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Abstract Comparison Principles and Multivariable Gronwall–Bellman Inequalities

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INTRODUCTION

A fundamental result pertaining to differential/integral equations theory is the so-called Gronwall–Bellman inequality asserting that, if $u: R_+ \to R$ is a continuous solution of

$$x(t) \leqslant f(t) + \int_0^t k(s)\, x(s)\, ds, \qquad t \in R_+ \tag{1}$$

(where $f: R_+ \to R$ and $k: R_+ \to R_+$ are continuous functions) then

$$u(t) \leqslant f(t) + \int_0^t k(s) \exp\left(\int_s^t k(r)\, dr\right) f(s)\, ds, \qquad t \in R_+;$$

see as basic references Coddington and Levinson [18, Chap. I, Problem 1], Bellman and Cooke [8, Chap. VII, Ex. 2], Lakshmikantham and Leela [31, Chap. I, Sect. 1.9], Corduneanu [22, Chap. I, Sect. 1.5]. During the last three decades, this result was extended in many directions, the most representative of them being, from our viewpoint, the multivariable ones. Concerning the linear extensions of this kind, let us mention as a first illustrative example, the 1973 Young's result [67] stating that, if $u: R^n_+ \to R$ is continuous and satisfies (1) modulo R^n_+ ($f: R^n_+ \to R$ and $k: R^n_+ \to R_+$ being continuous) then

$$u(t) \leqslant f(t) + \int_0^t k(s)\, v(s; t)\, f(s)\, ds, \qquad t \in R^n_+,$$

where $v(s; t)$ is the solution of the characteristic initial value problem

$$\begin{aligned} &(-1)^n\, v_{s_1 \cdots s_n}(s; t) = k(s)\, v(s; t), \qquad 0 \leqslant s \leqslant t \\ &v(s; t) = 1 \qquad \text{on} \quad s_i = t_i,\ 1 \leqslant i \leqslant n; \end{aligned} \tag{2}$$

a further vectorial extension of Young's result was performed in 1976 by Chandra and Davis [15], through a specific "resolvent" procedure. As a

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second illustrative example, one must mention the 1979 Bondge–Pachpatte theorem [12] stating in essence that any continuous solution $u\colon R^n_+ \to R_+$ of (1)(modulo R^n_+) with $f\colon R^n_+ \to R_+$ continuous and increasing, and $k\colon R^n_+ \to R_+$ continuous, satisfies

$$u(t) \leqslant f(t) \exp\left(\int_0^t k(s)\, ds\right), \qquad t \in R^n_+;$$

this result may be viewed as a n-variable extension of the so-called Wendroff inequality [7, Chap. IV, Sect. 30]. Concerning the nonlinear multivariable extensions of (1), let us first mention the 1974 Headley's theorem [27] asserting in essence that, if $u\colon R^n_+ \to R$ is continuous and satisfies

$$u(t) \leqslant f(t) + \int_0^t k(s, u(s))\, ds, \qquad t \in R^n_+ \tag{1'}$$

with $f\colon R^n_+ \to R$ continuous and $k\colon R^n_+ \times R \to R$ continuous and increasing with respect to its last argument then, for any t_0 in R^n_+,

$$u(t) \leqslant w_0(t) \qquad 0 \leqslant t \leqslant t_0,$$

where w_0 is the maximal solution on $[0, t_0]$ of the integral equation associated to (1′). (Of course, Headley's contribution may be also viewed as a nonlinear version of Young's result; the idea of the proof goes back to Viswanatham [59].) Second, note that a more abstract version of Headley's result were formulated in the 1970 Chandra–Fleishman paper [16]: letting $(X, \|\cdot\|, \leqslant)$ be an ordered Banach space and supposing the point $f \in X$ and the increasing completely continuous mapping $T\colon X \to X$ are such that, an increasing continuous function $\omega\colon R_+ \to R_+$ may be found with

$$\begin{aligned} &\| Tu - Tv \| \leqslant \omega(\| u - v \|), && u, v \in X, \\ &\omega(r) + \| T(0) \| + \| f \| \leqslant r, && r \geqslant s, \text{ for some } s > 0 \end{aligned} \tag{3}$$

then, any solution $u \in X$ of the operator inequality

$$x \leqslant f + Tx \tag{1''}$$

must satisfy $u \leqslant w$, where w is the maximal solution in X of the corresponding operator equation associated to (1″). Finally, as a further generalization of this result, let us mention the 1973 Krasnoselskii–Sobolev contribution [29], obtained through a specific "iterative" compactness method. Under these lines, it is our main objective in the present exposition to state and prove a couple of comparison results involving (abstract) increasing self-mappings of a metrizable uniform space—extending in this way the above quoted Chandra–Fleishman and Krasnoselskii–Sobolev statements–the basic instrument of our investigations being a special ordering procedure similar in essence to that indicated in [56]. As direct applications, some "functional" versions of the contributions we exposed before are given. At the same time, as indirect applications of our main results, two reduction principles concerning multivariable Gronwall–Bellman inequalities are formulated; it is worth noting at this moment that, as a rather surprising consequence of these principles, most of the (multivariable) Wendroff type extensions of (1) may be regarded, in the last analysis, as a particular case of this one-variable statement. It should also be underlined our main results may be put into a "purely" uniform framework; these aspects will be discussed elsewhere.

1. PRELIMINARIES

Let X be a nonempty set, and let $\leqslant$ be an *ordering* (i.e., a reflexive, antisymmetric, and transitive relation) on X. For any $x \in X$ denote $(\leqslant, x] = \{y \in X; y \leqslant x\}$ and $[x, \leqslant) = \{y \in X;\ x \leqslant y\}$; also, given any couple $x, y \in X$, $x \leqslant y$, put $[x, y] = (\leqslant, y] \cap [x, \leqslant)$ and call it the (order) *interval* between x and y. A sequence $(x_n;\ n \in N)$ in X will be said to be *increasing* when $x_i \leqslant x_j$ for $i \leqslant j$, and *bounded from above* in case $x_n \leqslant y$, $n \in N$, for some y in X. Furthermore, let $D = (d_i;\ i \in N)$ be a denumerable *sufficient* family of *semi-metrics* on X (in which case, the triplet $(X, D, \leqslant)$ will be termed an *ordered metrizable uniform space*). We shall say the sequence $(x_n;\ n \in N)$ in X, *D-converges* to $x \in X$ (and we write $x_n \to^D x$) when $d_i(x_n, x) \to 0$ as $n \to \infty$, for each $i \in N$. Of course, any D-convergent sequence is necessarily D-Cauchy (i.e., d_i-Cauchy, for all $i \in N$); in this context, X will be said to be *order complete* when each increasing D-Cauchy sequence converges. A subset Y of X will be termed *order closed* when the limit of any D-convergent increasing sequence in Y belongs to Y; also, the ambient ordering $\leqslant$ on X will be called *self-closed* (*anti self-closed*) in case $[x, \leqslant)$(resp. $(\leqslant, x]$) is order closed for any x in X, and *interval-closed*, when it is both self-closed and anti self-closed (or, equivalently, when each interval of X is order closed).

In what follows, we shall say $(y_n;\ n \in N)$ is a *subsequence* of $(x_n;\ n \in N)$ when a strictly increasing function k from N to itself may be found with $x_{k(n)} = y_n$, $n \in N$. Under such a convention, let us call the sequence $(x_n;\ n \in N)$ in X, *relatively compact* when any subsequence $(y_n;\ n \in N)$ of it contains a convergent subsequence. The importance of this notion is put into evidence by the following result–largely used in the sequel–closely related to that of Ward [63] (see also Krasnoselskii [28, Chap. I, Sect. 5]).

LEMMA 1. *Let the ordered metrizable uniform space* $(X, D, \leqslant)$ *be such that* $\leqslant$ *is interval closed. Then, the increasing sequence* $(x_n;\ n \in N)$ *in* X *is a relatively compact one, if and only if it converges to some element* x *of* X.

Proof. Let $(u_n;\ n \in N)$ and $(v_n;\ n \in N)$ be a couple of convergent subsequences of $(x_n;\ n \in N)$. If $u_n \to^D u$ and $v_n \to^D v$ then, by the interval-closedness property we immediately get $u \leqslant v \leqslant u$, that is, $u = v$. In other words, all convergent subsequences of $(x_n;\ n \in N)$ have the same limit, x. We claim $x_n \to^D x$. Indeed, suppose this assertion were false then, a couple $i \in N$, $\varepsilon > 0$ may be chosen so that, for each $n \in N$ there exists $m > n$ with $d_i(x_m, x) \geqslant \varepsilon$. It follows at once a subsequence $(y_n;\ n \in N)$ of $(x_n;\ n \in N)$ exists with the property $d_i(y_n, x) \geqslant \varepsilon$, $n \in N$, proving no convergent subsequence $(z_n;\ n \in N)$ of it (hence of $(x_n;\ n \in N)$) can have x as limit, contradicting the above conclusion. Q.E.D.

