JOURNAL OF APPROXIMATION THEORY 49, 196–199 (1987)

Note

Lipschitz Constants for the Bernstein Polynomials of a Lipschitz Continuous Function

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Communicated by Frank Deutsch

Received April 1, 1985

1. INTRODUCTION AND PRINCIPAL RESULT

Suppose a function f is continuous on $[0, 1]$. The nth $(n \geqslant 1)$ Bernstein polynomial of f is denoted and defined by

$$B_n(f; x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k}. \tag{1.1}$$

Obviously, $B_n(f)$ is a polynomial of degree $\leqslant n$ and its importance in approximation theory arises from the fact that $\lim_{n \to \infty} B_n(f; x) = f(x)$ uniformly on $[0, 1]$. For elementary properties of these polynomials see Davis [2, Chap. 6] and, for a more extensive treatment, Lorentz [3]. One of the outstanding properties of these polynomials is that they mimic the behaviour of the given function f to a remarkable degree. Thus if, in addition to being continuous on $[0, 1]$, f is convex, then $B_n(f)$ is also convex. Furthermore (see [2, Sect. 6.3]), for $n = 2, 3, 4,\ldots$ and for all $x \in [0, 1]$,

$$B_{n-1}(f; x) \geqslant B_n(f; x) \geqslant f(x). \tag{1.2}$$

As a consequence of this result we have, since the function $-x^\mu$, $0 < \mu \leqslant 1$, is convex on $[0, 1]$, that for $n = 1, 2, 3,\ldots,$

$$B_n(x^\mu; h) \leqslant h^\mu, \qquad 0 \leqslant h \leqslant 1. \tag{1.3}$$

In this note we shall assume only that the given function f is Lipschitz

0021-9045/87 $3.00

continuous of order μ, $0 < \mu \leqslant 1$, on $[0, 1]$. That is, there exists a constant $A \geqslant 0$ such that for every pair of points $x_1, x_2 \in [0, 1]$, we have

$$|f(x_1) - f(x_2)| \leqslant A\, |x_1 - x_2|^{\mu}. \tag{1.4}$$

The constant A, which depends upon f and μ, is known as the *Lipschitz constant* of f, and for f satisfying (1.4), we write $f \in \mathrm{Lip}_A\, \mu$. Obviously, if f is differentiable on $[0, 1]$, then f satisfies inequality (1.4) for all $\mu \in (0, 1]$. The principal result is the following theorem.

THEOREM 1. *If $f \in \mathrm{Lip}_A\, \mu$, then for all $n \geqslant 1$, $B_n(f) \in \mathrm{Lip}_A\, \mu$ also.*

An elementary proof of this theorem is given in the next section. The interesting thing about this result is that each of the Bernstein polynomials $B_n(f)$, for $n = 1, 2, 3,...$, has the same Lipschitz constant as the given function f when considered as being in the class of functions $\mathrm{Lip}\, \mu$. This is another example of the mimicing behaviour of the Bernstein polynomials. Two further small points should be noted. First, since $\lim_{n \to \infty} B_n(f; x) = f(x)$, for all $x \in [0, 1]$, the converse of Theorem 1 is true. Second, by choosing $f(x)$ to be Ax^{μ} and the points x_1, x_2 to be 0, 1, respectively, we see that the constant A cannot be diminished for any value of n.

There is a brief history to this theorem. Bloom and Elliott [1] showed that it was true when $\mu = 1$ and for $\mu \neq 1$ showed that $B_n(f) \in \mathrm{Lip}_A(\mu/4)$. Theorem 1 was conjectured and, in a private communication to the authors of [1], Dr. Dickmeis stated that the result was true as a consequence of the Peetre K-theory of interpolation between Banach spaces. We shall now give an elementary proof of Theorem 1.

