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Periodic Boundary-Value Problems with Cyclic Totally Positive Green's Functions with Applications to Periodic Spline Theory

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INTRODUCTION

Arguing from a few evident oscillatory characteristics shared by numerous vibrating physical systems, Gantmacher and Krein [1] showed that the influence functions of many of these systems were totally positive (TP), see Definition 1.1. Subsequently, they supplied formal proofs establishing that the Green's functions associated with several standard boundary-value problems (BVP) of Sturm-Liouville type were TP. This discovery then served as a keystone for the elucidation and clarification of much of the classical theory of these boundary-value problems. The work of Krein and Gantmacher, which also includes discussions of some special 4th order equations as well as the 2nd order Sturm-Liouville theory, has undergone several extensions and refinements. For intance, in Karlin [3], a family of $2k$th order linear differential operators is presented together with an appropriate set of separate boundary conditions for which the associated Green's functions are TP. For further generalizations the reader may consult Karlin [5, Vol. 2] and Karon [4].

The study of boundary-value problems based on TP considerations requires a precise knowledge of when strict inequality will hold in the system of determinants (1.2). For a deeper analysis, it is essential to extend the notion of TP to allow for coincidences among the points occurring in (1.2). The appropriate concept here—extended total positivity (ETP)—was introduced in Karlin [3, Chap. 2], see also Karlin [5]. The concept of ETP has proven essential in analyzing certain problems in the theory of inequalities and in generalized convexity theory as well as in the study of boundary-value

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problems, disconjugacy problems, and multiple-knot spline approximation problems.

Motivated by the studies above and the observation that numerous vibrating physical systems admit mathematical descriptions as boundary-value problems with periodic boundary conditions, it is natural to ask whether the Green's functions arising from periodic boundary-value problems share the total positivity properties exhibited in the one-sided case. It is intrinsic to the periodic case that the Green's function cannot be TP; however, some sign regularity is preserved and we shall present a class of differential operators together with boundary conditions of periodic type for which the Green's functions are *cyclic* totally positive (CTP), see Definitition 1.3. Moreover, with an eye to the applications, we give precise conditions under which the determinants associated with these Green's functions are positive.

The main theorems of this paper (Theorems 1.2–1.4) have immediate application to the theory of periodic splines (see §3) and to the spectral analysis of the boundary-value problems described by equations (1.7–1.8). These spectral properties will appear in a separate paper by the second author. Still another application with a physical flavor is included at the end of §1.

1. Terminology and Main Results

We review briefly several definitions of total positivity theory. The reader may refer to Karlin [3] for further elaboration of these concepts and their interrelationships.

To begin with let X and S be linearly ordered sets and $G(x, s)$ a real function defined on $X \times S$. The p-dimensional (open) simplex in $X^p = X \times X \times \cdots \times X$ (p copies of X) is denoted $\Delta_p(X)$:

$$\Delta_p(X) = \{\mathbf{x} = (x_1, \ldots, x_p) \mid x_1 < \cdots < x_p,\ x_i \in X\}.$$

The (relative) closure of this simplex is $\bar{\Delta}_p(X)$:

$$\bar{\Delta}_p(X) = \{\mathbf{x} = (x_1, \ldots, x_p) \mid x_1 \leqslant \cdots \leqslant x_p,\ x_i \in X\}.$$

When $X = S$ we abbreviate $\Delta_p(X)$ to Δ_p (and $\bar{\Delta}_p(X)$ to $\bar{\Delta}_p$). The determinant function

$$G_{[p]}(\mathbf{x}, \mathbf{s}) = G\begin{pmatrix} x_1, \ldots, x_p \\ s_1, \ldots, s_p \end{pmatrix} = \det \| G(x_i, s_j) \|_{i,j=1}^{p} \tag{1.1}$$

defined on $\Delta_p(X) \times \Delta_p(S)$ is called the *compound kernel of order p* induced by $G(x, s)$.

DEFINITION 1.1. In the above context, $G(x, s)$ is called *totally positive of order n* (TP_n) if

$$G_{[p]}(\mathbf{x}, \mathbf{s}) \geqslant 0 \tag{1.2}$$

for $p = 1, 2,..., n$ and for all

$$(\mathbf{x}, \mathbf{s}) \in \Delta_p(X) \times \Delta_p(S). \tag{1.3}$$

To define extended total positivity, we assume that X and S are (real) intervals and the $G(x, s)$ is smooth enough to allow the operations performed below. Notice, incidentally, that (1.1) would not serve as a useful definition for $G_{[p]}(\mathbf{x}, \mathbf{s})$ if either $\mathbf{x}$ or $\mathbf{s}$ were boundary points of their respective simplicies because then $G_{[p]}(\mathbf{x}, \mathbf{s}) = 0$ trivially. If $\mathbf{x} \in \bar{\Delta}_p(X)$ and $\mathbf{s} \in \bar{\Delta}_p(S)$ we shall write

$$G^*_{[p]}(\mathbf{x}, \mathbf{s}) = G^* \begin{pmatrix} x_1, ..., x_p \\ s_1, ..., s_p \end{pmatrix}$$

for the determinant defined as in (1.1) but with the following modifications:

(i) If $x_{\ell-1} < x_\ell = \cdots = x_{\ell+k-1} < x_{\ell+k}$ we replace the $(\ell + \nu)$th row of the determinant (1.1) by

$$\left[\frac{\partial^\nu}{\partial x^\nu} G(x_\ell, s_1),..., \frac{\partial^\nu}{\partial x^\nu} G(x_\ell, s_p)\right] \qquad (\nu = 1, 2,..., k-1);$$

(ii) if $s_{\ell-1} < s_\ell = \cdots = s_{\ell+k-1} < s_{\ell+k}$ we make a similar substitution in the columns of (1.1) this time introducing partial derivatives $\partial/\partial s$;

(iii) if coincidences occur in both x_i's and s_j's we perform both substitutions.

DEFINITION 1.2. Continuing in the context above, $G(x, s)$ is called *extended totally positive of order n* (ETP_n) if

$$G^*_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \qquad \text{(strict inequality)}$$

whenever

$$(\mathbf{x}, \mathbf{s}) \in \bar{\Delta}_p(X) \times \bar{\Delta}_p(S).$$

We turn next to the concept of cyclic total positivity and justify the statement made earlier that a Green's function, $G(x, s)$, associated with a boundary-value problem of periodic type cannot be TP in the usual sense. Indeed, suppose for simplicity that the associated differential operator has constant coefficients so the $G(x, s) = g(x - s)$ is, in fact, a translation kernel, where g is a 2π-periodic function of its argument. For convenience we shall deal exclusively with 2π-periodic functions; however, any other period would serve equally well. In this context it is natural to view g as defined on the circumference of the unit circle and view the system of points

$$\begin{Bmatrix} x_1 < \cdots < x_p < x_1 + 2\pi \\ s_1 < \cdots < s_p < s_1 + 2\pi \end{Bmatrix}$$

as invariant under rotations, i.e., as equivalent to the system

$$\begin{Bmatrix} x_\ell < \cdots < x_p < x_1 + 2\pi < \cdots < x_{\ell-1} + 2\pi \\ s_m < \cdots < s_p < s_1 + 2\pi < \cdots < s_{m-1} + 2\pi \end{Bmatrix}$$

for all choices of ℓ and m. In order to define a notion of "total positivity" which is compatible with the periodic structure of g, we must impose the conditions

$$G\begin{pmatrix} x_1, \ldots, x_p \\ s_1, \ldots, s_p \end{pmatrix} = G\begin{pmatrix} x_\ell, \ldots, x_p, x_1 + 2\pi, \ldots, x_{\ell-1} + 2\pi \\ s_m, \ldots, s_p, s_1 + 2\pi, \ldots, s_{m-1} + 2\pi \end{pmatrix} \tag{1.4}$$

on the compounds of G. A simple calculation reduces the right side of (1.4) to

$$(-1)^{p+1}\, G\begin{pmatrix} x_1, \ldots, x_p \\ s_1, \ldots, s_p \end{pmatrix}.$$

Clearly then (1.4) is automatically satisfied by all the *odd*-order compounds of G and by none of the *even* order compounds (except in the trivial case when they vanish identically.) These observations show that the periodic nature of g is compatible with a "total positivity" structure on its compounds only when the positivity conditions are imposed only on the odd order compounds. This situation justifies

DEFINITION 1.3. A function $G(x, s)$ defined on $[a, b) \times [a, b)$ (and which may be regarded as extended periodically in each variable) is called *cyclic totally positive of order* $2\ell + 1$ ($\mathrm{CTP}_{2\ell+1}$) if all the *odd*-order compounds

$$G_{[p]}(\mathbf{x}, \mathbf{s}) \geqslant 0 \qquad (p = 1, 3, \ldots, 2\ell + 1)$$

for $\mathbf{x}, \mathbf{s} \in \Delta_p([a, b))$.

