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Subunit Balls for Symbols of Pseudodifferential Operators

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In this work we shall study a definition of subunit ball for non-negative symbols [illegible]

straightened, by means of a canonical transformation, to contain and be contained in boxes of certain sizes, which we give in terms of the size of the symbol. After microlocalizing the symbol, in Section 3 we define classes of subunit symbols and study some of their basic properties. Then we define the subunit ball. In the last section the main structure theorems, in the $(n+n)$-dimensional elliptic case and in the $(1+1)$- and $(2+2)$-dimensional nonelliptic–nondegenerate cases are stated and proved.

1. INTRODUCTION

As discovered by Stein *et al.* in [14–16, 19], a subelliptic operator

$$L = -\sum_{i,\, j=1}^{n} a^{ij}(x) \frac{\partial^2}{\partial x^i\, \partial x^j} + \sum_{i=1}^{n} b^i(x) \frac{\partial}{\partial x^i} + c(x)$$

($a^{ij} = a^{ji}$, b^i, c real and smooth; the matrix $(a^{ij}(x))_{i,\, j} \geqslant 0$) is governed by a family of "non-Euclidean" balls $B_L(x, \rho)$. For instance, the fundamental solution $K(x, y)$ for L is comparable to $\delta(x, y)^2/\mathrm{Vol}(x, y)$, where

$$\delta(x, y) = \inf\{\rho;\, y \in B_L(x, \rho)\}$$

and

$$\mathrm{Vol}(x, y) = \mathrm{Vol}\, B_L(x, \rho) \qquad \text{with} \quad \rho = \delta(x, y)$$

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(see Nagel *et al.* [16], Sanchez-Calle [20], and Fefferman and Sanchez-Calle [13]). The number of eigenvalues of L up to size λ is comparable to

$$\int_M \frac{d\mu(x)}{\mu(B_L(x, \lambda^{-1/2}))}$$

(in the case M is a compact manifold without boundary and μ is a smooth measure on M) (see Fefferman and Phong [6]), and the sharp subelliptic estimate

$$c\,\|u\|^2_{(2\varepsilon)} \leqslant \|Lu\|^2 + \|u\|^2$$

is equivalent to the geometric condition

$$B_{\mathrm{E}}(x, \rho) \subset B_L(x, C\rho^{\varepsilon})$$

(here $C > 0$ is a universal constant and B_{E} is the Euclidean ball). (See Fefferman and Phong [6].) See also Christ [1], Fefferman and Kohn [8, 9], Fefferman *et al.* [10], and Nagel *et al.* [17] for applications to CR manifolds.

The non-Euclidean ball $B_L(x, \rho)$ may be defined as the set of points that can be reached in time ρ by a "subunit path" starting at x. A subunit path is one whose velocity vector $(\gamma^1, ..., \gamma^n)$ satisfies the matrix inequality:

$$(\gamma^i \gamma^j)_{ij} \leqslant (a^{ij})_{ij}.$$

The fundamental geometric fact about $B_L(x, \rho)$ is that it is comparable to a rectangular box after a suitable change of variables.

The purpose of this paper is to associate non-Euclidean balls in phase-space, B_p, to a subelliptic pseudodifferential operator (ψ *do*) with non-negative symbol $p(x, \xi)$. We hope these balls will play for the ψ *do*'s a role more or less analogous to that of the now-standard non-Euclidean balls for differential operators. In particular, we believe that they are closely related to the "testing boxes" of Fefferman [2].

Our ball $B_p((x^0, \xi^0), \rho)$ is defined as the set of points in phase-space that can be reached in time 1 by a "subunit path" for $\rho^2 p$. A path in phase-space will be called a subunit for a symbol $p \geqslant 0$ if its velocity vector at each time agrees with a Hamiltonian vector field generated by a symbol q that satisfies the 1st-order estimates

$$\text{(i)} \quad |\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leqslant (1 + |\xi|)^{1 - |\beta|} \qquad \text{for} \quad |\alpha| + |\beta| \leqslant 2$$

and the inequality

$$\text{(ii)} \quad q(x, \xi)^2 \leqslant p(x, \xi).$$

In (i) it is essential to restrict the formula to $|\alpha| + |\beta| \leqslant 2$.

We begin to study the geometrical properties of the non-Euclidean balls in phase-space by studying the ψ *do*'s on $\mathbf{R}^n$ for $n = 1, 2$. For $n = 1$ and for classes of, for example, $n = 2$, we gain a complete understanding of the non-Euclidean balls, which are comparable to rectangular boxes after a suitable canonical transformation. However, in $n = 2$, we give an example which exhibits a new phenomenon, "stratification," with no analogue for the familiar differential operator case. For a fixed (x, ξ), the ball $B_p((x, \xi), \rho)$ looks like a rectangular box unless ρ is comparable to one of a bounded number of critical radii $\rho_1, \rho_2, ..., \rho_N$. If $\rho \approx \rho_j$ then $B_p((x, \xi), \rho)$ no longer looks like a box, and moreover $B_p((x, \xi), 4\rho)$ is very large compared to $B_p((x, \xi), \rho/4)$. We conjecture that such behavior holds in the general case, with N bounded *a priori*.

We shall formulate the results, and prove them, for symbols in the class $S^2(1 \times M)$ (see Fefferman and Phong [4] and Fefferman [2]).

In the next section we shall recall some facts about that class, the Calderon–Zygmund (C.Z.) decomposition, and the subelliptic hypotheses.

Afterwards we shall proceed by defining the subunit symbols, establishing some basic properties, and defining the subunit ball. We shall also need some properties of algebraic functions for which we will only recall the statements of some of them, and will simply refer the reader to Fefferman and Narasimhan [11, 12] and Parmeggiani [18] for the statements and proofs. Algebraic functions arise naturally since the subelliptic hypothesis will enable us to suppose that the symbol p (suitably localized) is a polynomial of an *a priori* fixed degree (depending on the subellipticity), this being done when constructing subunit balls of sufficiently small radius ρ (to be specified below) and considering the Taylor polynomial of p (in the chosen localization block). The mistake will be seen to be negligible.

In the last two sections the $(1+1)$- and $(2+2)$-dimensional results will be stated and proved.

A final remark is in order: one might expect, since we are dealing with 2nd order symbols, orders of magnitude of the size of the subunit ball behaving strictly like squares or square-roots. This is not true. In fact, suppose we have in $\mathbf{R}^n \times \mathbf{R}^n$, $p(x, \xi) = \xi_1^2 + M^2 c$ on a block of sizes $1 \times M$, c being > 0 but not "too small" (so small as to prevent subellipticity). We will see that, given $(x^0, \xi^0) \in 1 \times M$,

$$B_p((x^0, \xi^0), 1) \approx \{|x_1 - x_1^0| \lesssim 1\} \times \{|x' - x^{0\prime}| \lesssim c^{1/4}\} \times \{|\xi - \xi^0| \lesssim M c^{1/4}\}.$$

The "anomalous" presence of $c^{1/4}$ is completely natural: in computing B_p, one has to perform (as we will see) a C.Z. decomposition of $1 \times M$ for the "potential" $M^2 c$. The relative blocks Q_j will have sizes $\delta_j \times M\delta_j$ and $M^2 c_{|Q_j}$ will be elliptic there, i.e.,

$$M^2 c \sim M^2 \delta_j^4,$$

i.e., $\delta_j \sim c^{1/4}$.

Since the subunit symbols for $M^2 c$ will have strength (i.e., size of their $\nabla_{(x,\xi)}$) $\sim (\delta_j, M\delta_j)$ this will also be the optimal displacement (i.e., size δ_j in the x-direction, size $M\delta_j$ in the ξ-direction) given by subunit symbols related to the $M^2 c$ part of p (the other being ξ_1^2 which implies a displacement of order 1 in the x_1 variable) when travelling on a subunit path up to time 1.

2. REDUCTION TO $S^2(1 \times M)$ CLASSES AND MAIN HYPOTHESES

Let $\mathbf{R}^n \times \mathbf{R}^n \simeq T^*\mathbf{R}^n$ and $p \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$ be a real, non-negative symbol of order 2, i.e.,

$$|\partial_x^\alpha \partial_\xi^\beta p(x,\xi)| \leqslant C_{\alpha\beta}(1+|\xi|)^{2-|\beta|}, \qquad \forall \alpha, \beta, \quad \forall (x,\xi) \in \mathbf{R}^n \times \mathbf{R}.$$

The corresponding ψ *do* is

$$(p(x,D)\,u)(x) = \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} p(x,\xi)\,\hat{u}(\xi)\,d\xi, \qquad u \in C_0^\infty(\mathbf{R}^n).$$

Here $\hat{u}$ denotes the Fourier transform of u.

Let now $\{Q_\nu\}$ be a partition of the phase space $\mathbf{R}^n \times \mathbf{R}^n$ into blocks of various sizes $\operatorname{diam}_x Q_\nu \times \operatorname{diam}_\xi Q_\nu$, centered at various points (x^ν, ξ^ν), satisfying

$$\operatorname{diam}_x Q_\nu = 1, \qquad \operatorname{diam}_\xi Q_\nu \sim |\xi^\nu|$$

when $|\xi^\nu| \geqslant 1$, and

$$\operatorname{diam}_x Q_\nu = 1, \qquad \operatorname{diam}_\xi Q_\nu = 1$$

otherwise (for instance, for $|\xi^\nu| \geqslant 1$,

$$Q_\nu = \{|x_j - x_j^\nu| \leqslant 1; j = 1, ..., n\} \times \left\{|\xi_j - \xi_j^\nu| \leqslant \frac{1}{3\sqrt{n}}|\xi^\nu|; j = 1, ..., n\right\}.$$

Then, when $|\xi^\nu| \geqslant 1$,

$$|\partial_x^\alpha \partial_\xi^\beta p_{|Q_\nu}| \leqslant C'_{\alpha\beta}\,|\xi^\nu|^{2-|\beta|},$$

hence, if M is a fixed number $\geqslant 1$, $|\xi^\nu| \sim M$, we have

$$|\partial_x^\alpha \partial_\xi^\beta p_{|Q_\nu}| \leqslant C_{\alpha\beta} M^{2-|\beta|}, \tag{1}$$

i.e., $p_{|Q_\nu} \in S^2(1 \times M)$, with new constants $C_{\alpha\beta}$.

It is important to notice that the seminorms $C_{\alpha\beta}$ do not depend on M.

Let us now change notations in the following way: our basic block will be denoted by Q, its sizes by $1 \times M$, and we suppose that $Q_\nu = Q$, denoting by Q^* (for now) the usual dilate of Q by 10^n. (We denote by $Q^{**} = (Q^*)^*$ the "double-dilate" of Q, by $\frac{1}{2}Q$ its "middle-half", and by $2Q$ its "double.")

We now localize p to Q_ν by means of a family of cutoff functions, $\{\phi_\nu(x, \xi)\}$, where the ϕ_ν are constructed by the appropriate dilate and translate of a fixed cutoff function, such that:

$$0 \leqslant \phi_\nu \leqslant 1, \qquad \phi_\nu \equiv 1 \quad \text{on } Q_\nu^{**}$$
$$\operatorname{supp} \phi_\nu \subset Q_\nu^{***}.$$

(Hence $\{\phi_\nu(x, \xi)\}$ belong uniformly to S^0.)

Moreover, we can choose the partition $\{Q_\nu\}$ to satisfy

$$\sum_\nu \chi_{Q_\nu^*} \leqslant C$$

(i.e., the uniformly bounded number of overlappings, χ_Q, being the characteristic function of the set Q).

Write

$$p_\nu(x, \xi) = \phi_\nu(x, \xi)\, p(x, \xi).$$

We formulate at this point the

Main Hypothesis 1. *$p_{|Q^{**}}$ satisfies a subelliptic estimate*: $\exists \varepsilon \in (0, 1]$, $\exists c_\varepsilon > 0$ *such that*

$$\text{(s.e.)} \qquad \max_{(x, \xi) \in B} p_{|Q^{**}}(x, \xi) \geqslant c_\varepsilon M^\varepsilon, \qquad \forall B \text{ testing box} \subset Q^{**}.$$

Let us recall the definition of a *testing box* (see Fefferman [2]):

Definition 2.1. Let $\Phi: (z, \zeta) \mapsto (x, \xi)$ be a canonical transformation mapping $\{|z|, |\zeta| \leqslant M^\delta\}$ into $\mathbf{R}^{2n}$ and satisfying the estimates

$$|\partial_{z,\zeta}^\alpha x| \leqslant C_\alpha M^{-\delta|\alpha|}, \qquad |\partial_{z,\zeta}^\beta \xi| \leqslant C_\beta M^{1-\delta|\beta|}$$

for $|\alpha|$, $|\beta| \geqslant 1$. If $Q_{2n}^0 = \{(x, \xi) \in \mathbf{R}^{2n}; |x_j| \leqslant 1, |\xi_j| \leqslant 1, j = 1, ..., n\}$ is the unit cube in $\mathbf{R}^{2n}$, then

$$B = \Phi(Q_{2n}^0)$$

is called a *testing box*.

In view of the Calderon–Zygmund decomposition we will have to perform, it is convenient to extend p_ν to all of $\mathbf{R}^n \times \mathbf{R}^n$, preserving (1) and (s.e.). We then construct from ϕ_ν the function $\tilde{\phi}_\nu$ satisfying the following properties:

$$0 \leqslant \tilde{\phi}_\nu \leqslant 1, \qquad \tilde{\phi}_\nu \equiv 1 \quad \text{on } Q_\nu^*, \qquad \operatorname{supp} \tilde{\phi}_\nu \subset Q_\nu^{**}.$$

Consider then

$$p'(x, \xi) = p_\nu(x, \xi) + (1 - \tilde{\phi}_\nu(x, \xi))\, M^\varepsilon c_\varepsilon.$$

Then $0 \leqslant p' \in S^2(M)$, i.e., it satisfies (1) $\forall (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$, $p' = p$ on Q^*, and also it satisfies (s.e.) $\forall B \subset \mathbf{R}^n \times \mathbf{R}^n$, B the testing box.

In fact, let us first note the following fact:

Given $a(x), b(x) \geqslant 0$, two bounded functions, then trivially

$$\tfrac{1}{2}(\sup a + \sup b) \leqslant \sup(a + b) \leqslant \sup a + \sup b.$$

Thus:

(i) $\forall B$ testing boxes such that $B \subset Q^{**}$,

$$\max_{(x, \xi) \in B} p'(x, \xi) \geqslant \tfrac{1}{2}\{\max_B(p(x, \xi)\, \phi_\nu(x, \xi)) + M^\varepsilon c_\varepsilon \max_B(1 - \tilde{\phi}_\nu(x, \xi))\}$$

(since $\phi_\nu \equiv 1$ on Q_ν^{**})

$$\geqslant \tfrac{1}{2} \max_{(x, \xi) \in B} p(x, \xi) \geqslant \tfrac{1}{2} c_\varepsilon M^\varepsilon$$

in view of the above fact.

(ii) $\forall B$ testing boxes such that $B \cap (\mathbf{R}^n \times \mathbf{R}^n \backslash Q^{**}) \neq \varnothing$,

$$\max_{(x, \xi) \in B} p'(x, \xi) \geqslant \tfrac{1}{2} c_\varepsilon M^\varepsilon \max_B (1 - \tilde{\phi}_\nu) = \tfrac{1}{2} c_\varepsilon M^\varepsilon.$$

Hence,

$$\text{(s.e.1)} \qquad \max_{(x, \xi) \in B} p'(x, \xi) \geqslant c_\varepsilon M^\varepsilon \qquad \forall B \text{ testing boxes}$$

with a new constant c_ε.

We call this extension p' or p by p again.

Let us summarize our present situation: We are dealing with $0 \leqslant p(x, \xi) \in C^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, a symbol in the class $S^2(M)$, i.e.,

$$|\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leqslant C_{\alpha\beta} M^{2-|\beta|}, \qquad \forall \alpha, \beta, \forall (x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$$

($C_{\alpha\beta}$ *independent* of $M \geqslant 1$, M to be fixed later on), satisfying the condition: $\exists \varepsilon \in (0, 1]$, $\exists c_\varepsilon > 0$ such that

$$\max_{(x, \xi) \in B} p(x, \xi) \geqslant c_\varepsilon M^\varepsilon \qquad \forall B \text{ testing box}$$

(we refer to this condition, from now on, as condition (s.e.)).

We are interested in analyzing p on a basic block Q of size $1 \times M$.

Remark 2.2. We chose to extend $p_{|Q}$ in this way, i.e., by adding a term $\sim M^\varepsilon$ for $\max\{|x - x^\nu|, (1/M)\, |\xi - \xi^\nu|\} := \operatorname{dist}((x, \xi), (x^\nu, \xi^\nu)) \geqslant 10$, because we are interested in applications of the kind "Theorem SAK" (see Fefferman [2, p. 199]), so allowing error terms, microlocally in size $1 \times M$, of magnitude $\sim$(const) $M^\varepsilon \|u\|_{L^2}$, for u microlocalized to such a size.

