname{ind} \Lambda$ and assume that B, the middle term of the almost split sequence $0 \to D \operatorname{Tr} A \to B \to^{f} A \to 0$, is a strongly projective module. Then there exists an n such that $A \notin \bigcup_{n<i<\infty} \mathbf{P}_i$. Therefore $\tau_{\mathbf{P}_i}(A) \subset f(\tau_{\mathbf{P}_i}(B))$ for all i, $n \leqslant i < \infty$. Hence $A_0 \subset f(\bigcap_{i<\infty} \tau_{\mathbf{P}_i}(B)) \subset f(\mathrm{r}\, B) \subset \mathrm{r}\, A$, i.e., A is a strongly preprojective module in $\operatorname{mod} \Lambda$.

For the other part of the proposition assume that $D \operatorname{Tr} A$ is not simple, but assume that $(D \operatorname{Tr} A)_0 \not\subset \mathrm{r}(D \operatorname{Tr} A)$. Then there exists a C in $\mathbf{P}_\infty$ and a simple module S in $\operatorname{mod} \Lambda$ with a nonzero morphism $C \to S$ factoring through $D \operatorname{Tr} A$.

$$\begin{array}{ccccc} C & \longrightarrow & D \operatorname{Tr} A & \longrightarrow & B \\ & \searrow & \downarrow & \swarrow & \\ & & S & & \end{array}$$

Since the morphism $D \operatorname{Tr} A \to S$ is not a splittable injective morphism, there exists a morphism $B \to S$ making the diagram commute. But then $\tau_C(B) \not\subset \mathrm{r}\, B$ which is a contradiction since $C \in \mathbf{P}_\infty$ and $\tau_{\mathbf{P}_\infty}(B) \subset \mathrm{r}\, B$. This shows that $D \operatorname{Tr} A$ is strongly preprojective if it is not simple.

We now prove the main result of this section.

THEOREM 10.8. *Let* Λ *be an artin algebra. If all preprojective modules in* $\operatorname{mod} \Lambda$ *are strongly preprojective, then* Λ *satisfies the irreducible mapping property for preprojective modules.*

Proof. We are going to prove the theorem by proving that if we have a preprojective module A with an irreducible morphism $A \to B$ with B in $\mathbf{P}_\infty$, there is a preprojective module which is not strongly preprojective. So assume there exists an $n < \infty$ with $A \in \mathbf{P}_n$ and an irreducible morphism $A \to B$ with B in $\mathbf{P}_\infty$. Choose n minimal with this property. By using Proposition 8.5 we then know that $D \operatorname{Tr} B$ is in $\mathbf{P}_j$ for some $j \leqslant n - 1$, and hence the middle term A' of the almost split sequence $0 \to D \operatorname{Tr} B \to A' \to B \to 0$ is preprojective. Now if A' was strongly preprojective, B would also be strongly preprojective by Proposition 10.6. This is a contradiction, so A' is preprojective but not strongly preprojective.

We do not know if the converse of the last theorem is true or false. In any event it would be interesting to know which artin algebras have the property that all preprojective modules are strongly preprojective.

We now state the dual of the two last results.

PROPOSITION 10.9. *Let A be a noninjective module in* $\operatorname{ind} \Lambda$. *If the middle term of the almost split sequence*

$$0 \to A \to B \to \operatorname{Tr} DA \to 0$$

is strongly preinjective in $\operatorname{mod} \Lambda$, *then A is strongly preinjective and* $\operatorname{Tr} DA$ *is also strongly preinjective if* $\operatorname{Tr} DA$ *is not simple.*

THEOREM 10.10. *Let Λ be an artin algebra. If all preinjective modules in* $\operatorname{mod} \Lambda$ *are strongly preinjective, then Λ satisfies the irreducible mapping property for preinjective modules.*

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