2. PROOF OF THEOREM 1

Let $x_1 \leqslant x_2$ be any two points of $[0, 1]$. We need to show that

$$|B_n(f; x_2) - B_n(f; x_1)| \leqslant A(x_2 - x_1)^{\mu},$$

given that f satisfies (1.4). From (1.1),

$$\begin{aligned} B_n(f; x_2) &= \sum_{j=0}^{n} \binom{n}{j} (1 - x_2)^{n-j} f\left(\frac{j}{n}\right) (x_1 + (x_2 - x_1))^j \\ &= \sum_{j=0}^{n} \binom{n}{j} (1 - x_2)^{n-j} f\left(\frac{j}{n}\right) \left\{ \sum_{k=0}^{j} \binom{j}{k} x_1^k (x_2 - x_1)^{j-k} \right\} \\ &= \sum_{j=0}^{n} \sum_{k=0}^{j} \frac{n!\, x_1^k (x_2 - x_1)^{j-k}\, (1 - x_2)^{n-j}}{k!\, (j-k)!\, (n-j)!} f\left(\frac{j}{n}\right). \end{aligned}$$

On inverting the order of summation and writing $k+l=j$, then

$$B_n(f; x_2) = \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!\, l!(n-k-l)!} x_1^k (x_2 - x_1)^l \times (1-x_2)^{n-k-l} f\left(\frac{k+l}{n}\right). \tag{2.1}$$

We now construct a similar double sum for $B_n(f; x_1)$. Again, from (1.1), we have

$$\begin{aligned} B_n(f; x_1) &= \sum_{k=0}^{n} \binom{n}{k} x_1^k f\left(\frac{k}{n}\right) ((x_2 - x_1) + (1 - x_2))^{n-k} \\ &= \sum_{k=0}^{n} \binom{n}{k} x_1^k f\left(\frac{k}{n}\right) \left\{ \sum_{l=0}^{n-k} \binom{n-k}{l} (x_2 - x_1)^l (1-x_2)^{n-k-l} \right\} \\ &= \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!\, l!(n-k-l)!} x_1^k (x_2 - x_1)^l \\ &\quad \times (1-x_2)^{n-k-l} f\left(\frac{k}{n}\right). \end{aligned} \tag{2.2}$$

On subtracting (2.2) from (2.1), we have

$$\begin{aligned} &|B_n(f; x_2) - B_n(f; x_1)| \\ &\quad = \left| \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!\, l!(n-k-l)!} x_1^k (x_2 - x_1)^l (1-x_2)^{n-k-l} \right. \\ &\qquad \left. \times \left\{ f\left(\frac{k+l}{n}\right) - f\left(\frac{k}{n}\right) \right\} \right| \\ &\quad \leqslant A \sum_{k=0}^{n} \sum_{l=0}^{n-k} \frac{n!}{k!\, l!\, (n-k-l)!} x_1^k (x_2 - x_1)^l (1-x_2)^{n-k-l} \left(\frac{l}{n}\right)^{\mu}, \end{aligned}$$

on using (1.4),

$$\begin{aligned} &\quad = A \sum_{l=0}^{n} \frac{(x_2 - x_1)^l\, n!}{l!\, (n-l)!} \left(\frac{l}{n}\right)^{\mu} \left\{ \sum_{k=0}^{n-l} \binom{n-l}{k} x_1^k (1-x_2)^{n-l-k} \right\} \\ &\quad = A \sum_{l=0}^{n} \binom{n}{l} (x_2 - x_1)^l \left(\frac{l}{n}\right)^{\mu} (x_1 + 1 - x_2)^{n-l} \\ &\quad = A B_n(x^{\mu}; x_2 - x_1), \qquad \text{by (1.1)}, \\ &\quad \leqslant A (x_2 - x_1)^{\mu}, \qquad \text{by (1.3)}. \end{aligned}$$

Thus we see that $B_n(f) \in \mathrm{Lip}_A\, \mu$, where A is the Lipschitz constant of f so that the theorem is proved.

References

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Printed in Belgium