We shall have to make further assumptions. Before doing that, we wish to recall the Calderon–Zygmund decomposition, mentioned above, introduced by Fefferman and Phong in [3, 4, 7], and to describe the consequences that will be used over and over in this work.

Let Q be our basic block in $\mathbf{R}^n \times \mathbf{R}^n$ of sizes $1 \times M$. Then $p_{|Q} \in S^2(Q)$ (see Fefferman [2]). Divide Q into 2^{2n} equal parts, divide each part in the same manner, etc., and retain the blocks Q_ν which fail to satisfy one of the following conditions:

$$\max_{|\alpha| + |\beta| \leqslant 3} \max_{(x, \xi) \in \lambda Q_\nu} |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \leqslant A(M\delta_\nu^2)^2\, \delta_\nu^{-|\alpha|} (M\delta_\nu)^{-|\beta|} \tag{2}$$

$$\operatorname{Vol}(Q_\nu) := |Q_\nu| \geqslant 1 \tag{3}$$

where we have denoted the sizes of Q_ν by $\delta_\nu = \operatorname{diam}_x Q_\nu$ and $M\delta_\nu = \operatorname{diam}_\xi Q_\nu$. Here λQ_ν is the dilate of Q_ν by a fixed constant λ; λ, A to be chosen later. From now on we will also denote by Q^* the double of Q and by Q' the dilate of Q by a suitable constant $k(\lambda)$ depending on λ. Note the following important fact:

Inequality (2) for $|\alpha| + |\beta| \geqslant 4$ is a trivial consequence of the fact that $p \in S^2(Q)$. Hence, for each Q_ν, $p_{|Q_\nu} \in S^2(Q_\nu)$.

DEFINITION 2.3 (Fefferman and Phong [4]). (i) $p \in S^2(Q_\nu)$ is said to be *elliptic* if

$$|p(x, \xi)| \geqslant c(M\delta_\nu^2)^2, \qquad (x, \xi) \in Q_\nu, \quad c > 0.$$

(ii) $p \in S^2(Q_\nu)$ is said to be *non-degenerate* on Q_ν if

$$\max_{|\alpha|+|\beta|=2} \max_{(x,\xi)\in Q_\nu} |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \, \delta_\nu^{|\alpha|}(M\delta_\nu)^{|\alpha|} \, (M\delta_\nu^2)^{-2} \geqslant \bar{C}$$

with

$$\bar{C} \geqslant C \max \left\{ C', \sum_{|\alpha|+|\beta|=3} \max_{Q_\nu} |\partial_x^\alpha \partial_\xi^\beta p(x, \xi)| \, \delta_\nu^{|\alpha|}(M\delta_\nu)^{|\beta|} \, (M\delta_\nu^2)^{-2} \right\}$$

where C, C' are large positive constants. (Note that when $(x, \xi) \in Q_\nu$, $p \in S^2(Q_\nu)$, denoting $\tilde{x} = (x - x^\nu)/\delta_\nu$, $\tilde{\xi} = (\xi - \xi^\nu)/M\delta_\nu$ with $(x^\nu, \xi^\nu) =$ center(Q_ν), then

$$\frac{1}{(M\delta_\nu^2)^2} \, p(\delta_\nu \tilde{x} + x^\nu, M\delta_\nu \tilde{\xi} + \xi^\nu) = P(\tilde{x}, \tilde{\xi})$$

is a smooth function—(i.e., its derivatives of any order are bounded uniformly in M, δ_ν)—on Q^0.)

One has the following

LEMMA 2.4 (Fefferman and Phong [4]). *The blocks $\{Q_\nu\}$ can be divided into three classes $\mathscr{R}_1$, $\mathscr{R}_2$, $\mathscr{R}_3$ with the following properties*:

(i) *p is elliptic on Q_ν if $Q_\nu \in \mathscr{R}_1$*;

(ii) *p is non-degenerate on Q_ν if $Q_\nu \in \mathscr{R}_2$*;

(iii) *$|Q_\nu| \sim 1$ if $Q_\nu \in \mathscr{R}_3$*.

It follows from the proof of the above lemma that a good choice of A is: $A \geqslant k(n)\, C_4 \lambda^{20}$, where $C_4 \geqslant \max_{|\alpha|+|\beta|=4} C_{\alpha\beta}$ and $k(n)$ is another *a priori* constant depending on the dimension.

Therefore we still have the freedom of choosing λ.

The main property of a non-degenerate symbol is contained in the following lemma.

LEMMA 2.5 (Fefferman and Phong [4]). *Let p be non-degenerate on a block Q centered at $(0, 0)$ of size $1 \times M$. Then either p is elliptic on Q, or else by a linear symplectic transformation T we may bring about*

$$(p \circ T)(y, \eta) = e(y, \eta)(\eta_1 - \theta(y, \eta'))^2 + b(y, \eta') \tag{4}$$

in a dilate of Q. Here $e \in S^0(Q)$ is elliptic and positive, $\theta \in S^1(Q)$ and $b \in S^2(Q)$ are real symbols with $b \geqslant 0$. T may be taken to satisfy $|y| + M^{-1}|\eta| \sim |x| + M^{-1}|\xi|$.

Remark 2.6. By picking $\lambda \geqslant \lambda_0$ fixed *a priori*, depending on the dimension and the $C_{\alpha,\beta}$ (given *a priori*, as we have seen), T can be chosen to be either the *identity* or a *canonical permutation of variables* at scale $1 \times M$, i.e., a map of the kind

$$\sigma_i\colon (x_1, ..., x_i, ..., x_n, \xi_1, ..., \xi_i, ..., \xi_n)$$
$$\mapsto \left(\frac{-\xi_i}{M}, ..., x_1, ..., x_n, Mx_i, ..., \xi_1, ..., \xi_n\right)$$

or of the kind

$$\sigma_i'\colon (x_1, ..., x_i, ..., x_n, \xi_1, ..., \xi_i, ..., \xi_n)$$
$$\mapsto (x_i, ..., x_1, ..., x_n, \xi_i, ..., \xi_1, ..., \xi_n).$$

The idea behind the above lemma is that $p \geqslant 0$ and non-degenerate implies $\partial_x^\alpha \partial_\xi^\beta p_{|Q_v}$ are large for either $|\alpha| + |\beta| = 0$ or 2.

The case $|\alpha| + |\beta| = 0$ is the elliptic case, and the case $|\alpha| + |\beta| = 2$ implies that $\partial_x^\alpha \partial_\xi^\beta p_{|Q_v}$, $|\alpha| + |\beta| = 2$, dominate the derivatives of order $|\alpha| + |\beta| = 3$, allowing the use of the Implicit Function Theorem in studying the set (we suppose that, say, $\partial^2(p \circ T)/\partial\eta_1^2$ is as large as possible)

$$\Sigma = \left\{(y, \eta);\ \frac{\partial(p \circ T)}{\partial\eta_1} = 0\right\},$$

which actually is, in the $|\alpha| + |\beta| = 2$ case, a manifold, as stated by the above lemma.

Definition 2.7 (Fefferman [2]). Suppose $\Phi\colon (y, \eta) \mapsto (x, \xi)$ is a canonical transformation defined on Q (whose center is, say, (y^0, η^0)). Denote by i the map $i\colon (y, \eta) \mapsto (y - y^0, M^{-1}(\eta - \eta^0))$ which carries Q to Q^0, the unit cube (we drop the subscript $2n$ when there is no risk of confusion). Define $(x^0, \xi^0) = \Phi(y^0, \eta^0)$. We say that Φ satisfies *natural estimates* if $i \circ \Phi \circ i^{-1}$ is a C^∞ map with derivatives of all orders bounded independent of M.[1] More generally, let Q_1, Q_2 be blocks in $\mathbf{R}^n \times \mathbf{R}^n$ and, for a

[1] Note that $i \circ \Phi \circ i^{-1}$ is a C^∞ diffeomorphism,

$$\Psi = i \circ \Phi \circ i^{-1}\colon Q^0 \to \mathbf{R}^n \times \mathbf{R}^n,$$

Im $\Psi \subset CQ^0$, some fixed dilate of Q^0.

fixed constant $C>0$, let $\Phi\colon Q_1 \to CQ_2$ (the dilate of Q_2 by C) be a canonical transformation. Let $i_{Q_j}\colon Q_1 \to Q^0$, $j=1, 2$, be the natural rescaling maps carrying Q_j to Q^0. We say that Φ is a *tame canonical transformation* if

$$i_{Q_2} \circ \Phi \circ i_{Q_1}^{-1}\colon Q^0 \to \mathbf{R}^n \times \mathbf{R}^n$$

is a C^∞-diffeomorphism with derivatives of all orders bounded uniformly in $\operatorname{diam}_x Q_j$ and $\operatorname{diam}_\xi Q_j$, for $j=1, 2$.

We have the following well-known lemma:

LEMMA 2.8. *Under the hypotheses of Lemma* 2.5, *suppose p is in the non-degenerate non-elliptic form* (4) *on* Q''''. *There exists a canonical transformation*

$$\Phi\colon (y, \eta) \mapsto (x, \xi), \qquad \Phi\colon Q''' \to Q''''$$

such that on Q'' *we have*

$$(p \circ \Phi)(y, \eta) = \tilde{p}(y, \eta) = \tilde{e}(y, \eta)\, \eta_1^2 + \tilde{b}(y, \eta') \tag{5}$$

with $\tilde{e}$, $\tilde{b}$ *having the same properties of e, b respectively.* Φ *satisfies* natural estimates. *By picking* λ (*larger than an a priori fixed number*) *the associated* C^∞*-map* Ψ *is a small perturbation of the identity in* $C^k(Q^0)$, $k \gg 1$ (*k fixed as large as we wish*). *Moreover,* $\forall (x^0, \xi^0) \in Q$, *by picking* λ, *we can suppose*

$$\Phi^{-1}(x^0, \xi^0) \in Q^*, \qquad \text{the double of } Q. \tag{6}$$

Remark 2.9. Given a symbol $p \in S^2(M)$, one can relate its properties with a P.D.E.'s properties by means of the Beals–Fefferman Calculus (see Fefferman and Phong [4, p. 291] or Fefferman [2, p. 187]).

We now summarize the properties of Fefferman and Phong's Calderon–Zygmund microlocalization which will be used here.

The basic block Q $1 \times M$ will be dyadically cut into smaller blocks $\{Q_\nu\}$ such that

$$\text{(CZ1):} \quad \begin{array}{ll} \text{(i)} & \text{either } p_{|Q'_\nu} \text{ is elliptic;} \\ \text{(ii)} & \text{or } p_{|Q'_\nu} \text{ is nonelliptic-nondegenerate;} \\ \text{(iii)} & \text{or } |Q_\nu| \sim 1. \end{array}$$

Moreover, $\{Q_\nu\}$ has the property

$$\text{(iv)} \quad Q'''_\nu \cap Q'''_\mu \neq \varnothing \Rightarrow \delta_\nu \sim \delta_\mu.$$

In (CZ1, ii) p is then written in the *normal form* (through a "nice" canonical transformation)

$$p(x,\xi)=e(x,\xi)\,\xi_1^2+p_1(x,\xi') \tag{7}$$

on Q''_v (with $e\in S^0(\delta_v\times M\delta_v)$, >0, elliptic, $0\leqslant p_1\in S^2(\delta_v\times M\delta_v))$).

Remark 2.10. The condition (s.e.1) rules out (CZ1, iii). In fact, on Q_v with $|Q_v|\sim 1$, $p_{|Q_v}$ is just bounded by *a priori* constants. This would therefore violate (s.e.1) (M being $\gg 1$).

Remark 2.11. Remarks 2.2 and 2.10 make it possible for us to assume the following:

$$\text{(t.e.)}\quad \min_{\mathbf{R}^n\times\mathbf{R}^n} p\geqslant 1.$$

This can be achieved by adding a 0th order positive elliptic symbol belonging to $S^0(\mathbf{R}^n\times\mathbf{R}^n)$ to the original symbol considered in the beginning. This hypothesis will allow us to Taylor-expand the symbol (suitably microlocalized on certain C.Z. blocks) so that it will be possible to assume that it is a polynomial of degree d (d depending on ε). Hence, given $0\leqslant p\in S^2(M)$, we shall consider $\tilde{p}=p+1$ and call it p again.

Note that $1\in S^2(\mathbf{R}^n\times\mathbf{R}^n)\cap S^2(M)$. Moreover, it is important to note that the C.Z. decomposition for $\tilde{p}$ is the same as that for p, since the addition of 1 doesn't affect either (s.e.1) or the ellipticity or the non-degeneracy (since $\partial_x^\alpha\partial_\xi^\beta 1\equiv 0$, $\forall\alpha,\beta$, $|\alpha|+|\beta|>0$). Another property which will be used is stated by the following lemma. (See Fefferman [2, p. 189] and Parmeggiani [18, p. 23].)

LEMMA 2.12. *Consider $0\leqslant p\in S^2(M)$ on a block Q of sizes $1\times M$ centered at* $(0,0)$ *such that $p_{|Q'}$ is microlocally subelliptic* (*i.e.*, (s.e.1) *holds*) *and p is in the form*

$$p_{|Q''}(x,\xi)=\xi_1^2+p_1(x,\xi'),$$

where p_1 is a polynomial in x_1 of degree d. Take an interval $I\subset\pi_{x_1}(Q')$ (the x_1-projection of Q') such that $|I|\sim 1$. Then $\bar{p}_1(x',\xi')=(\mathrm{Av}_{x_1\in I}p_1)(x',\xi')$ satisfies a (s.e.) *condition, i.e., $\exists c'_\varepsilon>0$ such that*

$$\max_{B'}\bar{p}_1(x',\xi')\geqslant c'_\varepsilon M^\varepsilon \tag{8}$$

$\forall B'$ testing box contained in $\mathbf{R}^{n-1}\times\mathbf{R}^{n-1}\cap\pi_{(x',\xi')}(Q'')$.

Remark 2.13. The fact that p_1 is a polynomial in x_1 is no restriction (by Remark 2.11).

The foregoing Lemma offers the opportunity of giving some examples of symbols which do not satisfy condition (s.e.):

EXAMPLE 2.14. Let $Q=\{(x,\xi)\in\mathbf{R}^2\times\mathbf{R}^2;\ |x|\leqslant 1,\ |\xi|\leqslant M\}$. Then $p_{|Q}(x,\xi)=$

$$\text{(i)}\quad \xi_1^2+x_1^2\xi_2^2\simeq\xi_1^2+M^2x_1^2x_2^2$$
$$\text{(ii)}\quad \xi_1^2+\xi_2^2\simeq\xi_1^2+M^2x_2^2$$

don't satisfy condition (s.e.).

Here $\simeq$ means that there exists a *tame* canonical transformation ϕ under which the two symbols are equivalent.

In fact, in both cases (i) and (ii), $\phi\colon (y,\eta)\mapsto(x,\xi)$ is defined by

$$\begin{cases} x_1=y_1, & x_2=\dfrac{1}{M}\xi_2 \\ \xi_1=\eta_1, & \eta_2=My_2 \end{cases}$$

Let us now set up testing boxes for which (s.e.) doesn't hold for p.

In the case $p_{|Q}(x,\xi)=\xi_1^2+M^2x_1^2x_2^2$, we can consider (with $0<\varepsilon'<\varepsilon$)

$$B=\Big\{(x,\xi)\in\mathbf{R}^2\times\mathbf{R}^2;\ |x_1|\leqslant\frac{2}{c_\varepsilon^{1/2}M^{\varepsilon'/2}},\ |\xi_1|\leqslant c_\varepsilon^{1/2}\frac{M^{\varepsilon'/2}}{2},$$
$$|\xi_2|\leqslant\frac{4}{c_\varepsilon}M^{1-\varepsilon'},\ |x_2|\leqslant\frac{c_\varepsilon}{4}M^{\varepsilon'-1}\Big\}.$$

Hence $\max_B p(x,\xi)\leqslant\frac{1}{2}c_\varepsilon M^{\varepsilon'}$ and (s.e.) doesn't hold.

In the case $p_{|Q}(x,\xi)=\xi_1^2+\xi_2^2$, we can consider

$$B=\Big\{(x,\xi)\in\mathbf{R}^2\times\mathbf{R}^2;\ |x_1|,\ |x_2|\leqslant\frac{2}{M^{\varepsilon/2}c_\varepsilon^{1/2}},\ |\xi_1|,\ |\xi_2|\leqslant\frac{M^{\varepsilon/2}c_\varepsilon^{1/2}}{2}\Big\}.$$

Again $\max_B p(x,\xi)\leqslant\frac{1}{2}c_\varepsilon M^\varepsilon$ and (s.e.) doesn't hold.

