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Mathematics

Advances in Mathematics 211 (2007) 284–326

www.elsevier.com/locate/aim

Propagation of boundary *CR* foliations and Morera type theorems for manifolds with attached analytic discs[☆]

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Received 30 January 2006; accepted 23 August 2006

Available online 24 October 2006

Communicated by Takahiro Kawai

Abstract

We prove that homologically nontrivial generic smooth $(2n - 1)$ -parameter families of analytic discs in \mathbb{C}^n , $n \leq 2$, attached by their boundaries to a *CR*-manifold Ω , test *CR*-functions in the following sense: if a smooth function on Ω analytically extends into any analytic discs from the family, then the function satisfies tangential *CR*-equations on Ω . In particular, we give an answer (Theorem 1) to the following long standing open question, so called strip-problem, earlier solved only for special families (mainly for circles): given a smooth one-parameter family of Jordan curves in the plane and a function f admitting holomorphic extension inside each curve, must f be holomorphic on the union of the curves? We prove, for real-analytic functions and arbitrary generic real-analytic families of curves, that the answer is “yes,” if no point is surrounded by all curves from the family. The latter condition is essential. We generalize this result to characterization of complex curves in \mathbb{C}^2 as real 2-manifolds admitting nontrivial families of attached analytic discs (Theorem 4). The main result implies fairly general Morera type characterization of *CR*-functions on hypersurfaces in \mathbb{C}^2 in terms of holomorphic extensions into three-parameter families of attached analytic discs (Theorem 2). One of the applications is confirming, in real-analytic category, the Globevnik–Stout conjecture (Theorem 3) on boundary values of holomorphic functions. It is proved that a smooth function on the boundary of a smooth strictly convex domain in \mathbb{C}^n extends holomorphically inside the domain if it extends holomorphically into complex lines tangent to a given strictly convex subdomain. The proofs are based on a universal approach, namely, on the reduction to a problem of propagation, from the boundary to the interior, of degeneracy of *CR*-foliations of solid torus type manifolds (Theorem 2.2).
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MSC: 32V10; 20E25

[☆] This work was partially supported by Israel Scientific Foundation, grant No. 279/02-01.
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Keywords: Holomorphic functions; CR-functions; Analytic discs; Analytic extension; Manifolds; Foliations; Coverings; Differential forms

1. Formulation of the problem, the main results and comments

1.1. Formulations of the problems and definitions

The results of this article are related to the following general problem.

Morera problem for CR-functions. Let $\Omega \subset \mathbb{C}^n$ be a CR-manifold of real dimension k , and let \mathcal{D} be a family of analytic discs in \mathbb{C}^n , such that the boundaries ∂D , $D \in \mathcal{D}$, cover Ω . Let f be a continuous or smooth function on Ω such that for each $D \in \mathcal{D}$ the restriction $f|_{\partial D}$ admits analytic extension in D . When does this imply that f is a CR-function in Ω ?

Recall that the differentiable manifold Ω is called CR-manifold if the dimension of maximal complex subspaces $T_p^{\mathbb{C}}(\Omega)$ of the tangent spaces are the same for all points $p \in \Omega$. A smooth function f on Ω is called CR-function if $\bar{\partial}_b f = 0$ for any tangential Cauchy–Riemann operator $\bar{\partial}_b$ in the complex tangent space.

The main result of this article is Theorem 2.2 which implies an answer to the above question in the cases $n = 1$, $k = 2$ and $n = 2$, $k = 3$. We consider smooth regular $(2n - 1)$ -parameter families of discs and obtain conditions for the families to detect CR-functions.

The condition for the family \mathcal{D} involves the homology or relative homology groups, depending on whether the parameterizing family is closed or not. For the planar case, $n = 1$, $k = 2$, the condition for the family is most simple: the closures of the discs must have no common point.

In the case $n = 1$, $k = 2$ our main result implies a general solution, in real-analytic category, of the following problem which was known for a long time and was solved till now only for special families of curves.¹

Strip-problem. Let γ_t be a continuous (smooth) one-parameter family of Jordan curves in the complex plane. Let f be a continuous (smooth) function on the union $\Omega = \bigcup \gamma_t$. Suppose that for each t the restriction f admits analytic extension inside the curve γ_t . When does this imply that f is analytic in Ω ?

This question naturally arose in relation with the works [7,17,41] and, to the knowledge of the author, first was explicitly formulated by Josip Globevnik in his talk [19].

Note that the strip-problem is, in a sense, a degenerate case when the attached analytic discs lie in the same complex plane to which the manifold Ω belongs.

For the dimensions $n = 2$, $k = 3$, i.e. for real hypersurfaces Ω in \mathbb{C}^2 , we prove quite general Morera type theorem for CR-functions on Ω . Using this theorem we prove, in real-analytic category, the following conjecture formulated by J. Globevnik and E.L. Stout in the article [25] (independently this question was posed by the author in the talk [1]).

¹ We require real-analyticity to guarantee nice structure of zero sets of certain functions exploited in our constructions (see Section 3.7). This assumption seems redundant and likely might be reduced to conditions of differentiability by more refined analytic arguments, in particular, by using the technique of currents (see the concluding remarks at the end of the article).

Globevnik–Stout conjecture (one-dimensional extension property). Let D be a strictly convex bounded domain in \mathbb{C}^n and $S \subset D$ be a (smooth convex) closed hypersurface, compactly belonging to D . Suppose that a continuous (smooth) function f on ∂D possesses the property: for any complex line L , tangent to S , the restriction $f|_{L \cap \partial D}$ analytically extends to $L \cap D$. Then f is the boundary value of a holomorphic function in D .

1.2. Formulations of the main results

First of all, let us describe in a more precise way the families of curves and analytic discs we are going to deal with.

Let M be a compact connected smooth oriented $(2n - 1)$ -manifold with boundary ∂M which, in particular, may be empty. For instance, for $n = 1$ the manifold M is a smooth curve, topologically equivalent either to the circle S^1 or to the closed segment $[0, 1]$. In this case, we will think of M as of the unit circle $M = S^1$ in the complex plane, or the segment $M = [0, 1]$.

By *analytic disc* in the complex space \mathbb{C}^n we understand a holomorphic diffeomorphic embedding

$$g: \Delta \mapsto \mathbb{C}^n,$$

of the unit disc Δ in the complex plane. The mapping g is assumed smooth up to $\partial \Delta$ and

$$g'(\zeta) \neq 0, \quad \forall \zeta \in \bar{\Delta}.$$

Sometimes we will use the term “analytic disc” for the image $D = g(\Delta)$.

Given a real manifold $\Omega \subset \mathbb{C}^n$, the analytic disc D is called *attached* to Ω if its boundary $\partial D = g(\partial \Delta) \subset \Omega$. It may happen that the entire analytic disc D or its portion belongs to Ω .

By the C^r -family of analytic discs parameterized by the manifold M , $\dim M = k < 2n$, we understand the family $D_t = g_t(\Delta)$, where g_t , $t \in M$, is a family of holomorphic embeddings (or even immersions) of the disc Δ , smoothly parametrized by the points in M . Here we assume that

$$G(\zeta, t) = g_t(\zeta), \quad (\zeta, t) \in \Delta \times M,$$

belongs to the class $C^r(\bar{\Delta} \times M)$, $r \geq 2$. If the manifold M is real-analytic and the mapping $G(\zeta, t)$ is real analytic in the closed domain $\bar{\Delta} \times M$ then we say that the mapping G parameterizes a *real-analytic family* of analytic discs. In most considerations we will need only differentiability of the manifold M and of the parameterization mapping G .

Denote Ω the set covered by the boundaries of the analytic discs:

$$\Omega = G(\partial \Delta \times M) = \bigcup_{t \in M} \gamma_t.$$

Regularity assumptions for G , which we will discuss below, provide that Ω is a smooth manifold. We also denote

$$\hat{\Omega} = G(\bar{\Delta} \times M) = \bigcup_{t \in M} \bar{D}_t.$$

Throughout the article we will use the notations:

$$\Sigma = \Delta \times M, \quad b\Sigma = \partial\Delta \times M, \quad \Sigma_0 = \Delta \times \partial M, \quad \partial\Sigma = b\Sigma \cup \Sigma_0.$$

In this article we consider the following two cases:

Case A, $n = 1$. In this case the analytic discs D_t are Jordan domains, bounded by Jordan curves $\gamma_t = \partial D_t$, in the complex plane and $\Omega \subset \mathbb{C}$. The manifold M has the dimension $2n - 1 = 1$ and therefore M is topologically either a circle S^1 or a closed segment $[0, 1] \subset \mathbb{R}$.

We will assume the regularity condition. The family $\{D_t\}_{t \in M}$ of Jordan domains will be called *regular* if the parameterizing mapping G has the minimal degeneracy:

1. $\text{rank } dG(p) = 2, \forall p \in \Sigma$.
2. $\text{rank } dG|_{b\Sigma}(p) = 2, \forall p \in b\Sigma \setminus \text{Crit}(G)$, where $\text{Crit}(G) \subset b\Sigma$ is the one-dimensional critical smooth manifold of G .
3. $\text{rank } dG|_{b\Sigma}(p) = 1, \forall p \in \text{Crit}(G)$, and $G(\text{Crit}(G)) \subset \partial\Omega$.

Under these conditions, the set Ω is a closed domain, its interior points are regular values for the restriction of G to the boundary manifold $b\Sigma$ and the critical values are located on the boundary. The preimage $G^{-1}(\partial\Omega) \cap b\Sigma$ contains the critical curve $\text{Crit}(G)$ and the mapping G is regular out of the critical curve.

Case B, $n = 2$. In this case $\Omega \subset \mathbb{C}^2$. We are interested in the case when Ω is a smooth real hypersurface in $\mathbb{C}^2 \cong \mathbb{R}^4$. Then by *regularity* of the family D_t is understood the following conditions:

1. $\text{rank } dG(p) = 4, \forall p \in \Sigma = \Delta \times M$.
2. $\text{rank } dG|_{b\Sigma}(p) = 3, \forall p \in b\Sigma \setminus \text{Crit}(G)$, where $\text{Crit}(G) \subset b\Sigma$ is the two-dimensional smooth critical manifold.
3. $\text{rank } dG|_{b\Sigma}(p) = 2, \forall p \in \text{Crit}(G)$, and $G(\text{Crit}(G)) \subset \partial\Omega$.

Thus, the non-boundary points of the manifold Ω are regular values for the mapping $G: b\Sigma \mapsto \Omega$, while the critical values lie in $\partial\Omega$.

Now we can formulate the main results.

The results for Case A (Ω is a domain in \mathbb{C})

In the case $n = 1$ the parametrizing manifold M is diffeomorphic to either unit circle S^1 or the segment $[0, 1]$. We assume that M is just one of these curves.

Theorem 1. Let $\{D_t\}_{t \in M}$, $M = S^1$ or $M = [0, 1]$, be a real-analytic regular closed family of Jordan domains in \mathbb{C} .

Assume that the closures \overline{D}_t have no common point:

$$(a) \bigcap_{t \in M} \overline{D}_t = \emptyset.$$

Let f be a real-analytic function in $\overline{\Omega}$ and assume that f satisfies the property:

(*) for each $t \in M$ the restriction $f|_{\partial D_t}$ admits holomorphic extension in D_t .

Then f is holomorphic in the interior of $\Omega = \bigcup_{t \in M} \partial D_t$ and extends holomorphically to $\hat{\Omega} = \bigcup_{t \in M} D_t$.

The condition (a) cannot be omitted.

The results for Case B (Ω is a hypersurface in \mathbb{C}^2)

The condition (a) in general form is the following.

Definition 1.1. Let $\partial M = \emptyset$. We say that the family of the analytic discs $\mathcal{D} = \{D_t\}_{t \in M}$, parametrized by the mapping $G(\zeta, t)$, $(\zeta, t) \in \Sigma = \Delta \times M$, is homologically nontrivial if the induced mapping of the relative homology groups

$$G: H_k(M, \partial M; \mathbb{R}) \cong H_k(\bar{\Sigma}, \bar{\Sigma}_0; \mathbb{R}) \mapsto H_k(G(\bar{\Sigma}), G(\bar{\Sigma}_0); \mathbb{R}),$$

$k = \dim M$, is not trivial, $G_* \neq 0$.

Let us give a geometric interpretation of the condition of homological nontriviality. For the case $\partial M = \emptyset$ it reads as follows: no d -cycle $c \subset \hat{\Omega} = \bigcup_{t \in M} \bar{D}_t$, $d = \dim M$, meeting any closed analytic discs \bar{D}_t , $t \in M$, is the boundary $c = \partial c'$ of a $(d+1)$ -cycle $c' \subset \hat{\Omega}$.

In the case $\partial M \neq \emptyset$ (“non-closed family”) the condition means the following: if a d -cycle $c \subset \hat{\Omega}$ intersects each closed disc \bar{D}_t , $t \in M$, then c is relatively homologically nontrivial, i.e. for no d -cycle $c_1 \subset \bigcup_{t \in \partial M} \bar{D}_t$ the union $c \cup c_1$ is a boundary of a $(d+1)$ -cycle $c' \subset \hat{\Omega}$.

Theorem 2. Let $\mathcal{D} = \{D_t\}_{t \in M}$ be a real-analytic regular family of analytic discs in \mathbb{C}^2 , parametrized by a real connected 3-dimensional real analytic compact manifold M with boundary (possibly empty). Denote $\Omega = \bigcup_{t \in M} \partial D_t$ a real-analytic 3-dimensional closed manifold in \mathbb{C}^2 . Assume that

(a) the family \mathcal{D} is homologically nontrivial.

Let f be a real-analytic function on Ω such that:

(*) for each $t \in M$ the restriction $f|_{\partial D_t}$ admits analytic extension in the analytic disc D_t .

Then f is a CR-function on Ω .

Recall that $\hat{\Omega}$ is defined as the union of all closed analytic discs \bar{D}_t .

Remark. 1. The condition (*) is equivalent to the following:

$$\int_{\partial D_t} f \omega = 0,$$

for any holomorphic differential 1-form ω in \mathbb{C}^2 .

2. In Section 2.2 we will show that in the planar case the homological nontriviality is equivalent to the condition (a) in Theorem 1 that the closed domains \bar{D}_t have no common point.

Since for $n > 1$ the condition of holomorphic extendibility is local (differential) as opposite to the case $n = 1$ when it is a global (integral) condition, the boundary values of holomorphic functions can be checked by complex two-dimensional sections and our result for \mathbb{C}^2 implies corresponding characterization of boundary values of holomorphic functions of n variables

for $n \geq 2$. So, Theorem 3 implies answer to the question of Globevnik and Stout [25] in real-analytic category.

Theorem 3. *Let $D \subset \mathbb{C}^n$ be a bounded strictly convex domain with real-analytic boundary and $S \Subset D$ be a real-analytic strictly convex closed hypersurface. Let f be a real-analytic function on ∂D such that for any complex line L tangent to S the restriction $f|_{L \cap \partial D}$ extends analytically in $L \cap D$. Then f is the boundary value of a function holomorphic in D and continuous in \bar{D} .*

Proof. If $n = 2$, then Theorem 3 is a particular case of Theorem 2, for a special family of sections by complex lines. In this case the parametrizing manifold is $M = S$. The analytic disc D_t , $t \in S$, is defined as the intersection

$$D_t = L_t \cap D$$

of the domain D with the complex line L_t tangent to S at the point t .