A close analysis of the notion we just introduced shows it would be desirable (for both theoretical and practical reasons) to express it in terms of the sequence itself. To this end, let us call the sequence $(x_n;\ n \in N)$ in X, *precompact* when for each $i \in N$, $\varepsilon > 0$, a finite subset $A = A_{i,\varepsilon}$ of N may be found so that, for every $n \in N$ there exists $p \in A$ with $d_i(x_n, x_p) < \varepsilon$. Now, as a completion of Lemma 1, we have

LEMMA 2. *Assume* $(X, D, \leqslant)$ *is such that* X *is order complete. Then, for each increasing sequence in* X, *relatively compact is identical with precompact.*

Proof. Necessity. Let $(x_n;\ n \in N)$ be an increasing relatively compact sequence in X which is not precompact. Then, a couple $i \in N$, $\varepsilon > 0$ may be chosen so that, for each finite subset A of N, an index $n \in N$ will exist with $d_i(x_n, x_p) \geqslant \varepsilon$, for all $p \in A$. It easily follows a subsequence $(y_n;\ n \in N)$ of $(x_n;\ n \in N)$ may be constructed such that $d_i(y_n, y_m) \geqslant \varepsilon$, $n < m$, proving $(y_n;\ n \in N)$ has no D-Cauchy (hence, by our hypothesis, no D-convergent) subsequences, contrary to our assumption.

Sufficiency. Let $(x_n;\ n \in N)$ be an increasing precompact sequence in X and let $(y_n;\ n \in N)$ be a subsequence of it. As $(y_n;\ n \in N)$ is precompact too, it clearly follows, by definition, that a subsequence $(u_n;\ n \in N)$ of it may be found with $d_1(u_n, u_m) < 1$, $n \leqslant m$; furthermore, by the precompactness of $(u_n;\ n \in N)$, a subsequence $(v_n;\ n \in N)$ of it may be found with $d_2(v_n, v_m) < \frac{1}{2}$, $n \leqslant m$, and so on. By a standard diagonal process one easily arrives at a D-Cauchy (hence, by our completeness hypothesis, a D-convergent) subsequence $(z_n;\ n \in N)$ of $(y_n;\ n \in N)$ and the proof is complete. Q.E.D.

As an interesting particular case, let (K, d) be a metric space and let $\leqslant$ be a *quasi-ordering* (i.e., a reflexive and transitive relation) on X. Putting for each $t \in K$, $\varepsilon > 0$, $S(t, \leqslant, \varepsilon) = \{s \in [t, \leqslant);\ d(t, s) < \varepsilon\}$, assume K may be represented as the union $K_1 \cup K_2 \cup \cdots$, where the family $\mathscr{K} = \{K_1, K_2, \ldots,\}$ satisfies

(H_1) *to every* $t \in K$ *there corresponds* $\alpha = \alpha(t) > 0$ *and* $i = i(t) \in N$ *with* $S(t, \leqslant, \alpha) \subset K_i$.

Also, let $(Y, \|\cdot\|)$ be a normed space and $\leqq$ an ordering on Y. A function $x: K \to Y$ will be said to be *continuous at the right* when to any $t \in K$ and $\varepsilon > 0$ there corresponds a $\delta = \delta(t, \varepsilon) > 0$ such that $s \in S(t, \leqslant, \delta)$ implies $\|x(t) - x(s)\| < \varepsilon$. Let X indicate the class of all continuous at the right functions x from K into Y with $\sup\{\|x(t)\|;\, t \in K_i\} < \infty$, $i \in N$. A standard ordered metrizable uniform structure on X is that introduced by the conventions

$$d_i(x, y) = \sup\{\|x(t) - y(t)\|;\, t \in K_i\}, \qquad i \in N,\ x, y \in X,$$

$$x \leqq y \text{ if and only if } x(t) \leqq y(t),\ t \in K.$$

LEMMA 3. *Let the ordering* $\leqq$ *on* Y *be self closed (resp. anti self-closed or interval-closed) then, so is the associated ordering* $\leqq$ *on* X. *In the same context, let* Y *be order complete. Then,* X *is order complete too.*

Proof. The first part of the statement is evident. To prove the second one, let $(x_n;\, n \in N)$ be an increasing D-Cauchy sequence in X. Clearly, $(x_n(t);\, n \in N)$ is an increasing Cauchy sequence in Y for each $t \in K$ so that, by the order completeness assumption, $x(t) = \lim_n x_n(t)$ exists for any $t \in K$. It remains to show x is an element of X. To do this, let $t \in K$ be arbitrary fixed and let $\alpha > 0$ and $i \in N$ be given by (H_1). Since $x_n(t) \to x(t)$ uniformly with respect to K_i, it follows that, given $\varepsilon > 0$, a $n = n(\varepsilon)$ may be found with

$$\|x_n(t) - x(t)\| < \varepsilon/3 \qquad \text{for all} \quad t \in K_i.$$

On the other hand, x_n being continuous at the right, a $\delta \in (0, \alpha)$ may be chosen so that

$$\|x_n(t) - x_n(s)\| < \varepsilon/3 \qquad \text{for all} \quad s \in S(t, \leqslant, \delta) \subset K_i.$$

By a classical triangular procedure we get

$$\|x(t) - x(s)\| < \varepsilon, \qquad s \in S(t, \leqslant, \delta)$$

proving x is continuous at the right and completing, in fact, our argument. Q.E.D.

Under the same general conventions, let us call a family $F \subset X$, *$\mathscr{K}$-quasi-order-equicontinuous* when for each $i \in N$, $\varepsilon > 0$, there exists a finite subset $H = H_{i,\varepsilon}$ in K_i and a number $\delta = \delta(i, \varepsilon) > 0$ such that

(H_2) *to every* $t \in K_i$ *there corresponds* $s \in H$ *with* $s \leqslant t$ *and* $d(s, t) < \delta$,

(H_3) *for any couple* (t, s) *like in* (H_2) *we have* $\|x(t) - x(s)\| < \varepsilon$ *for all* $x \in F$.

The usefulness of this notion is put into evidence by the following precompactness result (for the sake of simplicity we restricted our considerations to denumerable families).

LEMMA 4. *Let the increasing sequence* $(x_n; n \in N)$ *in* X *be* K*-quasi-order-equicontinuous and let in addition assume*

(H_4) $(x_n(t); n \in N)$ *is precompact in* Y *for all* $t \in K$.

Then, necessarily, $(x_n; n \in N)$ *is precompact in* $(X, D, \leqslant)$.

Proof. (Dieudonné [24, Chap. VIII, Sect. 5]). Let $i \in N$ and $\varepsilon > 0$ be given. By hypothesis, there exist a finite subset $H_{i,\varepsilon}$ in K_i and a number $\delta(i, \varepsilon) > 0$ such that (H_2) and (H_3) (with $\varepsilon/4$ in place of ε) hold. The subset $Z_{i,\varepsilon} = \{x_n(t); n \in N, t \in H_{i,\varepsilon}\}$ is precompact in Y so, a finite subset $Z^0_{i,\varepsilon}$ of $Z_{i,\varepsilon}$ exists with the property

for each $n \in N$ and $t \in H_{i,\varepsilon}$ there corresponds $y = y(n, t)$ in $Z^0_{i,\varepsilon}$ with $\|x_n(t) - y\| < \varepsilon/4$. (4)

Let G denote the family of all mappings from $H_{i,\varepsilon}$ to $Z^0_{i,\varepsilon}$ and, for any $g \in G$, put

$$L(g) = \{n \in N; \|x_n(t) - g(t)\| < \varepsilon/4, t \in H_{i,\varepsilon}\}.$$

By (4), N will be covered by the union of the sets $L(g)$, $g \in G$; moreover, by the above evaluations,

$$d_i(x_n, x_m) < \varepsilon, \qquad n, m \in L(g), g \in G$$

so that, if we take as $A_{i,\varepsilon}$ the (finite) subset of N having a single element in common with $L(g)$ for any g in G, our proof is finished. Q.E.D.

As a first remark about this result, assume $\leqq$ is interval-closed and Y is order complete then, by Lemma 1 the hypothesis (H_4) may be written as

(H_4') $(x_n(t); n \in N)$ *is convergent, for all* $t \in K$

while, by Lemmas 2 and 3, the conclusion just obtained can be rephrased as : $(x_n; n \in N)$ is convergent in $(X, D, \leqslant)$. At the same time, suppose $\leqslant$ is the *trivial* quasi-ordering on K then, the above statement coincides with Theorem 7.7.7 of Dieudonné we already quoted. Finally, a more general version of Lemma 4 could be obtained in case Y were taken as an ordered metrizable uniform space; we preferred, however, this normed variant for some technical reasons whose usefulness will become clear by our future developments.