Remark 1.1. If $G(x, s) = g(x - s)$ and $[a, b) = [0, 2\pi)$ and the conditions of Definition 1.3 are met it is customary (following Schoenberg) to call $g(u)$ a *cyclic Pólya frequency function* of order $2\ell + 1$.

Remark 1.2. If $G(x, s)$ is TP_n (or CTP_n) for all $n = 1, 2, \ldots$, we say that $G(x, s)$ is TP_∞ (or CTP_∞) or simply TP (or CTP).

The class of periodic boundary-value problems we shall discuss is defined as follows: Let

$$\gamma_1(x), \gamma_2(x), \ldots, \gamma_r(x)$$

be real functions defined on $[0, 2\pi]$ which are of continuity class $C^r[0, 2\pi]$ and satisfy

$$\int_0^{2\pi} \gamma_i(x)\, dx \neq 0 \qquad (i = 1, \ldots, r) \tag{1.5}$$

Define

$$\begin{aligned} D_i &= D + \gamma_i I \\ D_i^* &= -D + \gamma_i I \qquad (i = 1, \ldots, r;\ D = d/dx) \\ D_0 &= D_0^* = I \end{aligned} \tag{1.6}$$

and let $\tilde{D}_i$ be either a D_i or a D_i^*. Define an rth order differential operator $\tilde{L}_r$ by

$$\tilde{L}_r u = \tilde{D}_r \tilde{D}_{r-1} \cdots \tilde{D}_1 u \tag{1.7}$$

and associate to it the generalized periodic boundary conditions (BC)

$$\tilde{D}_j \cdots \tilde{D}_1 \tilde{D}_0 u(0) = \tilde{D}_j \cdots \tilde{D}_1 \tilde{D}_0 u(2\pi) \tag{1.8}$$

where $j = 0, 1, \ldots, r - 1$. We denote these BC by $\tilde{\mathscr{B}}_r$. It is also convenient to let $\tilde{\mathscr{B}}_r$ stand for the class of all $C^r[0, 2\pi]$-functions satisfying the BC $\tilde{\mathscr{B}}_r$, and to take $\tilde{\mathscr{B}}_r$ for the domain of the operator $\tilde{L}_r$.

Remark 1.3. If all the γ_i's are periodic, the generalized BC $\tilde{\mathscr{B}}_r$ are equivalent to the purely periodic ones:

$$u(0) = u(2\pi), \ldots, u^{(r-1)}(0) = u^{(r-1)}(2\pi).$$

Theorem 1.1 confirms that the BVP determined by (1.7)-(1.8) has a Green's function which we denote by $\tilde{G}_r(x, s)$.

THEOREM 1.1. *If $u \in \tilde{\mathscr{B}}_r$ and $\tilde{L}_r u = 0$ then $u = 0$.*

Proof. The proof follows easily from the standard factorizations,

$$\begin{aligned} D_i &= D + \gamma_i I = \omega_i^{-1} D \omega_i \\ D_i^* &= -D + \gamma_i I = -\,\omega_i D \omega_i^{-1} \end{aligned} \qquad (i = 1, \ldots, r) \tag{1.9}$$

where

$$\omega_i(x) = \exp\left(\int_0^x \gamma_i(t)\, dt\right). \tag{1.10}$$

Simply write $\tilde{L}_r u$ in factored form and integrate $\tilde{L}_r u = 0$ step-by-step using the BC $\tilde{\mathscr{B}}_r$ and the basic assumption (1.5) to show that all integration constants vanish. The last integration gives $u = 0$. ||

Theorems 1.2–1.4 delimit the precise TP properties of $\tilde{G}_r(x, s)$. Their proofs are presented in §2. Let $\Delta_p = \Delta_p([0, 2\pi))$ for the remainder of this paper.

THEOREM 1.2. *The Green's function* $\tilde{G}_r(x, s)$ *for the* BVP (1.7)–(1.8) *is* CTP_∞ *i.e.*

$$(\tilde{G}_r)_{[p]}(\mathbf{x}, \mathbf{s}) \geqslant 0 \qquad (p = 1, 3, 5,...) \tag{1.11}$$

for all $\mathbf{x}, \mathbf{s} \in \Delta_p$. *Moreover*, (1.11) *implies*

$$(\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) \geqslant 0 \qquad (p = 1, 3, 5,...) \tag{1.11'}$$

for all $\mathbf{x}, \mathbf{s} \in \bar{\Delta}_p$ *for which* $(G_r)^*_{[p]}(\mathbf{x}, \mathbf{s})$ *is defined* (*see Remark* 1.4).

It is crucial for the applications to determine precisely when strict inequality holds in (1.11) and (1.11′). Based on the results in the one-sided case (Karlin [3, Chap. 10, Sec. 8]), one expects the determinants in question to be positive only if the points $x_1 ,..., x_p , s_1 ,..., s_p$ "interlace" properly. In the periodic case, this "interlacing" must be interpreted in a rotationally invariant way. An appropriate "cyclic" interlacing may be defined as follows: Let $\mathbf{x} = (x_1 ,..., x_p)$, $\mathbf{s} = (s_1 ,..., s_p) \in \Delta_p$ and assume temporarily that $\{x_i\} \cap \{s_j\} = \phi$. Regard x_i and s_j as distributed on the circumference of the unit circle, ∂U. Let $\Lambda(\mathbf{x}, \mathbf{s})$ be the finite set of all closed subarcs of ∂U whose end points lie in $\{x_i\} \cup \{s_j\}$.

DEFINITION 1.4. If $\alpha \in \Lambda(\mathbf{x}, \mathbf{s})$ is a closed subarc of ∂U we define

$$\#(\alpha) = \left| \sum_{x_i \in \alpha} 1 - \sum_{s_j \in \alpha} 1 \right| ;$$

that is, $\#(\alpha)$ is the difference between the number of x's and s's which lie in α.

DEFINITION 1.5. In the preceeding context,

$$\Delta(\mathbf{x}, \mathbf{s}) = \max_{\alpha \in \Lambda(\mathbf{x}, \mathbf{s})} \#(\alpha). \tag{1.12}$$

In short, $\Delta(\mathbf{x}, \mathbf{s})$ is the largest number obtainable when one counts all possible blocks of x's and s's on ∂U attributing opposite signs (± 1) to the x's and s's respectively. Clearly $\Delta(\mathbf{x}, \mathbf{s})$ is a rotationally invariant "measure" of the interlacing of the coordinates of $\mathbf{x}$ and $\mathbf{s}$.

To relax the restriction $\{x_i\} \cap \{s_j\} = \phi$ suppose

$$x_k = \cdots = x_{k+\lambda} = \xi = s_\ell = \cdots = s_{\ell+\mu} \qquad (\mu, \lambda \geqslant 0).$$

We perturb these x's and s's slightly separating them all but not disturbing their location on ∂U relative to the remaining coordinates of $\mathbf{x}$ and $\mathbf{s}$, and repeat this procedure for any other coincidence points (such as ξ). Formation of all possible perturbations of this sort leads to, say, M different configurations

$$\mathbf{x}^{(m)} = \{x_i^{(m)}\}_{i=1}^p, \qquad \mathbf{s}^{(m)} = \{s_j^{(m)}\}_{j=1}^p \qquad (m = 1,..., M)$$

with corresponding $\Lambda^{(m)}$'s for which

$$\Delta(\mathbf{x}^{(m)}, \mathbf{s}^{(m)}) = \max_{\Lambda^{(m)}} \#(\alpha^{(m)})$$

is already defined.

Definition 1.6. In the setting above, if $\mathbf{x}, \mathbf{s}, \in \bar{\Delta}_p$ we define

$$\Delta(\mathbf{x}, \mathbf{s}) = \max_{1 \leqslant m \leqslant M} \Delta(\mathbf{x}^{(m)}, \mathbf{s}^{(m)}).$$

In effect, Definition 1.6 says that we count coincident x_i's and s_j's so as to permit as little "interlacing" as possible.

We are now in a position to state two sharp extensions of Theorem 1.2 whose proofs are considerably deeper.