In order to state the final set of hypotheses, we have first to establish some facts.

FACT 1. *Given p as above, satisfying* (s.e.1) *on a basic block* $1\times M$, Q, *we have that $p_{|Q_\nu''}$ satisfies* (s.e.ν), *where Q_ν is a block arising from the* C.Z. *decomposition of* Q:

$$\text{(s.e.}\nu)\qquad \max_{(x,\xi)\in B} p_{|Q_\nu''}\geqslant c_\varepsilon(M\delta_\nu^2)^\varepsilon,$$

$\forall B$ *testing box* $\subset Q_\nu''$.

In fact, $0<\delta_\nu\leqslant 1$, $\delta_\nu=\operatorname{diam}_x Q_\nu$.

Call $M_\nu = M\delta_\nu^2$. By (s.e.1) we also have $\delta_\nu \gg M^{-1/2}$. Now, on Q_ν, either $p_{|Q_\nu}$ is elliptic or it is non-degenerate. In the latter case we hence suppose[2]

$$p_{|Q_\nu}(x, \xi) = e(x, \xi)\, \xi_1^2 + p_1(x, \xi') + 1$$

(and actually for $(x, \xi) \in$ large dilate of $Q_\nu = Q_\nu''$).

We shall have to consider $\rho^2 p(x, \xi)$, where $0 < \rho \ll 1$ is a number on which we shall impose some conditions.

In order to understand $\rho^2 p_{|Q_\nu}$ we have to carry out a further C.Z. decomposition of Q_ν' (keeping the same parameters A, λ of the C.Z. decomposition giving rise to Q_ν).

Let us call $Q_{\nu\mu}$ the C.Z. blocks arising from this further decomposition. Hence $\delta_{\nu\mu} := \operatorname{diam}_x Q_{\nu\mu}$. Since

$$\rho^2 p_{|Q_\nu}(x, \xi) = e(x, \xi)(\rho\xi_1)^2 + \rho^2 p_1(x, \xi') + \rho^2,$$

we note—(recalling that the ellipticity constant of e is related to the non-degeneracy constant in Definition 2.3 (see Lemma 3.3 in Fefferman and Phong [4])—that now the following is true, in view of Remark 2.11, (we write $p_{|Q_\nu}$, but everything we say is still true on a *large* dilate of Q_ν, as usual):

$$\min \rho^2 p_{|Q_\nu} \geqslant \rho^2,$$

and in particular,

$$\min_{Q_\nu} \{\rho^2 p_1(x, \xi') + \rho^2\} \geqslant \rho^2.$$

In the construction of the subunit ball, we shall see that if $0 \leqslant p_i$, $i = 1, 2$, are symbols in $S^2(Q)$ such that $p_1 \sim p_2$ on Q (i.e., $\exists c_1, c_2 > 0$ such that on Q, $c_1 p_1 \leqslant p_2 \leqslant c_2 p_1$), then

$$B_{c_1 p_1} \subset B_{p_2} \subset B_{c_2 p_1}$$

(B_p is the phase-space subunit ball related to p to be defined in the next sections). Hence, since

$$\rho^2 p_{|Q_\nu}(x, \xi) = e(x, \xi)\, \rho^2 \xi_1^2 + \rho^2 p_1(x, \xi') + \rho^2$$

[2] In so writing we suppose Q_ν is centered at $(0, 0)$ and $\operatorname{diam}_x Q_\nu = 1$, $\operatorname{diam}_\xi Q_\nu = M_\nu$. This can be achieved by means of the symplectic dilation

$$\frac{x - x^\nu}{\delta_\nu} = y, \qquad \delta_\nu(\xi - \xi^\nu) = \eta$$

where (x^ν, ξ^ν) is the former center of Q_ν.

(see Remark 2.11) and $c \leqslant e(x, \xi) \leqslant C$, c, C depending only on *a priori* constants (i.e., a number of seminorms of the original p, A, and λ which are fixed *a priori*), we can consider (dropping the ρ^2 term added in the above formula)

$$\rho^2 p_{|Q_\nu}(x, \xi) = \rho^2 \xi_1^2 + \rho^2 p_1(x, \xi') \tag{9}$$

satisfying

$$(\text{t.e.}\rho) \quad \min \rho^2 p_{|Q_\nu} \geqslant \rho^2$$

Remark 2.15. In so doing, we preserve the fact that $\partial^2_{\xi_1} p_{|Q_\nu}$ is the largest among $\partial^\alpha_x \partial^\beta_\xi p_{|Q_\nu}$, $|\alpha| + |\beta| = 2$; i.e., ξ_1 is still the "fastest" variable among the x, ξ. Note also that in (s.e.ν) we have to change c_ε by a c_ε that is new but still fixed, depending on *a priori* constants.

Having $\rho^2 p$ in the form (9), we note that when we perform the C.Z. cutting procedure of Q_ν we shall stop at blocks of size at least $\sim \rho \times M_\nu \rho$, i.e., $\delta_{\nu\mu} \sim \rho$. In fact, $\partial^2_{\xi_1}(\rho^2 p_{|Q_\nu}) = 2\rho^2$ (see the nondegeneracy condition in Definition 2.3).

Hence $1 \gtrsim \delta_{\nu\mu} \gtrsim \rho$ *and it follows that the normal form will occur on blocks of size* $\sim \rho \times M_\nu \rho$. (See Remark 2.15.)

FACT 2. *Suppose* $\rho^2 p_{|Q_\nu}(x, \xi) = \rho^2 \xi_1^2 + \rho^2 p_1(x, \xi')$, Q_ν *of sizes* $1 \times M_\nu$. *Decompose* Q_ν *into* C.Z. *blocks* $\{Q_{\nu\mu}\}$ *relative to* $\rho^2 p_{|Q_\nu}$. *Suppose* $Q \in \{Q_{\nu\mu}\}$ *is a block such that* $\rho^2 p_{|Q}$ *is nonelliptic–nondegenerate. Then* $\operatorname{diam}_x Q \sim \rho$ *and* $\bar{\xi}_1 = \pi_{\xi_1}(\text{center}(Q))$ *is such that either* $|\bar{\xi}_1| \lesssim M_\nu \rho$ *or* $|\bar{\xi}_1| \sim M_\nu \rho$.

Proof. In view of the choice (9), we have $\partial^2_{\xi_1}(\rho^2 p) = 2\rho^2$ throughout Q_ν. The non-degeneracy condition in Definition 2.3 says that

$$\max_{|\alpha| + |\beta| = 2} \max_{Q} |\partial^\alpha_x \partial^\beta_\xi (\rho^2 p)(x, \xi)| \, (\operatorname{diam}_x Q)^{|\alpha|} \, (\operatorname{diam}_\xi Q)^{|\beta|}$$
$$\times (\operatorname{diam}_x Q \operatorname{diam}_\xi Q)^{-2} \geqslant \bar{C}.$$

Hence, with $\operatorname{diam}_x Q = \delta$, $\alpha = 0$, $\beta = 2$,

$$2\rho^2 \delta^{|\alpha|} (M_\nu \delta)^{|\beta|} \, (M_\nu \delta^2)^{-2} = 2\rho^2 M_\nu^{|\beta| - 2} \delta^{-2} = 2\rho^2 \delta^{-2} \geqslant \bar{C},$$

i.e., $\rho \gtrsim \delta$ and the cutting procedure can stop when $\delta \sim \rho$.

Suppose now $|\bar{\xi}_1| \gg M_\nu \rho$. It follows then that $\rho^2 \xi_1^2$ is elliptic on Q, which contradicts the fact that, on Q, $\rho^2 p$ is non-elliptic. ∎

We now want to have $M_\nu \rho^2 \gg 1$, so it must be $\rho \gg M_\nu^{-1/2}$.

We make the following *main assumptions*:

$$\text{(A1)} \quad \text{(s.e.1) holds with } \varepsilon \in (0, 1]:$$

we then set

$$\varepsilon_1 = \frac{\varepsilon}{4(2-\varepsilon)}, \qquad \varepsilon_0 = \frac{1}{8}\,\varepsilon\varepsilon_1 = \frac{\varepsilon^2}{32(2-\varepsilon)}.$$

Then $0 < \varepsilon_0 < \varepsilon_1 < \frac{1}{2}$ and $M_\nu^{-\varepsilon_1} < M_\nu^{-\varepsilon_0}$. Hence we take

$$\text{(A2}\nu\text{)} \quad M_\nu^{-\varepsilon_1} < \rho < M_\nu^{-\varepsilon_0};$$

$$\text{(A3}\nu\text{)} \qquad M_\nu > M_{\min},$$

where $M_{\min}$ depends on ε, c_ε, and the bounds of a finite number N, fixed *a priori*, of seminorms $C_{\alpha\beta}$;

$$\text{(A4)} \quad (\text{t.e.}\rho) \text{ holds.}$$

Conditions (A2ν), (A3ν), and (t.e.ρ) will allow us to take the Taylor expansion of p to make it possible to consider a polynomial symbol of (a priori) bounded degree.

We now state some consequences of the main assumptions (A1)–(A4).

We suppose $p_{|Q_\nu}(x, \xi) = \xi_1^2 + p_1(x, \xi')$, Q_ν of sizes $1 \times M_\nu$ centered at $(0, 0)$. Consider $\rho^2 p_{|Q_\nu}(x, \xi) = \rho^2 \xi_1^2 + \rho^2 p_1(x, \xi')$. Cut Q_ν into a family of dyadic blocks $\{Q_{\nu\mu}\}$, a C.Z. decomposition relative to $\rho^2 p_{|Q_\nu}$. Let $Q \in \{Q_{\nu\mu}\}$ of size $\rho \times M_\nu \rho$ be such that $\rho^2 p_{|Q}$ is nonelliptic–nondegenerate. It follows from Fact 2 that $\operatorname{diam}_x Q \sim \rho$, and that

$$|\partial_x^\alpha \partial_\xi^\beta(\rho^2 p_1(x, \xi'))| \leqslant C_{\alpha\beta}(M_\nu \rho^2)^2\, (M_\nu \rho)^{-|\beta|}\, \rho^{-|\alpha|}, \qquad \forall \alpha, \beta. \tag{10}$$

We have the following

Consequence 1. *$\rho^2 p_{1|Q}(x, \xi')$ can be Taylor-expanded on $4Q'''$. More precisely, there exists a polynomial $P_1(x, \xi')$, $\deg P_1 = d \leqslant D$ (an a priori fixed constant), and universal constants $c_1, c_2 > 0$, such that*

$$c_1 \rho^2 p_1(x, \xi') \leqslant P_1(x, \xi') \leqslant c_2\, \rho^2 p_1(x, \xi'), \qquad \forall (x, \xi) \in 4Q'''.$$

Proof. If $\bar{\xi}_1 \in \pi_{\xi_1}(\text{center}(Q))$, it follows from Fact 2 that $|\bar{\xi}_1| \lesssim M_\nu \rho$. Moreover, $\xi_1 \in \pi_{\xi_1}(Q) \Rightarrow |\xi_1 - \bar{\xi}_1| \leqslant M_\nu \rho$, so that also $|\xi_1| \lesssim M_\nu \rho$.

We can hence consider the symplectic scaling $\psi: (x, \xi) \mapsto (y, \eta)$,

$$\rho \xi_1 = \eta_1, \qquad \rho(\xi' - \bar{\xi}') = \eta', \quad \frac{1}{\rho}(x - \bar{x}) = y \tag{11}$$

where $\mathrm{center}(Q)=(\bar{x},\bar{\xi})$. (Note that ψ is globally defined.) Then

$$\tilde{Q}=\psi(Q),\quad \text{a block of sizes } 1\times M_\nu\rho^2,\qquad \psi(\bar{x},\bar{\xi})=(0;\rho\bar{\xi}_1,0).$$

Let us consider $\rho^2 p_1(x,\xi')$. From $p_1\in S^2(1\times 1\times M_\nu)$, it follows that

$$\text{(i)}\quad |\partial_x^\alpha\partial_\xi^\beta(\rho^2 p_1(x,\xi'))|\leqslant C_{\alpha\beta}(M_\nu^2\rho^2)\,M_\nu^{-|\beta|},$$

and from the C.Z. localization, it follows that

$$\text{(ii)}\quad |\partial_z^\alpha\partial_{\xi'}^\beta(\rho^2 p_{1\,|\,Q}(x,\xi'))|\leqslant C_{\alpha\beta}(M_\nu\rho^2)^2\,(M_\nu\rho)^{-|\beta|}\,\rho^{-|\alpha|}.$$

But for $|\alpha|+|\beta|\geqslant 2$,

$$M_\nu^{2-|\beta|}\rho^2\leqslant M_\nu^2\rho^2 M_\nu^{-|\beta|}\rho^{2-(|\alpha|+|\beta|)}=(M_\nu\rho^2)^2\,M_\nu^{-|\beta|}\rho^{-(|\alpha|+|\beta|)},$$

which is on the right-hand side of estimate (ii) above. Write $\tilde{\eta}'=(1/M_\nu\rho^2)\,\eta'$ and consider the function

$$f(y,\tilde{\eta}')=\frac{1}{(M_\nu\rho^2)^2}\rho^2 p_1\left(\bar{x}+\rho y,\, M_\nu\rho^2\left(\frac{1}{\rho}\tilde{\eta}'+\frac{1}{M_\nu\rho^2}\bar{\xi}'\right)\right).$$

f is then a smooth function on the unit cube in $\mathbf{R}^n\times\mathbf{R}^{n-1}$. For $|\alpha|+|\beta|\geqslant 2$ we have

$$|\partial_y^\alpha\partial_{\tilde{\eta}'}^\beta f(y,\tilde{\eta}')|\leqslant C_{\alpha\beta}\,\rho^{|\alpha|+|\beta|-2}.$$

We can therefore choose $d=|\alpha|+|\beta|$ *a priori* sufficiently large (depending on ε_0; the $C_{\alpha\beta}$'s are *a priori* constants depending on the original ψ *do*) so that, if

$$P_1(x,\xi')=\sum_{|\alpha|+|\beta|\leqslant d}\frac{1}{\alpha!\beta!}\partial_x^\alpha\partial_\xi^\beta(\rho^2 p_1)(\bar{x},\bar{\xi}')(x-\bar{x})^\alpha\,(\xi'-\bar{\xi}')^\beta$$

is the Taylor polynomial of $\rho^2 p_1$ at $(\bar{x},\bar{\xi}')$ of degree d, we have, because of (A2ν), (A3ν), (A4),

$$|\rho^2 p_1(x,\xi')-P_1(x,\xi')|\leqslant c_{d+1}M_\nu^{-(d+1)}\leqslant\tfrac{1}{2}\rho^2,\qquad \forall(x,\xi)\in 4Q'''.$$

By (t.e.ρ) we can then consider on Q''' the *equivalent* (in the sense $c_1\rho^2 p_1\leqslant P_1\leqslant c_2\rho^2 p_1$) symbol

$$\rho^2\xi_1^2+P_1(x,\xi'),\tag{12}$$

which we shall write again as $\rho^2\xi_1^2 + p_1(x, \xi')$, so that (A1)–(A4) are still satisfied (with new *a priori* constants). ∎

Remark. Consequence 1 will be used to replace $\rho^2 p_1$ by its d-degree Taylor polynomial in such a way that the subunit balls relative to $\rho^2 p$ and to $\rho^2\xi_1^2 + P_1$ will be equivalent, as explained in Lemma 3.12 below (see Section 3).

Hence $\rho^2 p_1$ can be supposed to be a polynomial P_1 of bounded degree such that

$$\tfrac{1}{2}\rho^2 \leqslant P_2 \leqslant C(M_\nu \rho^2)^2, \qquad P_1 \in S^2(\rho \times M_\nu \rho).$$

CONSEQUENCE 2. *Suppose the* C.Z. *cutting procedure stops at* Q *of sizes* $\sim \rho \times M_\nu \rho$. *Then* $\rho^2 p_{|Q}$ *satisfies a* (s.e.) *condition.*

Proof.

$$\max_{(x,\,\xi) \in B} \rho^2 p_{|Q}(x, \xi) \geqslant c_\varepsilon \rho^2 M_\nu^\varepsilon \geqslant c_\varepsilon (M_\nu \rho^2)^{\varepsilon/2},$$

since $M_\nu^{\varepsilon/2} \rho^{(2-\varepsilon)} \geqslant 1$ by (A2ν). (In fact, $\rho^{2-\varepsilon} \geqslant M_\nu^{-(2-\varepsilon)\,\varepsilon/4(2-\varepsilon)} = M_\nu^{-\varepsilon/4} \geqslant M_\nu^{-\varepsilon/2}$). ∎

We shall have to consider a C.Z. localization for $\mathrm{Av}_{x_1 \in I_\rho}(\rho^2 p_{1\,|\,Q})$ where $I_\rho \subset \pi_{x_1}(Q')$, $|I_\rho| \sim \rho$ (which is the same as considering $\mathrm{Av}_{y_1 \in I}(\tilde{P}_1(y, \eta'))$ for $(y, \eta') \in$ block of sizes $1 \times M_\nu \rho^2$).