2. THE MAIN RESULTS

Let X be an ordered metrizable uniform space under the denumerable sufficient family of semi-metrics $D=(d_i;\ i\in N)$ and the ordering $\leqslant$. Also, let Y be a subset of X and T a mapping from Y to itself. An important problem concerning these elements is that of determining the existential comparative (modulo $\leqslant$) connections between the subset Y_{0i} of all solutions in Y of the operator inequality

$$x\leqslant Tx \tag{OI}$$

and the subset Y_{oe} of all solutions in Y of the associated operator equation

$$x=Tx. \tag{OE}$$

Of course, it implicitly follows from our context that we are in fact interested in establishing a number of topological answers to the above formulated question, in which case, it is quite natural to accept as basic hypothesis

(i) $\leqslant$ *is interval-closed.*

Under these preparatory facts, the first main result of the present paper is

THEOREM 1. *Let the order-closed subset Y of X and the increasing mapping T from Y to itself be such that*

(ii) *Y_{oi} is not empty*

(iii) *each increasing sequence $(x_n;n\in N)$ in Y with $x_n\in T^{k(n)}(Y_{\mathrm{oi}})$, $n\in N$, for some (strictly) increasing sequence $(k(n);\ n\in N)$ in N, is relatively compact.*

Then, to any u in Y_{oi} there corresponds $v\in Y_{\mathrm{oe}}$ with the properties (a) *$u\leqslant v$,* (b) *if $w\in Y_{\mathrm{oi}}$ satisfies $v\leqslant w$ then $v=w$.*

Proof. First, let us observe that, without loss of generality one may suppose D is an increasing family ($d_i\leqslant d_j$ whenever $i\leqslant j$) because, otherwise, replacing it by the family $E=(e_i;\ i\in N)$ defined as

$$e_i=d_1+\cdots+d_i,\qquad i\in N,$$

the general hypothesis (i) as well as the specific assumption (iii) remain valid. Second, we claim for every couple $i\in N$, $\varepsilon>0$, the following assertion is true

for each $m\in N$ and $x\in T^m(Y_{\mathrm{oi}})$ there exist $n\geqslant m$ in N and $y\geqslant x$ in $T^n(Y_{\mathrm{oi}})$ such that, for every $p\geqslant n$ in N and $z\geqslant y$ in $T^p(Y_{\mathrm{oi}})$, $d_i(y,z)<\varepsilon$. (5)

Indeed, assuming (5) were not valid, a $m \in N$ and $x \in T^m(Y_{\mathrm{oi}})$ may be found with the property

> for every $n \geqslant m$ in N and $y \geqslant x$ in $T^n(Y_{\mathrm{oi}})$, a $p \geqslant n$ in N and a $z \geqslant y$ in $T^p(Y_{\mathrm{oi}})$ will exist with $d_i(y, z) \geqslant \varepsilon$.

It immediately follows that an increasing sequence $(y_n;\ n \in N)$ in Y and a (strictly) increasing sequence $(k(n);\ n \in N)$ in N may be constructed with

$$y_n \in T^{k(n)}(Y_{\mathrm{oi}}) \qquad \text{and} \qquad d_i(y_n, y_{n+1}) \geqslant \varepsilon \quad \text{for all } n \in N.$$

By (iii), $(y_n;\ n \in N)$ is necessarily relatively compact, hence D-convergent if we take (i) plus Lemma 1 into account, so that $d_i(y_n, y_{n+1}) \to 0$ as $n \to \infty$. The contradiction at which we arrived shows the assertion (5) is true. In this case, given the arbitrary fixed u in Y_{oi}, an increasing sequence $(x_n;\ n \in N)$ in Y and a (strictly) increasing sequence $(k(n);\ n \in N)$ in N may be chosen so as to satisfy $u \leqslant x_n \in T^{k(n)}(Y_{\mathrm{oi}})$, $n \in N$, plus

$$N \ni p \geqslant k(n) \qquad \text{and} \qquad T^p(Y_{\mathrm{oi}}) \ni y \geqslant x_n \quad \text{imply } d_n(y, x_n) < 1/2^n. \tag{6}$$

Now, by (i) + (iii) in conjunction with Lemma 1 it follows $x_n \to^D v$ for some v in Y. We claim v is the desired element. Indeed, let us first observe that, in view of the self-closedness property of our ordering,

$$u \leqslant x_n \leqslant v, \qquad n \in N \tag{7}$$

and therefore, $u \leqslant v$. As an immediate consequence of (7) we have $Tx_n \leqslant Tv$, $n \in N$, so that, combining with the fact that, by the evident relation

$$x_n \leqslant Tx_n \in T^{k(n)+1}(Y_{\mathrm{oi}}), \qquad n \in N$$

plus (6) it clearly follows $Tx_n \to^D v$, one arrives (by the anti-self-closedness property of our ordering) at the conclusion $v \leqslant Tv$, that is, $v \in Y_{\mathrm{oi}}$; moreover, as a further consequence of (7)

$$x_n \leqslant T^{k(n)}x_n \leqslant T^{k(n)}v \in T^{k(n)}(Y_{\mathrm{oi}}), \qquad n \in N$$

in which situation, again by (6), $T^{k(n)}v \to^D v$, which in turn implies

$$v \leqslant Tv \leqslant T^{k(n)}v \leqslant v, \qquad n \in N,$$

that is, $v \in Y_{\mathrm{oe}}$. Finally, suppose $v \leqslant w$ for some w in Y_{oi} then, observing that

$$v \leqslant T^{k(n)}w \in T^{k(n)}(Y_{\mathrm{oi}}), \qquad n \in N$$

one immediately gets by (6) that $T^{k(n)}w \to^{D} v$ and therefore, by (i),

$$w \leqslant T^{k(n)}w \leqslant v, \qquad n \in N,$$

completing the argument. Q.E.D.

Let us call the subset Z of X, *order-sequentially* (resp. *sequentially*) *relatively compact* when each increasing sequence (each sequence) in Z is relatively compact. Clearly, a sufficient condition guaranteeing the validity of (iii) is

(iii$_s$) *$T^k(Y)$ is order-sequentially relatively compact, for some index $k \in N$*

(resp.

(iii$'_s$) *$T^k(Y)$ is sequentially relatively compact, for some $k \in N$*)

in which case, as an useful variant of the first main result, we have (see also Turinici [58])

THEOREM 2. *Let the order-closed subset Y of X and the increasing mapping T from Y to itself be such that* (ii) plus (iii$_s$) (*resp.* (iii$'_s$)) *hold. Then, conclusions* (a + b) *of Theorem* 1 *remain valid.*

Let X, D and $\leqslant$ be as before. We shall say the subset Z of X is *order-bounded* (resp. *bounded*) when

$$\sup\{d_i(x, y); x, y \in Z, x \leqslant y\} < \infty, \qquad i \in N$$

(resp.

$$\sup\{d_i(x, y); x, y \in Z\} < \infty, \qquad i \in N);$$

remark at this moment that any sequentially relatively compact subset of X is necessarily a bounded one. A simple inspection of the reasonings involved in the proof of the first main result shows no boundedness property of this type was effectively required for the ambient subset Y or its iterates $T^k(Y)$, $k \in N$; however, under such an assumption, a more elegant proof of Theorem 1 (patterned after Krasnoselskii and Sobolev [29]) may be obtained. To be more precise, assume that, in addition to (ii) plus (iii), we accept

(iv) *$T^k(Y)$ is order-bounded, for some $k \in N$*

and let us define for every couple $i \in N$, $u \in Y_{oi}$,

$$g_i(u) = \inf_{n \geqslant k} \sup \{d_i(T^n x, T^n y); u \leqslant T^n x \leqslant T^n y, x, y \in Y_{oi}\}.$$

Clearly, g_i is decreasing on its existence domain, i.e.,

$$u \leqslant v \qquad \text{implies } g_i(u) \geqslant g_i(v), i \in N;$$

moreover, we claim that

$$\inf\{g_i(v); u \leqslant v \in Y_{oi}\} = 0, \qquad i \in N, u \in Y_{oi}. \tag{5'}$$

Indeed, supposing the assertion (5′) were not valid, a couple $i \in N$, $u \in Y_{oi}$ may be found with

$$g_i(v) > \beta,\ Y_{oi} \ni v \geqslant u \qquad \text{for some} \quad \beta > 0$$

or, in other words, for any $v \geqslant u$ in Y_{oi} and any $n \geqslant k$ in N, a pair $x, y \in Y_{oi}$ will exist with

$$v \leqslant T^n x \leqslant T^n y \qquad \text{and} \qquad d_i(T^n x, T^n y) > \beta;$$

by a finite induction procedure, one may easily construct the sequences $(x_n; n \in N)$ and $(y_n; n \in N)$ in Y_{oi} with

$$u \leqslant T^k x_1 \leqslant T^k y_1 \leqslant \cdots \leqslant T^{k+n-1} x_n \leqslant T^{k+n-1} y_n \leqslant \cdots,$$
$$d_i(T^{k+n-1} x_n, T^{k+n-1} y_n) > \beta, \qquad n \in N,$$

and therefore, observing that the first of these relations contradicts (via the ambient hypotheses plus Lemma 1) the second one, our assertion is proved. In such a situation, given the arbitrary fixed $u \in Y_{oi}$, a sequence $(u_n; n \in N)$ in Y_{oi} may be determined so that

$$u \leqslant Tu_1 \leqslant T^2 u_2 \leqslant \cdots$$

and

$$g_n(T^n u_n) < 1/2^n, \qquad n \in N. \tag{6'}$$

Now, by (i), (iii) and Lemma 1, $T^n u_n \to^D v$ for some $v \in Y$. We claim v satisfies the requirements (a) + (b). Indeed, it is clear that, by the self-closedness property of our ordering

$$u \leqslant T^n u_n \leqslant v, \qquad n \in N \tag{7'}$$

and thus $u \leqslant v$. As a consequence of (7′)

$$T^n u_n \leqslant T^{n+1} u_n \leqslant Tv, \qquad n \in N$$

so that (passing to limit and using the anti self-closedness property) $v \leqslant Tv$ that is, $v \in Y_{oi}$; moreover, by the evident relations

$$T^n u_n \leqslant T^{n+m} u_{n+m} \leqslant v \leqslant T^n v \leqslant T^{n+m} v, \qquad n \geqslant k, m \in N$$

one immediately gets by (6′)

$$d_n(T^{n+h(n)} u_{n+h(n)}, T^{n+h(n)} v) < 1/2^n, \qquad n \geqslant k$$

for some (strictly) increasing sequence $(h(n);\ n \in N)$ in N so that, necessarily, $T^n v \to^D v$ which in turn implies (by (i) again)

$$v \leqslant Tv \leqslant T^n v \leqslant v, \qquad n \in N,$$

proving $v \in Y_{\text{oe}}$. Finally, let w in Y_{oi} be such that $v \leqslant w$; by the same reasonings as above (with w in place of v) one gets $T^n w \to^D v$ so that, by our basic hypothesis, $w \leqslant T^n w \leqslant v$, $n \in N$, completing the argument.