Then

$$\hat{\Omega} = \bigcup_{t \in S} \overline{D_t} = \bar{D} \setminus D',$$

where D' is the domain bounded by S . If a 3-cycle c intersects each analytic disc D_t then any 4-cycle bounded by c must contain D' and hence is not contained in $\hat{\Omega}$. Therefore c is not homologous to zero in $\hat{\Omega}$ and the condition (a) of Theorem 2 holds. Therefore f is CR function on ∂D and hence is the boundary value of a holomorphic function in D .

If $n > 2$, then consider complex 2-planes Π intersecting S . For almost all Π the intersection $S \cap \Pi$ is a real-analytic hypersurface, contained in the intersection

$$D_\Pi = D \cap \Pi$$

which can be regarded as a strictly convex domain in \mathbb{C}^2 . The surface $S \cap \Pi$ is strictly convex.

Thus, we are in position of Theorem 2 applied to the domain $D_\Pi \subset \mathbb{C}^2$ and to the family of complex lines $L \subset \Pi$, tangent to the 3-surface S_Π . By Theorem 3 for $n = 2$ the function f is annihilated by any tangential differential operator \bar{X} , $X \in T_p^\mathbb{C}(\partial D \cap \Pi)$, $p \in \partial D \cap \Pi$. Due to the large supply of the complex 2-sections Π , we obtain $\bar{X}f = 0$ for any point $p \in \partial D$ and any vector $X \in T_p^\mathbb{C}(\partial D)$. Therefore f is CR function on ∂D and hence extends holomorphically in the domain D . \square

1.3. Generalization of Theorem 1: Characterization of complex curves in \mathbb{C}^2

Theorem 1 can be rephrased in terms of the graph Γ_f of the function f . Indeed, if F_t is the analytic extension of f from the boundary ∂D_t into domain D_t then the graph Γ_{F_t} of F_t over D_t is an analytic disc in \mathbb{C}^2 attached to Γ_f . Therefore Theorem 1 states that if the real 2-manifold $\Gamma_f \subset \mathbb{C}^2$ admits a nontrivial one-parameter family of attached analytic discs then Γ_f is a complex manifold.

This version of Theorem 1 can be generalized to compact manifolds which are not necessarily graphs.

For simplicity we formulate the result for periodic families of attached analytic discs.

Theorem 4. *Let $\Lambda \subset \mathbb{C}^2$ be a real-analytic 2-dimensional oriented compact manifold with nonempty boundary, $\partial\Lambda \neq \emptyset$. Suppose that Λ admits one-parameter regular homologically non-trivial real-analytic family D_t , $t \in S^1$, of attached analytic discs such that $\Lambda = \bigcup_{t \in S^1} \partial D_t$. Then Λ is a 1-dimensional complex manifold in \mathbb{C}^2 and $D_t \subset \Lambda$ for all t .*

By regularity we understand here that $\Lambda \setminus \partial\Lambda$ consists of regular values of the parametrizing mapping of the family $\{D_t\}$ on $b\Sigma$.

1.4. History of the problems

Strip-problem. The name “strip-problem” is due to the typical shape of domains swept out by one-parameter families of curves in the plane (see, e.g. Ehrenpreis’ book [16, p. 575]; for the strip-problem for more general PDE see [16, Chapter 9.5], and [4]). The analytic extendibility inside a planar Jordan curve can be formulated in terms of a complex moments condition, thus the question can be regarded as a version of Morera theorem, in which the lowering by one of the number of parameters for the family of the testing contours is compensated by the stronger condition of vanishing of *all* complex moments. We also refer the reader for this and related problems to [41,42].

In the paper [7], by Val’sky and the author, on Moebius-invariant function algebras in the unit disc, a lemma was proved about testing of analyticity by analytic extendibility into families of Jordan curves in the disc. The families were assumed invariant with respect to conformal automorphisms of the unit disc. The method used was averaging of a function with respect to rotations and applying the argument principle to the averaged function.

Globevnik [17] observed that replacing the averaging by computing the Fourier coefficients in the polar coordinates leads to an analogous test of analyticity for rotation-invariant families of curves. In the articles [20,21,23] he made several interesting observations on the phenomena. In [5] the result of Globevnik [17] was generalized for $U(n)$ -invariant families of boundaries of analytic discs in \mathbb{C}^n , with using decompositions into spherical harmonics in \mathbb{C}^n .

The above results used tools of harmonic analysis and therefore required group invariance of the testing families. However, even for (noncompact) group-invariant families of curves, when no information about the growth of functions in the question is known and Fourier analysis becomes inapplicable, simple natural questions remained unanswered.

For instance, the following question became a challenge: given a continuous or smooth function f in the strip $|\operatorname{Im} z| < 1$, does the analytic extendibility inside any inscribed circle of radius 1 imply that f is holomorphic in the strip?

The first result beyond harmonic analysis was obtained by Globevnik and the author [3]. The problem was completely solved for arbitrary one-parameter families of circles in the plane, though for functions $f(x, y)$ which are rational (quotient of two polynomials) in x, y . In spite of yet geometric restrictions for the curves (circles), the approach in [3] led to a new insight. The key point was reformulating the problem, originally one-dimensional, in \mathbb{C}^2 using the embedding $z \mapsto (z, \bar{z})$ of the real 2-plane into \mathbb{C}^2 . The functions, along with their analytic extensions, lift to the quadrics $(z_1 - a)(z_2 - \bar{a}) = r^2$ in \mathbb{C}^2 , which are the complexifications of the circles $|z - a| = r$. Then the proof in [3] is based on analysis of the dynamics, in the parameter t , of the quadrics, parameterized by $a = a(t)$ and $r = r(t)$, with respect to the zero varieties of the polynomials generating the rational function f .

Also, in [3] the case of real-analytic functions and arbitrary smooth families of circles was solved (independently, but in a special case, the same result was obtained by Ehrenpreis [15], with the help of Fourier analysis).

The next significant progress was due to Tumanov [39]. As in [3], he also started with the lifting the problem into \mathbb{C}^2 , but applied powerful tools of *CR*-theory, in particular, the edge of the wedge Ayrapetyan–Henkin’s theorem [11], to prove forced analytic extendibility of the lifted function f to a larger domain. Note, that in [3] such extension was provided automatically, as rational functions in z, \bar{z} always possess meromorphic extensions inside any circle.

As the result, in [39] the strip-problem was solved for continuous functions, albeit for more restricted in the sense of [3], families of circles, namely those with constant radius and centers on an interval. However, soon afterward, Tumanov [40] got rid of the above restrictions and came up with a proof for the case of continuous functions f and arbitrary smooth families of circles. Moreover, this proof, motivated by an argument of Hans Lewy, was much simpler than that in [39]. Recently Globevnik [24] generalized the geometric construction from [39] and used the reasoning from [40] to solve the strip-problem for special families of non-circular Jordan curves which are translates of a fixed axially symmetric Jordan curve, along the line orthogonal to the symmetry axis.

In Theorem 1 of this article we give the solution for generic families of general Jordan curves with no restriction of geometric type. Our approach rests on a reformulating of the original problem to the topological language, namely, as a question about *CR*-extensions of coverings of the 2-dimensional torus or cylinder, inside the solid torus or solid cylinder.

This reduction reveals topological or, better to say, topology-analytical nature of the problem, as well as the adequate tools for the solutions. As result, it allows to get rid of geometric restrictions for the Jordan domains in the question and solve the problem for general families. Some ingredients of the analytic parts of the proofs are close to those in the article [40] by Tumanov.

One-dimensional extension property. We refer the reader for boundary Morera theorems to the recent survey by Kytmanov and Myslivets [28], and an extended bibliography there. Here we will outline only some results which are mostly related to our paper.

It was observed in [7] that boundary values of holomorphic functions in the unit ball in \mathbb{C}^n can be characterized by analytic extendibility into sections of the ball by complex lines. Stout [35] generalized this result to arbitrary smooth domains, using complex Radon transform. In [6] the family of lines was reduced to the set of complex lines passing through a fixed open domain.

Nagel and Rudin [31,32] proved one-dimensional property for the ball in \mathbb{C}^n and the family of complex lines tangent to a smaller concentric ball. Globevnik [18,22] reduced the families of lines in the question. In [8] an analogous result was obtained for lines tangent to a fixed orbit of the Heisenberg group acting on the complex ball.

The fundamental work by Globevnik and Stout [25] contains many deep results on the subject (see also [26]). There the approach is mainly based on the complex Radon transform in its various versions, in particular, approximation by the complex plane waves. Tumanov [38] obtained similar characterizations of *CR*-functions on *CR*-manifolds of higher codimension.

Note, that of most interest are the families not containing small analytic discs. By small discs we understand the families of discs that can be shrunk to boundary points. In this case, the problem of testing *CR*-functions is much easier, at least for smooth functions, as the tangential *CR*-equation follow from Stokes formula applied to the shrinking discs.

An example of families without small discs is the family of intersections of a fixed domain with complex curves (for instance, complex lines, as in [31]), tangent to a fixed surface. In [25]

Globevnik and Stout conjectured that the Nagel–Rudin theorem should be true for two arbitrary enclosed convex domains $D' \Subset D$, when the family of analytic discs, testing boundary values of holomorphic functions on ∂D , consists of sections of the domain D by the complex lines tangent to the hypersurface $S = \partial D'$.

Dinh [14] confirmed the conjecture for smooth functions and under certain condition of “strongly non real-analyticity” of the above sections by the complex lines.

Recently, the conjecture of Globevnik and Stout was confirmed in affirmative by Baracco, Tumanov and Zampieri [12] for the family of extremal discs. The extremal discs are geodesics in Kobayashi metrics in the larger domain. The proof goes back to the idea of the proof in [39] and hence the extremal discs are needed for meromorphic lifts to tangent spaces, similarly to lifting circles to complex quadrics in C^2 by means of \bar{z} .

In this article we prove (Theorem 2) that in dimension $n = 2$ and under assumptions of sufficient smoothness (real-analyticity), no geometric restrictions for analytic discs and for character of the family are required, and the one-dimensional extension property, at least for smooth functions, is true for arbitrary generic family of attached analytic discs. The essential condition is rather topological and requires homological nontriviality of the family.

Since for $n > 1$ characterization of (smooth) boundary values of holomorphic functions are differential (local), as opposed to $n = 1$, Theorem 2, albeit is stated for $n = 2$, lead to various boundary Morera theorems in arbitrary dimensions. To apply Theorem 2 the families of attached analytic discs must have rich enough supply of subfamilies filling up 2-dimensional complex submanifolds. Theorem 3 is just an example of such kind of application.

Theorems 1 and 2 are proved by an universal approach which we explain in the following section. The remarkable papers of Alexander and Wermer [10] and of Stout [36] about linking numbers and analytic functions were a major influence in my discovering this approach.

2. Reduction of the problems to CR-extensions of coverings and foliations

2.1. Formulation of the equivalent result on extensions of foliations

We will formulate a single theorem which includes, in equivalent form, Theorems 1 and 2.

Let n be an integer and M be a compact connected C^r -smooth oriented real $(2n - 1)$ -manifold with the boundary ∂M , possibly empty.

As above, we denote

$$\Delta = \{z \in \mathbb{C}: |z| < 1\}.$$

For the unit circle in the plane we will use both notations, $\partial \Delta$ and S^1 , depending on the context, analytical or topological. We also use all the notations, Σ , $b\Sigma$, Σ_0 , $\partial \Sigma$, Ω , $\hat{\Omega}$, from Section 1. The real dimensions of the manifolds are

$$\dim_{\mathbb{R}} \Sigma = 2n + 1, \quad \dim_{\mathbb{R}} b\Sigma = 2n.$$

The dimensions of the manifolds $\Omega = G(b\Sigma)$ and $\hat{\Omega} = G(\bar{\Sigma})$ depend on the rank of the mapping G , which we are now going to specify.

In order to combine both cases A and B, discussed in the previous section, let us introduce the definition.

Definition 2.1. Let $k \leq 2n$ be an integer. We say that a smooth mapping

$$G: \overline{\Sigma} \mapsto \mathbb{C}^n$$

is $(2n, k)$ -regular if:

1. $\hat{\Omega} = G(\overline{\Sigma})$ is a closed domain in \mathbb{C}^n , $\text{rank}_{\mathbb{R}} dG(p) = 2n$ for all $p \in \Sigma$.
2. $\Omega = G(b\Sigma)$ is a smooth real k -dimensional manifold in $\mathbb{C}^n \cong \mathbb{R}^{2n}$, with boundary and $\text{rank}_{\mathbb{R}} dG|_{b\Sigma}(p) = k$ for all $p \in b\Sigma \setminus \text{Crit}(G)$, where $\text{Crit}(G) \subset b\Sigma$ is $(k-1)$ -dimensional critical manifold, and $\text{Crit}(G) \subset G^{-1}(\partial\Omega)$.
3. $\text{rank}_{\mathbb{R}} dG|_{b\Sigma}(p) = k-1$, for $p \in \text{Crit}(G)$.

In this article we are concerned with the two cases.

(A) $n = 1, k = 2$. In this case the image $\hat{\Omega} = G(\overline{\Sigma})$ has real dimension 2 and is a closed domain in \mathbb{C} , $G(b\Sigma) = \Omega$ is also a domain in \mathbb{C} , with the smooth boundary $\partial\Omega$ containing the critical values of G . The mapping $\zeta \mapsto G(\zeta, t)$ maps conformally the unit disc Δ onto the analytic disc $D_t \subset \mathbb{C}$. These analytic discs are attached to Ω which means that $\partial D_t \subset \Omega$.

The restriction $G|_{b\Sigma}$ of G to the 2-dimensional boundary manifold $b\Sigma$ is a finitely sheeted covering over $\Omega \setminus \partial\Omega$. The fibers correspond to the curves

$$\gamma_t = \partial D_t = G(\partial\Delta \times \{t\}),$$

passing through a fixed point in Ω .

The set $\text{Crit}(G) \subset G^{-1}(\partial\Omega)$ of critical points is a curve in $b\Sigma$. The set of critical values $G(\text{Crit}(G)) \subset \partial\Omega$ is the envelope of the family of the curves γ_t . The points from the envelope are the *sliding* points. At these points the velocity vector (the motion direction) of the family γ_t is proportional to the tangent vector to γ_t . Except the sliding points the boundary $\partial\Omega$ contains subarcs of the curves γ_t corresponding to “boundary” parameters $t \in \partial M$.

The condition that the interior of Ω consists of regular values of the mapping $G|_{b\Sigma}$ (does not contain sliding points) guarantees that this mapping is a foliation over the interior of Ω . This condition is important in our considerations.

(B) $n = 2, k = 3$. In this case $\hat{\Omega}$ has real dimension $2n = 4$, while Ω is a real 3-dimensional submanifold of \mathbb{C}^2 , contained in $\partial\hat{\Omega}$. The analytic discs D_t are attached to Ω . The union of their closures constitutes $\hat{\Omega}$.