Since $\rho^2 p_{1\,|\,Q}$ is a polynomial, we note that if I_ρ^1, I_ρ^2 are intervals contained in $\pi_{x_1}(Q')$ with $|I_\rho^1| \sim |I_\rho^2| \sim \rho$, $I_\rho^1 \cap I_\rho^2 \neq \varnothing$, then

$$\mathrm{Av}_{x_1 \in I_\rho^1}(\rho^2 p_1) \sim \mathrm{Av}_{x_1 \in I_\rho^2}(\rho^2 p_1)$$

(see Fefferman [2, p. 146]).

Consider $(\rho^2 p \circ \psi)(y, \eta)$ where ψ is defined in (11). Then we can suppose $\rho^2 p(y, \eta) = \eta_1^2 + p_1(y, \eta')$ on a block of sizes $1 \times M_\nu \rho^2$ with p_1 a non-negative polynomial of bounded degree. We now apply Lemma 2.12 to obtain the

CONSEQUENCE 3. $\bar{p}_1(y', \eta') = (\mathrm{Av}_{y_1 \in I} p_1)(y', \eta')$ *satisfies* (s.e.).

Here I is an interval corresponding to I_ρ above through the symplectic scaling ψ.

3. DEFINITION OF THE SUBUNIT BALL B_p

3.1. *Subunit Symbols*

Let $0 \leqslant p \in S^2(1 \times M)$, Q be a basic block of sizes $1 \times M$, $\subset \mathbf{R}^n \times \mathbf{R}^n$. Set Q^* to be the dilate of Q by 4.

DEFINITION 3.1. Let $q \in C^2(Q^{**}, \mathbf{R})$, $\operatorname{supp} q \subset \operatorname{int} Q^{**}$ be such that

$$\text{(i)} \quad |\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leqslant C_{\alpha\beta} M^{1-|\beta|}, \qquad |\alpha| + |\beta| \leqslant 2;$$

$$\text{(ii)} \qquad q(x, \xi)^2 \leqslant p(x, \xi) \qquad \forall (x, \xi) \in Q^{**}.$$

q is said to be a *subunit symbol for p on Q* (or a *subordinate symbol*).

We denote the set of subunit symbols for p on Q by

$$\tilde{\mathscr{S}}(p, Q, 2n).$$

Note that to check conditions (i) it suffices to check them for $|\alpha| + |\beta| = 0$ and $|\alpha| + |\beta| = 2$, the remaining estimates following by interpolation (rescaling matters to the unit cube and scaling things back).[3]

Since, given $1 \geqslant c > 0$,

$$q \in \tilde{\mathscr{S}}(p, Q, 2n) \Leftrightarrow cq \in \tilde{\mathscr{S}}(p, Q, 2n),$$

we decide to normalize subunit symbols in such a way that

$$\max_{0 \leqslant k \leqslant 2} \sum_{|\alpha| + |\beta| = k} C_{\alpha\beta} \leqslant 1.$$

Denote by

$$\mathscr{S}(p, Q, 2n)$$

the subset of $\tilde{\mathscr{S}}(p, Q, 2n)$ of the so-normalized subunit symbols.

[3] We recall here one of the interpolation inequalities used several times in the following: given $f \in C^2$, $f\colon \mathbf{R}^n \to \mathbf{R}$, suppose $\|f\|_{L^\infty} \leqslant P$, $\sum_{|\alpha|=2} \|D^\alpha f\|_{L^\infty} \leqslant Q$. Then

$$\sum_{|\alpha|=1} \|D^\alpha f\|_{L^\infty} \leqslant c(n) \sqrt{PQ},$$

with $c(n) > 0$ a universal constant independent of f and depending only on the dimension n.

Remark 3.2. We shall deal with p and $\rho^2 p$. By picking the constants A, λ in the C.Z. decomposition in an *a priori* way, we can make it possible to always be in the following situation:

$$\forall Q_\mu \text{ C.Z. block}, \qquad Q_\mu^{****} \subset Q'_\mu.$$

Remark 3.3. We shall have to localize further subunit symbols. To do that we fix $\chi \in C_0^\infty(\text{int } Q_{2n}^0)$, $0 \leqslant \chi \leqslant 1$, and define various cut-off functions (related to C.Z. subblocks of various sizes) as the translates and dilates by *a priori* constants of χ (occasionally denoted again by χ), so their derivatives are bounded by *a priori* constants.

Subunit symbols can be *localized* and *extended*, as explained in the following proposition.

PROPOSITION 3.4. *Given* $Q_\delta \subset Q$ *of size* $\delta \times M\delta$, $0 < \delta \leqslant 1$, *we have*

$$q \in \mathscr{S}(p, Q, 2n), \qquad p_{|Q'_\delta} \leqslant C(M\delta^2)^2 \Rightarrow c\chi q \in \mathscr{S}(p_{|Q_\delta}, Q_\delta, 2n) \tag{1}$$

for some a priori $c > 0$, *cutoff* χ, $\operatorname{supp} \chi \subset \operatorname{int} Q_\delta^{**}$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ *on* Q_δ^*; *conversely*,

$$p_{|Q'_\delta} \leqslant C(M\delta^2)^2, \qquad q \in \mathscr{S}(p_{|Q_\delta}, Q_\delta, 2n) \Rightarrow q \in \mathscr{S}(p, Q, 2n). \tag{2}$$

Proof. We just prove the statements for the set $\tilde{\mathscr{S}}$, since the case $\mathscr{S}$ follows by normalization (c is *a priori*; see Remark 3.3).

(1) From (ii) in Definition 3.1, it follows that

$$q(x, \xi)^2 \leqslant p(x, \xi) \leqslant C(M\delta^2)^2, \qquad \forall (x, \xi) \in Q_\delta^{**},$$

so that also

$$(\chi(x, \xi)\, q(x, \xi))^2 \leqslant p(x, \xi) \leqslant C(M\delta^2)^2$$

on Q_δ^{**}. Since $|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leqslant M^{1-|\beta|}$ and, for $|\alpha| + |\beta| = 2$,

$$M^{1-|\beta|} = (M\delta^2)\, \delta^{-|\alpha|}(M\delta)^{-|\beta|},$$

it follows that $q_{|Q_\delta^{**}}$ satisfies the estimates (i) and (ii) at scale $\delta \times M\delta$ of Definition 3.1, the estimates for $|\alpha| + |\beta| = 1$ following by interpolation. Now,

$$|\partial_x^\alpha \partial_\xi^\beta \chi(x, \xi)| \leqslant C^1_{\alpha\beta}(M\delta)^{-|\beta|}\, \delta^{-|\alpha|}, \qquad \forall \alpha, \beta.$$

By Leibniz rule we have

$$\begin{aligned}
&|\partial_x^\alpha \partial_\xi^\beta(\chi(x,\xi)\, q(x,\xi))| \\
&\quad = \left| \sum_{(\gamma,\sigma)\leqslant(\alpha,\beta)} \binom{\alpha}{\gamma}\binom{\beta}{\sigma} (\partial_x^{\alpha-\gamma}\partial_\xi^{\beta-\sigma}\chi(x,\xi))(\partial_x^\gamma \partial_\xi^\sigma q(x,\xi)) \right| \\
&\quad \leqslant \sum_{(\gamma,\sigma)\leqslant(\alpha,\beta)} \binom{\alpha}{\gamma}\binom{\beta}{\sigma} C^1_{\alpha-\gamma,\beta-\sigma}(M\delta)^{-|\beta-\sigma|}\,\delta^{-|\alpha-\gamma|} \\
&\qquad \times (M\delta^2)(M\delta)^{-|\sigma|}\,\delta^{-|\gamma|} \\
&\quad = \left\{ \sum_{(\gamma,\sigma)\leqslant(\alpha,\beta)} \binom{\alpha}{\gamma}\binom{\beta}{\sigma} C^1_{\alpha-\gamma,\beta-\sigma} \right\} M^{1-|\beta|}\delta^{2-(|\alpha|+|\beta|)} \\
&\quad = C_{\alpha\beta}(M\delta^2)(M\delta)^{-|\beta|}\,\delta^{-|\alpha|}, \qquad \forall |\alpha|+|\beta|\leqslant 2,
\end{aligned} \tag{13}$$

i.e., $\chi q \in \tilde{\mathscr{S}}(p_{|Q_\delta}, Q_\delta, 2n)$. Note that the $C_{\alpha\beta}$'s above are *a priori* constants.

(2) It follows trivially from $q(x,\xi)^2 \leqslant p(x,\xi)$ on Q_δ^{**}, the support condition (being int $Q_\delta^{**} \subset$ int Q^{**}), and the fact that $0<\delta\leqslant 1$ and, $\forall \alpha, \beta$, $|\alpha|+|\beta|\leqslant 2$, that

$$(M\delta^2)(M\delta)^{-|\beta|}\,\delta^{-|\alpha|} = M^{1-|\beta|}\delta^{2-(|\alpha|+|\beta|)} \leqslant M^{1-|\beta|}.$$

Hence $q \in \mathscr{S}(p, Q, 2n)$. ∎

Subunit symbols behave well under *tame* canonical transformations. Let

$$Q_\delta = \{x; |x_j - x_j^0| \leqslant \delta,\ j=1, ..., n\} \times \{\xi; |\xi_j - \xi_j^0| \leqslant M\delta,\ j=1, ..., n\}$$

and

$$\tilde{Q}_\delta = \{y; |y_j - y_j^0| \leqslant 1,\ j=1, ..., n\} \times \{\eta; |\eta_j - \eta_j^0| \leqslant M\delta^2,\ j=1, ..., n\}$$

(from now on we shall drop the index j when defining blocks of the above kind). Let $\phi\colon \tilde{Q}_\delta''' \to Q''''_\delta$ be a smooth, tame canonical transformation. Define

$$i_1\colon (y,\eta) \mapsto \left(y - y^0, \frac{1}{M\delta^2}(\eta - \eta^0) \right) = (\tilde{y}, \tilde{\eta})$$

$$i_2\colon (x,\xi) \mapsto \left(\frac{1}{\delta}(x - x^0), \frac{1}{M\delta}(\xi - \xi^0) \right) = (\tilde{x}, \tilde{\xi})$$

so that

$$i_1^{-1}: (\tilde{y}, \tilde{\eta}) \mapsto (y, \eta) = (y^0 + \tilde{y}, \eta^0 + M\delta^2 \tilde{\eta})$$
$$i_2^{-1}: (\tilde{x}, \tilde{\xi}) \mapsto (x, \xi) = (\delta \tilde{x} + x^0, M\delta \tilde{\xi} + \xi^0).$$

Consider also the symplectic scaling

$$s: \tilde{Q}_\delta \to \tilde{Q}_\delta^0 = \{(z, \zeta); |z| \leqslant \delta, |\zeta| \leqslant M\delta\}$$
$$s(y, \eta) = \left(\delta(y - y^0), \frac{1}{\delta}(\eta - \eta^0) \right) = (z, \zeta).$$

For arbitrary (z^0, ζ^0), the canonical transformation

$$\psi: (\tilde{Q}_\delta^0 + (z^0, \zeta^0))''' \to Q_\delta'''$$
$$\psi: (z, \zeta) \mapsto (\phi \circ s^{-1})(z - z^0, \zeta - \zeta^0)$$

is then tame, where $\tilde{Q}_\delta^0 + (z^0, \zeta^0) = \{(z, \zeta); |z - z^0| \leqslant \delta, |\zeta - \zeta^0| \leqslant M\delta\}$. In fact, denoting by $Q_\delta^2 := \tilde{Q}_\delta^0 + (z^0, \zeta^0)$, by i_3 the natural rescaling of Q_δ^2 to the unit cube Q^0, and by T_0 the translation $(z, \zeta) \mapsto (z - z^0, \zeta - \zeta^0)$, we have that

$$i_2 \circ \psi \circ i_3^{-1} = (i_2 \circ \phi \circ i_1^{-1}) \circ (i_1 \circ s^{-1} \circ T_0 \circ i_3^{-1}).$$

We then use the fact that ϕ is tame and that

$$i_1 \circ s^{-1} \circ T_0 \circ i_3^{-1}: (\tilde{z}, \tilde{\zeta}) \mapsto (z^0 + \delta \tilde{z}, \zeta^0 + M\delta \tilde{\zeta}) \mapsto (\delta \tilde{z}, M\delta \tilde{\zeta})$$
$$\mapsto \left(y^0 + \frac{\delta \tilde{z}}{\delta}, \eta^0 + M\delta^2 \tilde{\zeta} \right) \mapsto (\tilde{z}, \tilde{\zeta}).$$

Suppose

$$\phi(\tilde{Q}_\delta^{**}) \subset (Q_\delta^1)^{**} \tag{14}$$

so that

$$\psi((\tilde{Q}_\delta^0 + (z^0, \zeta^0))^{**}) \subset (Q_\delta^1)^{**}$$

holds and vice versa (since ψ is obtained from ϕ through an affine symplectic transformation and vice versa).

Here $Q_\delta^1 = \{(x, \xi); |x - x^0| \leqslant C\delta, |\xi - \xi^0| \leqslant CM\delta\}$, $C > 0$ depending only on ϕ. Then

PROPOSITION 3.5. *Given* $0 \leqslant p \in S^2(\delta \times M\delta)$,

$$\text{(i)} \quad q \in \mathscr{S}(p, Q^1_\delta) \Rightarrow c(q \circ \phi) \in \mathscr{S}(p \circ \phi, \tilde{Q}_\delta)$$

and equivalently

$$\text{(ii)} \quad q \in \mathscr{S}(p, Q^1_\delta) \Rightarrow c(q \circ \psi) \in \mathscr{S}(p \circ \psi, \tilde{Q}^0_\delta + (z^0, \zeta^0)).$$

$c > 0$ *is an a priori constant depending on* ϕ *(equiv., on* ψ*).*

Proof. (i) $\Leftrightarrow$ (ii) since ϕ and ψ are equivalent under an affine symplectic transformation. We shall henceforth prove only (i). In view of (14), we just need to check (i) and (ii) in Definition 3.1.

(ii) is trivial (since the symbol p behaves well under tame canonical transformations of the above kind). To check (i), it suffices to check it for $|\alpha| + |\beta| = 0, 2$, the intermediate cases following by interpolation.

Write

$$q \circ \phi = q \circ i_2^{-1} \circ (i_2 \circ \phi \circ i_1^{-1}) \circ i_1 := (q \circ i_2^{-1}) \circ \Phi \circ i_1$$

with

$$|\partial^\alpha_x \partial^\beta_\xi \Phi(x, \xi)| \leqslant C_{\alpha\beta}, \qquad \forall \alpha, \beta.$$

Given functions f, g, write by Df, Dg their Jacobian matrices and by $D^2 f$, $D^2 g$ their Hessian matrices. Then the chain rule takes the form $D(f \circ g) = Df\, Dg$ whence, with $h = f \circ g$,

$$D^2 h = D^2 f\, Dg \otimes Dg + Df\, D^2 g.$$

By $Df_{|y}$ we shall mean the y-column of Df. Using an $(n+n)$-block notation, we have

$$Di_1 = \operatorname{diag}(I_{n \times n}, (M\delta^2)^{-1} I_{n \times n}) = \operatorname{diag}\left(\frac{\partial \tilde{y}}{\partial y}, \frac{\partial \tilde{\eta}}{\partial \eta}\right), \qquad D^2 i_1 = 0.$$

For $|\alpha| = 2$,

$$\begin{aligned}\partial^\alpha_{\tilde{x}}(q \circ i_2^{-1}) = {} & \sum c(\gamma, i_1, ..., i_j)((\partial^\gamma_x q) \circ i_2^{-1})\, \partial^{\sigma_1}_{\tilde{x}} x_{i_1} \cdots \partial^{\sigma_j}_{\tilde{x}} x_{i_j} \\ & + \sum c(\gamma, \beta, i_1, ..., i_j, i_{j+1}, ..., i_{j+k}) \\ & \times ((\partial^\gamma_x \partial^\beta_\xi q) \circ i_2^{-1})\, \partial^{\sigma_1}_{\tilde{x}} x_{i_1} \cdots \partial^{\sigma_j}_{\tilde{x}} x_{i_j} \partial^{\nu_1}_{\tilde{x}} \xi_{i_{j+1}} \cdots \partial^{\nu_k}_{\tilde{x}} \xi_{i_{j+k}}\end{aligned}$$

with, in the first sum,

$$|\sigma_1| + \cdots + |\sigma_j| = |\alpha|, \qquad |\sigma_l| > 0, \quad \forall l, \qquad |\gamma| = j \leqslant 2 = |\alpha|$$

and, in the second sum,

$$|\sigma_1| + \cdots + |\sigma_j| + |\nu_1| + \cdots + |\nu_k| = |\alpha|, \qquad |\sigma_l|, |\nu_l| > 0, \quad \forall l,$$
$$|\gamma| = j, \quad |\beta| = k, \quad 0 \leqslant j, k \leqslant 1.$$

Hence

$$|\partial_{\tilde{x}}^{\alpha}(q \circ i_2^{-1})(\tilde{x}, \tilde{\xi})| \lessapprox (M\delta^2)\, \delta^{-|\gamma|}\delta^{|\alpha|} \lessapprox M\delta^2$$

since $\partial_{\tilde{x}}^{\nu}\xi_i \equiv 0$.