Theorem 1.3. *The Green's function $\tilde{G}_r(x, s)$ for the* BVP (1.7)–(1.8) *is* CTP_∞. *Moreover,*

$$(\tilde{G}_r)_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \qquad \text{if and only if} \quad \Delta(\mathbf{x}, \mathbf{s}) \leqslant r \tag{1.13}$$

where $\mathbf{x}, \mathbf{s} \in \Delta_p$, $p = 1, 3, 5,...,$ *and r is the order of $\tilde{L}_r$.*

Theorem 1.4. *Let $\tilde{G}_r(x, s)$ be the Green's function for the* BVP (1.7)–(1.8) *and let the points* $\mathbf{x} = (x_1, ..., x_p)$, $\mathbf{s} = (s_1, ..., s_p) \in \bar{\Delta}_p (p = 1, 3, 5,...)$ *be subject to the restrictions*:

(a) *Whenever α of the x_i's coincide with β of the s_j's we require $\alpha + \beta \leqslant r + 1$.*

(b) *No more than r consecutive x's or s's coincide.*

Then $(\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) \geqslant 0$ *and moreover*

$$(\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \quad \text{if and only if} \quad \Delta(\mathbf{x}, \mathbf{s}) \leqslant r \tag{1.14}$$

Remark 1.4. The conditions (a) and (b) are minimal requirements guaranteeing that the entries of the determinant in (1.14) have a meaning. Whenever an $(r-1)$th derivative occurs it is taken as a right derivative with respect to x and a left derivative with respect to s.

Theorems 1.1–1.4 for the periodic BVP (1.7)–(1.8) have analogs for the "odd" periodic BVP when the differential operator (1.7) is assigned the BC

$$\tilde{D}_j \cdots \tilde{D}_1 \tilde{D}_0 u(0) = -\tilde{D}_j \cdots \tilde{D}_1 \tilde{D}_0 u(2\pi) \qquad (j = 0, 1,..., r-1) \quad (1.8')$$

instead of the BC (1.8). In this case the *even* order determinants of $\tilde{G}_r(x, s)$ maintain fixed signs and the conclusions of Theorems 1.2–1.4 hold by simply replacing "$p = 1, 3, 5,...$" by "$p = 2, 4, 6,...$". The proofs for the odd periodic case are omitted being slight variants on those for the periodic case. Arguments hinging on the periodicity of $\tilde{G}_r(x, s)$ in the periodic case are replaced by arguments hinging on the intermediate-value theorem in conjuction with (1.8′) in the odd case. In particular, the restrictions (1.5) necessary for the existance of $\tilde{G}_r(x, s)$ in the periodic case become superfluous in the odd periodic case.

We close this section with the physical application mentioned in §0. Consider a physical segment—which we conceive of as the interval $[0, 2\pi]$—upon which directed forces of magnitudes and directions f_j are impressed at the points s_j $(j = 1, 2,..., 2\ell + 1)$. We assume the resulting displacement function $y(x)$ satisfies the differential equation $\tilde{L}_r y = dF$ and the BC $\tilde{\mathscr{B}}_r$ where $dF = \sum f_j \delta_j$ and δ_j is the δ-measure concentrating at s_j. An interesting question emerges: Is it possible to find a set of forces $\{f_j\}$ which when applied at the points $\{s_j\}$ produce arbitrarily prescribed displacements $\{y_i\}$ at a given set of points $\{x_i\}$ $(i = 1, 2,..., 2\ell + 1)$? The answer is yes if and only if $\Delta(\mathbf{x}, \mathbf{s}) \leqslant r$. This result is physically satisfying for it reveals that a set of forces $\{f_j\}$ will exist if and only if the points of application $\{s_j\}$ and the points of interpolation $\{x_i\}$ are sufficiently intersperced. The proof of our assertion follows easily from Theorem 1.3 and the standard representation for the displacement

$$y(x) = \sum_{j=1}^{2\ell+1} \tilde{G}_r(x, s_j) f_j .$$

2. Proofs

The proofs of Theorems 1.2–1.4 present difficulties not encountered in the corresponding proofs for the TP case. Sylvester's determinant identity which proves so useful in the TP case is of only marginal value here because the even order compounds in the CTP case do not maintain fixed signs. The two cases are similar to the extent that both employ frequent forward and backward induction arguments coupled with certain perturbation arguments.

We begin with the proof of Theorem 1.2 which will be done by induction on r, the order of $\tilde{L}_r$. If $r = 1$, (1.7)–(1.8) is either

$$\begin{aligned} D^*_1 u &= -u' + \gamma_1 u \\ u(0) &= u(2\pi) \end{aligned} \tag{2.1}$$

or

$$\begin{aligned} D_1 u &= u' + \gamma_1 u \\ u(0) &= u(2\pi) \end{aligned} \tag{2.1'}$$

Using the notation of (1.9) the BVP (2.1) has the Green's function

$$\tilde{G}_1(x, s) = \begin{cases} \dfrac{\alpha}{\alpha - 1} \exp(\omega_1(x) - \omega_1(s)) & x \leqslant s \\ \dfrac{1}{\alpha - 1} \exp(\omega_1(x) - \omega_1(s)) & s < x \end{cases} \tag{2.2}$$

where $\alpha = \exp(\omega_1(2\pi)) > 1$, and the BVP (2.1′) has the Green's function obtained by interchanging x and s in (2.2).

LEMMA 2.1. *The Green's function for the* BVP's (2.1) *and* (2.1′) *are* CTP_∞. *Moreover,*

$$(\tilde{G}_1)_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \qquad \textit{if and only if} \quad \Delta(\mathbf{x}, \mathbf{s}) \leqslant 1 \qquad p = 1, 3, 5, \ldots . \tag{2.3}$$

This result is an immediate corollary of Lemma 2.2.

LEMMA 2.2. *Let*

$$K(x, s) = \begin{cases} a & x \leqslant s \\ b & s < x \end{cases} \tag{2.4}$$

with $a, b \in \mathbb{R}$. *Then for* $\mathbf{x}, \mathbf{s} \in \Delta_p$ *and* $p = 1, 2, 3, \ldots$,

$$K_{[p]}(\mathbf{x}, \mathbf{s}) = \begin{cases} a(a-b)^{p-1} & \text{if} \quad x_1 \leqslant s_1 < \cdots < x_p \leqslant s_p \\ b(b-a)^{p-1} & \text{if} \quad s_1 < x_1 \leqslant \cdots \leqslant s_p < x_p \\ 0 & \text{otherwise} \end{cases} \tag{2.5}$$

Proof. Evidently, except under the special cases stated, either a pair of rows or columns of $K_{[p]}(\mathbf{x}, \mathbf{s})$ agree and hence $K_{[p]}(\mathbf{x}, \mathbf{s}) = 0$. In the special cases, the determinants in question are easily computed and the lemma confirmed. ||

From (2.2) we see that $\tilde{G}_1(x, s)$ has the form

$$\tilde{G}_1(x, s) = f(x) g(s) K(x, s)$$

where f, $g > 0$, $a = \alpha/(\alpha - 1) > 0$, and $b = 1/(\alpha - 1) > 0$. Clearly, $\tilde{G}_1(x, s)$ will be CTP_∞ if and only if $K(x, s)$ is and examination of (2.5) reveals that this occurs if and only if $a > 0$ and $b > 0$ which is the case here.

Remark 2.1. It is interesting to note that the kernel in (2.4) will be TP_∞ if and only if it is triangular with $a > 0$, $b = 0$ or vice versa.

The induction step in proving Theorem 1.2 is advanced by applying the *basic composition formula* (BCF) to the convolution formula (2.10) below. Recall the BCF states: If $K(x, y)$, $L(x, z)$, and $M(z, y)$ are Borel functions satisfying

$$K(x, y) = \int_Z L(x, z)\, M(z, y)\, d\sigma(z) \tag{2.6}$$

where $x \in X$, $y \in Y$, $z \in Z$; X, Y, and Z are linearly ordered subsets of $\mathbb{R}$; and $d\sigma(z)$ is a sigma-finite measure on Z, then

$$K_{[p]}(\mathbf{x}, \mathbf{y}) = \int_{\Delta_p(Z)} L_{[p]}(\mathbf{x}, \mathbf{z})\, M_{[p]}(\mathbf{z}, \mathbf{y})\, d\sigma(\mathbf{z}) \tag{2.7}$$

where $d\sigma(\mathbf{z}) = d\sigma(z_1) \cdots d\sigma(z_p)$. A proof of this formula appears in Karlin [3, page 17].