The restriction of the parameterization mapping G to the 4-dimensional manifold $b\Sigma$ is a foliation with the 3-dimensional base $\Omega \setminus \partial\Omega$ and the 1-dimensional fibers $G^{-1}(b)$, $b \in \Omega \setminus \partial\Omega$. The fibers correspond to the one-parameter family of the curves passing through one point in Ω .

Now we are ready to formulate the main result, which is just a general form of Theorems 1 and 2.

Theorem 2.2. Let $Q = (F, G): \overline{\Sigma} \mapsto \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}$ be a real-analytic CR-mapping, that is Q is holomorphic on each complex fiber $\Delta \times \{t\}$.

Suppose that:

- (i) The mapping G is $(2n, k)$ -regular;
(ii) The mapping Q degenerates on the boundary $b\Sigma$, meaning that

$$F = f \circ G$$

holds on $b\Sigma$ for some smooth function f on $\Omega = G(b\Sigma)$;

- (iii) The induced homomorphism G_* of the relative homology groups

$$G_*: H_{2n-1}(\Sigma, \Sigma_0; \mathbb{R}) \mapsto H_{2n-1}(G(\Sigma), G(\Sigma_0); \mathbb{R})$$

has a nonzero image, where $\Sigma_0 = \Delta \times \partial M$.

Suppose that one of the two cases takes place:

- (A) $n = 1, k = 2$;
(B) $n = 2, k = 3$ and $H_1(M, \partial M) = 0$.

Then Q is degenerate on the entire $(2n + 1)$ -manifold Σ , meaning that

$$F = \hat{f} \circ G$$

on Σ , for some smooth function \hat{f} , or, equivalently $\text{rank } dQ = 2n < \dim \Sigma = 2n + 1$ and $\text{codim}_{\mathbb{R}} Q(\Sigma) = 2$. The function f is CR-function in the interior of $\Omega = G(b\Sigma)$.

In the case when M is a closed manifold, i.e. $\partial M = \emptyset$, the relative homology groups in condition (iii) of Theorem 2.2 must be replaced by the corresponding homology groups $H_{2n-1}(\overline{\Sigma})$, $H_{2n-1}(G(\overline{\Sigma}))$.

Theorem 2.2 can be exposed as a theorem giving conditions of existence of an extension \hat{f} of the commutative diagram

$$\begin{array}{ccc} b\Sigma & & \\ \downarrow G & \searrow F & \\ \Omega = G(b\Sigma) & \xrightarrow{f} & \mathbb{C} \end{array}$$

to the commutative diagram

$$\begin{array}{ccc} \overline{\Sigma} & & \\ \downarrow G & \searrow F & \\ \hat{\Omega} = G(\overline{\Sigma}) & \xrightarrow{\hat{f}} & \mathbb{C} \end{array}$$

where F and G are CR-mappings.

2.2. Reduction of Theorems 1 and 2 to Theorem 2.2

Denote $F(\zeta, t)$ the analytic extension of the function $\zeta \mapsto f(G(\zeta, t))$ from the unit circle $\{|\zeta| = 1\}$ to the unit disc $\{|\zeta| < 1\}$. Such extension exists by the condition (*) of Theorems 1

and 2. Then the two functions F and G define a smooth mapping

$$Q = (F, G): \bar{\Sigma} = \bar{\Delta} \times M \mapsto \mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}.$$

Now we want to show that Theorem 2.2, applied to the functions (F, G) , is equivalent to Theorems 1 and 2, applied to the function f .

First of all, the conditions (i) of regularity for G are the same in Theorems 1 and 2, and in Theorem 2.2. This condition implies that G is a covering over $\Omega \setminus \partial\Omega$, i.e. there are finite constant number of curves through each interior point in Ω .

The condition (ii) in Theorem 2.2 holds because

$$F(\zeta, t) = f(G(\zeta, t)), \quad \zeta \in \partial\Delta, \quad t \in M,$$

by the construction of F .

Thus, to complete comparing the conditions we need only to check that properties (a) in Theorems 1 and 2, formulated in terms of analytic discs, imply the property (iii) in Theorem 2.2, formulated in terms of the induced homomorphisms of the homology groups.

Proposition 2.3. *The conditions (a) in Theorems 1 and 2 are equivalent to condition (iii) in Theorem 2.2.*

Proof. Consider first the case of closed M . Suppose that condition (a) of homological nontriviality holds. The homology groups of the compact manifolds $\Sigma = \bar{\Delta}l \times M$ and M are isomorphic and generated by the the fundamental class $[M]$ which is the homology class of the cycles $C = \{0\} \times M$ and M correspondingly.

The $(2n - 1)$ -cycle

$$C = \{0\} \times M$$

which we will call *central cycle*, will play an important role in the sequel.

The image cycle $c = G(C)$ intersects each analytic disc

$$D_t = G(\Delta \times \{t\})$$

and by (a) the cycle c represents a nonzero element in $H_{2n-1}(G(\bar{\Sigma}))$. Therefore $G_* \neq 0$ and hence the condition (iii) holds.

The case of nonclosed manifolds M and relative homologies is considered analogously.

It remains to check that in the planar case, $n = 1$, $k = 2$ and $\dim M = 1$, the condition of homological nontriviality converts to the more transparent condition of Theorem 1 of empty intersection of the closures of analytic discs \bar{D}_t .

Let us start with a simple remark. The domains \bar{D}_t have a common point, b , if and only if the curve $G^{-1}(b)$ intersects each closed disc

$$\Delta_t = \bar{\Delta} \times \{t\}.$$

Therefore if the common point b exists then the homology class $[G^{-1}(b)] \in H_1(\bar{\Sigma})$ is not 0.

However, G maps this curve to the point b , and it follows that $G_* = 0$. Thus, if (a) in Theorems 1, 2 does not hold, then (iii) in Theorem 2.2 fails.

In the opposite direction, assume that (a) is true, i.e., $\bigcap_{t \in M} \overline{D}_t = \emptyset$. We claim that then the G -images of $b\Sigma$ and Σ coincide:

$$\Omega = \hat{\Omega}.$$

Indeed, if not then some value b is taken by G in Σ but not in $b\Sigma$. Then the function

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{dG(\zeta, t)}{G(\zeta, t) - b}$$

is continuous in t . It evaluates the number of zeros in the disc Δ_t and therefore is constant. Since it is different from zero for least one value of t it is not zero for all t . Therefore G takes on the value b on each disc Δ which means that b belongs to all domains D_t . Contradiction.

Now suppose that (iii) fails. Then G maps the central cycle

$$C = \{0\} \times M$$

to a cycle $c = G(C)$, (relatively) homologically equivalent to zero in $\hat{\Omega}$.

Consider first the case $\partial M = \emptyset$ and $M = S^1$ in Theorem 1. Then the cycle c is homotopic to 0. Moreover, since the cycle c belongs to the interior of $\hat{\Omega} = \Omega$, it can be contracted, within the interior of Ω , to a point

$$b \in \Omega \setminus \partial\Omega.$$

The mapping G is a locally trivial foliation over $\text{int } \Omega$ (by the regularity condition, the critical values of G are located on $\partial\Omega = \partial\hat{\Omega}$). Applying the axiom about covering homotopy [37, Theorem 4.1] we obtain that the homotopy $c \sim \{b\}$ lifts up to a homotopy $C \sim C'$, where C' is a nontrivial 1-cycle in $\overline{\Sigma}$.

The new cycle C' projects, by the mapping G , to the point b :

$$G(C') = \{b\}.$$

The cycle C' is homotopic to the central cycle $C = \{0\} \times M$ and hence it must intersect each complex disc $\overline{\Delta} \times \{t\}$. But this is just another way of saying that

$$b \in \bigcap_{t \in M} \overline{D}_t.$$

Thus, the condition (a) of Theorem 1 fails.

In the case $M = [0, 1]$ in Theorem 1, the argument is analogous. Namely, if the condition (iii) in Theorem 1 fails and the cycle $c = G(C)$ is relatively trivial, then there is a smooth curve

$$c_1 \subset G(\Sigma_0) = \overline{D}_0 \cup \overline{D}_1$$

such that $c \cup c_1$ a boundary in the interior of $\hat{\Omega} = \Omega$. The new curve is the image $c \cup c_1 = G(C \cup C_1)$, of the union of the cycle C with a 1-chain (a curve)

$$C_1 \subset \Sigma_0 = \Delta_0 \times \Delta_1.$$

Then the curve $c \cup c_1$ is homotopic, in the interior of $\Omega = \hat{\Omega}$ to a point. We can perform the homotopy in two steps.

First, we deform $c \cup c_1$ so that the part c_1 contracts to a point within the set $\bar{D}_0 \cup \bar{D}_1$. Then c is deformed homotopically to a closed curve (cycle) $c' \subset \Omega$.

Again, by the axiom on covering homotopy, the cycle $C \cup C_1$ can be homotopically transformed in $\bar{\Sigma}$ so that C_1 contracts to a point within Σ_0 and the cycle C transforms to a cycle C' with end points belonging to Σ_0 . This new cycle intersects all the discs Δ_t .

Second, we contract c , within the interior of Ω , to an inner point $b \in \Omega$. Correspondingly, by the covering homotopy, C' can be transformed to a cycle $C'' \subset \Sigma$, which intersects each disc Δ_t , $t \in M$, and projects by G to the point b , i.e.,

$$G(C'') = \{b\}.$$

This says that for any $t \in M$

$$b \in G(\Delta_t) = D_t$$

and therefore the condition (a) in Theorem 1 fails. The equivalence of conditions (a) and (iii) is proved. \square

To complete the reduction we have to prove that the conclusion of Theorem 2.2 implies the conclusions of Theorems 1 and 2.

Indeed, Theorem 2.2 asserts the relation $F = \hat{f} \circ G$ in Σ . This means that F is constant on the level curves $G = \text{const}$ in Σ , or, in other words, that the values of the analytic extensions at a fixed point $z \in D_t$ do not depend on t . This immediately implies that f is analytic ($n = 1$) or CR-function ($n = 2$) in Ω .

An alternative proof is as follows. Suppose that Q is degenerate, $F = \hat{f} \circ G$ on $\bar{\Sigma}$ for some smooth function \hat{f} on $\hat{\Omega}$.

Let $n = 1$. The functions F and G are analytic in ζ and differentiating in $\bar{\zeta}$ the above relation between F and G yields

$$(\bar{\partial} \hat{f} \circ G) \overline{\partial_{\zeta} G} = 0$$

and therefore $\partial \hat{f} = 0$ on $\hat{\Omega}$. Thus, \hat{f} is analytic on $\hat{\Omega}$ and hence $f = \hat{f}|_{\Omega}$ is analytic on Ω .

Let $n = 2$ and the 3-manifold Ω is locally defined by the equation $\rho(z_1, z_2) = 0$, where ρ is a smooth real-valued function in a neighborhood of a point in Ω . Then

$$\rho(G_1, G_2) = 0.$$

Differentiating in the variable $\bar{\zeta}$ the latter identity and the identity

$$F(\zeta, t) = \hat{f}(G_1(\zeta, t), G_2(\zeta, t))$$

leads to a linear system, with the zero determinant due to the embedding condition $\partial_{\zeta}(G_1, G_2) \neq (0, 0)$.

Vanishing of the determinant can be written as

$$\bar{\partial} \hat{f} \wedge \bar{\partial} \rho = 0,$$

which is just the *CR*-condition for the function $\hat{f} = f$ in Ω .

This completes the reduction.

2.3. Necessity of the conditions

Before going to the proof of Theorem 2.2, let us show necessity of conditions in Theorems 1, 2 and, correspondingly, in their topological versions Theorem 2.2.

Let us consider the case A, $n = 1$, $k = 2$, when the manifold Ω is a domain in the plane.

The main condition (a) in Theorems 1 and 2 is that the intersection of the closed analytic discs from the family is empty. The following example, demonstrating the importance of this condition, belongs to Globevnik [17,20]. His example is given by function

$$f(z) = z^n / \bar{z}, \quad n > 1,$$

where n is large enough to provide desirable smoothness at the origin.

The function \bar{z} extends meromorphically from any circle in the plane inside the disc, with a simple pole at the center, and with a simple zero, which is inside the disc if and only if when the point 0 is outside.

It follows that f extends analytically, without poles, inside any circle enclosing, or even containing, the origin. For any family of circles surrounding the origin, the corresponding closed discs have the common point 0, so the condition (a) fails. Nevertheless, function f is not holomorphic.

In the context of Theorem 2.2, the above example translates as follows:

$$M = S^1, \quad G(\zeta, t) = Re^{it} + r\zeta,$$

where R, r are nonnegative numbers, corresponding to rotation of a circle of radius r around a circle of radius R , and may serve a simplest model for understanding the constructions in this article.

The influence of the property of having or nonhaving a fixed point inside all the curves, to detecting analyticity in the sense of Theorem 1, was studied in details by Globevnik in [20]. He gave there other interesting examples of the above type. For instance, Globevnik showed that rotations of equilateral triangle around its center does not detect analyticity but this is not true for nonsymmetric triangles.

Define

$$F(\zeta, t) = \zeta e^{it} \frac{(Re^{it} + r\zeta)^n}{re^{it} + R\zeta}, \quad n > 1,$$

on the solid torus $\{|\zeta| \leq 1\} \times S^1$. In this case the sets Ω and $\hat{\Omega}$ are

$$\begin{aligned}\Omega &= G(\partial\Delta \times S^1) \\ &= \{|R-r| \leq |z| \leq R+r\}, \\ \hat{\Omega} &= G(\Delta \times S^1) \\ &= \{\max(R-r, 0) \leq |z| \leq R+r\}.\end{aligned}$$

The regularity condition hold as the Jacobian of G equals to

$$\frac{\partial(G, \bar{G})}{\partial(\psi, t)} = 2iRr \sin(t - \psi)$$

and it vanishes if and only if $\psi - t = 0$ or π , i.e. on the circles of radii $R+r$ and $r-R$ centered at 0 which constitute the boundary of the annulus Ω .

If $r < R$ then F has pole at $\zeta = (-r/R)e^{it} \in \Delta$ and the condition $(*)$ fails. However, if $R \leq r$ then this pole is outside of Δ and $(*)$ holds. At the same time, the condition (iii) obviously fails because the nontrivial cycle

$$\{(-(R/r)e^{it}, e^{it})\}$$

is mapped by G to the point 0 and hence that the induced mapping G_* of the homology groups is trivial. Correspondingly, the conclusion of Theorem 1 fails, too, as F and G are functionally independent in the solid torus and the mapping (F, G) is nondegenerate.

Consider the case B (Theorem 2), related to testing CR -functions on hypersurfaces. It is easy to see that the condition (a) cannot be removed. The corresponding example exists in any dimension. Indeed, let $\Omega = S^{2n-1}$ be the unit sphere in \mathbb{C}^n . Consider the analytic discs which are intersections of the complex unit ball $B^{2n} \subset \mathbb{C}^n$ with complex lines L passing through the origin. The boundaries of the discs are circles, covering the unit sphere (Hopf foliation). The condition (a) does not hold, as 0 is the common point of the discs. Correspondingly, the assertion of Theorem 2 is not true as any function constant on the circles $L \cap S^{2n-1}$ but real valued and nonconstant on the sphere S^{2n-1} provides a counterexample.