Since, as we had already occasion to say, a sufficient condition for (iv) to be valid is (iii$'_s$), the above reasoning is in effect to Theorem 2 but not in general to *Theorem* 1. Regarding this last aspect, it would be not without importance to ask whether the method we developed here might be applied to nonmetrizable uniform structures; a partial answer to this question will be given elsewhere.

Returning to the hypothesis (iii), essential to the present discussion, let us remark its particular form (iii$_s$) may be viewed as a "spatial" (strong) restriction of it so that it is of practical interest to determine what happens when (iii) is replaced by its "temporal" (weak) restriction

(iii$_t$) *each increasing sequence* $(T^n x;\, n \in N)$ *in* Y *with* $x \in Y_{\text{oi}}$, *is a relatively compact one.*

To do this, we have to introduce the notions below. Given the mapping U from Y to itself, let us call it *continuous at the left* when for each x in Y and each increasing sequence $(x_n;\ n \in N)$ in Y with $x_n \to^D x$ and $x_n \leqslant x$, $n \in N$, we have $Ux_n \to^D Ux$. Also, let us say U has an *order uniqueness property* when $x \leqslant y$ and $x = Ux$, $y = Uy$ imply $x = y$ (i.e., any two fixed points of U are either identical or incomparable). Under these conventions, the second main result of the present note is (cf. also Dugundji and Granas [25, Chap. I, Sect. 4]).

THEOREM 3. *Let the order-closed subset* Y *of* X *and the increasing mapping* T *from* Y *to itself be such that* (ii) + (iii$_t$) *as well as*

(v) *T is continuous at the left*

(vi) *T has an order uniqueness property*

hold. Then, conclusions (a) + (b) *of the main result remain valid.*

Proof. Let u in Y_{oi} be arbitrary fixed. By (iii$_t$) plus Lemma 1, $T^n u \to^D v$ for some $v \in Y$. Clearly, $T^n u \leqslant v$, $n \in N$, so that, by the left-continuity assumption (v), $T^{n+1} u \to^D Tv$, proving $v \in Y_{\text{oe}}$. Let w in Y_{oi} be such that $v \leqslant w$. By the above reasonings $T^n w \to^D v'$ for some $v' \in Y_{\text{oe}}$; on the other hand, by (i), $T^n w \leqslant v'$, $n \in N$, and this proves $v \leqslant v'$. Combining this fact with (vi), one gets $v = v'$ and hence $w \leqslant v$, completing the proof. Q.E.D.

An interesting feature of the above statements is given by the fact that (although implicitly embodied into the hypothesis (iii) or its variants) no explicit (order) completeness property for the ambient ordered metrizable uniform space were assumed so that, to complete our treatment and, at the same time, to cover some useful particular cases, it would be necessary to discuss this eventuality. Assume therefore in the following that, in addition to the basic hypothesis (i) we admit

(vii) *X is order complete*

then, in view of Lemma 2, a more appropriate formulation of the main results might be obtained if one replaces in (iii), (iii$_s$), (iii$_t$), the word "relatively compact" by "precompact." Particularly, if we restrict our considerations to Theorem 3 above, the following remark turns out to be in effect in many concrete situations. Let $f\colon R_+ \to R_+$ be an increasing function; after a terminology suggested by [54] we shall say f has the property (P) provided that

$$f^n(t) \to 0 \qquad \text{as} \quad n \to \infty \quad \text{for all } t > 0,$$

where f^n indicates its nth iterate (note that, by a lemma due to Matkowski [33] we necessarily have in such a case $f(t) < t$, for all $t > 0$ (and hence $f(0) = 0$)). Now, Y and T being as before, let us denote

$$f_i(t) = \sup\{d_i(Tx, Ty);\, x, y \in Y, x \leqslant y, d_i(x, y) \leqslant t\},\ t \in R_+,\, i \in N.$$

Then we claim the hypothesis

(v′) *f_i has the property* (P) *for all $i \in N$*

is a sufficient one for the validity of (iii$_t$) + (v) + (vi). Indeed, letting $u \in Y_{\text{oi}}$ be arbitrary fixed, put $a_i = d_i(u, Tu)$, $i \in N$, and observe that

$$d_i(T^n u, T^{n+1} u) \leqslant f_i^n(a_i), \qquad i, n \in N,$$

a relation which in turn implies, by (v′)

$$d_i(T^n u, T^{n+1} u) \to 0 \qquad \text{as} \quad n \to \infty, \quad \text{for all } i \in N.$$

Let $i \in N$ and $\varepsilon > 0$ be arbitrary fixed. By the above relation, a $m = m(i, \varepsilon) \in N$ may be found with $d_i(T^m u, T^{m+1} u) \leqslant \varepsilon - f_i(\varepsilon) \leqslant \varepsilon$; combining this with the definition of f_i, one gets $d_i(T^{m+1} u, T^{m+2} u) \leqslant f_i(\varepsilon)$ so that, by the triangle property, $d_i(T^m u, T^{m+2} u) \leqslant \varepsilon$. Again using the definition of f_i, we have $d_i(T^{m+1} u, T^{m+3} u) \leqslant f_i(\varepsilon)$ so that, by the same procedure as above, $d_i(T^m u, T^{m+3} u) \leqslant \varepsilon$, and so on. By a finite induction one easily arrives at $d_i(T^m u, T^{m+n} u) \leqslant \varepsilon$, $n \in N$, proving (iii$_t$) and therefore, the assertion follows because (v) + (vi) are almost trivial in our case.

In concluding this section, let us remark that the comparison theorems we formulated before may be interpreted either as maximality results modulo Y_{oi} in which case, via Theorem 1 of Turinici [58] they appear as a particular version of the maximality principle stated in [56] (see also the variant indicated in [57]) or as fixed point results modulo Y in which situation (under a continuity assumption similar to (v)) they may be viewed as metrizable uniform versions of some topological contributions in this area due to Wallace [61], Ward [63], Smithson [49], and Turinici [55] (see also, from a more abstract perspective, Tarski [51], Abian and Brown [2], Bakhtin [6]). On the other hand, suppose X is a complete Fréchet space under a denumerable sufficient family of seminorms $S = \{|\cdot|_i;\ i \in N\}$ and let X_+ be a closed cone in X then, defining an ordering structure by

$$x \leqslant y \qquad \text{if and only if } y - x \in X_+$$

the general hypotheses (i) + (vii) of this section are clearly fulfilled; in particular, when S reduces to a single element (resp. a norm on X) Theorem 1 reduces (under the supplementary assumption (iv)) to the above quoted Krasnoselskii–Sobolev result, while Theorem 3 reduces to the Chandra–Fleishman result quoted in the introductory part of the paper (see also Azbelev and Tsaljuk [5]). Some concrete examples of such cones may be found in Krasnoselskii [28, Chap. I] (cf. also Vulikh [60, Chap. III]). Finally, suppose the self-mapping T were decreasing then, evidently, T^2 is increasing so that (modulo the remaining hypotheses) a number of appropriate comparison results concerning the couple (OI) + (OE) (with T^2 in place of T) may be given; some topological aspects of the problem were discussed by Seda [44] (see also Pelczar [41], Abian [1], Kurepa [30], and Taskovič [52] for an abstract ordered set viewpoint).