We associate with the BVP (1.7)–(1.8) two related problems:

$$\begin{aligned} \tilde{L}_{r-1}u &= \tilde{D}_{r-1} \cdots \tilde{D}_1 u \\ \tilde{D}_j \cdots \tilde{D}_1\tilde{D}_0 u(0) &= \tilde{D}_j \cdots \tilde{D}_1\tilde{D}_0 u(2\pi) \qquad (j = 0, 1,..., r-2) \end{aligned} \tag{2.8}$$

with Green's function $\tilde{G}_{r-1}(x, s)$ and

$$\begin{aligned} L_1^{(r)}u &= \tilde{D}_r u \\ \tilde{D}_{r-1}u(0) &= \tilde{D}_{r-1}u(2\pi) \end{aligned}$$

with Green's function $\tilde{G}_1^{(r)}(x, s)$. Making a standard interpretation we have the convolution formula

$$\tilde{G}_r(x, s) = \int_0^{2\pi} \tilde{G}_{r-1}(x, \xi)\, \tilde{G}_1^{(r)}(\xi, s)\, d\xi. \tag{2.10}$$

There is a companion relation to (2.10) which we will need later. Let $\hat{L}_{r-1} = \tilde{D}_r \cdots \tilde{D}_2$ and associate to it the BC $\hat{\mathscr{B}}_{r-1}$ based on the $r-2$ operators $\tilde{D}_{r-1}, \ldots, \tilde{D}_2$ and denote its Green's function by $\hat{G}_{r-1}(x, s)$. Let $\tilde{G}_1(x, s)$ be the Green's function for the BVP $\tilde{L}_1$, $\tilde{\mathscr{B}}_1$. Then

$$\tilde{G}_r(x, s) = \int_0^{2\pi} \tilde{G}_1(x, \xi)\, \hat{G}_{r-1}(\xi, s)\, d\xi. \tag{2.10$'$}$$

Assume now that (1.11) of Theorem 1.2 holds for all Green's functions associated with differential operators of type (1.7)–(1.8) whose orders are $<r$. Then both Green's functions on the right side of (2.10) are CTP_∞ and hence by the BCF the same is true of $\tilde{G}_r(x, s)$. This establishes (1.11) of Theorem 1.2 since the case $r = 1$ holds by Lemma 2.1. The final conclusion of Theorem 1.2, namely, (1.11′) follows directly from (1.11) and a standard argument of total positivity theory (consult Karlin [3], Chapter 2 Theorem 2.2 and its proof).

A similar induction argument in conjunction with the following general result proves (1.13) of Theorem 1.3 and hence establishes that theorem.

THEOREM 2.1. *Let $K(x, \xi)$ and $L(\xi, s)$ be sign-consistent of order p (SC_p); that is, there are signs $\epsilon_p(K)$ ($+1$ or -1) and $\epsilon_p(L)$ so that*

$$\epsilon_p(K)\, K_{[p]}(\mathbf{x}, \boldsymbol{\xi}) \geqslant 0, \qquad \epsilon_p(L)\, L_{[p]}(\boldsymbol{\xi}, \mathbf{s}) \geqslant 0$$

for all $\mathbf{x}, \boldsymbol{\xi}, \mathbf{s} \in \Delta_p$. *Assume further that*

$$\begin{aligned} \epsilon_p(K)\, K_{[p]}(\mathbf{x}, \boldsymbol{\xi}) > 0 \quad &\textit{if and only if} \quad \Delta(\mathbf{x}, \boldsymbol{\xi}) \leqslant 1 \qquad (\leqslant r-1) \\ \epsilon_p(L)\, L_{[p]}(\boldsymbol{\xi}, \mathbf{s}) > 0 \quad &\textit{if and only if} \quad \Delta(\boldsymbol{\xi}, \mathbf{s}) \leqslant r-1 \quad (\leqslant 1). \end{aligned}$$

Then

$$M(x, s) = \int_0^{2\pi} K(x, \xi)\, L(\xi, s)\, d\xi.$$

is SC_p *with* $\epsilon_p(M) = \epsilon_p(K)\, \epsilon_p(L)$ *and*

$$M_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \qquad \textit{if and only if} \quad \Delta(\mathbf{x}, \mathbf{s}) \leqslant r \tag{2.11}$$

Proof. The BCF

$$M_{[p]}(\mathbf{x}, \mathbf{s}) = \int_{\Delta_p} K_{[p]}(\mathbf{x}, \boldsymbol{\xi})\, L_{[p]}(\boldsymbol{\xi}, \mathbf{s})\, d\boldsymbol{\xi} \tag{2.12}$$

shows at once that $M(x, s)$ is SC_p and that $\epsilon_p(M) = \epsilon_p(K)\, \epsilon_p(L)$. Only (2.11) requires discussion. Suppose $M_{[p]}(\mathbf{x}, \mathbf{s}) > 0$. By (2.12) and our hypothesis there is a point $\boldsymbol{\xi} \in \Delta_p$ such that

$$\Delta(\mathbf{x}, \boldsymbol{\xi}) \leqslant 1 \qquad \text{and} \qquad \Delta(\boldsymbol{\xi}, \mathbf{s}) \leqslant r - 1 \tag{2.13}$$

However, the definition of $\Delta(\cdot, \cdot)$ readily yields the transitivity relation

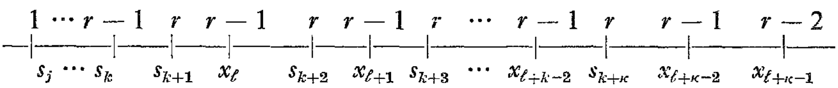
$$\Delta(\mathbf{a}, \mathbf{b}) \leqslant \Delta(\mathbf{a}, \mathbf{c}) + \Delta(\mathbf{c}, \mathbf{b})$$

and hence $\Delta(\mathbf{x}, \mathbf{s}) \leqslant r$ in view of (2.13).

Conversely, suppose $\Delta(\mathbf{x}, \mathbf{s}) \leqslant r$. If, in fact, $\Delta(\mathbf{x}, \mathbf{s}) \leqslant r - 1$ the set

$$\{\boldsymbol{\xi} = (\xi_1, ..., \xi_p) \mid x_i - \epsilon < \xi_i < x_i + \epsilon \ (i = 1, ..., p)\}$$

has positive p-dimensional Lebesgue measure and for $\epsilon > 0$ suitably small $\Delta(\mathbf{x}, \boldsymbol{\xi}) \leqslant 1$ and $\Delta(\boldsymbol{\xi}, \mathbf{s}) \leqslant r - 1$ so by (2.12) $M_{[p]}(\mathbf{x}, \mathbf{s}) > 0$. The extreme case $\Delta(\mathbf{x}, \mathbf{s}) = r$ requires a more delicate argument. In this case we can find an arc $\alpha_0 \in \Lambda(\mathbf{x}, \mathbf{s})$, $\alpha_0 = [s_j, x_{l+\kappa}]$ which contains an excess of s's over x's with points distributed as indicated in the following figure:



Perturbing x_ℓ to x_ℓ',..., $x_{\ell+\kappa-1}$ to $x'_{\ell+\kappa-1}$ with

$$s_{k+t} < x'_{\ell+t} < s_{k+t+1} \qquad (t = 0, ..., \kappa + 1)$$

and leaving the remaining x_i's fixed, we secure a new sequence $\mathbf{x}'$ with the properties: The arcs $\alpha \in \Lambda = \Lambda(\mathbf{x}, \mathbf{s})$ are in an obvious correspondence with the arcs $\alpha' \in \Lambda' = \Lambda(\mathbf{x}', \mathbf{s})$ so that

(i) $\#(\alpha') \leqslant r$ for all $\alpha' \in \Lambda'$,

(ii) $\alpha \in \Lambda, \#(\alpha) \leqslant r - 1 \Rightarrow \#(\alpha') \leqslant r - 1$,

(iii) $\alpha' \cap \alpha_0 \neq \emptyset \Rightarrow \#(\alpha') \leqslant r - 1$ i.e. if the arc α' overlaps the (fixed) arc α_0, then $\#(\alpha') \leqslant r - 1$.

Properties (i)–(iii) imply that we may reduce step-by-step the number of arcs on ∂U where $\#(\alpha) = r$ in such a way that we finally secure a point $\mathbf{x}'' \in \Delta_p$:

(a) $\#(\alpha'') \leqslant r - 1$ for all $\alpha'' \in \Lambda'' = \Lambda(\mathbf{x}'', \mathbf{s})$,

(b) $\Delta(\mathbf{x}, \mathbf{x}'') \leqslant 1$,

(c) $\mathbf{x}''$ is gotten from $\mathbf{x}$ by "perturbations" as indicated above.

Thus the set of positive measure

$$\{\boldsymbol{\xi} = (\xi_1, ..., \xi_p) \mid x_i'' - \epsilon < \xi_i < x_i'' + \epsilon\}$$

satisfies $\Delta(\mathbf{x}, \boldsymbol{\xi}) \leqslant 1$ and $\Delta(\boldsymbol{\xi}, \mathbf{s}) \leqslant r - 1$ for small $\epsilon > 0$. As before the BCF shows that $M_{[p]}(\mathbf{x}, \mathbf{s}) > 0$. ||

Theorems 1.2 & 1.3 are now established and will be used to prove Theorem 1.4. Only relation (1.14) remains in doubt:

$$(\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \quad \text{if and only if} \quad \Delta(\mathbf{x}, \mathbf{s}) \leqslant r \tag{1.14}$$

Proof of Theorem 1.4.