3. Proof of Theorem 2.2

3.1. Informal comment on the proof of Theorem 2.2

The problem under consideration may be viewed as a version of an argument principle for boundaries of small dimensions (of codimensions greater than 1). Let us explain what do we mean by that.

A simplest one-dimensional analog of Theorem 2.2 could be the following simple fact: given a smooth mapping

$$G: \bar{\Delta} \mapsto I$$

of the unit disc onto a segment $I = [\alpha, \beta] \subset \mathbb{R}$, whose restriction to $\partial\Delta$ is a smooth covering over the interval (α, β) , and a function F from the disc-algebra, smooth up to the boundary, then $F = \text{const}$ provided F takes equal values on the intersections $G^{-1}(b) \cap \partial\Delta$, $b \in I$.

Here is the proof in the form we need (a similar argument was used in [7,13,17]). Suppose there exists $b \in F(\bar{\Delta}) \setminus F(\partial\Delta)$. Then by the argument principle

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{dF}{F-b} = \#\{z \in \Delta: F(z) = b\} > 0.$$

On the other hand, from the condition we have $F = f \circ G$ on $\partial\Delta$ for some smooth function f and then

$$\frac{1}{2\pi i} \int_{\partial\Delta} \frac{dF}{F-b} = \deg G \cdot \frac{1}{2\pi i} \int_I \frac{df}{f-b} = 0,$$

because the Brouwer degree of the mapping $G: \partial\Delta = S^1 \mapsto I$ is zero, $\deg G = 0$.

The obtained contradiction says that $F(\bar{\Delta}) = F(\partial\Delta)$ which implies $f = \text{const}$, as otherwise, the curve $F(\partial\Delta)$ must contain an open set, due to the openness property.

The above argument can be viewed as a simple model of the idea of the proof of our main result, Theorem 2.2. However, the main difference and the main difficulty is that in our case the function F depends on the additional parameter t and the fibers of the foliation G do not belong to the complex discs where F is analytic. Moreover, the fibers are transversal to the discs. Our goal is to prove that then, still, F is constant on the G -fibers and hence the mapping $Q = (F, G)$ is degenerate: Q maps the level curves $G^{-1}(b)$ to points.

The topological interpretation of the above toy model might be the following: if the analytic function F collapses the boundary circle to a curve then F collapses the unit discs as well, and the images of the disc and its boundary coincide: $F(\bar{\Delta}) = F(\partial\Delta)$.

Theorems 2.2 may be interpreted in the similar topological terms. Take for example, the case $n = 1$, $k = 2$, corresponding to the strip-problem for a periodic family of Jordan curves.

The mapping G from Theorem 2.2 maps the 2-dimensional torus T^2 to a planar domain, Ω . Now, the function F is constant on the level sets of G , as $F = f \circ G$, and therefore the composite mapping Q , defined on the solid torus,

$$Q = (F, G): \Sigma \mapsto \mathbb{C}^2 \cong \mathbb{R}^4$$

collapses the boundary torus T^2 to a manifold diffeomorphic to a planar domain (which is the graph of the function f over Ω .)

The image $Q(\Sigma)$ of the interior is a set in \mathbb{R}^4 attached to the 2-dimensional image of the boundary torus T^2 , and this image is “flat,” i.e. is topologically equivalent to a planar domain. In Theorem 2.2 we prove in that if the mapping Q is homologically nontrivial, meaning that Q induces nontrivial homomorphism of the homology group, then Q collapses the interior of the solid torus as well it does with its boundary. In other words, the image of the 3-dimensional solid torus Σ is 2-dimensional and is contained in the image of the boundary and $Q(\bar{\Sigma}) = Q(\partial\Sigma)$, similarly to our model example.

3.2. The plan of the proof of Theorem 2.2

We assume, to the contrary of the assertion, that Q is nondegenerate and our goal will be arriving to a contradiction.

Step 1. A key point is Lemma 3.3 where we prove the symmetry relation (2) of linking numbers which are periods (winding numbers) of one function on the zero sets of the other. Lemma claims that certain logarithmic residue type integrals are equal. This equality is true for functions analytic in ζ variable and having the important property: they take equal values at the points on the boundary manifold $b\Sigma$, belonging to the same G -fiber.

Lemma 3.3 is proved by the standard technique from residues theory, assuming removing neighborhoods of singularities, applying Stokes formula to differential forms on the remaining manifold and then shrinking the removed neighborhoods.

The key point is that the constancy of functions on the G -fibers on $b\Sigma$ provides cancellation of a surface term in the Stokes formula and this leads to the needed symmetry relation.

Step 2. In the second, analytic, part of the proof we study the minors J of the Jacobi matrix $J(F, G)$ in Σ . First, we observe that the zero sets of J (the Q -critical sets) contain the central cycle $C = \{0\} \times M$. Since we assumed that Q is not degenerate, there are minors J which are not identically zero and hence the zero sets $J^{-1}(0)$ are essentially $(2n - 1)$ -dimensional.

The final part of the proof, for instance in the case $\partial M = \emptyset$ (Theorems 1 and 3), is as follows.

Take for simplicity the case $n = 1$. We apply the symmetry relation proved in Lemma 3.3 to the functions J and $G - b$. However, Lemma 3.3 requires that both functions identify points on $b\Sigma$ from the same G -fiber. The function $G - b$ certainly satisfies this property, but J may not. Nevertheless, it appears that the “phase” part, $\Theta = J/\bar{J}$, does satisfy the required property away from zeros of J . This is enough to write the symmetry relation for J and $G - b$, as the change of the argument of Θ is just the doubled change of the argument of J .

Now we use the symmetry relation to obtain contradiction with the assumption that J does not vanish identically.

First, take $b \notin \hat{\Omega} = G(\bar{\Sigma})$. Then $G^{-1}(b) = \emptyset$ and the integral over $G^{-1}(b)$ in the symmetry relation (2) vanishes simply because the set of integration is empty.

On the other hand, condition (iii) guarantees that the image $c = G(C)$ of the nontrivial loop $C = \{0\} \times M$, is homologically nontrivial in $\hat{\Omega}$. Since $C \subset J^{-1}(0)$ then $c \subset G(J^{-1}(0))$.

Then b can be chosen out of $\hat{\Omega}$ and so that the loop c has a nonzero index, $\text{ind}_b c \neq 0$, with respect to the point b and we have the desired contradiction because then the second integral in (2), over $G(J^{-1}(0))$, is different from zero.

This contradiction implies that $J = 0$ identically which is just the conclusion of Theorem 2.2.

In the case $\partial M \neq \emptyset$ (Theorems 2 and 4) we slightly modify the above argument. First, an additional surface term appears in Stokes formula, corresponding the boundary of M . Then the symmetry relation delivers a jump-function which counts algebraic number of intersections with the image $G(J^{-1}(0))$ of the critical set.

The homological condition (iii) provides that there exists a path, L , which crosses the union of the discs D_t , $t \in M$, without meeting the discs with $t \in \partial M$ and has the nonzero intersection index with $G(C) \subset G(J^{-1}(0))$.

The final argument rests on determining the relative homological class of a cycle of codimension 1 by intersection indices with a transversal curve L . The contradiction comes from comparing the total algebraic number of the intersection indices (jumps of the counting func-

tion) which is different from 0 with the total variation of the counting function along the path L which appears to be 0.

3.3. The Brouwer degree of the mapping G

Let us study properties of the mapping G .

We start with the planar case $(2n, k) = (2, 2)$ and with the case of the closed manifold M . In this case

$$M = S^1, \quad \Sigma = \Delta \times S^1, \quad b\Sigma = \partial\Sigma = S^1 \times S^1 = T^2.$$

Lemma 3.1. *The Brouwer degree of the mapping*

$$G: T^2 \mapsto G(T^2) = \Omega$$

is zero, $\deg G = 0$.

Proof. The image $G(T^2)$ is a compact domain $\Omega = G(T^2) \subset \mathbb{C}$.

Embed Ω to the Riemann sphere as a compact subset $\tilde{\Omega} \subset S^2$. Then we can view the mapping G as a mapping to S^2 :

$$G: T^2 \mapsto \Omega \subset \tilde{\Omega} \subset S^2.$$

The 2-dimensional homology groups of the 2-dimensional compact manifolds without boundaries are

$$H_2(T^2; \mathbb{Z}) = H_2(S^2; \mathbb{Z}) = \mathbb{Z}$$

and the Brouwer degree is defined by the relation

$$G_*(\mu_1) = \deg G \cdot \mu_2,$$

where μ_1, μ_2 are generators (fundamental classes) of the corresponding homology groups $H_2(T^2; \mathbb{Z})$ and $H_2(S^2; \mathbb{Z})$. However, the image of G is a compact subdomain of the sphere S^2 and is contractible, hence $G_*(\mu_1) = 0$ and therefore $\deg G = 0$. \square

We will need a local version of Lemma 3.1.

Corollary 3.2. *If $O \subset \Omega$ is a submanifold, then the local Brouwer degree of the restricted mapping*

$$G: T^2 \setminus G^{-1}(O) \mapsto \Omega \setminus O$$

is zero.

Proof. Recall that locally the Brouwer degree is defined as the algebraic number of points in the preimage $G^{-1}(b)$ of a regular value $b \in \Omega$, i.e.

$$\deg G = \sum_{G(p)=b} \text{sign } dG_p.$$

Clearly, this number preserves after simultaneous removing a set O from $\Omega = G(T^2)$ and its full G -preimage $G^{-1}(O)$ from T^2 . \square

3.4. Linking numbers

Everywhere in this section we assume that $\partial M = \emptyset$, but combine both cases $(2n, k) = (2, 2)$ and $(2n, k) = (4, 3)$, so that Ω is either a planar domain or 3-dimensional closed manifold in \mathbb{C}^2 . Note that if $\partial M = \emptyset$ then $b\Sigma$ coincides with the topological boundary, $b\Sigma = \partial\Sigma$.

Denote β_{MB} the Martinelli–Bochner $(2n-1)$ -differential form β_{MB} (Martinelli–Bochner kernel) in the space \mathbb{C}^n (see, e.g. [9,36]). It will be convenient to denote the coordinates in \mathbb{C}^n by z_2, \dots, z_{n+1} . Then

$$\beta_{\text{MB}} = \frac{1}{(2\pi i)^n} |z|^{-2n} \omega'(\bar{z}) \wedge \omega(z),$$

where $\omega'(\bar{z}) = \sum_{j=2}^{n+1} (-1)^j \bar{z}_j d\bar{z}[j]$ and $\omega(z) = dz_2 \wedge \dots \wedge dz_{n+1}$ and $[j]$ means that the differential $d\bar{z}_j$ is skipped.

The Martinelli–Bochner form coincides with the $(2n-1)$ -surface form in the real Euclidean space R^{2n} and integration in this form is well related with computing degrees of mappings, and linking numbers (see e.g., in the close context, [10,36]).

Define the $2n$ -form in \mathbb{C}^{n+1} :

$$\beta = \frac{1}{2\pi i} \frac{dz_1}{z_1} \wedge \beta_{\text{MB}}.$$

In particular, for $n = 1$ we have

$$\beta = \frac{1}{(2\pi i)^2} \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2}.$$

The form β is closed in \mathbb{C}^{n+1} with the deleted linear spaces $\{z_1 = 0\}$ and $\{z' = (z_2, \dots, z_{n+1}) = 0\}$.

The following lemma is a key one in the proof of Theorem 2.2. It asserts a symmetry of linking numbers associated to the level sets of functions, compatible with the foliation G . By G -compatibility we understand the constancy on the G -fibers.

Lemma 3.3. *Let $G: \bar{\Sigma} \mapsto \mathbb{C}^n$ be the mapping from Theorem 2.2, and $\partial M = \emptyset$. Let J be a nonzero C^1 -function*

$$J: \bar{\Sigma} \mapsto \mathbb{C}$$

having the following properties:

1. The zero set $J^{-1}(0)$ is a $(2n - 1)$ -chain in $\overline{\Sigma}$.
2. The set $J^{-1}(0) \cap b\Sigma$ contains the critical set $\text{Crit}(G)$ of the mapping G on $b\Sigma$.
3. The quotient

$$\Theta = \frac{J}{\overline{J}} = \exp(2i \arg J) : (\partial \Delta \times M) \setminus J^{-1}(0) \mapsto S^1$$

is G -compatible, i.e., Θ is representable, outside of zeros of J , in the form

$$\Theta = \sigma \circ G, \quad (1)$$

for some smooth function σ defined on $G(b\Sigma \setminus J^{-1}(0))$.

Then

- (a) for any regular value $b \in \mathbb{C}^n$ of the mapping G , such that $J^{-1}(0) \cap G^{-1}(b) = \emptyset$, holds:

$$2 \int_{J^{-1}(0)} (G - b)^* \beta_{\text{MB}} = \frac{1}{2\pi i} \int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\overline{J}}{\overline{J}} \right); \quad (2)$$

- (b) the chain $c = G(J^{-1}(0))$ is a cycle, i.e. $\partial c = \emptyset$.

Proof. We use the usual technique from the theory of residues, which involves deleting neighborhoods of singular sets, applying Stokes formula to the remaining manifold and then shrinking the neighborhoods (e.g. [10,36]).

First of all, we will complete the set $J^{-1}(0)$ to make it G -compatible on $b\Sigma$, i.e. union of G -fibers. For this aim, consider the full G -preimage

$$N_0 = [G^{-1}(G(J^{-1}(0)))] \cap b\Sigma.$$

Define N the compliment of the zeros of J on $b\Sigma$ in N_0 :

$$N = N_0 \setminus (J^{-1}(0) \cap b\Sigma).$$

By the construction, the union

$$(J^{-1}(0) \cap b\Sigma) \cup N = N_0$$

is G -compatible on $b\Sigma$. The set N is either empty, or, by the regularity conditions for G , it is finite union of smooth manifolds of the dimensions less or equal to $2n - 1$.

The first step of the proof of the formula (2) is to construct a family of shrinking neighborhoods, in Σ , of the singular sets $J^{-1}(0)$ and $G^{-1}(b)$.

Define

$$A_\varepsilon = \{|J| < \varepsilon\}, \quad B_\varepsilon = \{|G - b| < \varepsilon\}.$$

Additionally, construct a family N_ε of ε -neighborhoods of the set N .

Now remove the constructed neighborhoods from Σ and denote the remainder:

$$\Sigma_\varepsilon = \Sigma \setminus (A_\varepsilon \cup B_\varepsilon \cup N_\varepsilon).$$

Define the orientation on the manifolds ∂A_ε , ∂B_ε and ∂N_ε by the inward, with respect to Σ_ε , normal vector.

Define on $\overline{\Sigma}_\varepsilon$ the $2n$ -form:

$$\mathcal{E} = \Phi^* \beta = \frac{1}{2\pi i} \frac{d\Theta}{\Theta} \wedge (G - b)^* \beta_{\text{MB}},$$

where $\Phi = (\Theta, G - b)$. The differential form \mathcal{E} has no singularities in $\overline{\Sigma}_\varepsilon$ and is closed there because the form β is closed.