3. MULTIVARIABLE GRONWALL–BELLMAN INEQUALITIES

Let $n \in N$ be a positive integer and let R^n_+ denote the standard positive cone in R^n, endowed with one of the usual norms (e.g., that introduced by the familiar scalar product $\langle \cdot, \cdot \rangle$ in R^n) and the natural ordering. Also, $m \in N$ being another positive integer, let $\|\cdot\|$ indicate one of the usual norms in R^m and $\leqslant$ the ordering on R^m defined as

$$(s_1, ..., s_m) \leqslant (t_1, ..., t_m) \qquad \text{when} \quad s_i \leqslant t_i,\ i \in I \text{ and } s_j \geqslant t_j,\ j \in J,$$

where $\{I, J\}$ is a partition of $\{1, ..., m\}$ (the cases I or J is empty being not excluded). Now, let X^m_n indicate the class of all continuous functions from

R^n_+ to R^m. An useful Fréchet structure on X^m_n is that indicated by the family of seminorms $S(A)=\{|\cdot|_i;\ i\in N\}$ introduced by the convention

$$|x|_i=\sup\{\|x(t)\|;\ 0\leqslant t\leqslant a_i\},\qquad i\in N,\ x\in X^m_n,$$

$A=(a_i;\ i\in N)$ being a cofinal sequence in R^n_+ (to any $t\in R^n_+$ there corresponds $i\in N$ with $t\leqslant a_i$); also, a natural ordering structure on X^m_n is that indicated by

$$x\leqslant y\qquad\text{if and only if}\quad x(t)\leqslant y(t),\quad t\in R^n_+.$$

It is a simple exercise to verify X^m_n is complete (hence order complete) and $\leqslant$ as well as $\geqslant$ (its dual) are closed in Nachbin's sense [34, Appendix] hence interval closed. Furthermore, let X^0_n denote the class of all continuous functions from R^n_+ to R_+. Defining as before (by deleting the sign $\|\cdot\|$) a Fréchet structure and (with $\leqslant$ taken as the usual ordering on R_+) an ordering structure on X^0_n, it is clear that the above (order) completeness and (interval) closedness properties continue to hold in our case. Finally, given s, $t\in R^n_+$, $s\leqslant t$, and $x\in X^m_n$, by $\int_s^t x(r)\,dr$ we shall mean the n-fold integral $\int_{[s,t]} x(r)\,dr$.

Under these preparatory facts, let $x\longmapsto k(x)$ be an increasing map from X^m_n to itself, and $f\in X^m_n$ a given element. Consider the multivariable Gronwall–Bellman inequality

$$x(t)\leqslant f(t)+\int_0^t k(x)(s)\,ds,\qquad t\in R^n_+.\tag{GBI}$$

As in the preceding section, we are interested in determining the existential comparative connections between the solutions in X^m_n of (GBI) and the solutions in X^m_n of the associated multivariable Volterra equation

$$x(t)=f(t)+\int_0^t k(x)(s)\,ds,\qquad t\in R^n_+.\tag{VE}$$

In this direction, as an immediate application of the first main result, the following theorem about the couple (GBI) + (VE) may be formulated.

THEOREM 4. *Assume there is a cofinal sequence $(a_i;\ i\in N)$ of vectors in R^n_+, a sequence $G=(g_i;\ i\in N)$ in X^0_n and a sequence $(h_i;\ i\in N)$ of mappings from X^0_n to itself, with the properties*

(viii) *to any $i\in N$ there corresponds $j\in N$ such that, for every x in X^m_n with $\|x(t)\|\leqslant g_j(t)$, $0\leqslant t\leqslant a_j$ we have $\|k(x)(t)\|\leqslant h_j(g_j)(t)$, $0\leqslant t\leqslant a_i$,*

(ix) *for each $i\in N$, at least one couple (i,j) with $j\in N$ taken as in* (viii) *satisfies*

$$\|f(t)\|+\int_0^t h_j(g_j)(s)\,ds\leqslant g_i(t),\qquad 0\leqslant t\leqslant a_i.$$

Assume also that (GBI) *has at least a solution in the subset* $X_n^m(G)$ *consisting of all* x *in* X_n^m *with* $\|x(t)\| \leqslant g_i(t)$, $0 \leqslant t \leqslant a_i$, $i \in N$. *Then, to any solution* $u \in X_n^m(G)$ *of* (GBI) *there corresponds a solution* $v \in X_n^m(G)$ *of* (VE) *with* $u \leqslant v$ *and, moreover, for each solution* $w \in X_n^m(G)$ *of* (GBI) *distinct from* v, *the relation* $v \leqslant w$ *does not hold.*

Proof. Denote by T the mapping from X_n^m into itself defined by the right hand of (GBI). Clearly, T is increasing; moreover, we claim $X_n^m(G)$ is invariant under T. Indeed, let $x \in X_n^m(G)$ and $i \in N$ be arbitrary fixed; taking $j \in N$ as in (viii) we have, by the definition of $X_n^m(G)$

$$\|k(x)(t)\| \leqslant h_j(g_j)(t), \qquad 0 \leqslant t \leqslant a_i \tag{8}$$

so that, by (ix)

$$\|Tx(t)\| \leqslant \|f(t)\| + \int_0^t h_j(g_j)(s)\,ds \leqslant g_i(t), \qquad 0 \leqslant t \leqslant a_i$$

proving our assertion. Observing that, as another consequence of (8)

$$\|Tx(t) - Tx(s)\| \leqslant \|f(t) - f(s)\| + \int_{I(s,t)} h_j(g_j)(r)\,dr, \quad 0 \leqslant s, t \leqslant a_i$$

(where $I(s, t)$ stands for the symmetric difference between $[0, s]$ and $[0, t]$) an immediate application of Lemma 4 (modulo the trivial quasi-ordering) tells us $T(X_n^m(G))$ is order-sequentially precompact. This shows all conditions of the first main result (more precisely, of Theorem 2) are fulfilled and conclusion follows. Q.E.D.

As an interesting particular case, let us analyse the situation

$$k(x)(t) = K\left(t, x(a(t)), \int_0^t H(s, x(s))\,ds\right), \quad t \in R_+^n, x \in X_n^m,$$

where $K(t, u, v)$ is continuous from $R_+^n \times R^m \times R^m$ to R^m and increasing with respect to u and v, $H(t, u)$ is continuous from $R_+^n \times R^m$ to R^m and increasing with respect to u, and $a(t)$ is continuous from R_+^n to itself. Assume that

$$\|K(t, u, v)\| \leqslant p(t)(\|u\| + \|v\|), \qquad t \in R_+^n, u, v \in R^m,$$

$$\|H(t, u)\| \leqslant q(t)\|u\|, \qquad t \in R_+^n, u \in R^m$$

($p, q \in X_n^0$ being increasing) and let $(a_i;\ i \in N)$ be an increasing cofinal sequence in R_+^n satisfying, for each $i \in N$,

$$0 \leqslant t \leqslant a_i \quad \text{implies} \quad 0 \leqslant a(t) \leqslant a_{i+1}$$

then, putting

$$h_i(y)(t)=p(a_i)\left(y(a(t))+q(a_i)\int_0^t y(s)\,ds\right),\quad t\in R^n_+,\, y\in X^0_n,\, i\in N$$

condition (viii) of the above theorem will be clearly fulfilled (with $j=i+1$) while (ix) reduces to

$$\begin{aligned}\|f(t)\|&+p(a_{i+1})\int_0^t g_{i+1}(a(s))\,ds+p(a_{i+1})\,q(a_{i+1})\\&\times\int_0^t\int_0^s g_{i+1}(r)\,dr\,ds\leqslant g_i(t),\qquad 0\leqslant t\leqslant a_i,\, i\in N,\end{aligned}\tag{9}$$

a condition that, actually, may be fulfilled in a large number of concrete situations. It will follow then by the above result that, if the sequence $G=(g_i;\, i\in N)$ in X^0_n were constructed so as to satisfy (9) then, any solution in $X^m_n(G)$ of the (integro-functional) multivariable Gronwall–Bellman inequality

$$x(t)\leqslant f(t)+\int_0^t K(s, x(a(s)), \int_0^s H(r, x(r))\,dr)\,ds,\qquad t\in R^n_+\tag{IFGBI}$$

is necessarily bounded above by a certain (maximal) solution in $X^m_n(G)$ of the corresponding (integro-functional) multivariable Volterra equation

$$x(t)=f(t)+\int_0^t K(s, x(a(s)), \int_0^s H(r, x(r))\,dr)\,ds,\qquad t\in R^n_+\tag{IFVE}$$

for a number of related contributions in this direction we refer to Ashirov and Mamedov [4] as well as Turinici [58]. A dual form of this statement is the following: suppose (IFGBI) has at least a solution in $X^m_n(G)$ then—modulo the remaining hypotheses—(IFVE) possesses at least a solution in $X^m_n(G)$; note that, under such a perspective, the corresponding formulation of Theorem 4 might be interpreted as a multivariable "monotone" counterpart of Corduneanu's existence result [21] (cf. also Pelczar [42]) in the "nonanticipative" case ($a(t)\leqslant t,\ t\in R^n_+$) and, respectively, Oberg's existence result [36] (see also Skripnik [48]) in the "anticipative" case ($a(t)\nleqslant t,\ t\in R^n_+$). Of course, a rather prohibitive feature of the above reasonings is the existence of the a priori evaluation involved in the definition of $X^m_n(G)$ for the solution to which we are going to apply this comparation procedure because we do not dispose in general, of such an evaluation; more exactly, the usual device is to start from a *certain* solution u in X^m_n of (GBI or (IFGBI) and to obtain for this function an evaluation of the form $u\leqslant v$ where v is a solution in X^m_n of (VE) or (IFVE). However, this evaluation is in many concrete situations a perfectly feasible fact, whenever one substitutes in (ix) (or in (9) for the particular case just considered) the function $\|f(t)\|$ by $\max(\|f(t)\|, \|u(t)\|)$; note that, by such a procedure it is possible to arrive at Headley's result [27] if we restrict the domain of the functions involved in the above example to a compact $[0, b]$ with $b \in R^n_+$ (see also Westphal [64], Rasmussen [43], Pachpatte [39, and others).