That $(\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) > 0 \Rightarrow \Delta(\mathbf{x}, \mathbf{s}) \leqslant r$ is easily established: Indeed, in the contrary situation, we could find $\mathbf{x}, \mathbf{s} \in \bar{\Delta}_p$ with $\Delta(\mathbf{x}, \mathbf{s}) \geqslant r + 1$ and $(\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) \geqslant 2a > 0$. By the standard process of "separating knots" we could then find $\mathbf{x}', \mathbf{s}' \in \Delta_p$ which are perturbations of $\mathbf{x}, \mathbf{s} \in \bar{\Delta}_p$ and satisfy $(\tilde{G}_r)_{[p]}(\mathbf{x}', \mathbf{s}') \geqslant a > 0$. Also, exercising a little care in the separation process, we may select $\mathbf{x}'$ and $\mathbf{s}'$ so that $\Delta(\mathbf{x}', \mathbf{s}') \geqslant r + 1$. This situation is now incompatible with Theorem 1.3.

The reverse implication in (1.14) (i.e., $\Delta(\mathbf{x}, \mathbf{s}) \leqslant r \Rightarrow (\tilde{G}_r)^*_{[p]}(\mathbf{x}, \mathbf{s}) > 0$) requires an elaborate series of induction arguments which we proceed to outline: For $r = 1$, (1.14) is true by Theorem 1.3. We assume by induction that (1.14) is true for all Green's functions corresponding to differential operators of the type considered in Theorem 1.4 with orders $\leqslant r - 1$. We prove (1.14) for $\tilde{L}_r$'s as follows. For such differential operators of order r and for $p = 1$ (1.14) is true (by the induction hypothesis for orders $\leqslant r - 1$ and (2.10)). We assume by a further induction that (1.14) is true for all odd $p \leqslant 2\ell - 1$ ($\ell \geqslant 1$) and prove that (1.14) holds for $p = 2\ell + 1$. This last step is accomplished by a final induction; namely, let

$$\mu = \mu(\mathbf{x}, \mathbf{s}) = \text{the number of distinct points in } \{x_i\} \cup \{s_j\}.$$

For the case $p = 2\ell + 1$, if $\mu = 2(2\ell + 1)$ then (1.14) is certainly true by Theorem 1.3. We assume by induction that (1.14) holds for $\mu \geqslant \lambda$ ($\lambda \geqslant 2$) and prove (1.14) holds for $\mu \geqslant \lambda - 1$. Once this is done the proof will be complete.

In order for our proof to have substance there must be either repeated x's or s's. (Otherwise, the proof is easy with the aid of the BCF and (2.10).) Assume for definiteness that repeated s's occur. The induction argument on μ is rather involved and is presented in a series of lemmas.

LEMMA 2.2. *Let*

$$h(s) = G^* \begin{pmatrix} x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-1}, s, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix}$$

where s_j is the last number of some repeated s group with $\tau \geqslant 2$ elements. Suppose $\mu(\mathbf{x}, \mathbf{s}) \geqslant \lambda - 1$. Then

$$h(s) = \sum_{i=1}^{2\ell+1} a_i G(x_i, s) \qquad \text{and} \qquad \sum_{i=1}^{2\ell+1} a_i^2 > 0. \tag{2.14}$$

CONVENTION. We shall write G for $\tilde{G}_r$ in the rest of this article.

Equation (2.14) is simply the expansion of $G^*(:::)$ by its jth column. For example if $x_1 = x_2 = x_3$, for ease of notation, we have written

$$a_1 G(x_1, s) + a_2 G(x_2, s) + a_3 G(x_3, s) + \cdots$$

in place of

$$a_1 G(x_1, s) + a_2 \frac{\partial G}{\partial x}(x_1, s) + a_3 \frac{\partial^2 G}{\partial x^2}(x_1, s) + \cdots$$

and so forth.

Remark 2.2. In the following proof we adopt the notational convention that if

$$h(s) = G^*_{[p]}(\mathbf{x}, \mathbf{s})$$

we shall write $\Delta(h)$ for $\Delta(\mathbf{x}, \mathbf{s})$ and $\mu(h)$ for $\mu(\mathbf{x}, \mathbf{s})$.

Proof. For s close enough to s_j the $(\mathbf{x}, \mathbf{s})$-sequence associated with $h(s)$ satisfies $\Delta(h) \leqslant r$ and $\mu(h) \geqq \lambda$ if $s \neq s_j$. By the induction hypotheses for such s we have

$$G^* \begin{pmatrix} x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-1}, s, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix} > 0$$

Removing $x_k \leqslant s_{j-1} = s_j < x_{k+1}$ from the original $\mathbf{x}$, $\mathbf{s}$ sequence leaves two new sequences $\mathbf{x}'$, $\mathbf{s}'$ (say) of length $2\ell - 1$ with $\Delta(\mathbf{x}', \mathbf{s}') \leqslant r$. Thus, also

$$G^* \begin{pmatrix} x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-2}, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix} > 0.$$

By Sylvester's determinant identity (see Karlin [3, p. 3]), with s close to s_j as prescribed above and if $x_k \leqslant s_{j-1} = s_j < x_{k+1}$,

$$0 < G^* \begin{pmatrix} x_1, \ldots, x_{j-1}, x_j, x_{j+1}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-1}, s, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix} G^* \begin{pmatrix} x_1, \ldots, x_{k-1}, x_{k+2}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-2}, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix}$$

$$= G^* \begin{pmatrix} x_1 \ldots, x_k, x_{k+2}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix} G^* \begin{pmatrix} x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-2}, s, \ldots, s_{2\ell+1} \end{pmatrix}$$

$$- G^* \begin{pmatrix} x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-1}, s_{j+1}, \ldots, s_{2\ell+1} \end{pmatrix} G^* \begin{pmatrix} x_1, \ldots, x_k, x_{k+2}, \ldots, x_{2\ell+1} \\ s_1, \ldots, s_{j-2}, s, \ldots, s_{2\ell+1} \end{pmatrix}$$

$$= a_{k+1} G^*(:::) - a_k G^*(:::). = \sum_{i=1}^{2l+1} c_i G(x_i, s)$$

It follows that $a_k^2 + a_{k+1}^2 > 0$ as was to be proved. ||

The observation (2.14) will be used repeatedly during our induction on μ. Suppose we could find $\mathbf{x}^0$, $\mathbf{s}^0 \in \Delta_{2\ell+1}$ as follows:

$$\begin{aligned} \Delta(\mathbf{x}^0, \mathbf{s}^0) &\leqslant r \\ \mu(\mathbf{x}^0, \mathbf{s}^0) &= \lambda - 1 \\ G^*_{[2\ell+1]}(\mathbf{x}^0, \mathbf{s}^0) &= 0 \end{aligned} \tag{*}$$

We shall show that (*) leads to a contradiction; and hence that the induction step on μ may be advanced.

LEMMA 2.4. *Suppose that* both $\mathbf{x}^0$ *and* $\mathbf{s}^0$ *contain repeated elements. Then* (*) *is untenable.*

Proof. Choose k and p such that $x_{k-1}^0 = x_k^0$ and $s_{p-1}^0 = s_p^0$ where x_k^0 and s_p^0 are the last members of their respective coincidences. Consider

$$h(s) = G^* \begin{pmatrix} x_1^0, \ldots, x_{k-1}^0, x_k^0, x_{k+1}^0, \ldots, x_{2\ell+1}^0 \\ s_1^0, \ldots, s_{p-1}^0, s, s_{p+1}^0, \ldots, s_{2\ell+1}^0 \end{pmatrix}$$

and

$$g(s) = G^* \begin{pmatrix} x_1^0, \ldots, x_{k-1}^0, \tilde{x}_k, x_{k+1}^0, \ldots, x_{2\ell+1}^0 \\ s_1^0, \ldots, s_{p-1}^0, s, s_{p+1}^0, \ldots, s_{2\ell+1}^0 \end{pmatrix}$$

where $\tilde{x}_k \neq x_k^0$ is a fixed value near enough to x_k^0 so that $\Delta \leqslant r$ prevails for the sequences $\{s_j^0\}$ and $\{\{x_j^0\}_{i\neq k}, \tilde{x}_k\}$.