For $|\beta| = 2$,

$$\partial_{\tilde{\xi}}^{\beta}(q \circ i_2^{-1})(\tilde{x}, \tilde{\xi}) = \sum c(\gamma, i_1, ..., i_j)((\partial_{\xi}^{\gamma} q) \circ i_2^{-1})\, \partial_{\tilde{\xi}}^{\sigma_1}\xi_{i_1} \cdots \partial_{\tilde{\xi}}^{\sigma_j}\xi_{i_j}$$

(since $\partial_{\tilde{\xi}}^{\nu} x_i \equiv 0$) with

$$|\gamma| = j \leqslant 2, |\sigma_1| + |\sigma_2| + \cdots + |\sigma_j| = |\beta|, \qquad |\sigma_l| > 0 \quad \forall l.$$

Hence

$$|\partial_{\tilde{\xi}}^{\beta}(q \circ i_2^{-1})(\tilde{x}, \tilde{\xi})| \lessapprox (M\delta^2)(M\delta)^{-|\gamma|}\,(M\delta)^{|\beta|} \lessapprox M\delta^2.$$

For $|\alpha| = |\beta| = 1$,

$$\partial_{\tilde{x}}^{\alpha}\,\partial_{\tilde{\xi}}^{\beta}(q \circ i_2^{-1})(\tilde{x}, \tilde{\xi}) = \sum c(\gamma, \mu, i_1, i_2)((\partial_x^{\gamma}\,\partial_{\xi}^{\mu} q) \circ i_2^{-1})\, \partial_{\tilde{x}}^{\sigma} x_{i_1}\, \partial_{\tilde{\xi}}^{\nu}\xi_{i_2}$$

with

$$j = |\gamma| = 1, \qquad |\mu| = 1 = k, \qquad |\sigma| = |\alpha| = 1, \qquad |\nu| = |\beta| = 1$$

since

$$\partial_{\tilde{x}}^{\nu}\xi_i = \partial_{\tilde{\xi}}^{\sigma} x_i = \partial_{\tilde{x}\tilde{\xi}}^{\sigma\nu} x_i = \partial_{\tilde{x}\tilde{\xi}}^{\sigma\nu}\xi_i \equiv 0.$$

Hence

$$|\partial_{\tilde{x}}^{\alpha}\,\partial_{\tilde{\xi}}^{\beta}(q \circ i^{-1})(\tilde{x}, \tilde{\xi})| \lessapprox (M\delta^2)\,\delta^{-1}(M\delta)^{-1}\,\delta(M\delta) = M\delta^2$$

and this is true $\forall \alpha, \beta$, $|\alpha| + |\beta| \leqslant 2$.

We know that $|D\Phi|$, $|D^2\Phi|$ are bounded uniformly in $M\delta$ and δ. Now,

$$D(q\circ i_2^{-1}) = \left(\frac{\partial(q\circ i_2^{-1})}{\partial\tilde{x}}, \frac{\partial(q\circ i_2^{-1})}{\partial\tilde{\xi}}\right)$$

$$D\Phi = \begin{pmatrix} \dfrac{\partial\tilde{x}}{\partial\tilde{y}} & \dfrac{\partial\tilde{x}}{\partial\tilde{\eta}} \\ \dfrac{\partial\tilde{\xi}}{\partial\tilde{y}} & \dfrac{\partial\tilde{\xi}}{\partial\tilde{\eta}} \end{pmatrix}.$$

For $|\alpha| = 2$,

$$\begin{aligned}\partial_y^\alpha(q\circ\phi)(y,\eta) &= D^2(q\circ i_2^{-1})(D\Phi\, Di_{1\,|\,y})\otimes(D\Phi\, Di_{1\,|\,y})\\ &\quad + D(q\circ i_2^{-1})\, D^2\Phi\, Di_{1\,|\,y}\otimes Di_{1\,|\,y}\end{aligned}$$

whence $|\partial_y^\alpha(q\circ\phi)(y,\eta)| \lesssim M\delta^2 + M\delta^2 \sim M\delta^2$.

For $|\beta| = 2$,

$$\begin{aligned}\partial_\eta^\beta(q\circ\phi)(y,\eta) &= D^2(q\circ i_2^{-1})(D\Phi\, Di_{1\,|\,\eta})\otimes(D\Phi\, Di_{1\,|\,\eta})\\ &\quad + D(q\circ i_2^{-1})\, D^2\Phi\, Di_{1\,|\,\eta}\otimes Di_{1\,|\,\eta}\end{aligned}$$

whence $|\partial_\eta^\beta(q\circ\phi)(y,\eta)| \lesssim M\delta^2(M\delta^2)^{-2} + M\delta^2(M\delta^2)^{-2} \sim (M\delta^2)^{-1}$.

For $|\alpha| = |\beta| = 1$,

$$\begin{aligned}\partial_y^\alpha\,\partial_\eta^\beta(q\circ\phi)(y,\eta) &= D^2(q\circ i_2^{-1})(D\Phi\, Di_{1\,|\,y})\otimes(D\Phi\, Di_{1\,|\,\eta})\\ &\quad + D(q\circ i_2^{-1})\, D^2\Phi\, Di_{1\,|\,y}\otimes Di_{1\,|\,\eta}\end{aligned}$$

whence $|\partial_y^\alpha\,\partial_\eta^\beta(q\circ\phi)(y,\eta)| \lesssim M\delta^2(M\delta^2)^{-1} + M\delta^2(M\delta^2)^{-1} \sim 1$.

The case $|\alpha| + |\beta| = 0$ being trivial, we have, for $c > 0$, a universal constant,

$$|\partial_y^\alpha\,\partial_\eta^\beta(q\circ\phi)(y,\eta)| \leqslant C_{\alpha\beta}(M\delta^2)^{1-|\beta|}, \qquad |\alpha| + |\beta| \leqslant 2$$

and $c(q\circ\phi) \in \mathscr{S}(p\circ\phi, \tilde{Q}_\delta)$. ∎

Suppose now $\phi\colon \tilde{Q}_\delta''' \to Q_\delta''''$ as above. Suppose $\phi(\tilde{Q}_\delta^{**}) \subset Q_\delta^{1**}$ and $\phi^{-1}(Q_\delta^{1**}) \subset \tilde{Q}_\delta^{1**}$ with $Q_\delta^{1**} \subset Q_\delta''''$. Combining Propositions 3.4 and 3.5 gives

Corollary 3.6.

$$q \in \mathscr{S}(p, Q_\delta^1) \Rightarrow c(q\circ\phi) \in \mathscr{S}(p\circ\phi, \tilde{Q}_\delta^1)$$

and

$$(q \circ \phi) \in \mathscr{S}(p \circ \phi, \tilde{Q}^1_\delta) \Rightarrow c_1 q \in \mathscr{S}(p, Q^1_\delta)$$

for universal constants $c, c_1 > 0$. *Under similar natural assumptions on* ψ, *the same holds for* q *and* $q \circ \psi$.

Corollary 3.6 is crucial since it says that subunit geometry is preserved under tame canonical transformations. (Remark that subunit geometry is preserved by definition under affine canonical transformations like the symplectic scaling s and $s^{-1} \circ T_0$.)

Another property which will be crucial in the next sections is the following:

Lemma 3.7. *Let* Q *be one of either our basic blocks or a block arising from a* C.Z. *decomposition, centered at* $(0, 0)$, *and let*

$$p_{|Q'}(x, \xi) = \xi_1^2 + p_1(x, \xi').$$

$\forall q \in \mathscr{S}(q, Q, 2n)$, *q can be written in the form*

$$q(x, \xi) = q_1(x, \xi) + q_2(x, \xi), \tag{15}$$

where $cq_1 \in \mathscr{S}(\xi_1^2, Q^1, 2n)$, $cq_2 \in \mathscr{S}(p_1, Q^1, 2n)$, *with* $0 < c \leqslant 1$ *a universal constant. Here* Q^1 *is a block, whose sizes are comparable to those of* Q, $\mathrm{center}(Q^1) = \mathrm{center}(Q)$, *such that*

$$Q \subset Q^1 \subset Q^{1**} = Q^{***}.$$

In particular,

$$q_{1\,|\,Q^{**}} \in \mathscr{S}(\xi_1^2, Q), \qquad q_{2\,|\,Q^{**}} \in \mathscr{S}(p_1, Q).$$

Proof. From $q(x, \xi)^2 \leqslant \xi_1^2 + p_1(x, \xi')$ it follows that

$$q(x, 0, \xi')^2 = q(x, \xi)^2_{|\xi_1 = 0} \leqslant p_1(x, \xi).$$

By Taylor's formula we have

$$q(x, \xi) = Q(x, \xi)\, \xi_1 + q(x, 0, \xi') \tag{16}$$

with

$$Q(x, \xi) = \int_0^1 (\partial_{\xi_1} q)(x, t\xi_1, \xi')\, dt.$$

Take now $\chi \in C_0^\infty(\mathbf{R}^n \times \mathbf{R}^n)$, $0 \leqslant \chi \leqslant 1$, $\chi \equiv 1$ on $Q^{**} \supset \operatorname{supp} q$, $\operatorname{supp} \chi \subset Q^{***}$ (so that χ satisfies the natural estimates associated with Q, as in Remark 3.3). Then

$$\begin{aligned}\chi(x, \xi)\, q(x, \xi) = q(x, \xi)&\\ &= \chi(x, \xi)\, Q(x, \xi)\, \xi_1 + \chi(x, \xi)\, q(x, 0, \xi')\\ &:= q_1(x, \xi) + q_2(x, \xi).\end{aligned}$$

Clearly $q_i \in C^2$, $i = 1, 2$, and, by normalization, they belong to $\mathscr{S}(\xi_1^2, Q^1)$, $\mathscr{S}(p_1, Q^1)$ respectively. ∎

Corollary 3.8. *Under the above hypotheses, suppose further, in* $\mathbf{R}^2 \times \mathbf{R}^2$,

$$p_{|Q'}(x, \xi) = \xi_1^2 + e(x, \xi_2)(\xi_2 - \theta(x_1, x_2))^2 + V(x_1, x_2)$$

with $0 < c \leqslant e \leqslant C$; e, V, θ *real*; $V \geqslant 0$; *and* $e \in S^0(Q)$, $\theta \in S^1(Q)$, $V \in S^2(Q)$. *Then for* Q_1 *such that* $Q_1^{**} = Q^{***}$, $\operatorname{center}(Q_1) = \operatorname{center}(Q)$, *and* $\operatorname{size}(Q_1) \sim \operatorname{size}(Q)$,

$$q_2(x, \xi) = q_2^1(x, \xi) + q_2^2(x, \xi)$$

where

$$c q_2^1 \in \mathscr{S}(e(\xi_2 - \theta)^2, Q_1), \qquad c q_2^2 \in \mathscr{S}(V, Q_1),$$

for $0 < c \leqslant 1$ *a universal constant.*

Proof. It follows immediately from the fact that

$$q_{2\,|\,Q^{**}} \in \mathscr{S}(e(\xi_2 - \theta)^2 + V, Q),$$

and by Taylor expanding with respect to

$$\Sigma = \{(x, \xi);\, \xi_2 = \theta(x_1, x_2)\}. \quad ∎$$

Denote now by H_q the Hamiltonian vector field associated with the Hamiltonian $q(x, \xi)$, where q is *subordinate* to p on Q, where Q is a block of sizes $\delta \times M\delta$, centered at $(0, 0)$ (for simplicity).

Let $(x^0, \xi^0) \in Q$ and let $\gamma(t) = \exp(tH_q)(x^0, \xi^0)$.

Lemma 3.9. $\forall (x^0, \xi^0) \in Q$, $\forall t \in [0, 1]$, $\gamma(t) \in Q^*$.

Proof. The Hamilton's equations are:

$$\begin{cases}\dot{x}(t) = (\partial_\xi q)(x, \xi), & x(0) = x^0\\ \dot{\xi}(t) = -(\partial_x q)(x, \xi), & \xi(0) = \xi^0.\end{cases}$$

By Taylor's formula and $q \in \mathscr{S}(p, Q)$, it follows that

$$|x(t) - x^0| \leqslant |t|\ \delta, \qquad |\xi(t) - \xi^0| \leqslant |t|\ M\delta$$

so that

$$\gamma(t) = (x(t), \xi(t)) \in Q^*, \qquad \forall t \in [0, 1]. \quad \blacksquare$$

Remark 3.10. Since $q \in \mathscr{S}(p, Q) \Leftrightarrow -q \in \mathscr{S}(p, Q)$, we can flow forwards and backwards along $\exp(tH_q)$ and consider only trajectories defined for $t \in [0, 1]$.

3.2. *The Definition of the Subunit Ball $B_p((x^0, \xi^0), 1)$*

We now define the subunit ball associated with a non-negative 2nd order symbol $0 \leqslant p$ defined on a suitable dilate of a basic block Q of sizes $1 \times M$, centered for simplicity, at $(0, 0)$. Consider a C.Z. decomposition of Q into subblocks Q_ν of various sizes $\delta_\nu \times M\delta_\nu$ (as always, centered at various points (x^ν, ξ^ν)).

Given $(x^0, \xi^0) \in Q$, then $(x^0, \xi^0) \in Q_\delta$, for a certain δ.

DEFINITION 3.11. Define by

$$T(p, Q_\delta) = \{\gamma: [0, 1] \mapsto \mathbf{R}^n \times \mathbf{R}^n;\ \exists q \in \mathscr{S}(p, Q_\delta^*),\ \dot{\gamma}(t) = H_q(\gamma(t))\} \tag{17}$$

the set of subunit trajectories. Define by $\Gamma(t; x^0, \xi^0)$ *a subunit broken path starting at* (x^0, ξ^0) if $\exists \{t_k\}_{k=0}^L$, a partition of $[0, 1]$, $t_0 = 0$, $t_L = 1$, and $\{\gamma_k\}_{k=1}^L$, $\gamma_k \in T(p, Q_\delta)$ such that $\gamma_k(t_k) = \gamma_{k+1}(t_k)$ and

$$\Gamma_{|[t_k, t_{k+1}]} = \gamma_{k+1\,|\,[t_k, t_{k+1}]}, \qquad k = 0, ..., L-1.$$

The p-subunit ball centered at (x^0, ξ^0) of radius 1 is the set of $(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n$ such that (x, ξ) can be reached through a broken subunit trajectory starting at (x^0, ξ^0):

$$B_p((x^0, \xi^0), 1) = \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ \exists \Gamma \text{ subunit broken path with } (x, \xi) = \Gamma(1; x^0, \xi^0)\}. \tag{18}$$

Define the p-subunit ball of radius ρ, $0 < \rho \leqslant 1$, to be

$$B_p((x^0, \xi^0), \rho) := B_{\rho^2 p}((x^0, \xi^0), 1). \tag{19}$$

The reasons for such a choice of $B_p((x^0, \xi^0), \rho)$ will be clear when we discuss the case in which $p_{|Q_\delta}$ is elliptic.

Suppose now $0 \leqslant p_1, p_2 \in S^2(Q)$ with $p_1 \sim p_2$, then we immediately have the following

LEMMA 3.12. *$q \in \mathscr{S}(p_1, Q) \Rightarrow cq \in \mathscr{S}(p_2, Q)$, and $q \in \mathscr{S}(p_2, Q) \Rightarrow \tilde{c}q \in \mathscr{S}(p_1, Q)$, for universal constants $\tilde{c}, c > 0$. Hence, if $c_1 p_1(x, \xi) \leqslant p_2(x, \xi) \leqslant c_2 p_1(x, \xi)$ $\forall (x, \xi) \in Q''''$, then*

$$B_{c_1 p_1} \subset B_{p_2} \subset B_{c_2 p_1}.$$

Remark 3.13. We want to comment about the definition of the subunit ball of radius ρ. If L is a 2nd-order differential operator, one has that

$$B_L(x, \rho) \approx B_{\rho^2 L}(x, 1).$$

On the other hand, a definition of a phase-case subunit ball of radius ρ by means of broken paths defined on the interval $[0, \rho]$ is not the right one (see also Fefferman [2]).