Passing to the second part of our developments, let us specify another usual Fréchet structure on X^m_n is that defined by the family of seminorms $S(A, G) = \{|\cdot|_i;\ i \in N\}$ introduced by the Bielecki procedure [10]

$$|x|_i = \sup\{\|x(t)\|/g_i(t); 0 \leqslant t \leqslant a_i\}, \qquad i \in N, x \in X^m_n,$$

$A = (a_i;\ i \in N)$ being a cofinal sequence in R^n_+ and $G = (g_i;\ i \in N)$ a sequence in X^0_n with

$$g_i(t) > 0, \qquad 0 \leqslant t \leqslant a_i, i \in N;$$

clearly, $(X^m_n, S(A))$ and $(X^m_n, S(A, G))$ are equivalent as Fréchet spaces and therefore the order completeness and interval closedness properties, valid for the first of these structures, will remain as such for the second one. Also, letting $(R^{2n}_+)_+$ indicate the subset of all (t, s) in R^{2n}_+ with $s \leqslant t$, denote by Y^m_n (resp. Y^0_n) the class of all continuous functions from $(R^{2n}_+)_+$ to $R^m(R_+)$; of course, a corresponding Fréchet as well as ordering structure may be introduced on Y^m_n (Y^0_n) by the same way as that indicated, at the beginning of this section, for X^m_n (resp. X^0_n). Now, let $x \longmapsto k(x)$ be an increasing map from X^m_n to Y^m_n and $f \in X^m_n$ a given element. Consider the multivariable Gronwall–Bellman inequality

$$x(t) \leqslant f(t) + \int_0^t k(x)(t, s)\, ds, \qquad t \in R^n_+. \tag{GBI$'$}$$

As above, we are interested to compare the solutions in X^m_n of this inequality with the solutions in X^m_n of the corresponding multivariable Volterra equation

$$x(t) = f(t) + \int_0^t k(x)(t, s)\, ds, \qquad t \in R^n_+ \tag{VE$'$}$$

in this direction, as a consequence of the second main result, we have

THEOREM 5. *Suppose there exist a cofinal sequence* $(a_i; i \in N)$ *in* R^n_+, *a sequence* $(g_i; i \in N)$ *in* X^0_n *satisfying the above positivity condition, a sequence* $(h_i; i \in N)$ *of mappings from* X^0_n *to* X^0_n *and a sequence* $(\lambda_i; i \in N)$ *in* $[0, 1)$ *with the properties*

(x) $x, y \in X^m_n$, $x \leqslant y$, $a \in X^0_n$, $i \in N$, $\|x(t) - y(t)\| \leqslant a(t)$, $0 \leqslant t \leqslant a_i$ *imply* $\|k(x)(t, s) - k(y)(t, s)\| \leqslant h_i(a)(t, s)$, $0 \leqslant s \leqslant t \leqslant a_i$

(xi) $\int_0^t h_i(\tau g_i)(t, s)\,ds \leqslant \lambda_i \tau g_i(t)$, $0 \leqslant t \leqslant a_i$, $\tau \geqslant 0$, $i \in N$.

Then, to every solution $u \in X_n^m$ *of* (GBI′) *there corresponds a solution* $v \in X_n^m$ *of* (VE′) *with* $u \leqslant v$ *and, moreover, for each solution* $w \in X_n^m$ *of* (GBI′) *distinct from* v, *relation* $v \leqslant w$ *does not hold.*

Proof. Denote by T the mapping from X_n^m into itself given by the second part of (GBI′), and let $x, y \in X_n^m$ be such that $x \leqslant y$ and $|x-y|_i \leqslant \tau$ for some $\tau \geqslant 0$, $i \in N$. Then, clearly,

$$\|x(t)-y(t)\| \leqslant \tau g_i(t), \qquad 0 \leqslant t \leqslant a_i$$

so that, by $(x)+(xi)$,

$$\begin{aligned}\|Tx(t)-Ty(t)\| &\leqslant \int_0^t \|k(x)(t, s)-k(y)(t, s)\|\,ds \\ &\leqslant \int_0^t h_i(\tau g_i)(t, s)\,ds \\ &\leqslant \lambda_i \tau g_i(t), \qquad 0 \leqslant t \leqslant a_i,\end{aligned}$$

that is, $|Tx-Ty|_i \leqslant \lambda_i \tau$ and therefore, Theorem 3 (under its "contractive" form) applies. Q.E.D.

A simple inspection of the above hypotheses shows that, due to the interval-restrictive condition (x), Theorem 5 may be effectively applied especially to nonanticipative multivariable Gronwall–Bellman inequalities of the preceding form. A particular case of practical interest is that corresponding to the choice

$$k(x)(t, s) = K(t, s, x(s)), \qquad (t, s) \in (R_+^{2n})_+,\ x \in X_n^m,$$

where $K(t, s, u)$ is continuous from $(R_+^{2n})_+ \times R^m$ to R^m, increasing with respect to u, and satisfies the order type Lipschitz condition

$$\|K(t, s, u)-K(t, s, v)\| \leqslant L(t, s)\|u-v\|, \quad (t, s) \in (R_+^{2n})_+,\ u, v \in R^m,\ u \leqslant v,$$

L being an element of Y_n^0 then, defining the sequence $(h_i; i \in N)$ of mappings from X_n^0 to Y_n^0 by

$$h_i(y)(t, s) = \mu_i\, y(s), \quad (t, s) \in (R_+^{2n})_+,\ y \in X_n^0,\ i \in N,$$

where

$$\mu_i = \sup\{L(t, s); 0 \leqslant s \leqslant t \leqslant a_i\}, \qquad i \in N,$$

condition (x) will clearly take place, while (xi) reduces to

$$\mu_i \int_0^t g_i(s)\,ds \leqslant \lambda_i\, g_i(t), \qquad 0 \leqslant t \leqslant a_i,\ i \in N,$$

a condition which is fulfilled, e.g., by the sequence of functions

$$g_i(t) = \exp\langle b_i, t\rangle, \qquad t \in R^n_+, i \in N,$$

where $(b_i = (\beta_{i1}, ..., \beta_{in});\ i \in N)$ is a sequence of vectors in R^n_+ satisfying $\beta_{ik} > 0$, $1 \leqslant k \leqslant n, i \in N$, as well as $\mu_i \leqslant \lambda_i \beta_{i1} \cdots \beta_{in}, i \in N$. An interesting linear variant of this case may be constructed as follows. Letting $Z^{(m)}_n$ indicate the class of all (m, m) matrices over Y^0_n, put

$$K(t, s, u) = C(t, s)\, u,\ (t, s) \in (R^{2n}_+)_+, \qquad u \in R^m,$$

where $C(t, s) = (c_{ij}(t, s);\ 1 \leqslant i, j \leqslant m)$ is an element of $Z^{(m)}_n$; clearly the above Lipschitz condition is fulfilled by any couple u, v in R^m and therefore, the mapping T appearing in the right member of the corresponding (GBI′) possesses a global uniqueness property. As an immediate consequence of this fact, the conclusion of Theorem 5 may be written as: let $u \in X^m_n$ be a solution of the (linear) multivariable Gronwall–Bellman inequality

$$x(t) \leqslant f(t) + \int_0^t C(t, s)\, x(s)\, ds, \qquad t \in R^n_+ \tag{LGBI}$$

then, it necessarily satisfies

$$u(t) \leqslant f(t) + \int_0^t H(t, s) f(s)\, ds, \qquad t \in R^n_+, \tag{10}$$

where $H(t, s) = (h_{ij}(t, s);\ 1 \leqslant i, j \leqslant m)$, the resolvent kernel, is the unique solution in $Z^{(m)}_n$ of the (linear) matrix Volterra integral equation

$$Z(t, s) = C(t, s) + \int_s^t C(t, r)\, Z(r, s)\, dr, \qquad (t, s) \in (R^{2n}_+)_+. \tag{LMVE}$$