Therefore the Stokes formula yields:

$$\int_{\partial \Sigma_\varepsilon} \mathcal{E} = \int_{\Sigma_\varepsilon} d\mathcal{E} = 0.$$

The boundary of Σ_ε consists of the four parts equipped by the induced orientations:

$$\partial \Sigma_\varepsilon = \partial A_\varepsilon \cup \partial B_\varepsilon \cup \partial N_\varepsilon \cup (b\Sigma \setminus (A_\varepsilon \cup B_\varepsilon \cup N_\varepsilon))$$

and therefore the latter identity reads as

$$\int_{b\Sigma \setminus (A_\varepsilon \cup B_\varepsilon \cup N_\varepsilon)} \mathcal{E} - \int_{\partial A_\varepsilon} \mathcal{E} - \int_{\partial B_\varepsilon} \mathcal{E} - \int_{\partial N_\varepsilon} \mathcal{E} = 0. \quad (3)$$

The signs minus before the last three integrals appear because the induced orientations on ∂A_ε and ∂B_ε , required by Stokes formula, are opposite to the above defined orientations.

Now we take the limit of the integrals, when $\varepsilon \rightarrow 0$.

First, let us check that the added set N contributes nothing, i.e. the integral over ∂N_ε in (3) disappears when $\varepsilon \rightarrow 0$. Indeed, since N is not contained in $J^{-1}(0)$ then by real-analyticity $\dim(\overline{N} \cap J^{-1}(0)) < 2n - 1$. On the other hand, the intersection $\overline{N} \cap G^{-1}(b)$ with the level curve of G is not more than finite.

The differential form \mathcal{E} is the wedge product of the Martinelli–Bochner type form and the logarithmic derivative type form. The first factor has singularities at zeros of functions J and the second one—on the curve $G = b$. We see from the dimensions of these singularities on N that \mathcal{E} is integrable (in a regularized sense) on N and since $\dim N < 2n$, $\deg \mathcal{E} = 2n$, the integral of \mathcal{E} over N is zero. Since the neighborhoods N_ε shrink to N , the last integral in the left-hand side in (3) goes to zero as ε goes to zero.

To understand the limits of other integrals in (3), write the differential form under the integral (3) as

$$\mathcal{E} = \frac{1}{2\pi i} \left(\frac{dJ}{J} - \frac{d\overline{J}}{\overline{J}} \right) \wedge (G - b)^* \beta_{\text{MB}} = \Phi^* \beta - \overline{\Phi}^* \beta,$$

where $\Phi = (J, G - b)$, $\bar{\Phi} = (\bar{J}, G - b)$ and

$$\beta = \frac{1}{2\pi i} \frac{dz_1}{z_1} \wedge \beta_{\text{MB}}.$$

The mapping Φ maps $J^{-1}(0)$ to $\{0\} \times (G - b)(J^{-1}(0))$ and $G^{-1}(b)$ to $J(G^{-1}(b)) \times \{0\}$ and hence change of variables $z = \Phi(\zeta, t) \in \mathbb{C}^2$, i.e. $z_1 = J$, $z' = G - b$, in (3) leads, after letting $\varepsilon \rightarrow 0$, to

$$2 \int_{J^{-1}(0)} (G - b)^* \beta_{\text{MB}} - \frac{1}{2\pi i} \int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right) = \int_{b\Sigma'} \Xi, \quad (4)$$

where

$$b\Sigma' = b\Sigma \setminus (J^{-1}(0) \cup N \cup G^{-1}(b)) = b\Sigma \setminus (N_0 \cup G^{-1}(b)).$$

Here we have used that the forms $(2\pi i)^{-1} dz_1/z_1$ and β_{MB} are correspondingly the arc and the surface measures on the unit circle and the unit sphere in the spaces of the variables z_1 and z' . The sign minus before the second integral comes from the orientation consideration. The factor 2 in the left-hand side is due to the relation

$$\frac{d\bar{J}}{\bar{J}} = -\frac{dJ}{J}$$

on $|J| = \varepsilon$.

Formula (2) will be proved if we would show that the right-hand side in (4) is zero. We will prove this separately for the case when Ω is a domain in \mathbb{C} (the case A, $(2n, k) = (2, 2)$) and for the case when Ω is a hypersurface in \mathbb{C}^2 (the case B, $(2n, k) = (4, 3)$).

In the first case, $(2n, k) = (2, 2)$ we use Corollary 3.2 from Lemma 3.1. By regularity condition, the mapping G is a covering of $b\Sigma \setminus \text{Crit}(G)$ over $\Omega \setminus \partial\Omega$.

The set $b\Sigma'$ is G -compatible because it is obtained by removing from $b\Sigma$ the full G -preimages, N_0 and $G^{-1}(b)$. It does not contain the critical $\text{Crit}(G)$ because this set is removed along with $J^{-1}(0)$. Hence G is a covering on $b\Sigma'$. By Corollary 3.2 from Lemma 3.1, we have that

$$\nu = \deg G|_{b\Sigma'} = 0.$$

By the condition 2, $\Theta = \sigma \circ G$ away from zeros of J . Then the change of variables $G(\zeta, t) = z'$ in the integral yields:

$$\int_{b\Sigma'} \Xi = \nu \cdot \frac{1}{2\pi i} \int_{G(b\Sigma')} \frac{d\sigma}{\sigma} \wedge \beta_{\text{MB}}(z' - b) = 0,$$

because $\nu = 0$.

For $(2n, k) = (4, 3)$, the vanishing of the integral on the right-hand side of (3) follows even more simply. In this case, the mapping $G: b\Sigma \mapsto \Omega$ is a smooth foliation of 4-dimensional manifold over a 3-dimensional manifold, with one-dimensional fibers. The differential form

$$\eta = \frac{d\sigma}{\sigma} \wedge \beta_{\text{MB}}(z' - b)$$

has degree 4 and therefore $\eta = 0$ on the 3-dimensional manifold Ω . Then the pull back of the form η to $b\Sigma$ satisfies $\mathcal{E} = G^*(\eta) = 0$ and therefore the integral of \mathcal{E} over $b\Sigma' \subset b\Sigma$ vanishes.

Now we have completed the proof of the main part, (a) of Lemma 3.3. Notice that the above proof may be briefly exposed by using the language of currents (see [10]).

It remains to check the assertion (b). To this end, take in (2) the point b belonging to the unbounded component V of $\mathbb{C}^n \setminus G(\overline{\Sigma})$. Then the right-hand side in (2) equals zero and hence, after changing of variables $z' = G(\zeta, t)$ in the left-hand side, we obtain

$$\int_c \beta_{\text{MB}}(z' - b) = 0, \quad b \in V.$$

It can be proved by many ways that vanishing of Martinelli–Bochner type integral in a neighborhood of infinity implies that the surface of integration is closed, $\partial c = \emptyset$.

Now the proof of Lemma 3.3 is completed. \square

Remarks. 1. The proof of Lemma 3.3 may be briefly exposed by using the language and the technology of currents (see [10]).

2. In the proof of Theorem 2.2 for case of closed manifold M we will use formula (2) in the situation when $G^{-1}(b) = \emptyset$ and therefore the right-hand side in (2) is zero.

3.5. The Jacobi determinants J , the function Θ , and the central cycle C , for the case $(2n, k) = (2, 2)$

Our immediate goal is to construct a functions J and satisfying the conditions of Lemma 3.3, and vanishing on the central cycle $C = \{0\} \times M$. The function we are going to construct will also carry information about degeneracy of the mapping Q .

Such function turns out to be the Jacobian of the mapping $Q = (F, G)$ in the angular variable ψ and the local coordinate t on M . The needed properties follow from the functional relation between F and G on $b\Sigma$, while vanishing on the central cycle $C = \{\zeta = 0\}$ in $\overline{\Sigma}$ comes from the vanishing at $\zeta = 0$ of the tangent vector field to the unit circle. This vector field acts on analytic functions as the complex derivation $\partial_\psi = i\zeta \partial_\zeta$.

Let us do the corresponding computations. Choose the basis $\partial_\psi, \partial_t$ in the tangent space to $b\Sigma$. Here $\zeta = e^{i\psi}$ and t is the local coordinate on M . If $M = [0, 1]$ then take $t \in [0, 1]$ and if $M = S^1$ then t can be taken the angular variable on the unit circle, $M = S^1 = \{e^{it}\}$.

We will use the notation ∇ for the column:

$$\nabla = \text{column}[\partial_\psi, \partial_t].$$

In this notation, the Jacobi matrix for the mapping G becomes $[\nabla G, \nabla \overline{G}]$.

Lemma 3.4. *Define*

$$J_+ = \det[\nabla F, \nabla \bar{G}], \quad J_- = \det[\nabla G, \nabla F].$$

Thus, J_+ is obtained from the Jacobi matrix of G by replacing ∇G by ∇F , while J_- —by replacing $\nabla \bar{G}$ by ∇F .

Then on $b\Sigma$ the relation holds:

$$J_+ = \det[\nabla G, \nabla \bar{G}](\partial f \circ G), \quad J_- = \det[\nabla G, \nabla \bar{G}](\bar{\partial} f \circ G). \quad (5)$$

Here $\partial, \bar{\partial}$ are the derivatives in z and \bar{z} , respectively.

Proof. We start with the link $F = f \circ G$ between F and G (condition (ii) of Theorem 2.2) on the manifold $b\Sigma = \partial\Delta \times M$. Differentiation in local coordinates on $b\Sigma$ and the chain rule lead to the linear system:

$$[\nabla G, \nabla \bar{G}][\partial f \circ G, \bar{\partial} f \circ G] = \nabla F. \quad (6)$$

Then (5) follows immediately from the Cramer's rule when solve the linear system (6) for $[\partial f \circ G, \bar{\partial} f \circ G]$. \square

In the sequel we will be exploiting only the “minus” minor, J_- , as it possesses needed orientation properties. The function $J(\zeta, t) = J_-(\zeta, t)$ is just the Jacobian

$$J = \frac{\partial(F, G)}{\partial(\psi, t)} = \partial_\psi F \partial_t G - \partial_\psi G \partial_t F,$$

where $\zeta = |\zeta|e^{i\psi}$.

On the manifold $S^1 \times M$, the function J also can be understood as the Poisson bracket $J = \{F, G\}$. The function J is defined in the entire $\Delta \times M$ and can be expressed there in terms of complex derivatives in ζ :

$$J(\zeta, t) = i\zeta(\partial_\zeta F \partial_t G - \partial_\zeta G \partial_t F). \quad (7)$$

Consider a smooth G -level curve $\Gamma = G^{-1}(b)$, parametrized by $\zeta = \zeta(t)$. Differentiating the identity $G(\zeta(t), t) = b$ in t and taking into account that $\partial_\zeta G \neq 0$ we obtain for the directional derivative ∂^G along the G -level curves:

$$\partial^G F = \frac{d}{dt} F(\zeta(t), t) = \partial_\zeta F \zeta'(t) + \partial_t F = -\frac{\partial_\zeta F \partial_t G - \partial_\zeta G \partial_t F}{\partial_\zeta G}$$

and therefore J is related to the directional derivative by

$$J = -i\zeta \partial_\zeta G \partial^G F.$$

Thus, vanishing of J identically in Σ means that $F = \text{const}$ on the G -level curves and therefore F is a function of G .

Lemma 3.5. *The functions $J = J_-$ possesses the properties:*

1. $J(0, t) = 0$ for all $t \in M$.
2. The zero set $J^{-1}(0)$ contains the critical set $\text{Crit}(G) = \{\det[\nabla G, \nabla \bar{G}] = 0\} \subset b\Sigma$.
3. On $b\Sigma$, away from zeros of J , the relation holds:

$$\Theta = \frac{J}{\bar{J}} = \sigma \circ G,$$

where

$$\sigma = -\frac{\bar{\partial} f}{\partial f}.$$

Proof. The property 1 follows from (7). The property 2 follows from the representation of J in formula (5). The property 3 also follows from (5) because the Jacobian $\det[\nabla G, \nabla \bar{G}]$ obviously takes purely imaginary values. \square

Thus, the function J satisfies all the conditions of Lemma 3.3, except, maybe, the assumption 1 about a nice structure of the zero set $J^{-1}(0)$.

3.6. The functions J in the case $(2n, k) = (4, 3)$

Constructing Jacobi minors. Now we need a version of Lemma 3.5 for the case B, when $n = 2$ and Ω is a hypersurface in \mathbb{C}^2 , $\dim \Omega = 2n - 1 = 3$. Recall that in this case, the regularity condition says that $\text{rank } dG|_{\partial \Delta \times M} = 3$.

Consider the complex Jacobi matrix

$$[\nabla G_1, \nabla G_2, \nabla F]$$

contains 3 columns and 4 rows. The rows correspond to the derivatives with respect to ψ, t_1, t_2, t_3 , where t_k are local coordinates on M .

Let us first specify how should we understand the differentiation in local coordinates t_k .

Any 3-dimensional compact orientable manifold M is parallelizable (the tangent bundle is trivial), see e.g. [30]. This means that there exist three linearly independent smooth tangent vector fields $X_1, X_2, X_3 \in T(M)$ on M .

In a neighborhood of any point in M we define parametrization t_k via coordinates in the basic X_1, X_2, X_3 in the tangent plane, so that the vector field X_k corresponds to the differentiation in t_k , $X_k = \partial_{t_k}$, $k = 1, 2, 3$.

The degeneracy of the mapping $Q = (F, G)$ would be shown if we prove that all 3×3 minors J^0, J^1, J^2, J^3 are identically zero, where J^0 is obtained from the above Jacobi matrix by removing the first row (containing the derivatives with respect to ψ) and J^k , $k = 1, 2, 3$, are obtained by deleting the row $(\partial_{t_k} G_1, \partial_{t_k} G_2, \partial_{t_k} F)$.

Lemma 3.6. $J^1, J^2, J^3 \equiv 0$ imply $J^0 \equiv 0$.

Proof. Denote

$$h_\psi = \partial_\psi(G_1, G_2, F), \quad h_\alpha = \partial_{t_\alpha}(G_1, G_2, F), \quad \alpha = 1, 2, 3.$$

Suppose $J^0(\zeta_0, t_0) \neq 0$. Then $J^0(\zeta, t) \neq 0$ for (ζ, t) in a neighborhood of (ζ_0, t_0) and the vectors h_1, h_2, h_3 are linearly independent over \mathbb{C} for such (ζ, t) .

On the other hand, the condition implies that each system consisting of the vector h_ψ and any two vectors from the set h_1, h_2, h_3 , is \mathbb{C} -linearly dependent. Since the last two vectors in the above triple are linearly independent, we have

$$h_\psi \in \Pi_{\alpha, \beta} = \text{span}\{h_\alpha, h_\beta\}, \quad \alpha, \beta = 1, 2, 3, \alpha \neq \beta.$$

But due to the linear independence of the vectors h_1, h_2, h_3 we have $\Pi_{1,2} \cap \Pi_{1,3} \cap \Pi_{2,3} = \{0\}$. The vector h_ψ belongs to this intersection and hence

$$h_\psi = (\partial_\psi G_1, \partial_\psi G_2, \partial_\psi F) = (0, 0, 0).$$

The derivatives are evaluated at points (ζ, t) in an open set in Σ . However it is impossible as the functions $G_1(\zeta, t), G_2(\zeta, t)$ are nonconstant analytic (in $\zeta = re^\psi$) functions. \square

Lemma shows that, to prove the degeneracy of the mapping (F, G) , it suffices to check vanishing the minors J^1, J^2, J^3 . The advantage of these minors is that their first row is $\partial_\psi(G, F)$ and therefore vanishes at $\zeta = 0$, the fact which is crucially important to us. The minor J^0 does not have this property.