In fact, in the case $p_{|Q}$ is elliptic, we expect the subunit ball to have sizes comparable to those of Q. We will see that this is not the case, according to a definition which uses trajectories defined on $[0, \rho]$.

Moreover, we want to have that $\{(y, \eta); \eta = 0\} \cap B_p((x, 0), \rho)$ is essentially $B_p(x, \rho)$ when $p(x, \xi) = \sum a^{jk}(x)\, \xi_j \xi_k$, i.e., in the differential operator case. In that case we can suppose, after a C.Z. localization in the base space (see Fefferman [2, p. 182, Lemma 2] and the following pages)

$$p(x, \xi) = e(x)\, \xi_1^2 + \sum_{n \geqslant j, k \geqslant 2} \tilde{a}^{jk}(x_1, x')\, \xi'_j \xi'_k,$$

i.e., p is in non-degenerate normal form (the factor e is elliptic).

When considering $B_{\rho^2 p}((x^0, 0), 1)$, we perform a C.Z. decomposition of Q in $\mathbf{R}^n \times \mathbf{R}^n$ relative to $\rho^2 p$. For blocks Q_ν for which $Q''_\nu \cap \{\xi = 0\} \neq \emptyset$, it will then be true that $\delta_\nu \sim \rho$, because of non-degeneracy. At this scale, for differential operators, the usual subunit analysis and the pseudodifferential one, will agree, when $\xi = 0$.

Denote by $B_p((x^0, \xi^0), t = \rho)$ the subunit ball defined through broken paths parametrized by $[0, \rho]$. Then consider, for some point $\gamma_0 \in Q$ and $Q \in \mathscr{S}(p, Q)$, the path

$$\gamma(t) = \exp(tH_q)(\gamma_0), \qquad t \in [0, \rho], \quad 0 < \rho \leqslant 1.$$

Then $t = s\rho$ with $s \in [0, 1]$, and we can write

$$\gamma(t) = \sigma(s) = \exp(sH_{\rho q})(\gamma_0).$$

Now, $(\rho q)^2 \leqslant \rho^2 p$ and, on Q that we suppose to be of sizes $1 \times M$,

$$|\partial_x^\alpha \partial_\xi^\beta (\rho q)(x, \xi)| \leqslant C_{\alpha\beta} M^{1-|\beta|}, \qquad |\alpha| + |\beta| \leqslant 2.$$

It follows that $\rho q \in \mathscr{S}(\rho^2 p, Q)$.

Suppose now that $p_{|Q}$, Q of size $1 \times M$, is elliptic, i.e.,

$$\exists c > 0, \qquad p(x, \xi) \geqslant c M^2, \qquad \forall (x, \xi) \in \text{some dilate of } Q.$$

Take $(x^0, \xi^0) \in Q$ and consider a C.Z. decomposition of Q relative to $\rho^2 p_{|Q}$. Since $\rho^2 p(x, \xi) \geqslant c\rho^2 M^2$ on Q, ellipticity will occur on blocks Q_{δ_j}, whose sizes $\delta_j \times M\delta_j$ are such that $1 \geqslant \delta_j \sim \rho^{1/2}$ (see Definition 2.3).

Say that $(x^0, \xi^0) \in Q_\delta$, one of these blocks. It will be seen that

$$B_{\rho^2 p}((x^0, \xi^0), 1) \approx \{ |x - x^0| \lesssim \delta \} \times \{ |\xi - \xi^0| \lesssim M\delta \}$$

while

$$B_p((x^0, \xi^0), t = \rho) \approx \{ |x - x^0| \lesssim \rho \} \times \{ |\xi - \xi^0| \lesssim M\rho \} \subsetneqq B_{Cp}((x^0, \xi^0), \rho).$$

It might then seem that a scaling factor ρ^4, when considering $\rho^4 p$, would be the right one. This is not true, since it would contradict what was said above in the case p is a differential operator.

We conclude the section with the following immediate corollary of the proof of Lemma 3.9:

LEMMA 3.14. *Let* $(x^0, \xi^0) \in Q$ *and let* $\Gamma(t; x^0, \xi^0)$ *be a subunit broken path starting at* (x^0, ξ^0). *Then*

$$\Gamma(t; x^0, \xi^0) \in Q^*, \qquad \forall t \in [0, 1].$$

Remark 3.15. $\Gamma(t, x^0, \xi^0)$ is *Lipschitz-continuous*. This follows from the definition of Γ and the fact that $\forall q \in \mathscr{S}(p, Q)$, Q of size $1 \times M$,

$$M^{-1} |\nabla_x q|, |\nabla_\xi q| \leqslant 1.$$

4. SOME PROPERTIES OF SMOOTH FUNCTIONS

We shall have to use a number of properties of smooth functions and functions defined as solutions to polynomial equations.[4] We will make use of them simply by referring the reader to Parmeggiani [18] for precise

[4] In the following, every constant C, c, $c(n, d)$, c_1, c_2, c_3, c_4, c_5, C_α, is a universal constant.

statements and proofs. For the convenience of the reader we just recall in this section three of the properties and state two fundamental theorems proved by Fefferman and Narasimhan in [11, 12].

The following lemma shows how to construct cut-off functions having "controlled" gradient:

LEMMA 4.1. *Suppose* $0<\delta\leqslant 1$; $c_1, c_2>0$. *There exists* $\psi\in C_0^\infty(\mathbf{R})$ *such that* $\operatorname{supp}\psi\subset(-c_2\delta^{1/4}, c_2\delta^{1/4})$,

$$\psi(x)^2\leqslant c_1^2\delta, \qquad \partial_x\psi(x)\equiv\frac{c_1}{c_2}\delta^{1/4}=c_3\delta^{1/4} \tag{20}$$

for $x\in[-\frac{1}{2}c_2\delta^{1/4}, \frac{1}{2}c_2\delta^{1/4}]$,

$$|\partial_x^\alpha\psi(x)|\leqslant C_\alpha\delta^{(1/2)-(\alpha/4)}, \qquad 0\leqslant\alpha\leqslant 2. \tag{21}$$

We shall need bounds in the following situation: suppose we have functions $F(x,\xi)$, $P(x,\xi)$ such that $F^2\leqslant P$ pointwise. How big can $\partial_x F$ be?

LEMMA 4.2. *Let* Q *be the unit cube in* $\mathbf{R}^{2n}$, *centered at* $(0,0)$. *Let* $F\in C_0^2(\operatorname{int} Q)$, $0\leqslant P\in C(Q)$, *be such that*

$$F(x,\xi)^2\leqslant P(x,\xi), \qquad \forall(x,\xi)\in Q,$$

and

$$|\partial_x^\alpha\partial_\xi^\beta F(x,\xi)|\leqslant C_{\alpha\beta}\leqslant 1, \qquad |\alpha|+|\beta|\leqslant 2.$$

Then, with $Q=I\times I$, I *the unit cube centered at the origin in* $\mathbf{R}^n$, $\forall\xi^0\in I$ *we have*

$$\max_{x\in I}|\nabla_x F(x,\xi^0)|\leqslant C(\max_{x\in I}P(x,0))^{1/4}+|\xi^0|, \tag{22}$$

C *being a universal constant (i.e., also independent of* ξ).

The next lemma is about smooth algebraic functions.

LEMMA 4.3. *Let* $Q=Q_1\times I$ *be the unit cube, centered at the origin, in* $\mathbf{R}^{n+1}$, *with coordinates* $(x,y)\in\mathbf{R}^n\times\mathbf{R}$. *Let* $P(x,y)$ *be a polynomial of a priori bounded degree* d, *with* $|\partial_y P|\geqslant C>0$, $\forall(x,y)\in Q^*$, *and* $\|P\|_{L^\infty(Q^*)}\leqslant C_*$, *for fixed constants* C, $C_*>0$. *Let* $y=f(x)$ *be the solution to* $P(x,y)=0$ *on* Q^*, *with* $f\in C^\infty(\frac{1}{2}Q_1^*)$, $\|f\|_{L^\infty(Q_1)}\leqslant 2$. *Consider, for fixed*

$y \in \mathbf{R}$, the polynomial in $X \in \mathbf{R}$, $P_y(X) = (y - X)^2$, and the associate function $p_y(x) = (y - f(x))^2$. Then

$$\mathrm{Av}_{x \in Q}\, p_y(x) \sim \max_{x \in Q} p_y(x) \tag{23}$$

and

$$\|\partial_x p_y\|_{L^\infty(Q)} \leqslant C\, \|p_y\|_{L^\infty(Q)}, \tag{24}$$

where C and the constants in the equivalence do not depend on y.

Furthermore, if $x_1 \mapsto f(x_1, x_2)$ is a polynomial of a priori bounded degree in x_2, then the same holds true for $(y - (f(x_1, x_2) - (\mathrm{Av}_{|x_1| \leqslant 1} f)(x_2)))^2$.

Suppose $\Gamma = \{(x, f(x)) \in \mathbf{R}^2;\, P(x, f(x)) = 0\}$, P a polynomial (of *a priori* bounded degree) as above. Given another polynomial (of *a priori* bounded degree) $V(x, y)$, we need properties of the above kind for the function $V(x, f(x))$. Looking at the above facts, one might conjecture that $V(x, f(x))$ satisfies a Bernstein's inequality. As proved in the paper of Fefferman and Narasimhan [11], $V(x, f(x))$ does satisfy important inequalities, among which is Bernstein's inequality.

We now state the theorem about $V(x, f(x))$ (see [11]):

THEOREM 4.4. *Let $\Gamma = \{(x, y) \in \mathbf{R}^2;\, y = f(x)$ and $|x| \leqslant 1\}$, where $P(x, f(x)) = 0$ for a polynomial $P(x, y)$. Assume:*

(i) $|f(x)| \leqslant 1$ *for* $|x| \leqslant 1$;

(ii) *$P(x, y)$ has degree at most D*;

(iii) $|P(x, y)| \leqslant C$ *for* $|x|, |y| \leqslant 1$;

(iv) $|\partial_y P(x, y)| \geqslant c > 0$ *for* $(x, y) \in \Gamma$.

Then, with $g(x) = V(x, f(x))$, for a polynomial $V(x, y)$ of degree d:

(a) $$\max_{|x| \leqslant 1} |g(x)| \leqslant C_* \max_{|x| \leqslant 1/2} |g(x)|;$$

(b) $\max_{|x| \leqslant 1} |g'(x)| \leqslant C_* \max_{|x| \leqslant 1} |g(x)|$ *(Bernstein's inequality)*;

(c) $$\max_{|x| \leqslant 1} |g(x)| \leqslant C_* \int_{-1}^{1} |g(x)|\, dx,$$

with C_* depending only on d, D, C, c.

Thus, g behaves like a polynomial of one variable. Note that if $f(x_1, x_2)$ is a smooth algebraic function, polynomial of bounded degree in x_2, and solution to $P(x_1, x_2, f(x_1, x_2)) = 0$ (P satisfying hypotheses like in

Lemma 4.3), and $V(x_1, x_2, y)$ is another polynomial (as above), the same conclusions of Theorem 4.4 hold for $g(x_1, x_2) = V(x_1, x_2, f(x_1, x_2))$ (g is actually a polynomial in x_2) where g' is substituted by ∇g and $\int_{-1}^{1} |g|$ is now the average of g in x_1, x_2.

All this will be crucial when studying the subunit geometry of the symbol

$$p(x_1, x_2, \xi_2) = (\xi_2 - \theta(x_1, x_2))^2 + V(x_1, x_2)$$

(on a C.Z. block Q), where we can suppose θ is a polynomial in x_2 and a smooth algebraic function in x_1, $V(x_1, x_2) = p(x_1, x_2, \theta(x_1, x_2))$, p a polynomial symbol (by this, we mean that, when rescaling matters to the unit cube in $\mathbf{R}^3$, the corresponding θ and p are polynomials of bounded degree and bounded maximum-norms, and algebraic functions, in the corresponding rescaled variables). We shall refer to them as *rescaled polynomials* and *rescaled algebraic functions* or simply as *polynomials* and *algebraic functions* respectively.

Finally we have to know what happens to $(\theta(x_1, x_2) - (\mathrm{Av}_{x_1} \theta)(x_2))^2$ in the case θ is an algebraic function in x_1, x_2, and not a polynomial in x_2. The right quantity to consider in this case is not the "continuous" average in x_1, but rather a discrete average on an *a priori* choice of N points $x_1^1, ..., x_1^N$. This is described below in a theorem of Fefferman and Narasimhan proved in [12].

In order to state the theorem, we need to make some assumptions: we let Q be the unit cube centered at 0 in $\mathbf{R}^n$; let $P_1, ..., P_k$ be polynomials on $\mathbf{R}^n$ with real coefficients $(1 \leqslant k < n)$. Assume the following:

$$\text{(I)} \qquad \deg P_j \leqslant D, \quad \max_Q |P_j| \leqslant C \qquad \text{for} \quad j = 1, ..., k;$$

$$\text{(II)} \quad P_1(0) = \cdots = P_k(0) = 0, \quad \text{and} \quad \left| \det \left(\left(\frac{\partial P_j}{\partial x_i} \right)_{1 \leqslant i,\, j \leqslant k} \right) \right| \geqslant c > 0 \quad \text{at } 0.$$

Set

$$V = \{ x \in \mathbf{R}^n ; P_1(x) = \cdots = P_k(x) = 0 \}$$

and define

$$\pi \colon V \to \mathbf{R}^{n-k},$$

the projection of $(x_1, ..., x_n)$ to $(x_{k+1}, ..., x_n)$. Let F be a polynomial of degree $\leqslant D$ on $\mathbf{R}^n$, and let Q_ρ be the cube of side ρ centered at 0 in $\mathbf{R}^{n-k}$.

THEOREM 4.4′ (Fefferman and Narasimhan [12]). *There are constants ρ_*, $C_* > 0$, depending only on n, C, D, c above, with the following properties:*

(A) *The local inverse* $\pi^{-1}\colon Q_{\rho_*} \to V$ *is well defined and smooth.*

(B) *If* $f(y) = F \circ \pi^{-1}(y)$ *for* $y \in Q_{\rho_*}$, *then we have the estimates (Bernstein's inequalities)*

$$\max_{Q_\rho} |\nabla f| \leqslant \frac{C_*}{\rho} \max_{Q_\rho} |f|,$$

$$\frac{1}{\operatorname{Vol} Q_\rho} \int_{Q_\rho} |f| \geqslant \frac{1}{C_*} \max_{Q_\rho} |f|,$$

$$\max_{Q_{2\rho}} |f| \leqslant C_* \max_{Q_\rho} |f|,$$

valid for $0 < \rho < \frac{1}{2}\rho_*$.

Theorem 4.4′ can be used in the case of discrete averages: let $P(x, y, t)$ be a polynomial of degree $\leqslant D$ in variables $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, $t \in \mathbf{R}$. Assume $|P(x, y, t)| \leqslant C$ and $(\partial P/\partial t)(x, y, t) > c > 0$ on the unit cube $\{|x|, |y|, |t| \leqslant 1\}$.

Assume $\theta(x, y)$ satisfies $|\theta(x, y)| \leqslant 1$, $P(x, y, \theta(x, y)) = 0$ for $|x|, |y| \leqslant 1$. Let now $x \in \mathbf{R}^m$, $y \in \mathbf{R}^n$, $t_0, t_1, ..., t_N \in \mathbf{R}$ be variables, and let $y_1, ..., y_N \in \mathbf{R}^n$ be fixed points with $|y_j| \leqslant \frac{1}{10}$.

Define $P_0(x, y, t_0, ..., t_N) = P(x, y, t_0)$, $P_j(x, y, t_0, ..., t_N) = P(x, y_j, t_j)$ for $1 \leqslant j \leqslant N$.

Then $\det((\partial P_j/\partial t_i)_{0 \leqslant i,\, j \leqslant N}) > c' > 0$ on the unit cube. The common zeros of $P_0, ..., P_N$ in the unit cube are

$$V = \{(x, y, t_0, ..., t_N);\ t_0 = \theta(x, y),\ t_j = \theta(x, y_j) \text{ for } 1 \leqslant j \leqslant N\}.$$

If $\pi\colon (x, y, t_0, ..., t_N) \mapsto (x, y)$ projects V to $\mathbf{R}^{n+m}$, then

$$\pi^{-1}\colon (x, y) \mapsto (x, y, \theta(x, y), \theta(x, y_1), ..., \theta(x, y_N)).$$

Thus, if $F(x, y, t_0, t_1, ..., t_N)$ is a polynomial of degree $\leqslant D$, then the above theorem shows that

$$\begin{aligned} f(x, y; y_1, ..., y_N) &= F \circ \pi^{-1}(x, y; y_1, ..., y_N) \\ &= F(x, y, \theta(x, y), \theta(x, y_1), ..., \theta(x, y_N)) \end{aligned}$$

satisfies Bernstein's inequality with constants depending only on C, c, D, m, n, N. Hence:

COROLLARY. *The function*

$$F(x, y, \theta(x, y), ..., \theta(x, y_N)) = \left(\theta(x, y) - \frac{1}{N} \sum_{j=1}^{N} \theta(x, y_j) \right)^2$$

satisfies Bernstein's inequalities of Theorem 4.4′.