Indeed, it is a simple exercise to verify (see, e.g., Tricomi [53, Chap. I, Sect. 1.3]) the right part of (10) is, by (LMVE) the (unique) solution of

$$x(t) = f(t) + \int_0^t C(t, s)\, x(s)\, ds, \qquad t \in R^n_+ \tag{LVE}$$

and this proves our assertion. Note at this moment, a more direct way of studying the above inequality is that of using the successive approximation method for (LVE) starting with a solution of (LGBI); the idea of this method goes back to Chu and Metcalf [17]. At the same time, let us observe that, if we take

$$C(t, s) = P(t)\, Q(s), \qquad (t, s) \in (R^{2n}_+)_+$$

with $P(t) = (p_{ij}(t);\ 1 \leqslant i, j \leqslant m)$ and $Q(t) = (q_{i,j}(t);\ 1 \leqslant i, j \leqslant m)$, (m, m) matrices over X^0_n, the above inequality (10) becomes

$$u(t) \leqslant f(t) + P(t) \int_0^t H(t, s)\, Q(s) f(s)\, ds, \qquad t \in R^n_+, \tag{10′}$$

where $H(t, s)$ is a solution in $Z_n^{(m)}$ of the linear matrix equation

$$Z(t, s) = I + \int_s^t Q(r)\, P(r)\, Z(r, s)\, dr \qquad (t, s) \in (R_+^{2n})_+ \quad \text{(LMVE}'\text{)}$$

and therefore, the corresponding variant of Theorem 3 reduces to the Chandra–Davis result [15]; see also Snow [50], Nagumo and Simoda [33], Young [67], Berruti Onesti [9], and Walter [62, Chap. III, Sect. 19]. Finally, as a more technical example of this kind, let us consider the linear multivariable Gronwall–Bellman inequality

$$\begin{aligned} x(t_1,..., t_n) \leqslant f(t_1,..., t_n) &+ \sum_{(i)} \int_0^{t_i} K^{(i)}(t_1,..., t_n; 0,..., s_i,..., 0) \\ &\times x(t_1,..., s_i,..., t_n)\, ds_i \\ &+ \sum_{(i,j)} \int_0^{t_i} \int_0^{t_j} K^{(i,j)}(t_1,..., t_n; 0,..., s_i,..., s_j,..., 0) \\ &\times x(t_1,..., s_i,..., s_j,..., t_n)\, ds_i ds_j + \cdots \\ &+ \int_0^{t_1} \cdots \int_0^{t_n} K^{(1,...,n)}(t_1,..., t_n; s_1,..., s_n) \\ &\times x(s_1,..., s_n)\, ds_1 \cdots ds_n,\ (t_1,..., t_n) \in R_+^n, \qquad \text{(LGBI}'\text{)} \end{aligned}$$

where $\sum_{(i)}$ comprises $\binom{n}{1} = n$ terms, $\sum_{(i,j)}$ comprises $\binom{n}{2} = n(n-1)/2$ terms, etc., and $K^{(i)}$, $K^{(i,j)}$,..., $K^{(1,...,n)}$ are elements of $Z_n^{(m)}$. Formally, (LGBI′) appears as a generalization of (LGBI); however, a close analysis shows it is in fact reductible to the above quoted inequality. Indeed, regarding (LGBI′) as a linear Gronwall–Bellman inequality with respect to the variable t_1 and making use of (10) with $n = 1$, it follows that any solution of it must also satisfy the linear inequality

$$\begin{aligned} x(t_1,..., t_n) \leqslant f_1(t_1,..., t_n) &+ \sum_{(i)}{}' K_1^{(i)}(t_1,..., t_n; 0,..., s_i,..., 0) \\ &\times x(t_1,..., s_i,..., t_n)\, ds_i \\ &+ \sum_{(i,j)} \int_0^{t_i} \int_0^{t_j} K_1^{(i,j)}\, (t_1,..., t_n; 0,..., s_i,..., s_j,..., 0) \\ &\times x(t_1,..., s_i,..., s_j,..., t_n)\, ds_i ds_j + \cdots \\ &+ \int_0^{t_1} \cdots \int_0^{t_n} K_1^{(1,...,n)}(t_1,..., t_n; s_1,..., s_n) \\ &\times x(s_1,..., s_n)\, ds_1 \cdots ds_n,\ (t_1,..., t_n) \in R_+^n, \qquad \text{(LGBI}_1'\text{)} \end{aligned}$$

where $\sum_{(i)}'$ indicates $\sum_{(i)}$ without its first term and $K_1^{(i)}$, $K_1^{(i,j)}$,..., $K_1^{(1,...,n)}$ are defined in terms of $K^{(i)}$, $K^{(i,j)}$,..., $K^{(1,...,n)}$ and some additional expressions involving the resolvent kernel associated to the first part of $\sum_{(i)}$; furthermore, treating (LGBI_1') as a linear Gronwall–Bellman inequality with respect to t_2 one arrives (again by (10) with $n=1$) at a new linear inequality of the form (LGBI′) in which two terms of $\sum_{(i)}$ were deleted, etc. Continuing in this way, one concludes, after $\binom{n}{1}=n$ steps, that any solution of (LGBI′) must satisfy the linear inequality

$$\begin{aligned} x(t_1,\ldots,t_n) \leqslant f_n(t_1,\ldots,t_n) &+ \sum_{(i,j)} \int_0^{t_i}\int_0^{t_j} K_n^{(i,j)}(t_1,\ldots,t_n;0,\ldots,s_i,\ldots,s_j,\ldots,0) \\ &\times x(t_1,\ldots,s_i,\ldots,s_j,\ldots,t_n)\,ds_i\,ds_j \\ &+ \int_0^{t_1}\cdots\int_0^{t_n} K_n^{(1,\ldots,n)}(t_1,\ldots,t_n;s_1,\ldots,s_n) \\ &\times x(s_1,\ldots,s_n)\,ds_1\cdots ds_n,\ (t_1,\ldots,t_n)\in R_+^n. \end{aligned} \tag{LGBI_n'}$$

Now, treating successively (LGBI_n') as a linear Gronwall–Bellman inequality with respect to the couples $(t_1,t_2),\ldots,(t_{n-1},t_n)$ and making use of (10) with $n=2$, one arrives, after $\binom{n}{2}=n(n-1)/2$ steps, at another integral inequality of the form (LGBI′), in which $\sum_{(i,j)}$ were deleted, and so on. Consequently, after $\binom{n}{1}+\cdots+\binom{n}{n-1}=2^n-2$ steps, it will follow that any solution in X_n^m of (LGBI′) must satisfy an inequality of the form (LGBI) and our assertion is proved. Of course, these reasonings do not make any distinction between ($\leqslant$) and ($=$) in the above reduction process so that, the solution of the Volterra integral equation associated to the last inequality is nothing but the solution of the Volterra integral equation associated to (LGBI′). A number of concrete forms of this solution (under some restrictive conditions about our kernels) were given by DeFranco [23]; see also Conlan and Diaz [19] or, from a more particular viewpoint, Ghoshal and Masood [26].

4. A Reduction Principle

The comparison results we formulated in the preceding section are, in a certain sense, the best possible ones, because, as already noted, the solutions of (VE) (resp. (VE′)) appear as maximal elements among the solutions of (GBI) ((GBI′)). However, in many concrete situations, these solutions are, technically speaking, difficult to be handled, the most visible disadvantages being the rather complicated Zorn procedure for the construction of a maximal solution of (VE) and/or the intervention of the multivariable iterated integrals in the process of building up a solution of (VE′). For these reasons, some "approximate" comparative evaluations for a solution of (GBI) (resp. (GBI′)) via one-variable techniques were welcomed. It must be noted at this time that, although the statements given below are formulated in the cone $(X_n^m)_+ = (X_n^0)^m$ of all continuous functions from R_+^n to R_+^m, their corresponding extensions to the whole space X_n^m are almost immediate; some of these questions will be treated elsewhere.

In the following, it is convenient to express a point of R_+^n as a couple (t, τ) where $t \in R_+$, $\tau \in R_+^{n-1}$. Given any $x \in (X_n^0)^m$, we shall denote by $x(\tau)$ the element of $(X_1^0)^m$ defined as

$$x(\tau)(t) = x(t, \tau), \qquad t \in R_+, \tau \in R_+^{n-1};$$

in this context, an element $x \in (X_n^0)^m$ will be termed *quasi-increasing* when $\sigma \leqslant \tau$ implies $x(\sigma) \leqslant x(\tau)$ in $(X_1^0)^m$. Let $x \longmapsto k_1(x)$ be an increasing map from $(X_1^0)^m$ to itself and f_1 an element of $(X_1^0)^m$. We shall say (f_1, k_1) is a normal couple when (P_1) the set $V(f_1, k_1)$ of all solutions in $(X_1^0)^m$ of (VE) is not empty, (P_2) to any solution $u_1 \in (X_1^0)^m$ of (GBI) there corresponds a $v_1 \in V(f_1, k_1)$ with $u_1 \leqslant v_1$. Sufficient conditions assuring this property were made precise, in fact, by the results of the preceding section. Under these conventions, let $x \longmapsto k(x)$ be an increasing map from $(X_n^0)^m$ to itself and f a quasi-increasing element of $(X_n^0)^m$. The following "sectional" comparison principle may be stated and proved.

THEOREM 6. *Suppose there exists a family $(K_{(\tau)}; \tau \in R_+^{n-1})$ of increasing mappings from $(X_1^0)^m$ to itself, such that*

(xii) $\int_0^\tau k(x)(s, \sigma)\, d\sigma \leqslant K_{(\tau)}(x(\tau))(s)$, $s \in R_+$, $\tau \in R_+^{n-1}$, *for each quasi-increasing x in $(X_n^0)^m$,*

(xiii) *the couple $(f(\tau), K_{(\tau)})$ is normal, for any $\tau \in R_+^{n-1}$.*

Then, for each solution $u \in (X_n^0)^m$ of (GBI) *one has*

$$u(t, \tau) \leqslant v_{(\tau)}(t), \qquad t \in R_+, \tau \in R_+^{n-1} \tag{11}$$

with $v_{(\tau)}$ belonging to $V(f(\tau), K_{(\tau)})$ for any $\tau \in R_+^{n-1}$.