Let us fix some of the minors, for instance, $J = J^3$. This minor corresponds to deleting from Jacobi matrix the last row with the derivatives with respect to t_3 :

$$J = \begin{vmatrix} \partial_\psi G_1 & \partial_\psi G_2 & \partial_\psi F \\ \partial_{t_1} G_1 & \partial_{t_1} G_2 & \partial_{t_1} F \\ \partial_{t_2} G_1 & \partial_{t_2} G_2 & \partial_{t_2} F \end{vmatrix}.$$

Remind that the local parameters on M are chosen so that X_1 and X_2 are differentiations in the first two coordinates $X_1 = \partial_{t_1}, X_2 = \partial_{t_2}$.

To the end of this section, in order not to overload notations, we will denote by the symbol “det” the minors including the rows with the derivatives $\partial_\psi, \partial_{t_1}, \partial_{t_2}$, but not ∂_{t_3} .

Properties of the function J . We have chosen one of the minors J^1, J^2, J^3 , namely $J = J^3$, but what follows applies for any minor J^α .

Suppose that $J = J^3$ is not identically zero. We start with checking that J/\bar{J} is G -compatible on $b\Sigma$ away from the zeros of J .

It would be more convenient to perform computations for arbitrary n and then set $n = 2$. Let $\rho = \rho(z, \bar{z})$ be a smooth real-valued defining function in a neighborhood U of the hypersurface Ω :

$$\Omega = \{z \in \mathbb{C}^n: \rho(z, \bar{z}) = 0\}, \quad \nabla \rho \neq 0.$$

The function f is assumed to be extended as a smooth function in U .

Consider the tangential CR-operators on Ω :

$$\bar{\partial}_{\mu,v} = \bar{\partial}_\mu \rho \bar{\partial}_v - \bar{\partial}_v \rho \bar{\partial}_\mu.$$

Here ∂_μ and $\bar{\partial}_v$ are derivatives in z_μ and \bar{z}_μ correspondingly. The system $\bar{\partial}_{\mu,v}$, $\mu < v$, forms a basis in the complex tangent $\bar{\partial}$ -bundle $T^{0,1}(\Omega)$.

We start again with the main relation on the boundary manifold $b\Sigma = \partial\Delta \times M$:

$$F = f \circ G$$

and differentiate it in the local coordinates ψ, t :

$$\sum_{j=1}^n (\partial_j f \circ G) \nabla G_j + \sum_{j=1}^n (\bar{\partial}_j f \circ G) \nabla \bar{G}_j = \nabla F. \quad (8)$$

The extra relation between the gradients comes from the equation of the hypersurface Ω :

$$\rho(G_1, \dots, G_n, \bar{G}_1, \dots, \bar{G}_n) = 0.$$

Differentiating yields:

$$\sum_{j=1}^n (\partial_j \rho \circ G) \nabla G_j + \sum_{j=1}^n (\bar{\partial}_j \rho \circ G) \nabla \bar{G}_j = 0. \quad (9)$$

Fix μ such that $\bar{\partial}_\mu \rho \neq 0$ in a neighborhood of a point in Ω . Express $\nabla \bar{G}_\mu$ from (9) and substitute in (8). After grouping terms we obtain:

$$\sum_{j=1}^n [(\partial_{\mu,j} f) \circ G] \nabla G_j + \sum_{j \neq \mu} [(\bar{\partial}_{\mu,j} f \circ G)] \nabla \bar{G}_j = \bar{\partial}_\mu \rho \nabla F,$$

where $\partial_{\mu,j} = \bar{\partial}_\mu \rho \partial_j - \partial_j \rho \bar{\partial}_\mu$.

The columns (gradients) are vectors with $2n$ components. Solve by the Cramer's rule the resulting $2n \times (2n - 1)$ system for the $2n - 1$ unknowns $(\bar{\partial}_{\mu,v} f) \circ G$, $v \neq \mu$:

$$[\bar{\partial}_{\mu,v} f \circ G] \det[\nabla G, \nabla \bar{G}[\mu]] = \bar{\partial}_\mu \rho \det[\nabla G_1 \cdots \nabla G_n, \nabla \bar{G}_1 \cdots [\mu] \cdots \nabla F \cdots \nabla \bar{G}_n], \quad (10)$$

where ∇F is inserted in the place of the v th column $\nabla \bar{G}_v$ and $[\mu]$ means that the μ th column is skipped.

Let us remind that we agreed to denote by the symbol “det” the minor obtained by deleting the 4th row corresponding to t_3 . Denote $J_{\mu,v}$ the determinant in the right-hand side of (10):

$$J_{\mu,v} = \det[\nabla G; \nabla \bar{G}_1 \cdots [\mu] \cdots \nabla F \cdots \nabla \bar{G}_n].$$

Then on $b\Sigma$ formula (10) implies:

$$J_{\mu,v} = K_\mu (\bar{\partial}_{\mu,v} f \circ G),$$

where

$$K_\mu = (1/\bar{\partial}_\mu \rho) \det[\nabla G, \nabla \bar{G}[\mu]].$$

The function $J_{\mu,v}$ vanishes on the central cycle $C = \{0\} \times M$ because the first line of the determinant is formed by the derivatives $\partial_\psi F$, $\partial_\psi G_j$, $\partial_\psi \bar{G}_j$, and all these derivatives vanish at $\zeta = 0$.

Now suppose that some function $J = J_{\mu,v}$ is not identically zero function and check the properties 1, 2 of Lemma 3.5.

First of all, $J_{\mu,v}(0, t) = 0$ because the first line of the determinants consists of the derivatives in ψ which vanish at $\zeta = 0$, as $\partial_\psi = i\zeta \partial_\zeta$ on holomorphic functions.

To prove that the “phase” part J/\bar{J} takes the same values at points on $b\Sigma$ from the same G -fibers, we will check that the function K_μ takes only real values. To this end, let us write, taking into account that $\partial_\mu \rho = \bar{\partial}_\mu \rho$ and $\nabla \bar{G} = \nabla \bar{G}$:

$$\bar{K}_\mu = (1/\partial_\mu \rho) \det[\nabla \bar{G}, \nabla G[\mu]].$$

Now express the column $\nabla \bar{G}_\mu$ from the relation (9) and substitute to the above determinant. The determinants with equal columns vanish and we obtain:

$$\bar{K}_\mu = (1/\partial_\mu \rho) \det[\nabla \bar{G}_1 \cdots (-\partial_\mu \rho / \bar{\partial}_\mu \rho) \nabla G_k \cdots \nabla \bar{G}_n; \nabla G[\mu]].$$

Cancel $\partial_\mu \rho$ and rearrange the determinant by permutations of the columns:

$$\begin{aligned} \bar{K}_\mu &= (1/\bar{\partial}_\mu \rho)(-1)^{n-1} \det[\nabla \bar{G}[\mu]; \nabla G] = (1/\bar{\partial}_\mu \rho)(-1)^q \det[\nabla G, \nabla \bar{G}[\mu]] = (-1)^q K_\mu, \\ q &= (n-1)(n+2)/2. \end{aligned}$$

We are interested in the case $n = 2$ and therefore $q = 2$. Then $\bar{K}_\mu = K_\mu$ and therefore K_μ is real-valued function.

For $n = 2$, the CR -dimension of Ω equals 1 and the basic tangential $\bar{\partial}$ -operator is,

$$\bar{\partial}_b = \bar{\partial}_{1,2} = \bar{\partial}_1 \rho \bar{\partial}_2 - \bar{\partial}_2 \rho \bar{\partial}_1.$$

The functions $J_{\mu,v}$, corresponding to the indices $(\mu, v) = (1, 2)$ and $(\mu, v) = (2, 1)$, coincide and are equal to

$$J_{1,2} = J_{2,1} = J = \det[\nabla G_1, \nabla G_2; \nabla F].$$

As in the previous section, we define

$$\Theta = \frac{J}{\bar{J}}.$$

We have

$$J = \begin{cases} K_1(\bar{\partial}_{1,2} f \circ G), & \bar{\partial}_1 \rho \neq 0, \\ K_2(\bar{\partial}_{2,1} f \circ G), & \bar{\partial}_2 \rho \neq 0, \end{cases} \quad (11)$$

on $b\Sigma$. We have shown that the functions K_1 and K_2 are real, hence on the manifold $b\Sigma$ holds

$$\Theta = \sigma \circ G,$$

where

$$\sigma = -\frac{\bar{\partial}_{1,2}f}{\overline{\partial_{1,2}f}} = -\frac{\bar{\partial}_{2,1}f}{\overline{\partial_{2,1}f}}.$$

We have used here that $\bar{\partial}_{1,2}f = -\bar{\partial}_{2,1}f$. Let us note also that the function $J(\zeta, t) = J_{1,2}(\zeta, t)$ is analytic in ζ , as well as the function J constructed in the previous section.

The representation (11) and definitions of the functions K_1 and K_2 imply that the set $J^{-1}(0) \cap b\Sigma$ contains the critical set $\text{Crit}(G)$ of G on $b\Sigma$.

Thus, the constructed function J has all the properties from Lemma 3.5. Therefore Lemma 3.3 applies to the function J , after checking the structure of the set $J^{-1}(0)$ (condition 1 in Lemma 3.3).

3.7. The structure of the set $J^{-1}(0)$ and the regularization of the integrals

In the previous we have constructed the Jacobians J for both cases $n = 1$ and $n = 2$ and shown that in each case the function J satisfies Lemma 3.3. However Lemma 3.3 assumes that the zero set $J^{-1}(0)$ is a $(2n - 1)$ -chain for integrals of corresponding differential forms over $J^{-1}(0)$ be defined.

Thus, to apply the symmetry relation proved in Lemma 3.3 we have to understand the structure of the zero set of the functions $J(\zeta, t)$. This is essentially the only point where we use the assumption about real-analyticity of the mapping G and of the function f (and therefore of its analytic extension, F).

In this case the Jacobian $J(\zeta, t)$ is real-analytic in $\bar{\Delta} \times M$, holomorphic in ζ in the unit disc Δ , and the structure of the zero set $J^{-1}(0)$ is well understood.

Namely, for each value of the parameter t the function $J(\zeta, t)$ has finite number of zeros $\zeta = \zeta_j(t)$, unless $J(\cdot, t) \equiv 0$. The zeros constitute a collection of points in the disc $\Delta_t = \Delta \times M$ each of which changes continuously till it reaches the boundary circle $|\zeta| = 1$.

If for some t_0 the zeros are not isolated then $J(\zeta, t_0) = 0$ identically. In this case we will call the disc Δ_{t_0} *zero-disc*. Nevertheless, by real-analyticity, the zero discs are isolated.

Thus, the zero set $J^{-1}(0)$ consists of finite number of smooth $(2n - 1)$ -manifolds $C_j = \{(\zeta(t), t) : t \in M\}$, $\partial C_j \subset b\Sigma$, union with a collection (maybe empty) of zero discs:

$$J^{-1}(0) = \left(\bigcup_j C_j \right) \cup Z(T),$$

where

$$Z(T) = \bigcup_{t \in T} \bar{\Delta}_t = \bar{\Delta} \times T.$$

For $t \notin T$ the section $J^{-1}(0) \cap \{t = t_0\} = \{\zeta_1(t), \dots, \zeta_j(t)\}$ is finite and the function $J(\cdot, t)$ has a constant multiplicity κ_j at its zero $\zeta_j(t)$.

The set T , parametrizing the zero discs, can be defined as

$$T = \bigcap_{s=1}^{\infty} \{t \in M: \partial_{\zeta}^s J(0, t) = 0\}$$

and is an analytic set. We want to prove that the integral over the part $Z(T)$ of the critical set $J^{-1}(0)$ contributes nothing in the integral over $J^{-1}(0)$ in formula (2).

Removing the zero discs in the case A, $(2n, k) = (2, 2)$. Consider the case $n = 1$. Then M is a curve, the parameter t is one-dimensional, and since $J(\zeta, t)$ is a real-analytic nonzero function, it can vanish on at most finite number of discs, that is $\#T < \infty$. Let $T = \{t_1, \dots, t_N\}$, where by t_i we understand (real) coordinates of the corresponding points in T . By real-analyticity, J can be represented as

$$J(\zeta, t) = q(t)J_0(\zeta, t).$$

Here $q(t) = (t - t_1)^{k_1} \dots (t - t_N)^{k_N}$ and the function $J_0(\zeta, t)$ possesses all the properties of the function J except that the zero set $J_0^{-1}(0)$ contains no disc Δ_t .

Since the parameters t_1, \dots, t_N are real, we have

$$\Theta = \frac{J}{\bar{J}} = \frac{J_0}{\bar{J}_0},$$

where J_0 has on each disc $\bar{\Delta}_t$ only finite number of zeros of finite order. Then J_0 satisfies the conditions of Lemma 3.3 and formula (2) holds for J_0 . Thus, in the presence of zero disc we compute integrals in formula (2) for the “regular” part, J_0 of J .

Removing the zero discs in the case B, $(2n, k) = (4, 3)$. In this case $\dim M = 3$ and the dimension of the analytic subset $T \subset \Omega$ can be 0, 1 or 2. By the (real) dimension we understand the maximal dimensions of Whitney strata (see [27]).

If $\dim T = 0$ then $\dim Z(T) = 2$. Since in (2) we integrate 3-form, the cycle $G(Z(T))$ is negligible.

Suppose $\dim T = 1$. Then $Z(T)$ contains 3-chains of the form $c = \bar{\Delta} \times \gamma$, where $\dim \gamma = 1$. This chains might contribute in (2).

However, the condition $H_1(M) = 0$ in Theorem 2.2 implies that γ is homological (relatively homological) to zero in M .

Then the cycle $G(c)$ is homological to zero in the image $G(\bar{\Sigma})$ and since the Martinelli–Bochner form is closed, the integral over the cycle $G(c)$ in (2) is zero.

Finally, if $\dim T = 2$ and $\gamma \subset T$ is a stratum of the pure dimension 2, then locally, in a neighborhood of a point $t_0 \in M$, we have

$$\gamma = \{q(t) = 0\},$$

where $q(t)$ is real-valued. Then, as in the case $n = 1$, locally

$$J(\zeta, t) = q(t)J_0(\zeta, t)$$

where $J_0(\zeta, t)$ has only isolated zeros on discs $\bar{\Delta}_t$.

Since $J/\bar{J} = J_0/\bar{J}_0$, then we conclude that the set $c = \bar{\Delta} \times \gamma$ is removable, by the same argument as for $n = 1$.

4. End of the proof of Theorem 2.2 for the case $\partial M = \emptyset$

Everywhere in this section $n = 1$ or $n = 2$.