$\mu \geqslant \lambda$ for these sequences if we also make $\{\tilde{x}_k\} \cap (\{x_i^0\} \cup \{s_i^0\}) = \emptyset$. By the induction hypothesis and (*), $h(s)$ vanishes to a higher order at s_p^0 than $g(s)$ does. (In the extreme case where s_p^0 equals some x_q^0 and $\alpha + \beta = r + 1$

—see the conditions (a, b) of Theorem 1.4—our convention for evaluating $\partial^{r-1}/\partial x^{\alpha-1}\,\partial s^{\beta-1}$ shows that $h(s)$ vanishes to a higher order than $g(s)$ to the left of $s_p{}^0$.) Moreover, $g(s)$ and $h(s)$ exhibit the same sign for s smaller than, but sufficiently close to, $s_p{}^0$ provided of course $\tilde{x}_k$ is near to $x_k{}^0$. Thus,

$$f(s) = h(s) - \epsilon g(s)$$

vanishes at least once at $s_p{}^* < s_p{}^0$ but arbitrarily near $s_p{}^0$ for $\epsilon > 0$ suitably small. Then $f(s)$ vanishes at least at the points

$$\{s_j{}^*\}_{j=1}^{2\ell+1} = \{s_1{}^0,\ldots, s_{p-1}^0, s_p{}^*, s_{p+1}^0,\ldots, s_{2\ell+1}^0\}.$$

Explicitly,

$$f(s) = \sum_{\substack{i=1\\ i\neq k}}^{2\ell+1} a_i G(x_i{}^0, s) + a_k[G(x_k{}^0, s) - G(\tilde{x}_k, s)] \tag{2.16}$$

where

$$a_i = G^*\begin{pmatrix} x_1{}^0,\ldots, x_{i-1}^0, x_{i+1}^0,\ldots, x_{2\ell+1}^0 \\ s_1{}^0,\ldots, s_{p-1}^0, s_{p+1}^0,\ldots, s_{2\ell+1}^0 \end{pmatrix} -$$

$$\times\ \epsilon\, G^*\begin{pmatrix} x_1{}^0,\ldots, x_{i-1}^0, x_{i+1}^0,\ldots, \tilde{x}_k,\ldots, x_{2\ell+1}^0 \\ s_1{}^0,\ldots, s_{p-1}^0, s_{p+1}^0,\ldots \qquad , s_{2\ell+1}^0 \end{pmatrix}$$

for $i \neq k$ and

$$a_k = (1-\epsilon)\, G^*\begin{pmatrix} x_1^0,\ldots, x_{k-1}^0, x_{k+1}^0,\ldots, x_{2\ell+1}^0 \\ s_1{}^0,\ldots, s_{p-1}^0, s_{p+1}^0,\ldots, s_{2\ell+1}^0 \end{pmatrix}$$

By Lemma 2.3 not all the a's can vanish if $\epsilon > 0$ is suitably small. Equation (2.16) gives $2\ell + 1$ linear equations $f(s_j{}^*) = 0$ for the a_i's. The determinant of this system

$$G^*\begin{pmatrix} x_1{}^0,\ldots, x_{2\ell+1}^0 \\ s_1{}^*,\ldots, s_{2\ell+1}^* \end{pmatrix} - G^*\begin{pmatrix} x_1{}^0,\ldots, \tilde{x}_k,\ldots, x_{2\ell+1}^0 \\ s_1{}^*,\ldots, \qquad s_{2\ell+1}^* \end{pmatrix} \neq 0 \tag{2.17}$$

provided $\epsilon < 1$ because

$$0 = f(s_p{}^*) = G^*\begin{pmatrix} x_1{}^0,\ldots, x_{2\ell+1}^0 \\ s_1{}^*,\ldots, s_{2\ell+1}^* \end{pmatrix} - \epsilon G^*\begin{pmatrix} x_1{}^0,\ldots, \tilde{x}_k,\ldots, x_{2\ell+1}^0 \\ s_1{}^*,\ldots, \qquad s_{2\ell+1}^* \end{pmatrix} \tag{2.18}$$

and clearly (2.18) coupled with the fact that both determinants in (2.18) are nonzero (by the induction hypothesis) imply that inequality must hold in

(2.17). The nonvanishing of (2.17) together with $\sum a_i^2 > 0$ is a contradiction. ||

Lemma 2.4 allows us to assume (without loss of generality) that the x_i^0's are distinct. Reasoning almost as above, we could also prove

LEMMA 2.5. *If* $\{x_i^0\} \cap \{s_j^0\} \neq \emptyset$ *then* (*) *is untenable.*

In review, the preceeding lemmas allow us to augment (*) with the additional assumptions:

$$0 \leqslant x_1^0 < x_2^0 < \cdots < x_{2\ell+1}^0 < 2\pi$$

$$\{s_j^0\} \text{ and } \{x_i^0\} \text{ are disjoint}$$

The disjoint condition is used repeatedly below, without explicit mention, to guarantee that the differentiations performed on the Green's function $G(x, s)$ are permissible.

The remainder of the proof rests on a careful examination of zeros of certain functions, the most important one being

$$h(s) = G^* \begin{pmatrix} x_1^0, \ldots, & & & x_{2\ell+1}^0 \\ s_1^0, \ldots, & s_{\sigma+\tau-1}^0, & s, & s_{\sigma+\tau+1}^0, \ldots, & s_{2\ell+1}^0 \end{pmatrix} \tag{2.19}$$

where $s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau}^0$ $(\tau \leqslant r)$ is a repeated s-value.

LEMMA 2.6. *$h(s)$ has an isolated zero at $s_{\sigma+1}^0$ with multiplicity $\geqslant \tau + 1$; in particular,*

$$h(s_{\sigma+1}^0) = \cdots = h^{(\tau)}(s_{\sigma+1}^0) = 0.$$

Proof. Since $\tau \leqslant r$ and no x_i^0 equals $s_{\sigma+1}^0$, $h(s)$, $h'(s), \ldots, h^{(\tau)}(s)$ are all continuous near $s_{\sigma+1}^0$. Trivially by (*), $h(s_{\sigma+1}^0) = \cdots = h^{(\tau-1)}(s_{\sigma+1}^0) = 0$ and by the induction hypothesis $s_{\sigma+1}^0$ is an isolated zero of $h(s)$. If s increases through $s_{\sigma+1}^0$, $h(s)$ changes sign from $(-1)^{\tau-1}$ to $+1$ because $G(x, s)$ is CTP. Thus if τ is odd, $h(s)$ does not change sign at $s_{\sigma+1}^0$ and this requires $h^{(\tau)}(s_{\sigma+1}^0) = 0$ precisely because τ is odd. Similarly, we deduce if τ is even that $h^{(\tau)}(s_{\sigma+1}^0) = 0$. ||

For ease of exposition the rest of the proof is presented in three cases.

Case I. There are at least two distinct repeated s-values.

For argument's sake suppose

$$s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau}^0 < s_{\alpha+1}^0 = \cdots = s_{\alpha+\beta}^0 \qquad \tau, \beta \geqslant 2)$$

are two repeated s-values and that x_γ^0 is defined by

$$x_\gamma^0 < s_{\alpha+1}^0 = s_{\alpha+\beta}^0 < x_{\gamma+1}^0 .$$

Choose points $s'_{\alpha+1}$ and $s''_{\alpha+1}$ in the interval $(\max(s_\alpha{}^0, x_\gamma{}^0), s^0_{\alpha+1})$ and consider

$$g(s) = G^* \begin{pmatrix} x_1{}^0, \ldots, & & & & & & & & & & \cdots & x^0_{2\ell+1} \\ s_1{}^0, \ldots, & s^0_{\sigma+\tau-1}, & s, & s^0_{\sigma+\tau+1}, \ldots, & s_\alpha{}^0, & s'_{\alpha+1}, & s''_{\alpha+1}, & s^0_{\alpha+1}, \ldots, & s^0_{\alpha+\beta-2}, & s^0_{\alpha+\beta+1}, \ldots, & & s^0_{2\ell+1} \end{pmatrix}$$

which satisfies $\Delta(g(s)) \leqslant r$ for s near $s^0_{\sigma+1}$ and $\mu(g(s)) \geqslant \lambda + 1$. Let $h(s)$ be the function defined by (2.19), then by Lemma 2.6 and the induction hypothesis $h(s)$ vanishes to higher order at $s^0_{\sigma+1}$ than $g(s)$ and $\operatorname{sgn} h(s) = \operatorname{sgn} g(s)$ near $s^0_{\sigma+1}$. Thus for $\epsilon > 0$ and small

$$f(s) = h(s) - \epsilon g(s)$$

must vanish at least at

$$s^*_{\sigma+1} < s^0_{\sigma+1} = \cdots = s^0_{\sigma+\tau-1} < s^*_{\sigma+\tau-1} \tag{2.20}$$

where $s^*_{\sigma+1}$ and $s^*_{\sigma+\tau-1}$ may be made arbitrarily near $s^0_{\sigma+1}$ if ϵ is small enough.