5. DESCRIPTION OF $B_p((X^0, \xi^0), 1)$ FOR A SYMBOL p

5.1. *The Elliptic* $(n+n)$*-Dimensional Case*

We are now in a position to describe the subunit ball for a symbol $p(x, \xi) \in S^2(M)$ satisfying the Main Assumptions (A1) through (A4) of Section 2. We therefore suppose $0 \leqslant p(x, \xi) \in S^2(1 \times M)$, localized to a basic block Q of sizes $1 \times M$, centered at the origin of $\mathbf{R}^n \times \mathbf{R}^n$. Suppose $(x^0, \xi^0) \in Q$. By performing a C.Z. localization, we first consider the case in which the restriction of p to a C.Z. block, at which the cutting procedure stops, is *elliptic* and, calling that block Q_δ, of sizes $\delta \times M\delta$, $(x^0, \xi^0) \in Q_\delta$. Note that, by Remark 2.10, the other case we have to consider is the *nonelliptic–nondegenerate* case, i.e., after a tame canonical transformation, $p(x, \xi)$ can be written as $\xi_1^2 + p_1(x, \xi')$.

Hence we now suppose

$$p_{|Q''_\delta}(x, \xi) \sim (M\delta^2)^2$$

(the equivalence constants being *a priori* constants).

So, consider $(x^0, \xi^0) \in Q_\delta$. Let $\varphi_j(\tilde{x}, \tilde{\xi})$, $j = 1, 2, ..., 2n$, be the functions constructed in [18, Corollary 4.4], $(\tilde{x}, \tilde{\xi}) \in (-2\delta, 2\delta)^{2n}$.

Consider, for fixed $(\bar{x}, \bar{\xi}) \in Q^*_\delta$, the subunit symbols, for $j = 1, 2, ..., 2n$,

$$q_j(x, \xi) = cM\varphi_j\left(x - \bar{x}, \frac{\xi - \bar{\xi}}{M} \right). \tag{25}$$

That the q_j's are subunit symbols follows from the estimates in [18, Corollary 4.4] (c serves to normalize the derivatives of the q_j's).

Hence, for $j = 1, 2, ..., 2n$,

$$q_j(x, \xi)^2 \leqslant c(M\delta^2)^2 \leqslant p_{|Q''_\delta}(x, \xi),$$

$$\partial_{x_j} q_j(x, \xi) \equiv c_3 M\delta, \qquad j = 1, ..., n,$$

$$\partial_{x_i} q_j(x, \xi) \equiv \partial_{\xi_i} q_j(x, \xi) \equiv \partial_{\xi_j} q_j(x, \xi) \equiv 0, \qquad \text{for} \quad i \neq j,$$

and

$$\partial_{\xi_{j-n}} q_j(x, \xi) \equiv c_3 \delta, \qquad j = n+1, ..., 2n,$$
$$\partial_{\xi_{i-n}} q_j(x, \xi) \equiv \partial_{x_{i-n}} q_j(x, \xi) \equiv \partial_{x_{j-n}} q_j(x, \xi) \equiv 0, \qquad \text{for} \quad i \neq j,$$

for

$$(x, \xi) \in Q_\delta(\bar{x}, \bar{\xi}) = \{(x, \xi) \in Q''_\delta;\ |x - \bar{x}| \leqslant \delta,\ |\xi - \bar{\xi}| \leqslant M\delta\},$$

and we have

$$|\partial_x^\alpha \partial_\xi^\beta q_j(x, \xi)| \leqslant (M\delta^2)(M\delta)^{-|\beta|}\, \delta^{-|\alpha|}, \qquad 0 \leqslant |\alpha| + |\beta| \leqslant 2, \quad \forall j.$$

Consider then H_{q_j}, the associated Hamiltonian vector field.

It follows that $\forall (x, \xi) \in Q_\delta(x^0, \xi^0)$ we can find subunit symbols q_j as above so that

$$H_{q_j}(x, \xi) = -c_4 M\, \delta \frac{\partial}{\partial \xi_j}, \qquad j = 1, ..., n,$$

$$H_{q_j}(x, \xi) = c_4\, \delta \frac{\partial}{\partial x_{j-n}}, \qquad j = n+1, ..., 2n,$$

thus allowing us to flow in all the coordinate directions, through broken paths $\Gamma(t; x^0, \xi^0)$ having the above H_{q_1}'s as velocity fields. We can therefore fill in, for $t \sim 1$, a box of the kind

$$\{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x - x^0| \lesssim \delta,\ |\xi - \xi^0| \lesssim M\delta\},$$

whence we conclude

$$B_1 = \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x - x^0| \lesssim \delta,\ |\xi - \xi^0| \lesssim M\delta\} \subset B_p((x^0, \xi^0), 1).$$

We now want to show that the subunit ball is contained in a box B_2 whose sizes are comparable to those of B_1, with center$(B_2) = (x^0, \xi^0)$.

To do that we just note that if $(x(t), \xi(t)) = \Gamma(t; x^0, \xi^0)$, i.e., a subunit broken path starting at (x^0, ξ^0) (see Definition 3.11), applying Lemma 3.9 (actually the corresponding Lemma 3.14 for subunit broken paths) to $\Gamma(t; x^0, \xi^0)$ gives that the best possible displacement along subunit paths is:

$$|x - x^0| \leqslant C\delta, \qquad |\xi - \xi^0| \leqslant CM\delta$$

for a universal constant $C > 0$. Hence

$$B_p((x^0, \xi^0), 1) \subset \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x - x^0| \lesssim \delta,\ |\xi - \xi^0| \lesssim M\delta\} = B_2.$$

We have therefore proved the

THEOREM 5.1. *Suppose* Q_δ *is a* C.Z. *block, of sizes* $\delta \times M\delta$, *on which* $p(x, \xi)$ *is* elliptic. *Suppose* $(x^0, \xi^0) \in Q_\delta$. *Then*

$$B_p((x^0, \xi^0), 1) \approx \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x - x^0| \lessapprox \delta,\ |\xi - \xi^0| \lessapprox M\delta\}.$$

(Note that the choice of λ, the dilation parameter, and of the normalization constants yields $B_p((x^0, \xi^0), 1) \subset Q'_\delta$).

Theorem 5.1 agrees with the definition of B_p, in the case p is elliptic, given in Fefferman [2, pg. 203].

Using Theorem 5.1 we can now complete the argument in Remark 3.13. In fact, we have the following

COROLLARY 5.2. *Same hypotheses as in Theorem* 5.1. *Then, using the notations of Remark* 3.13,[5]

$$\text{(i)} \qquad B_p((x^0, \xi^0), \rho) \approx \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x - x^0| \lessapprox \rho^{1/2}\delta,\ |\xi - \xi^0| \lessapprox \rho^{1/2} M\delta\},$$

$$\text{(ii)} \quad B_p((x^0, \xi^0), t = \rho) \approx \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x - x^0| \lessapprox \rho\delta,\ |\xi - \xi^0| \lessapprox \rho M\delta\}.$$

Proof. Point (ii) follows immediately from the construction in Theorem 5.1, for $t \sim \rho$.

About point (i), we have $p_{|Q_\delta}(x, \xi) \sim (M\delta^2)^2$. Hence to understand $\rho^2 p_{|Q_\delta}$, we localize it to blocks of sizes $\rho^{1/2}\delta \times M\rho^{1/2}\delta$. Call $Q_\delta(\rho)$ the one containing (x^0, ξ^0). Then $\rho^2 p_{|Q_\delta(\rho)}(x, \xi)$ is elliptic, since its order of magnitude is $(M(\rho^{1/2}\delta)^2)^2$.

Repeating on $Q_\delta(\rho)$ the construction of Theorem 5.1 yields

$$B_{\rho^2 p}((x^0, \xi^0), 1) \approx \{(x, \xi);\ |x - x^0| \lessapprox \rho^{1/2}\delta,\ |\xi - \xi^0| \lessapprox M\rho^{1/2}\delta\},$$

thus proving the corollary and the conclusion of Remark 3.13. ∎

5.2. *The Nonelliptic–Nondegenerate* (1 + 1)*-Dimensional Case*

We now describe the subunit ball in the nonelliptic–nondegenerate case. We shall obtain the description in three steps:

[5] Recall that $B^1 \approx B^2$, for blocks B^1, B^2, when there exist constants $C_1, C_2 > 0$, independent of the sizes of B^1 and B^2, such that $B^1_{C_1} \subset B^2 \subset B^1_{C_2}$, B^1_C being the dilate of B^1 by the constant $C > 0$.

(1) First we will study B_p for $p(x, \xi) = \xi^2 + M^2 f(x)$, where $0 \leqslant f$ is a polynomial of *a priori* bounded degree d.

(2) Next, we will examine $B_{\rho^2 p}$ for $p(x, \xi) = \xi^2 + M^2 f(x)$. Now $0 \leqslant f \in S^2(Q_v)$. Choosing ρ suitably small, as specified in (A2v) (Section 2), we will be able to Taylor expand $f(x)$ (see Consequence 1, Section 2), reducing matters henceforth to case (1) above.

(3) Finally, we have the general case:

$$p(x, \xi) = e(x, \xi)(\xi - \theta(x))^2 + M^2 f(x) \sim (\xi - \theta(x))^2 + M^2 f(x).$$

We reduce this to case (2) above, through Φ, the tame canonical transformation of Lemma 2.8, and through Lemma 3.12. We shall have that (see (ii) in Theorem 5.5 below)

$$B_{\rho^2 p}((x^0, \xi^0), 1) \approx \Phi(B_{\rho^2(\eta^2 + M^2 f(y))}(\Phi^{-1}(x^0, \xi^0), 1))$$

where $\Phi(y, \eta) = (x, \xi) = (y, \eta + \theta(y))$. By $\approx$ we mean that, denoting

$$B = \Phi(B_{\rho^2(\eta^2 + M^2 f(y))}(\Phi^{-1}(x^0, \xi^0), 1)),$$

and by B_C the box B dilated by the positive constant C, we have that there exist universal constants $C_1, C_2 > 0$ such that

$$B_{C_1} \subset B_{\rho^2 p}((x^0, \xi^0), 1) \subset B_{C_2}.$$

We therefore suppose Q is a block of sizes $1 \times M$ centered at the origin in $\mathbf{R} \times \mathbf{R}$, and, on Q'', $p(x, \xi) = \xi^2 + M^2 f(x)$.

So, suppose for now, $0 \leqslant f$, a polynomial of *a priori* bounded degree d (depending on the subellipticity exponent) on Q'.

THEOREM 5.3. *Let* $(x^0, \xi^0) \in Q$ *and let* $0 \leqslant p$ *satisfy assumptions* (A1) *through* (A4). *Suppose, on* Q', $p(x, \xi) = \xi^2 + M^2 f(x)$ (*a nonelliptic–non-degenerate normal form*), *where* $0 \leqslant f$ *is a polynomial of a priori bounded degree d. Define* $\sigma(f) := \mathrm{Av}_{|x| \leqslant 1} f$. *We can suppose* $\sigma(f) \leqslant 1$. *Then*

$$B_p((x^0, \xi^0), 1) \approx \{x \in \mathbf{R}; |x - x^0| \lessapprox 1\} \times \{\xi \in \mathbf{R}; |\xi - \xi^0| \lessapprox |\xi^0| + M\sigma(f)^{1/4}\}.$$

Proof. We shall prove that $B_1 \subset B_p \subset B_2$, where B_1, B_2 are boxes of comparable size, centered at (x^0, ξ^0). We start with the inclusion $B_1 \subset B_p$.

Take $\chi(x, \xi)$, $\chi \in C_0^\infty(\mathbf{R} \times \mathbf{R})$, $0 \leqslant \chi \leqslant 1$,

$$\operatorname{supp} \chi \subset \{(x, \xi); |x - x^0| \leqslant 2, |\xi - \xi^0| \leqslant 2M\},$$
$$\chi \equiv 1 \text{ on } \{|x - x^0| \leqslant 1, |\xi - \xi^0| \leqslant M\} := Q(x^0, \xi^0).$$

Then $\forall\alpha, \beta$,

$$|\partial_x^\alpha \partial_\xi^\beta \chi(x, \xi)| \leqslant C_{\alpha\beta} M^{-|\beta|}$$

($C_{\alpha\beta}$ being *a priori* constants).

Let

$$q_1(x, \xi) = c\xi\chi(x, \xi).$$

Then $q_1 \in \mathscr{S}(p, \frac{3}{2}Q)$, for an *a priori* suitable choice of $c > 0$. (More precisely, $q_1 \in \mathscr{S}(p, Q(x^0, \xi^0))$. Note that sizes$(Q(x^0, \xi^0)) \sim$ sizes(Q).)

In fact,

$$q_1(x, \xi)^2 \leqslant c^2 \xi^2 \chi(x, \xi)^2 \leqslant \xi^2$$

and

$$\begin{aligned}
|\partial_x^2 q_1(x, \xi)| &= c\, |\xi\, \partial_x^2 \chi(x, \xi)| \leqslant M \qquad \text{(choice of } c\text{)},\\
|\partial_\xi^2 q_1(x, \xi)| &= c\, |\partial_\xi \chi(x, \xi) + \xi\, \partial_\xi^2 \chi(x, \xi)|\\
&\leqslant c(M^{-1} + MM^{-2} C_{02}) \leqslant M^{-1} \qquad \text{(choice of } c\text{)},\\
|\partial_{x\xi}^2 q_1(x, \xi)| &\leqslant c(|\partial_x \chi(x, \xi)| + |\xi|\, |\partial_{x\xi}^2 \chi(x, \xi)|)\\
&\leqslant c(1 + MC_{11} M^{-1}) \leqslant 1 \qquad \text{(choice of } c\text{)}.
\end{aligned}$$

We can therefore consider, for $(x, \xi) \in Q(x^0, \xi^0)$, the subunit vector field

$$H_{q_1}(x, \xi) = \partial_\xi q_1(x, \xi) \frac{\partial}{\partial x} - \partial_x q_1(x, \xi) \frac{\partial}{\partial \xi} \sim \frac{\partial}{\partial x}.$$

The same construction clearly holds true on blocks $Q(x^1, \xi^1)$, $\forall (x^1, \xi^1) \in Q(x^0, \xi^0)$. Hence, $\forall (x^1, \xi^1) \in Q(x^0, \xi^0)$, for $t_0 \sim 1$,

$$[-t_0 + x^1, x^1 + t_0] \times \{\xi^1\} \subset B_p((x^0, \xi^0), 1).$$

We now exploit the contribution of ξ^0.

Let $\chi \in C_0^\infty(\mathbf{R} \times \mathbf{R})$, $0 \leqslant \chi \leqslant 1$,

$$\chi(x, \xi) = \begin{cases} 1, & \operatorname{dist}((x, \xi), (x^0, \xi^0)) \leqslant \dfrac{1}{2} \dfrac{|\xi^0|}{M} \\ 0, & \operatorname{dist}((x, \xi), (x^0, \xi^0)) \geqslant \dfrac{2}{3} \dfrac{|\xi^0|}{M} \end{cases} \tag{26}$$

(we recall that $\operatorname{dist}((x, \xi), (x^0, \xi^0)) = \max\{|x - x^0|, M^{-1}|\xi - \xi^0|\}$). Then

$$|\partial_x^\alpha \partial_\xi^\beta \chi(x, \xi)| \leqslant C_{\alpha\beta} \left(\frac{|\xi^0|}{M}\right)^{-|\alpha|} |\xi^0|^{-|\beta|}.$$

Consider

$$q_2(x, \xi) = c\,|\xi^0|\,(x - x^0)\,\chi(x, \xi).$$

Then $\operatorname{supp} q_2 \subset Q(x^0, \xi^0)$. We have

$$q_1(x, \xi)^2 \leqslant c^2\,|\xi^0|^2 \frac{|\xi^0|^2}{M^2}\,\chi(x, \xi)^2 \leqslant \frac{1}{9}\,|\xi^0|^2\,\chi(x, \xi)^2 \leqslant \xi^2,$$

since, on $\operatorname{supp} \chi$,

$$|\xi| \geqslant |\xi^0| - |\xi - \xi^0| \geqslant \tfrac{1}{3}|\xi^0|.$$

About the derivatives of q_2:

$$\begin{aligned}
|\partial_x^2 q_2(x, \xi)| &= c\,|\xi^0|\,|\partial_x \chi(x, \xi) + (x - x^0)\,\partial_x^2 \chi(x, \xi)| \\
&\leqslant c\,|\xi^0| \left(\frac{M}{|\xi^0|} + \frac{|\xi^0|}{M}\,\frac{M^2}{|\xi^0|^2}\right) \leqslant M; \\
|\partial_\xi^2 q_2(x, \xi)| &= c\,|\xi^0|\,|(x - x^0)\,\partial_\xi^2 \chi(x, \xi)| \\
&\leqslant c\,|\xi^0|\,\frac{|\xi^0|}{M}\,|\xi^0|^{-2} \leqslant M^{-1}; \\
|\partial_{x\xi}^2 q_2(x, \xi)| &= c\,|\xi^0|\,|\partial_\xi \chi(x, \xi) + (x - x^0)\,\partial_{x\xi}^2 \chi(x, \xi)| \\
&\leqslant c\,|\xi^0| \left(|\xi^0|^{-1} + \frac{|\xi^0|}{M}\,\frac{M}{|\xi^0|^2}\right) \leqslant 1.
\end{aligned}$$

Hence q_2 is subordinate to p.