Our strategy is as follows. We assume that the mapping Q is nondegenerate. Then the Jacobi minors J constructed in Sections 3.5 and 3.6 are not identically zero. Using the symmetry relation in Lemma 3.3, for the case when the curve $G^{-1}(b)$ is just empty, we obtain that the integral of the differential form $(G - b)^* \beta_{\text{MB}}$ over the set $J^{-1}(0)$ is zero. On the other hand, this set contains the nontrivial cycle C which is mapped by G to a nontrivial cycle in the image. Hence the integral can be made nonzero by appropriate choice of the point b and we obtain contradiction with the assumption that Q is nondegenerate.

Let us start realizing this program. Let J be the function constructed in Sections 3.5 and 3.6 and assume that $J \neq 0$. Now we want to apply Lemma 3.3 to the function J .

All the conditions of Lemma 3.3 are fulfilled. Removing the zero discs according to the regularizations procedure in Section 3.7, we represent the $(2n - 1)$ -chain $J^{-1}(0)$ as the union of connected smooth manifolds with multiplicities:

$$J^{-1}(0) = \bigcup_j \kappa_j C_j.$$

One of these manifolds is just the cycle $C' = C_{j_0}$. The multiplicities κ_j are the winding numbers (indices) of the function J at its zeros with respect to ζ (see Section 3.7).

Let $b \in \mathbb{C}^n$ is such that $J^{-1}(0) \cap G^{-1}(b) = \emptyset$. Apply Lemma 3.3:

$$2 \int_{J^{-1}(0)} (G - b)^* \beta_{\text{MB}} = \frac{1}{2\pi i} \int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right).$$

After change of variables $(z_1, z') = (J(\zeta, t), G(\zeta, t))$ in the first integral, the latter identity transforms to

$$2 \sum_j \kappa_j \int_{G(C_j)} \beta_{\text{MB}}(z' - b) = \frac{1}{2\pi i} \int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right). \quad (12)$$

Lemma 4.1. *In the case $(2n, k) = (2, 2)$ and $(2n, k) = (4, 3)$, the numbers κ_j are positive.*

Proof. Each chain $C_j \subset J^{-1}(0)$ is defined, near generic points, by the equation $\zeta = \zeta_j(t)$, $t \in M$.

In Sections 3.5 and 3.6 we have constructed the functions $J = J_-(\zeta, t)$ and $J = J_{1,2}$, for the cases $(2, 2)$ and $(4, 3)$ correspondingly. In both cases these functions are analytic in ζ , for the determinants defining J do not contain antiholomorphic columns.

Hence the winding numbers $\kappa_j = \kappa_j(t)$ of the mapping $\zeta \mapsto J(\zeta, t)$ on a small circle $|\zeta - \zeta_j(t)| = \varepsilon$, at an isolated zeros $\zeta = \zeta_j(t)$, are positive integers, equal to the multiplicities of the zero of the holomorphic function J . If $\zeta_j(t)$ consists of the boundary zeros, $|\zeta_j(t)| = 1$ then κ_j

must be taken $1/2$ of the multiplicity of the zero $\zeta_j(t)$ because the integration in (11) is performed within the unit disc. \square

Remark 4.2. Lemma 4.1 is the only point where we use the restriction $n \leq 2$ for the dimension. This restriction means that in both cases, A and B, of Theorem 2.2, the CR-dimension of the manifold Ω is one. In this case, the Jacobians J in Sections 3.5 and 3.6 are sense-preserving mappings and the winding numbers κ_j are positive, which is an important point in the proof.

Now we can finish the proof. We know that J vanishes on a homologically nontrivial cycle C in $\overline{\Sigma}$:

$$C \subset J^{-1}(0),$$

so that C is one of the cycles C_j constituting the chain $J^{-1}(0)$.

Let $b \notin G(\overline{\Sigma})$. Apply the identity (12) to that value b . Since b is not in the image of G then $G^{-1}(b) = \emptyset$ and hence the integral over the curve $G^{-1}(b)$ is zero:

$$\int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\overline{J}}{\overline{J}} \right) = 0. \quad (13)$$

On the other hand, according to Lemma 3.6, the image cycles $G(C_j)$ are all co-oriented and the multiplicities κ_j are all positive, $\kappa_j > 0$.

Moreover, by condition (iii) of Theorem 2.2, the cycle $c = G(C)$ is not homological to zero in $G(\overline{\Sigma})$. However it is homological to zero in \mathbb{R}^{2n} and therefore $c = \partial c'$ for some $2n$ -dimensional cycle c' in \mathbb{R}^{2n} . This cycle does not belong entirely to $G(\overline{\Sigma})$ and therefore the point $b \in \mathbb{C}^n \setminus G(\overline{\Sigma})$ can be chosen so that $c \neq 0$ in $H_{2n-1}(\mathbb{R}^{2n} \setminus \{b\})$.

We have the isomorphisms of the homology groups with coefficients in \mathbb{R} :

$$H_{2n-1}(\overline{\Sigma}) \cong H_{2n-1}(M) \cong \mathbb{R} \cong H_{2n-1}(\mathbb{R}^{2n} \setminus \{b\}).$$

Martinelli–Bochner form $\beta_{\text{MB}}(z' - b)$ is a closed nonexact form in $\mathbb{R}^{2n} \setminus \{b\}$, representing the generator in de Rham cohomology group $H^{2n-1}(\mathbb{R}^{2n} \setminus \{b\}, \mathbb{R}) \cong \mathbb{R}$. By the duality, the integral

$$\int_{G(C_j)} \beta_{\text{MB}}(z' - b) = m_j,$$

where

$$m_j = \{G(C_j)\} \in H_{2n-1}(\mathbb{R}^{2n} \setminus \{b\})$$

is the homology class of the cycle $G(C_j)$. Since the cycles $G(C_j)$ all are co-oriented, we have $m_j \geq 0$ and $m_j = 0$ if and only if the cycle $G(C_j)$ is homological to zero in $\mathbb{R}^{2n} \setminus \{b\}$.

The cycle C is one of the C_j 's, i.e. $C = C_{j_0}$. Then $m_{j_0} > 0$ because, as we saw, $c = G(C)$ is a homologically nontrivial cycle in $\mathbb{R}^{2n} \setminus \{b\}$ and hence the total sum

$$\sum_j \kappa_j \int_{(G-b)(C_j)} \beta_{\text{MB}} \geq \kappa_{j_0} m_{j_0} > 0.$$

Together with (13) this inequality contradicts to (12).

We have assumed that $J \neq 0$ and arrived to contradiction. Therefore $J \equiv 0$. Now are ready to obtain the final conclusion of Theorem 2.2.

The final step. *Proof that Q is degenerate, F is constant of G -fibers and f is a CR function.*

The contradiction we just have obtained says that the minor J (the case A, $n = 1$) or the minors J^0, J^1, J^2, J^3 (the case B, $n = 2$) of the Jacobi matrix $J(F, G)$ with respect to variables ψ, t vanish identically in our main manifold $\bar{\Sigma}$. Observe that since the functions F, G are analytic in ζ , the minors built from the derivatives in ζ vanish as well.

In the case $n = 1$, when the parameter t is one-dimensional, we obtain that $\text{rank}_{\mathbb{C}} J(F, G) < 2$ and Q is degenerate in accordance with the claim of Theorem 2.2.

Moreover, $J = 0$ implies that $F = \hat{f} \circ G$ in Σ for some function \hat{f} . Indeed, we have shown in Section 3.5 that the Jacobian J is proportional to the directional derivative of F along the level curves $G = \text{const}$ and hence $J = 0$ means that $F = \text{const}$ on G -fibers.

Also the relation between J and the $\bar{\partial}$ -derivatives of the function f proved in Section 3.5 implies

$$\bar{\partial} f \circ G = 0$$

on $b\Sigma$ which means that f is holomorphic in $\Omega = G(b\Sigma)$.

Now turn to the case $n = 2$. Again, vanishing the minors J^k , $k = 0, \dots, 3$, implies that $\text{rank}_{\mathbb{C}} J(F, G_1, G_2) < 3$ and hence Q is degenerate. Moreover, since due to the regularity condition, $\nabla G_1, \nabla G_2$ are linearly independent over \mathbb{C} at any point in Σ , we have $\nabla F \in \text{span}\{\nabla G_1, \nabla G_2\}$. If $\zeta = \zeta(s), t_k = t_k(s)$, $k = 1, 2, 3$; s is one-dimensional parameter, is a level curve $\Gamma = G^{-1}(b)$, then the directional derivative along the level curve is

$$\partial^G F = \zeta'(s) \partial \zeta F + t_1'(s) \partial_{t_1} F + t_2'(s) \partial_{t_2} F + t_3'(s) \partial_{t_3} F.$$

Since $\partial^G G_1 = \partial^G G_2 = 0$, we conclude from the above linear dependence of the gradients:

$$\partial^G F = 0$$

and therefore $F = \text{const}$ on any G -fiber.

Finally, the identity (11) implies that for tangential $\bar{\partial}$ -derivative holds

$$\bar{\partial}_b f \circ G = 0$$

on $b\Sigma$, and this means that f is a CR-function on Ω .

5. End of the proof of Theorem 2.2 for the case $\partial M \neq \emptyset$

The proof for the case of nonclosed parameterizing manifold M follows the same line as in the case of M with the empty boundary.

Again, we assume that the Jacobian J in the case $n = 1$ or one of the three minors J^1, J^2, J^3 in the case $n = 2$ is not identical zero. In all cases, we denote this determinant by J . Our goal is to arrive to a contradiction which would allow to conclude that all the above minors vanish identically. Then all the conclusions of Theorem 2.2 follow by the same arguments as in the previous section.

The main difference with the case of the previous section, where the parametrizing manifold M was closed, is that now we have to deal with the relative homology groups.

Our basic manifold now is also a bit different. Namely, we set

$$Y = G(\bar{\Sigma}) \setminus G(\bar{\Sigma}_0), \quad \Sigma_0 = \Delta \times \partial M$$

and define

$$\Sigma' = G^{-1}(Y).$$

Thus, we take only that part of $\Delta \times M$ which is the full G -preimage of its image. Note, that by the condition the set Y is nonempty and connected. The topological boundary of Σ' is

$$\partial \Sigma' = (\partial \Sigma \cap G^{-1}(\bar{Y})) \cup G^{-1}(\partial Y).$$

Taking into account that, in turn, $\partial \Sigma = (\partial \Delta \times M) \cup (\Delta \times \partial M) = b\Sigma \cup \Sigma_0$ and that $G^{-1}(Y) \cap \Sigma_0 = \emptyset$, we obtain that the boundary of the new manifold is

$$\partial \Sigma' = (G^{-1}(Y) \cap b\Sigma) \cup (G^{-1}(\partial Y) \cap \Sigma_0).$$

Thus, we assume that the mapping $Q = (F, G)$ is not degenerate and, as we have mentioned already, construct the Jacobian minor J and the function $\Theta = J/\bar{J}$, as in Sections 3.5 and 3.6. Our main assumption about Q is that the function J is not identically zero.

Assume initially that the zero set $J^{-1}(0)$ does not contain entire discs Δ_r . Proceed as in the case $\partial M = \emptyset$ (Lemma 3.3). Namely, we delete from Σ small neighborhoods of the zero sets and apply Stokes formula to the remainder. Again, we show as in Lemma 3.3, that the surface integral contributes nothing, because of degeneracy of the mapping (J, G) on $\partial \Sigma'$. However, an extra surface term appear, corresponding to the extra part of the boundary. As the result, we obtain:

$$2 \int_{J^{-1}(0)} (G-b)^* \beta_{\text{MB}} = \frac{1}{2\pi i} \int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right) - \frac{1}{2\pi i} \int_{G^{-1}(E)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right) \wedge (G-b)^* \beta_{\text{MB}}, \quad (14)$$

where

$$E = \partial Y \cap G(\Sigma_0)$$

and correspondingly $G^{-1}(E)$ is the $2n$ -dimensional extra part of the boundary of Σ' , coming from ∂M .

Note that the set $J^{-1}(0)$ is not necessary closed cycles any longer, but it is a cycle relatively to $G(\Sigma_0)$. Then we will use duality arguments based on counting of intersection indices, in place of computing periods by integration of Martinelli–Bochner type integrals as in the case $\partial M = \emptyset$.

Write (14) as

$$\chi(b) = Z(b) + N(b),$$

where $\chi(b)$ stands for the left-hand side in (14) and

$$Z(b) = \frac{1}{2\pi i} \int_{G^{-1}(b)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right), \quad N(b) = -\frac{1}{2\pi i} \int_{G^{-1}(E)} \left(\frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}} \right) \wedge (G - b)^* \beta_{\text{MB}}.$$

By change of variables,

$$\chi(b) = \frac{1}{\pi i} \sum_j \kappa_j \int_{G(C_j)} \beta_{\text{MB}}(z' - b),$$

where C_j are connected components of the $(2n - 1)$ -chain $J^{-1}(0)$ (as in the case $\partial M = \emptyset$), and κ_j are their multiplicities with respect to J . According to Lemma 4.1 the numbers κ_j are positive.

Lemma 5.1. *The function $Z(b)$ is integer valued as long as $J^{-1}(0) \cap G^{-1}(b) = \emptyset$.*

Proof. By the construction in Lemma 3.5 (property 2), the function $\Theta = J/\bar{J}$ is G -compatible on $b\Sigma = \partial\Delta \times M$, i.e. it takes same values at the end points of the curves $G^{-1}(b)$. Then the integral of the logarithmic derivative equals to the variation of the argument of Θ along the curve $G^{-1}(b)$ and is integer:

$$Z(b) = \frac{1}{2\pi i} \int_{G^{-1}(b)} \frac{d\Theta}{\Theta} = (1/2\pi) \text{Var}_{G^{-1}(b)} \arg \Theta \in \mathbb{Z}. \quad \square$$

Lemma 5.2. *The function $N(b)$ is continuous in the domain Y .*

Proof. By the construction the sets $G^{-1}(Y)$ and $G^{-1}(E)$ are disjoint. Therefore when $b \in Y$ then the differential form $(G - b)^* \beta_{\text{MB}}$ has no singularities on the surface of integration, $G^{-1}(E)$. The 1-form $dJ/J - d\bar{J}/\bar{J}$ has removable singularities at isolated zeros of J on the $2n$ -surface $G^{-1}(E)$ and does not depend on b . It follows that the integral defining the function $N(b)$ depends continuously on $b \in Y$. \square

Let $J^{-1}(0) = \bigcup_j C_j$ be the decomposition into connected chains. We know that the set $J^{-1}(0)$ contains the central cycle $C_{j_0} = C = \{0\} \times M$, for which the condition (iii) yields that the homology class $[G(C)] \in H_{2n-1}(G(\Sigma), G(\Sigma_0))$ is not 0.

The portion of $G(\Sigma_0)$ in Y is just the set E and the homological nontriviality of $G(C)$ means that $\partial G(C) \subset E$ and $G(C)$ cannot be made the boundary of a $2n$ -cycle in $Y \cup E$ by adding a $(2n - 1)$ -chain contained in $G(\Sigma_0)$. In other words, the $(2n - 1)$ -chain $G(C)$ has its boundary in the set E , but is not homologous to any $(2n - 1)$ -chain in E . The chain $G(C)$ intersects each domain $D_t = G(\Delta_t)$, $t \in M$.