A little reflection shows that

$$\Delta(g(s)) \leqslant r \qquad \text{if and only if} \quad \Delta(h(s)) \leqslant r$$

indeed that $\Delta(g(s)) = \Delta(h(s))$ for any s. Consequently, $f(s)$ must vanish at

$$s^*_{\alpha+1} < s^0_{\alpha+1} = \cdots = s^0_{\alpha+\beta-2} < s^*_{\alpha+\beta-2} \tag{2.21}$$

where $s^*_{\alpha+1}(s^*_{\alpha+\beta-2})$ may be made arbitrarily near $s^0_{\alpha+1}(s^0_{\alpha+\beta-2})$.

Finally $f(s)$ has the obvious zeros

$$s_j{}^0 \; (j \neq \sigma + 1, \ldots, \sigma + \tau;\ \alpha + 1, \ldots, \alpha + \beta - 2) \tag{2.22}$$

In total then $f(s)$ vanishes at least $2\ell + 2$ times and by (2.20)–(2.22) we can select a subset of $2\ell + 1$ of these zeros, say $\mathbf{s}^* = \{s_j{}^*\}_{j=1}^{2\ell+1}$, such that

$$f(s_j{}^*) = 0 \; (j = 1, \ldots, 2\ell + 1), \qquad \Delta(\mathbf{x}^0, \mathbf{s}^*) \leqslant r, \quad \text{and } \mu(\mathbf{x}^0, \mathbf{s}^*) \geqslant \lambda.$$

The linear system

$$f(s_j{}^*) = \sum_{i=1}^{2\ell+1} a_i G(x_i{}^0, s_j{}^*) = 0 \qquad (j = 1, \ldots, 2\ell + 1)$$

has positive determinant $G^*_{[2\ell+1]}(\mathbf{x}^0, \mathbf{s}^*)$ (by the induction hypothesis), while, at the same time, not all the a_i's vanish as can be seen by their explicit representations in terms of $G^*(:::)$ as in the proof of Lemma 2.4. This contradiction establishes that Case I and (*) are incompatible.

Case II. There is only one repeated s-value,

$$s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau}^0$$

and at least two further simple s-values.

(a) Suppose

$$x_\gamma^0 < s_\rho^0 < \cdots < s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau}^0 < \cdots < s_\omega^0 < x_{\ +1} .$$

Define

$$g(s) = G^* \begin{pmatrix} x_1^0, \ldots, x_{\gamma-1}^0, x_{\gamma+2}^0, \ldots & & \ldots, x_{2\ell+1}^0 \\ s_1^0, \ldots, s_{\rho-1}^0, s_{\rho+1}^0, \ldots, s_{\sigma+\tau-1}^0, s, s_{\sigma+\tau+1}^0, \ldots, s_{\omega-1}^0, s_{\omega+1}^0, \ldots, s_{2\ell+1}^0 \end{pmatrix}$$

One verifies easily that $\Delta(g(s)) \leqslant r$ for $s \in (x_\gamma^0, x_{\gamma+1}^0)$, that $h(s)$ vanishes to higher order at $s_{\sigma+1}^0$ than $g(s)$ does, and that $\operatorname{sgn} g(s) = \operatorname{sgn} h(s)$ near $s_{\sigma+1}^0$. It follows, then, as before that the function $f(s) = h(s) - \epsilon g(s)$ must vanish for

$$s_{\sigma+1}^* < s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau-1}^0 < s_{\sigma+\tau-1}^* \tag{2.23}$$

with $s_{\sigma+1}^*(s_{\sigma+\tau-0}^*)$ near $s_{\sigma+1}^0(s_{\sigma+\tau-1}^0)$. Furthermore since $g(s)$ does not vanish at s_ρ^0 or s_ω^0 while $h(s)$ vanishes simply there, we can find two zeros s_ρ^* and s_ω^* near s_ρ^0 and s_ω^0. Using these zeros with the ones $s_j^0 (j \neq \rho, \omega, \sigma + 1, \ldots, \sigma + \tau)$ and an appropriate selection of all but one of the zeros from (2.23) we reach a contradiction as in Case I.

(b) Suppose

$$x_\gamma^0 < s_\rho^0 < \cdots < s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau}^0 < x_{\gamma+1}^0 < \cdots < x_\lambda^0 < s_{\sigma+\tau+1}^0$$

and define $h(s)$ as in (a) and

$$g(s) = G^* \begin{pmatrix} x_1^0, \ldots, x_{\gamma-1}^0, x_{\gamma+1}^0, \ldots, x_{\lambda-1}^0, & x_{\lambda+1}^0, \ldots, & x_{2\ell+1}^0 \\ s_1^0, \ldots, s_{\rho-1}^0, s_{\rho+1}^0, \ldots, s_{\sigma+\tau-1}^0, s, s_{\sigma+\tau+2}^0, \ldots, & s_{2\ell+1}^0 \end{pmatrix}$$

Just as in (a) we can find zeros of $f(s) = h(s) - \epsilon g(s)$:

$$s_{\sigma+1}^* < s_{\sigma+1}^0 = \cdots = s_{\sigma+\tau-1}^0 < s_{\sigma+\tau-1}^*$$

$$s_\rho^* \quad \text{near} \quad s_\rho^0$$

One sees readily that $\Delta(g(s)) \leqslant \Delta(h(s))$ for any s; hence, if $h(s) \not\equiv 0$ near $s_{\sigma+\tau+1}^0$ then the same is true of $g(s)$ and we reach a contradiction as in (a).

In case $h(s) \equiv 0$ near $s_{\sigma+\tau+1}^0$, it must vanish identically in $[x_{\gamma+j}^0, s_{\sigma+\tau+1}^0]$ for some $2 \leqslant j \leqslant \lambda - \gamma$ with $h(x_{\gamma+j}^0 - \eta) > 0$ for small $\eta > 0$: Indeed the

$\Delta(h(s))$ values are constant on the intervals $(x^0_{\gamma+i}, x^0_{\gamma+i+1})$ and one easily confirms that $\Delta(h(x^0_{\gamma+1}+\eta)) \leqslant r$ for small $\eta > 0$. Since $\operatorname{sgn} g(s) = \operatorname{sgn} h(s)$ in $[s^0_{\sigma+\tau}, s^0_{\tau+\sigma+1}]$, it follows that $f(s)$ must have a zero $s^*_{\sigma+\tau+1}$ in $(x^0_{\gamma+1}, s^0_{\sigma+\tau+1})$ for small $\epsilon > 0$. Because $s^*_{\sigma+\tau+1} \in (x^0_{\gamma+1}, s^0_{\sigma+\tau+1})$ it is easily confirmed that

$$\Delta\begin{pmatrix} x_1{}^0 \cdots & & x^0_{2\ell+1} \\ s_1{}^0 \cdots s_\rho{}^* \cdots s^*_{\sigma+\tau+1} & \cdots & s^0_{2\ell+1}\end{pmatrix} \leqslant r$$

and we reach a contradiction as in Case I.

(c) If $s_\sigma{}^0 < x_\rho{}^0 < \cdots < x_\gamma{}^0 < s^0_{\sigma+1} = \cdots = s^0_{\sigma+\tau} < \cdots < s_\omega{}^0 < x^0_{\gamma+1}$ we can proceed in a similar fashion to (b).

(d) If $s_\sigma{}^0 < x_\rho{}^0 < \cdots < x_\gamma{}^0 < s^0_{\sigma+1} = \cdots = s^0_{\sigma+\tau} < x^0_{\gamma+1} < \cdots < x_\lambda{}^0 < s^0_{\sigma+\tau+1}$ we can proceed as in (c) to get $s_\sigma{}^*$ and as in (b) to get $s^*_{\sigma+\tau+1}$.

The cases (a)–(d) are exhaustive and thus the contradiction involved shows that Case II is incompatible with (*).

Case III. There is only one repeated s-value, $s^0_{\sigma+1} = \cdots = s^0_{\sigma+\tau}$, and at most one further s-value.