Proof. Let u be a solution in $(X_n^0)^m$ of (GBI). Suppose in addition u is quasi-increasing then, by the above hypothesis (xii) we get for each $\tau \in R_+^{n-1}$

$$u(\tau)(t) \leqslant f(\tau)(t) + \int_0^t \left(\int_0^\tau k(u)(s, \sigma)\, d\sigma \right) ds$$

$$\leqslant f(\tau)(t) + \int_0^t K_{(\tau)}(u(\tau))(s)\, ds, \ t \in R_+$$

so that, by (xiii) one immediately arrives at (11). Now, if u were not quasi-increasing then, replacing it by

$$u^*(t) = f(t) + \int_0^t k(u)(s)\, ds, \qquad t \in R_+^n,$$

it is clear this new function is a quasi-increasing solution in $(X_n^0)^m$ of (GBI) and the conclusion follows from the preceding discussion combined with the evident relation $u \leqslant u^*$. Q.E.D.

As an interesting particular case when $m=1$, let us consider the choice

$$k(x)(t) = h(t)\, F(x(t)), \qquad t \in R_+^n,\ x \in X_n^0,$$

where h is an element of X_n^0 and F is a continuous increasing function from R_+ to itself then, putting

$$K_{(\tau)}(y)(t) = H_{(\tau)}(t)\, F(y(t)) \qquad t \in R_+,\ y \in X_1^0,\ \tau \in R_+^{n-1},$$

where

$$H_{(\tau)}(t) = \int_0^\tau h(t, \sigma)\, d\sigma, \qquad t \in R_+,\ \tau \in R_+^{n-1},$$

condition (xii) will evidently take place and consequently, under the acceptance of (xiii), any solution $u \in X_n^0$ of the scalar Gronwall–Bellman inequality

$$x(t) \leqslant f(t) + \int_0^t h(s)\, F(x(s))\, ds, \qquad t \in R_+^n \tag{SGBI}$$

satisfies, for any $\tau \in R_+^{n-1}$, an evaluation of the form (11), $v_{(\tau)}$ being a solution of the one-variable nonlinear Volterra integral equation

$$y(t) = f(\tau)(t) + \int_0^t H_{(\tau)}(s)\, F(y(s))\, ds, \qquad t \in R_+. \tag{SVE$_1$}$$

For example, assume F is strictly increasing, $F(0)=0$, and

$$0 < f(t, \tau) \leqslant M(\tau) + \int_0^t g_{(\tau)}(s)\, ds, \qquad t \in R_+,\ \tau \in R_+^{n-1},$$

where $(g_{(\tau)};\ \tau \in R_+^{n-1})$ is a family of elements belonging to X_1^0 and $(M(\tau);\ \tau \in R_+^{n-1})$ is a family of strict positive numbers; then, every solution $y \in X_1^0$ of (SVE$_1$) satisfies

$$y(t) \leqslant M(\tau) + \int_0^t ((g_{(\tau)}(s)/F(f(s, \tau)) + H_{(\tau)}(s))\, F(y(s))\, ds, \quad t \in R_+$$

in which case, by a standard procedure (see, e.g., Bihari [11])

$$v_{(\tau)}(t) \leqslant G^{-1}\left(G(M(\tau)) + \int_0^t (g_{(\tau)}(s)/F(f(s, \tau)))\, ds + \int_0^t H_{(\tau)}(s)\, ds\right)$$

for all $t \geqslant 0$ with

$$G(M(\tau)) + \int_0^t (g_{(\tau)}(s)/F(f(s, \tau)))\, ds + \int_0^t H_{(\tau)}(s)\, ds < G(\infty)$$

the function $G\colon (0, \infty) \to R$ being defined as

$$G(t) = \int_a^t (1/F(s))\,ds, \qquad t > 0 \text{ for some } a > 0.$$

Note that, under such a circumstance, the corresponding variant of Theorem 6 extends a series of Wendroff type inequalities due to Singare and Pachpatte [47]; see also Mamedov, Ashirov, and Atdaev [32, Chap. II, Sect. 2] or, from a more particular viewpoint, Shih and Yeh [46], Bondge and Pachpatte [13], Yeh [65], Shastri and Kasture [45], Abramovich [3], and others.

Let $x \longmapsto k_1(x)$ be an increasing map from $(X_1^0)^m$ into itself and f_1 an element of $(X_1^0)^m$. We shall say (f_1, k_1) is a *strongly normal couple* when it is normal and (P_3) $v_1 \leqslant w_1$ for each solution $v_1 \in (X_1^0)^m$ of (VE) and every solution $w_1 \in (X_1^0)^m$ of (GBI), with $(\leqslant)$ replaced by its dual $(\geqslant)$. Now, as a completion of the above result, we have

THEOREM 7. *Under the same general hypotheses, assume there exists a family* $(K_{(\tau)};\ \tau \in R_+^{n-1})$ *of increasing mappings from* $(X_1^0)^m$ *to itself such that,* (xii) *plus*

(xiii′) the couple $(f_{(\tau)}, K_{(\tau)})$ *is strongly normal for any* $\tau \in R_+^{n-1}$

hold and let in addition the element w *in* $(X_n^0)^m$ *be such that*

(xiv) $f(t, \tau) + \int_0^t K_{(\tau)}(w(\tau))(s)\,ds \leqslant w(t, \tau),\ t \in R_+,\ \tau \in R_+^{n-1}.$

Then, we necessarily have $u \leqslant w$, *for each solution* $u \in (X_n^0)^m$ *of* (GBI).

Proof. By the reasonings of the above theorem we have the evaluation (11) where $v_{(\tau)}$ is a solution in $(X_1^0)^m$ of (VE) with (f, k) replaced by $(f(\tau), K_{(\tau)})$. This fact, together with (xiii)′ plus (xiv), establishes our assertion. Q.E.D.

As a particular case useful in applications, let us take

$$k(x)(t) = P(t)\Big(x(t) + \int_0^t Q(s)\,x(s)\,ds\Big), \qquad t \in R_+^n,\ x \in (X_n^0)^m$$

with $P(t) = (p_{ij}(t);\ 1 \leqslant i, j \leqslant m)$ and $Q(t) = (q_{ij}(t);\ 1 \leqslant i, j \leqslant m)$, (m, m) matrices over X_n^0 then, defining

$$K_{(\tau)}(y)(t) = H_{(\tau)}(t)\,y(t), \qquad t \in R_+,\ y \in (X_1^0)^m,\ \tau \in R_+^{n-1},$$

where

$$H_{(\tau)}(t) = \int_0^\tau P(t, \sigma)\left(I + \int_0^t \int_0^\sigma Q(r, \rho)\,dr\,d\rho\right)d\sigma, \qquad t \in R_+,\ \tau \in R_+^{n-1}$$

hypothesis (xii) will be fulfilled if we restrict our considerations to the increasing elements of (X_n^0); note that, a sufficient condition that such a restrictive statement of (xii) be effective is that f be increasing. Moreover, by the discussion of the preceding section (xiii′) takes evidently place. Now, let us put

$$w(t, \tau)=\left(\exp\left(\int_0^t H_{(\tau)}(s)\,ds\right)\right)f(t, \tau), \qquad t\in R_+,\ \tau\in R_+^{n-1}$$

with f taken as above, and assume $t\longmapsto H_{(\tau)}(t)$ and $t\longmapsto \int_0^t H_{(\tau)}(s)\,ds$ are permutable as (m, m) matrix functions then (cf. also Coddington and Levinson [18, Chap. III, Sect. 4])

$$\begin{aligned} f(t, \tau)&+\int_0^t K_{(\tau)}(w(\tau))(s)\,ds\\ &=f(t, \tau)+\int_0^t H_{(\tau)}(s)\exp\left(\int_0^s H_{(\tau)}(r)\,dr\right)f(s, \tau)\,ds\\ &\leqslant\left(I+\int_0^t H_{(\tau)}(s)\exp\left(\int_0^s H_{(\tau)}(r)\,dr\right)ds\right)f(t, \tau)\\ &=\left(\exp\left(\int_0^t H_{(\tau)}(s)\,ds\right)\right)f(t, \tau)=w(t, \tau)\end{aligned}$$

and (xiv) will be satisfied too. By the above theorem it will follow that any solution $u\in(X_n^0)^m$ of the (vector) linear Gronwall–Bellman inequality

$$x(t)\leqslant f(t)+\int_0^t P(s)\left(x(s)+\int_0^s Q(r)\,x(r)\,dr\right)ds, \qquad t\in R_+^n \quad \text{(LGBI)}$$

is necessarily bounded above by the function w defined as before. Note that, when $m=1$, this conclusion is nothing but the Pachpatte result [38] proved by a specific "differential" procedure; see also Bondge, Pachpatte, and Walter [14], Yeh and Shih [66], Corduneanu [20], and Pachpatte [37, 40]. At the same time, it is not without interest to specify that an appropriate matrix version of this example may be identified with a similar one due to Chandra and Davis [15]. A number of useful applications of these results to hyperbolic partial differential equatons may be found in the above quoted Pachpatte's papers.

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