Lemma 5.3. *There exists a connected continuous curve, $L \subset \mathbb{C}^n \setminus \overline{G(\Sigma_0)}$, with end points $p, q \in \mathbb{C}^n \setminus \overline{G(\Sigma)}$ intersecting the $(2n - 1)$ -chain $G(C)$ with the intersection index $m \neq 0$.*

Proof. Denote for brevity $X = G(\Sigma)$, $A = G(\Sigma_0)$. The chain $G(C)$ represents a nonzero element

$$0 \neq [G(C)] \in H_{2n-1}(X, \bar{A}).$$

By Poincaré–Lefschetz duality (see e.g. [34, Chapter 6]), there exists an element

$$0 \neq [L] \in H_1(\mathbb{C}^n \setminus \bar{A}, \mathbb{C}^n \setminus \bar{X})$$

with nonzero intersection index,

$$\text{ind}(L \cap G(C)) \neq 0.$$

The 1-chain L has the required properties. \square

The final step. *Proof that $J \equiv 0$.*

The constructed 1-chain L enters and leaves the set Y , avoiding the set $E \subset G(\Sigma)$ and crossing the chain $G(C)$ with a nonzero intersection index. Reversing, if needed, orientation of the path $L \subset Y$ we can assume that the intersection index $m = m_{j_0} > 0$.

By Sokhotsky–Plemelj theorem for the Martinelli–Bochner type integrals (see e.g. [29]), each time $b \in L$ intersects transversally some chain $G(C_j)$, the integral $\chi(b)$ changes for $+1$ or -1 , depending on the index of the intersection. If m_j is the total variation of $\chi(b)$, resulted from the crossing the cycle $G(C_j)$, then numbers m_j are of the same sign since all the cycles $G(C_j)$ are co-oriented.

Thus, the algebraic number of jumps the function $\chi(b)$, $b \in L$, makes, after passing through Y , is at least $\kappa_{j_0} m_{j_0}$:

$$\#\text{jumps}_{b \in L} \chi(b) \geq \kappa_{j_0} m_{j_0} > 0.$$

But $\chi(b) = N(b) + Z(b)$, $b \in Y$, and the function $N(b)$ changes continuously. On the other hand, by Lemma 5.1, the function $Z(b)$ is integer valued. Therefore $Z(b)$ changes only when b crosses the set $J^{-1}(0)$ which results in a jump of value of $\chi(b)$. Thus, the total algebraic number of jumps of $\chi(b)$ and $Z(b)$ are equal and hence:

$$\#\text{jumps}_{b \in L} Z(b) = \#\text{jumps}_{b \in L} \chi(b) \geq \kappa_{j_0} m_{j_0} > 0.$$

However, $G^{-1}(b) = \emptyset$ when $b \notin Y$. Therefore the function $Z(b)$ takes the value 0 before the argument b enters the set Y and after it has left Y . In other words, if $b_0, b_1 \notin Y$ are the starting and end points of the path L then $Z(b_0) = Z(b_1) = 0$ and therefore

$$\text{Var}_{b \in L} Z(b) = 0.$$

This contradiction says that $J = 0$.

It remains to explain how to get rid of the zero discs. For the case $(2n, k) = (2, 2)$ the argument from Section 3.7 works as it does not depend on whether M is closed or not.

Let $(2n, k) = (4, 3)$ and let T and $Z(T)$ are sets defined in Section 3.7. When $\dim T = 0$ and $\dim T = 2$, the zero discs are removable for the same reasons as in Section 3.7. When $\dim T = 1$

and $Z(T)$ contains 3-chain of the form $c = \bar{\Delta} \times \gamma$, then by the condition $H_1(M, \partial M) = 0$ the relative 1-cycle γ is relatively homologically zero. This implies that the total intersection index of the 1-chain L with the relative 3-cycle $G(c)$ is zero and therefore the cycle $G(c)$ contributes nothing in the result of our computation.

The proof of Theorem 2.2 is completed.

6. Proof of Theorem 4

We restrict ourselves by the proof for the case $\partial M = \emptyset$, i.e. $M = S^1$. Let $Q(\zeta, t) = (F(\zeta, t), G(\zeta, t))$, $(\zeta, t) \in \Sigma = \Delta \times S^1$, be a parametrization of the family D_t in the formulation of Theorem 4. We have $Q(b\Sigma) = \Lambda$ because the curves ∂D_t , $t \in S^1$, cover Λ . By regularity, the mapping

$$Q: b\Sigma \mapsto \Lambda$$

is finitely sheeted covering by the torus $b\Sigma = T^2$ of $\Lambda \setminus \partial\Lambda$. The image $Q(\Sigma)$ is the union of the discs D_t and its dimension is at most 3.

We want to prove that the mapping $Q = (F, G)$ is degenerate in Σ , and correspondingly, $Q(\Sigma)$ is 2-dimensional and entirely contained in Λ .

We do so by slight modification of the arguments in the proof of Theorem 2.2. First of all we need to prove the symmetry relation analogous to (2).

To this end we will check first that, as in Lemma 3.5, the Jacobian

$$J = \det[\nabla F, \nabla G] = \frac{\partial(F, G)}{\partial(\psi, t)},$$

with respect to local coordinates ψ, t on $b\Sigma$ has constant “phase” on the Q -fibers $Q^{-1}(b) \cap b\Sigma$, $b \in \mathbb{C}^2$. More precisely, we want to prove that J/\bar{J} is constant on the above fibers, outside of the zeros of J .

Lemma 6.1. *Let $u_1, u_2 \in b\Sigma$ belong to the same Q -fiber: $Q(u_1) = Q(u_2)$. If $J(u_1) \neq 0$ then $J(u_2) \neq 0$ and*

$$\frac{J(u_1)}{\overline{J(u_1)}} = \frac{J(u_2)}{\overline{J(u_2)}}.$$

Therefore on $b\Sigma \setminus J^{-1}(0)$ holds

$$\frac{J}{\bar{J}} = \sigma \circ Q,$$

for some function σ .

Proof. Each pair of vectors in \mathbb{C}^2

$$\{\partial_\psi Q(u_j), \partial_t Q(u_j)\}, \quad j = 1, 2,$$

belongs to the tangent space $T\Lambda_b$ at the point $b = Q(u_1) = Q(u_2)$. Since the Jacobian J does not vanish at the points u_1, u_2 , each pair constitutes a basis in the 2-dimensional real space $T\Lambda_b$.

Let \mathcal{A} be the (real) transition matrix from one basis to another. The Jacobians are related by

$$\frac{\partial(F, G)}{\partial(\psi, t)}(u_1) = \det \mathcal{A} \cdot \frac{\partial(F, G)}{\partial(\psi, t)}(u_2),$$

that is $J(u_1) = \det \mathcal{A} \cdot J(u_2)$ and lemma follows as $\det \mathcal{A}$ is real and nonzero and hence it cancels when one divides J by \bar{J} . \square

Now we need an analog of Lemma 3.1.

Lemma 6.2. *The Brouwer degree of the mapping $Q: T^2 = b\Sigma \mapsto \Lambda \subset \mathbb{C}^2$ equals zero.*

Proof. Lemma 6.2 follows from the condition $\partial\Lambda \neq \emptyset$ and from the fact that any smooth mapping of a compact closed manifold to a compact manifold with nonempty boundary has the Brouwer degree 0. Indeed, the image $Q(T^2)$ can be slightly homotopically contracted inside Λ so that some points in Λ would have empty preimage with respect to a mapping Q' , homotopically equivalent to Q . Therefore, by local definition of the degree, $\deg Q = \deg Q' = 0$. \square

Now assume that J is not identically zero and correspondingly $J^{-1}(0)$ is a 1-chain (we remove the zero discs exactly as we did in Section 3.7). We need the analog of Lemma 3.3 in the form

Lemma 6.3. *Let ω be a closed differential 1-form in a neighborhood of $Q(\bar{\Sigma})$. Then*

$$\int_{J^{-1}(0)} Q^* \omega = 0. \quad (15)$$

Proof. We follow the proof of Lemma 3.3. Define

$$\Theta = \frac{J}{\bar{J}}.$$

The form

$$\mathcal{E} = \frac{d\Theta}{\Theta} \wedge Q^* \omega$$

is closed in $\bar{\Sigma} \setminus J^{-1}(0)$. The differential form

$$\frac{d\Theta}{\Theta} = \frac{dJ}{J} - \frac{d\bar{J}}{\bar{J}}$$

computes the current $d\Theta/\Theta = 2[J^{-1}(0)]$. Lemmas 6.1 and 6.2 imply that \mathcal{E} integrates to zero on $b\Sigma \setminus J^{-1}(0)$ and then Stokes formula leads to identity (15), in the same way as in Lemma 3.3. \square

Now we can complete the proof of Theorem 4. Since $J(0, t) = 0$, the zero set $J^{-1}(0)$ contains the central cycle $C = \{0\} \times S^1$. The image $Q(C)$ meets each analytic disc $Q(\Delta \times S^1)$ and by the

condition of homological nontriviality the cycle $Q(C) \subset Q(\Sigma)$ is not homologous to zero. By de Rham duality, there exists a closed differential 1-form ω on $Q(\Sigma)$ such that

$$\int_{Q(C)} \omega \neq 0. \quad (16)$$

The final argument is as in Section 4. Namely, the mapping J is holomorphic in ζ and hence all cycles $G(C_j)$, where $J^{-1}(0) = \bigcup C_j$, are co-oriented. Hence the integrals of ω over $Q(C_j)$ are of the same sign and since the cycle C is one of C_j 's, (15) implies

$$\int_{Q(C)} \omega = 0.$$

This contradicts to (16). Therefore

$$i\zeta \frac{\partial(F, G)}{\partial(\zeta, t)} = \frac{\partial(F, G)}{\partial(\psi, t)} = 0$$

identically.

The vectors

$$\partial_\psi(F, G)(u), \quad \partial_t(F, G)(u), \quad u \in b\Sigma,$$

span the tangent space $T\Lambda_b$, $b = Q(u) = (F(u), G(u))$. We have shown that the Jacobian of $Q = (F, G)$ vanishes identically and hence the above two vectors are linearly dependent over \mathbb{C} . Therefore the real 2-dimensional tangent space $T\Lambda_b$ at each point $b \in \Lambda$ is a complex line in \mathbb{C}^2 . This proves that Λ is a 1-dimensional complex manifold. By the uniqueness theorem for analytic functions, the attached analytic discs entirely belong to the manifold: $D_t \subset \Lambda$, $t \in S^1$. Theorem 4 is proved.

7. Concluding remarks

Comments on the proof

The proof of Theorem 2.2 for the case $\partial M \neq \emptyset$ given in Section 5 is also valid for the case $\partial M = \emptyset$. As a matter of fact, for the case of closed manifold M both proofs coincide as one can count the number of nontrivial $(2n - 1)$ -cycles either by computing logarithmic residue type integrals, as in Section 4, or by counting of intersection indices in transversal direction, as in Section 5.

Some ingredients of the proof in Section 5 are close to the argument, presented in a different form and for the case of circles in the plane, in Tumanov's article [40].

Let us comment on this. In [40] (as well as in [3,39]) the Schwarz function of the circles (the meromorphic extension of \bar{z} from the circle in the corresponding disc) is a key tool. Globevnik observed in [24] that the construction in [40], works also for families parameterized by mapping of the form $G(\zeta, t) = g(\zeta) + it$, $t \in \mathbb{R}$, where g is a conformal mapping of the unit disc, having the symmetry $g(\bar{\zeta}) = \overline{g(\zeta)}$. This allowed him to obtain the corresponding test of analyticity for a special families of non-circular curves.

In fact, the Schwarz function in the above articles plays the role similar to that of Jacobian J in our approach. To explain this, observe that the Schwarz function of the circle coincides with \bar{z} on the circle $|z - a| = r$ and therefore is G -compatible for the parameterizing mapping $G(\zeta, t) = a(t) + r(t)\zeta$. This function develops a simple pole at the center of the circle which can be turned to zero by taking the reciprocal function. It results in the function satisfying the two conditions: it is G -compatible on $b\Sigma$ and vanishes on the central cycle C . Existence and construction of nontrivial functions with these two properties is just the key ingredient in our proof.

Nevertheless, the only domains, for which the Schwarz function has a simple pole, are discs. In fact, there is a larger class of domains with meromorphic Schwarz functions, so called quadrature domains (see e.g. [33]), but this class is pretty restrictive, too, as even simple domains, like ellipses, do not belong to it. So, the approach based on the Schwarz functions does not seem to work for general domains and rather works for discs or some other quite special families.

The key point in our proof is that the needed properties (G -compatibility and vanishing on a nontrivial cycle in Σ) are provided by the Jacobian determinants J , which encode the holomorphy or being CR of f . The Jacobi determinants are intrinsically produced both by the family of curves (i.e. by the parametrizing function G) and by the analytic extensions F of the original function f . This circumstance allows to deal with arbitrary functions G and makes the proof work for general Jordan curves.

Smoothness assumptions

Our construction involves derivatives and therefore requires at least smoothness both of the family of the analytic discs and, which is perhaps even worse, of the function f .

Moreover, we use integration of differential forms over the zero sets $J^{-1}(0)$ and assume even stronger condition of real-analyticity to provide nice structure of the zero sets and the integrals being defined.

In this article we have focused on the very constructions and therefore assume all objects under consideration maximally smooth. However, the technology of *currents* allows integration of differential forms over “bad” sets and hence, as it was noticed in the beginning of the article, may allow to extend our approach for differentiable, say C^∞ , functions f .

Another possible way of relaxing smoothness assumptions might be the following. Observe that the function f in Theorem 1 (the strip-problem) appeared, after Theorem 1 is proved, real analytic in Ω because it is holomorphic there. Therefore one could try to derive real-analyticity of f from the original conditions, under the initial minimal assumption of continuity of f .

It seems plausible that it might be done by using argument from [39] based on the edge of the wedge theorem. This argument might lead to real analyticity of f in a narrow open curved strip in Ω which would be enough to conclude that all the zero discs in $J^{-1}(0)$ are isolated. The latter is important for integrals of differential forms in our proofs to be defined. It should be mentioned, either, that the discussed method may work only for the strip-problem, since in higher dimensions (for instance, in Globevnik–Stout conjecture) one cannot expect *a posteriori* real analyticity of f . We plan to return to the problem of relaxing the smoothness assumptions elsewhere.

The case $n > 2$

The restriction $n \leq 2$ for the dimension is used only at one, albeit crucial, point, namely for checking that the multiplicities κ_j are of the same sign. This is provided by analyticity of the Jacobians J in ζ , which is true just in the cases $n = 1$, $\dim_{\mathbb{R}} \Omega = 2$, and $n = 2$, $\dim_{\mathbb{R}} \Omega = 3$.

Nevertheless, the method developed in this article can be applied, *mutatis mutandis*, in higher dimensions by considering minors of less dimensions. Then the proof works for families of attached analytic discs satisfying certain conditions for less dimensional subfamilies. We are going to return to the higher-dimensional case elsewhere.

The main results of this article were announced in the short note [2].

Acknowledgments

I am greatly indebted to Josip Globevnik for numerous stimulating and inspiring discussions of the strip-problem. I thank Alexander Tumanov for informing me about the preprint of his article [40]. I am grateful to Steve Shnider who read the first version of the manuscript and made several useful remarks.

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