Under these circumstances, we may assume that $2\ell + 1 \leqslant r - 1$; this results from

LEMMA 2.7. *Suppose* $\mathbf{x}^0$ *and* $\mathbf{s}^0$ *are such that there exists an interval* $[x_i{}^0, x^0_{i+1}]$ *which contains a repeated s-value which appears* $r - 1$ *times in* $\mathbf{s}^0$. *Then* (*) *is untenable.*

Proof. Let $s^0_{\sigma+1} = \cdots = s^0_{\sigma+r-1}$ be such a repeated s-value in $[x_i{}^0, x^0_{i+1}]$. By Lemma 2.6 $s^0_{\sigma+1}$ is an isolated zero of $h(s)$ of multiplicity at least r. But on $[x_i{}^0, x^0_{i+1}]$, $h(s)$ is a solution of $(\tilde{L}_r h)(s) = 0$—while $\tilde{L}_r$ is a differential operator of type W of Pólya—and so can have at most $r - 1$ zeros counting multiplicities (cf. Karlin [3, Ch. 6 § 4]) unless it vanishes indentically. Thus $h(s) \equiv 0$ on $[x_i, x_{i+1}]$ and this contradicts the induction assumption since $\mu(h(s)) \geqslant \lambda$ for s near $s^0_{\sigma+1}$. ||

In view of Lemma 2.7, we may assume $\tau \leqslant r - 2$ so that $2\ell + 1 \leqslant \tau + 1 \leqslant r - 1$. This inequality quarantees that the compound kernels of order $2\ell + 1$ appearing in the BCF applied to (2.10′) are all well defined and that

$$(\hat{G}_{r-1})^*_{[2\ell+1]}(\xi, \mathbf{s}^0) > 0 \qquad \text{for all} \quad \xi \in \Delta_{2\ell+1}.$$

Since $(\tilde{G}_1)_{[2\ell+1]}(\mathbf{x}^0, \xi) > 0$ on the set

$$x_i{}^0 < \xi_i < x^0_{i+1} \qquad (x^0_{2\ell+2} = 2\pi;\ i = 1,\ldots, 2\ell + 1),$$

it follows that $(\tilde{G}_r)^*{}_{[2\ell+1]}(\mathbf{x}^0, \mathbf{s}^0) > 0$ which contradicts (*). Thus (*) is also incompatible with Case III.

Since Cases I–III are exhaustive, (*) itself must be false and hence the backward induction step of μ is advanced and Theorem 1.4 is thereby established. ||

3. Periodic Splines

Theorems 1.3 and 1.4 allow the solution of several spline interpolation problems. We restrict the discussion here to splines with simple knots (hence only Theorem 1.3 will be used) but the obvious generalizations to multiple-knot splines are easily established using Theorem 1.4. The theory outlined here is similar to that in Karlin [3], (see also references therein), so all proofs are omitted.

Definition 3.1. A function $p(x)$ is called a *cyclic generalized spline* (CGS) with knots $\mathbf{s} = \{s_j\}_{j=1}^{2\ell+1} \in \Delta_{2\ell+1}$ provided

(i) $(\tilde{L}_r p)(x) = 0$ on $[s_i, s_{i+1}] (i = 0, 1,..., 2\ell + 1, \quad s_0 = 0, s_{2\ell+2} = 2\pi)$

(ii) $p(x) \in C^{r-2}[0, 2\pi]$

(iii) $p(x)$ satisfies the BC $\tilde{\mathscr{B}}_r$.

Briefly, $p(x)$ is a piecewise solution of the differential equation $\tilde{L}_r u = 0$ whose derivatives of $(r - 1)$th orders may jump at the knots.

A standard argument proves

Lemma 3.1. *Any* CGS $p(x)$ *is of the form*

$$p(x) = \sum_{j=1}^{2\ell+1} a_j G(x, s_j)$$

for certain constants a_j.

This lemma yields in turn the following important interpolation result.

Theorem 3.1. *Let* $\{x_i\}_{i=1}^{2\ell+1} \in \Delta_p$ *satisfy* $\Delta(\{x_i\}, \{s_j\}) \leqslant r$ *and let* $\{y_k\}_{k=1}^{2\ell+1}$ *be any set of real (or complex) numbers. Then there exists a unique* CGS $p(x)$ *with knots* $\{s_j\}$ *interpolating the* $\{y_k\}$ *at* $\{x_i\}$: $p(x_i) = y_i$ $(i = 1, 2,..., 2\ell + 1)$.

If the differential operator defining the CGS's is self-adjoint a number of special optimality results obtain. Henceforth we assume that $p(x)$ is a CGS arising from the self-adjoint BVP.

$$M_{2r}u = D_1^* \cdots D_r^* D_r \cdots D_1 u$$

$$D_j^* \cdots D_r^* D_r \cdots D_1 D_0 u(0) = D_j^* \cdots D_r^* D_r \cdots D_1 D_0 u(2\pi) \qquad (j = 2,..., r)$$

$$D_k \cdots D_0 u(0) = D_k \cdots D_0 u(2\pi) \qquad (k = r, r - 1,..., 0)$$

whose BC are denoted $\tilde{\mathscr{B}}_r\mathscr{B}_r$. We also set

$$L_r = D_r \cdots D_1$$

and denote its natural BC by $\mathscr{B}_r$. The optimality properties of these CGS result from the orthogonality relation of

LEMMA 3.2. *Let $f \in \tilde{\mathscr{B}}_r\mathscr{B}_r$ and let p be the unique* CGS *having the knots $\{s_j\}$ that interpolates f on $\{s_j\}$. Then p is orthogonal to $f - p$ in the sense that*

$$\int_0^{2\pi} (L_r p)(x)(L_r(f-p))(x)\,dx = 0.$$

Remark 3.1. It is important to emphasize that the knots $\{s_j\}$ and the points of interpolation $\{x_i\}$ agree in Lemma 3.2. The orthogonality relation crucially requires this. On the other hand, the basic interpolation result (Theorem 3.1) permits the points of interpolation and the knots to differ.

As a consequence of Lemma 3.2, we have the following optimality theorems.

THEOREM 3.2. *Let $f \in \tilde{\mathscr{B}}_r\mathscr{B}_r$ and let p be the unique* CGS *interpolating f on $\{s_j\}$. Then*

$$\int_0^{2\pi} [(L_r p)(x)]^2\,dx \leqslant \int_0^{2\pi} [(L_r f)(x)]^2\,dx$$

and equality holds iff $p = f$.

Remark 3.2. Observe that p depends only on the values of f on $\{s_j\}$ and in no other way on f. Therefore, we can express the result in the form

$$\min_{g \in \mathscr{B}_r\mathscr{B}_r(f)} \int_0^{2\pi} [L_r(g)(x)]^2\,dx = \int_0^{2\pi} [(L_r p)(x)]^2\,dx$$

where the minimum is over all functions $g \in C^{r-2}[0, 2\pi]$ which interpolate f on $\{s_j\}$ and satisfy the BC $\tilde{\mathscr{B}}_r\mathscr{B}_r$.

THEOREM 3.3. *Suppose the hypothesis of Theorem* 3.2 *prevails. Let p be the unique* CGS *interpolating f on $\{s_j\}$. If q is any other* CGS *with knots confined to $\{s_j\}$, then*

$$\int_0^{2\pi} [L_r(f-p)(x)]^2\,dx \leqslant \int_0^{2\pi} [L_r(f-q)(x)]^2\,dx$$

with equality holding iff $q = p$.

Remark 3.3. The preceeding results on optimality also hold if the number of knots is *even*. Indeed, Theorem 3.1 holds for the differential operator M_{2r} in the special case $\{x_i\} = \{s_j\}$ $(i, j = 1, 2,..., 2\ell)$: Let $G_r(x, s)$ be the Green's function for the boundary-value problem L_r, $\mathscr{B}_r$ and let $G_r^*(x, s)$ be the Green's function for the adjoint boundary-value problem L_r^*, $\mathscr{B}_r^*$. If $G_{2r}(x, s)$ is the Green's function for the self-adjoint problem $M_{2r} = L_r^* L_r$, $\mathscr{B}_r^* \mathscr{B}_r$, then

$$G_{2r}(x, s) = \int_0^{2\pi} G_r^*(x, \xi)\, G_r(\xi, s)\, d\xi.$$

Thus, the determinant of the relevant system of linear equations involved in the proof of Theorem 3.1 is

$$(G_{2r})_{[2\ell]}(\vec{\mathbf{x}}, \vec{\mathbf{x}}) = \int_{\Delta_{2\ell}} (G_r)_{[2\ell]}(\vec{\boldsymbol{\xi}}, \vec{\mathbf{x}})(G_r)_{[2\ell]}(\vec{\boldsymbol{\xi}}, \vec{\mathbf{x}})\, d\boldsymbol{\xi}$$

which is easily seen to be positive (the computation at the top of page 388 shows that $(G_r)_{[2\ell]}(\vec{\boldsymbol{\xi}}, \vec{\mathbf{x}}) \not\equiv 0$). In particular, in the self-adjoint case, the Green's function $G_{2r}(x, s)$ is positive definite as well as CTP_∞.

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