Consider

$$H_{q_2}(x, \xi) = \partial_\xi q_2(x, \xi)\,\frac{\partial}{\partial x} - \partial_x q_2(x, \xi)\,\frac{\partial}{\partial \xi} \sim -c\,|\xi^0|\,\frac{\partial}{\partial \xi}$$

in the middle half of $\operatorname{supp} \chi$. (Note that $|\xi^0| \lesssim M$, and, by normalization, $c\,|\xi^0| \leqslant M$).

Now, by means of $\gamma_1(t; x^0, \xi^1) = \exp(tH_{q_1})(x^0, \xi^1)$, $|t| \leqslant |t_0| \sim 1$, we reach a point $\bar{x}$ of maximum for f:

$$f(\bar{x}) = \max_{|t| \leqslant |t_0|} f(\gamma_1^1(t)) \sim \operatorname{Av}_{|x| \leqslant 1} f = \sigma(f) \sim \operatorname{Av}_{x \in I} f, \qquad \forall I,\ |I| \sim 1,$$

since f is a polynomial on Q'. Moreover, f being a polynomial, there exists $I \subset [x^0-1, x^0+1]$, $\bar{x} \in I$, $|I| \sim 1$, such that

$$\min_{x \in I} f(x) \geqslant \tfrac{1}{2}\sigma(f).$$

Hence $f(x) \sim \sigma(f)$, $\forall x \in I$. Using Lemma 4.1, we can then construct $\varphi \in C_0^\infty(\mathbf{R})$, supp $\varphi \subset I$, diam(supp $\varphi) \sim \sigma(f)^{1/4}$ $(\leqslant 1)$ with

$$\varphi(x)^2 \leqslant c\sigma(f), \qquad \partial_x \varphi(x) \equiv c\sigma(f)^{1/4} \ \forall x \in \text{middle half of supp } \varphi,$$

and

$$|\partial_x^\alpha \varphi(x)| \leqslant C_\alpha \sigma(f)^{(1/2)-(\alpha/4)}, \qquad 0 \leqslant \alpha \leqslant 2.$$

Take $\psi \in C_0^\infty(\mathbf{R})$, $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $\{|x| \leqslant 1\}$, supp $\psi \subset \{|x| \leqslant 2\}$. We can then construct $q_3 \in \mathscr{S}(p, Q(x^0, \xi^0))$.

For a suitable *a priori* constant $c>0$,

$$q_3(x, \xi) = cM\varphi(x)\, \psi\left(\frac{\xi-\xi^0}{M}\right).$$

We have

$$q_3(x, \xi)^2 \leqslant c^2 M^2 \sigma(f) \leqslant p(x, \xi)$$

on supp q_3,

$$|\partial_x^2 q_3(x, \xi)| = cM\psi\left(\frac{\xi-\xi^0}{M}\right) |\partial_x^2 \varphi(x)| \leqslant cM\sigma(f)^{(1/2)-(1/2)} \leqslant M;$$

$$|\partial_\xi^2 q_3(x, \xi)| = cM\, |\varphi(x)| \left| \partial_\xi^2 \left(\psi\left(\frac{\xi-\xi^0}{M}\right)\right)\right| \leqslant cC_2 M\sigma(f)^{1/2}\, M^{-2} \leqslant M^{-1};$$

$$|\partial_{x\xi}^2 q_3(x, \xi)| = cM\, |\partial_x \varphi(x)| \left| \partial_\xi \left(\psi\left(\frac{\xi-\xi^0}{M}\right)\right)\right|$$
$$\leqslant cMC_1 C_2 \sigma(f)^{(1/2)-(1/4)}\, M^{-1} \leqslant 1.$$

Then, we consider

$$H_{q_3}(x, \xi) = \partial_\xi q_3(x, \xi) \frac{\partial}{\partial x} - \partial_x q_3(x, \xi) \frac{\partial}{\partial \xi} \sim -M\sigma(f)^{1/4} \frac{\partial}{\partial \xi}$$

$\forall (x, \xi) \in \tfrac{1}{2}$supp $\varphi \times \{|\xi-\xi^0| \leqslant M\}$. Let

$$\gamma_3(t; \gamma(t_1)) = \exp(tH_{q_3})(\gamma(t_1)), \qquad t_1 \sim 1.$$

By flowing along $\gamma_1(\pm t)$, $\gamma_3(\pm t)$, for $t \sim 1$, we can fill in the region

$$\{(x, \xi) \in \mathbf{R} \times \mathbf{R};\ |x - x^0| \lessapprox 1,\ |\xi - \xi^0| \lessapprox M\sigma(f)^{1/4}\}.$$

Let $\gamma_2(t; x^1, \xi^0) = \exp(tH_{q_2})(x^1, \xi^0)$. Thus, flow along $\gamma_2(\pm t; x^1, \xi^0)$ and $\gamma_1(\pm t; x^1, \xi^1)$, $|x^1 - x^0| \leqslant 1$, $|\xi^1 - \xi^0| \leqslant \frac{1}{3}|\xi^0|$, up to time $t_1 \sim 1$, to fill in the region

$$R_1 = \{|x - x^0| \lessapprox 1\} \times \{|\xi - \xi^0| \lessapprox |\xi^0|\}.$$

Then, $\forall (x^1, \xi^1) \in R_1$, use $\gamma_1(\pm t; x^1, \xi^1)$, $\gamma_3(\pm t; \bar{x}, \xi^1)$ to fill in, for $t \sim 1$,

$$B_1 = \{(x, \xi) \in \mathbf{R} \times \mathbf{R};\ |x - x^0| \lessapprox 1,\ |\xi - \xi^0| \lessapprox |\xi^0| + M\sigma(f)^{1/4}\}.$$

Hence

$$B_1 \subset B_p((x^0, \xi^0), 1).$$

To have the other inclusion, we first note that

$$p(x, \xi) \lessapprox \xi^2 + M^2\sigma(f) = \tilde{p}(x, \xi) \qquad \text{on } Q.$$

Then, by Lemma 3.12, it suffices to prove the inclusion $B_{\tilde{p}} \subset B_2$. To estimate the best displacement along subunit broken paths we need the following

Lemma 5.4. *Let $\Gamma(t; x^0, \xi^0)$ be a subunit broken path, relative to a block $Q \subset \mathbf{R}^n \times \mathbf{R}^n$, starting at $(x^0, \xi^0) \in Q$, for $t \in [0, 1]$. Denote*

$$(x(t), \xi(t)) = (\Gamma_1(t; x^0, \xi^0), \Gamma_2(t; x^0, \xi^0)) = \Gamma(t; x^0, \xi^0).$$

Suppose that $\exists \sigma > 0$ such that, $\forall q \in \mathscr{S}(p, Q)$,

$$|\partial_x q(\Gamma(t; x^0, \xi^0))| \leqslant M\sigma + |\Gamma_2(t; x^0, \xi^0) - \Gamma_2(0; x^0, \xi^0)|. \tag{27}$$

Then

$$|\Gamma_2(1; x^0, \xi^0) - \Gamma_2(0; x^0, \xi^0)| \leqslant eM\sigma.$$

Proof. $t \mapsto \Gamma(t; x^0, \xi^0)$ is an absolutely continuous function for $t \in [0, 1]$. By definition of the subunit broken path, there exists a partition $0 = t_0 < t_1 < \cdots < t_L = 1$ of $[0, 1]$, and subunit paths γ_k, $k = 0, 1, ..., L-1$, satisfying:

$$\begin{cases} \dot{\gamma}_k(t) = H_{q_k}(\gamma_k(t)), & t \in [t_k, t_{k+1}] \\ \gamma_k(t_k) = (x_k, \xi_k) = \gamma_{k-1}(t_k), \end{cases}$$

$q_k \in \mathscr{S}(p, Q)$, $k = 0, 1, ..., L-1$, and

$$\Gamma(t; x^0, \xi^0) \equiv \gamma_k(t), \qquad \forall t \in [t_k, t_{k+1}].$$

We then have, for $I_i = (t_i, t_{i+1})$, $t \in \bigcup_{i=0}^{L-1} I_i$

$$\dot{\Gamma}_2(t; x^0, \xi^0) = -\sum_{i=0}^{L-1} \partial_x q_i(\Gamma(t; x^0, \xi^0))\, \chi_{I_i}(t),$$

where χ_{I_i} is the characteristic function of the interval I_i (recall that $\partial_x q_i \in C_0^1$). Then

$$\begin{aligned}
&|\Gamma_2(t; x^0, \xi^0) - \Gamma_2(0; x^0, \xi^0)| \\
&\quad = \left| \int_0^t \dot{\Gamma}_2(\tau; x^0, \xi^0)\, d\tau \right| \leqslant \int_0^t |\dot{\Gamma}_2(\tau; x^0, \xi^0)|\, d\tau \\
&\quad \leqslant M\sigma \int_0^t \sum_{i=0}^{L-1} \chi_{I_i}(\tau)\, d\tau + \int_0^t \sum_{i=0}^{L-1} \chi_{I_i}(\tau)\, |\Gamma_2(\tau; x^0, \xi^0) - \xi^0|\, d\tau \\
&\quad \leqslant M\sigma t + \int_0^t |\Gamma_2(\tau; x^0, \xi^0) - \xi^0|\, d\tau.
\end{aligned}$$

From Gronwall's inequality it follows then that

$$|\Gamma_2(t; x^0, \xi^0) - \xi^0| \leqslant M\sigma t e^t,$$

whence

$$|\Gamma_2(1; x^0, \xi^0) - \xi^0| \leqslant eM\sigma. \quad \blacksquare$$

In our case, estimate (27) follows from Lemma 4.2. In fact, for any $q \in \mathscr{S}(p, Q)$,

$$|\partial_x q(x, \xi^0)| \leqslant C(|\xi^0| + M\sigma(f)^{1/4}),$$

whence

$$|\Gamma_2(1; x^0, \xi^0) - \xi^0| \leqslant C(|\xi^0| + M\sigma(f)^{1/4}).$$

Since for any subunit symbol we have

$$|\partial_\xi q(x, \xi)| \leqslant 1,$$

we also have that

$$|\Gamma_1(1; x^0, \xi^0) - x^0| \leqslant 1.$$

Hence

$$B_p((x^0, \xi^0), 1) \subset \{(x, \xi) \in \mathbf{R} \times \mathbf{R};\, |x - x^0| \leqslant 1,$$
$$|\xi - \xi^0| \leqslant C(|\xi^0| + M\sigma(f)^{1/4})\}. \quad \blacksquare$$

Having Theorem 5.1 and Theorem 5.3, we may now pass to the general $(1+1)$-dimensional, nonelliptic–nondegenerate case. Hence let $\tilde{Q}_v$ be a C.Z. block of sizes $1 \times M_v$, centered at $(0, 0)$, on which (actually, as always, on $\tilde{Q}_v'''$)

$$p(z, \zeta) = e(z, \zeta)(\zeta - \theta(z))^2 + M_v^2 \tilde{V}(z).$$

As explained at the end of Section 2, since $c \leqslant e(z, \zeta) \leqslant C$ and $p(z, \zeta) \leqslant AM_v^2$, we have

$$p(z, \zeta) \sim (\zeta - \theta(z))^2 + M_v^2 V(z),$$

with $|\partial_z^2 V| \leqslant 1$, $|V| \leqslant 1$.

Using $\Phi^{-1}(z, \zeta) = (y, \eta) = (z, \zeta - \theta(z))$, the tame canonical transformation of Lemma 2.8, we can consider $(p \circ \Phi)(y, \eta) = \eta^2 + M_v^2 V(y)$ on Q_v, centered at $(0, 0)$, such that $\Phi\colon Q_v'' \to \tilde{Q}_v'''$ and $\Phi(y^0, \eta^0) = (z^0, \zeta^0)$, the center of our ball.

Now $M_v^2 V \in S^2(Q_v)$, $V \geqslant 0$. In order to be able to Taylor expand, we consider $\rho^2(p \circ \Phi)$. We shall hence state a theorem about the subunit ball of radius ρ.

Recall from Section 2 that $M_v^{-\varepsilon_1} < \rho < M_v^{-\varepsilon_0}$, so that $\rho^2(p \circ \Phi)$ still satisfies (A1) through (A4).

We are therefore in the following situation: Q_v is of size $1 \times M_v$, centered at the origin in $\mathbf{R} \times \mathbf{R}$, and $\rho^2(p \circ \Phi)(y, \eta) = \rho^2 \eta^2 + M_v^2 \rho^2 V(y)$.

As already explained in Section 2, we perform a further C.Z. cutting procedure in Q_v, in order to understand $\rho^2(p \circ \Phi)$. Then

$$Q_v = \bigcup_\mu Q_{\mu v}, \qquad Q_{\mu v} \text{ of sizes } \delta_\mu \times M_v \delta_\mu.$$

On each $Q_{\mu v}$, $\rho^2(p \circ \Phi)$ will be either elliptic or nonelliptic–nondegenerate.

Also, $1 \gtrsim \delta_\mu \gtrsim \rho$ (as shown in Section 2). Then $(y^0, \eta^0) \in Q \in \{Q_{\mu v}\}$. If $(y^0, \eta^0) \in Q$ on which $\rho^2(p \circ \Phi)$ is elliptic, we apply Theorem 5.1.

Suppose instead $\rho^2(p \circ \Phi)$ is nonelliptic–nondegenerate. It follows form Fact 2 (Section 2) that $\delta := \operatorname{diam}_x Q \sim \rho$ then, and $\bar{\eta} = \pi_\xi(\text{center}(Q))$ is such that

$$|\bar{\eta}| \lesssim M_v \rho \qquad \text{or} \qquad |\bar{\eta}| \sim M_v \rho. \tag{28}$$

We can then apply Consequence 1 (Section 2) to conclude that we may Taylor expand $V(y)$ (on $2Q'$). Let $\tilde{f}(y)$ be its Taylor polynomial of (*a priori*) bounded degree d. Note that now

$$M_v^2 \rho^2 V(y) \lessapprox M_v^2 \rho^4, \qquad y \in \pi_y(Q'),$$

hence

$$V(y) \lessapprox \rho^2, \qquad y \in \pi_y(Q'). \tag{29}$$

Note that we also have

$$\max_{y \in \pi_y(Q')} V(y) \sim \max_{y \in \pi_y(Q')} \tilde{f}(y). \tag{30}$$

Hence on Q (containing (y^0, η^0)) we have:

$$\rho^2(p \circ \Phi)(y, \eta) \sim \rho^2\eta^2 + M_v^2 \rho^2 \tilde{f}(y).$$

By (28) we can use the symplectic scaling, with $\bar{y} = \pi_y(\text{center}(Q))$,

$$\psi : (y, \eta) \mapsto (x, \xi), \qquad \xi = \rho\eta, \qquad x = \frac{y - \bar{y}}{\rho}.$$

Let $\tilde{Q} = \psi(Q)$. $\tilde{Q}$ is then a block of sizes $1 \times M_v \rho^2$,

$$\psi(\bar{y}, \bar{\eta}) = (0, \rho\bar{\eta} = \bar{\xi}) = \text{center}(\tilde{Q}).$$

Call $f(x)$ the polynomial $(1/\rho^2)\, \tilde{f}(\bar{y} + \rho x)$. Then, on $\tilde{Q}'$,

$$\rho^2(p \circ \Phi \circ \psi^{-1})(x, \xi) \sim \xi^2 + M_v^2 \rho^4 f(x),$$

where f is a non-negative polynomial of *a priori* bounded degree d (note that $f \in S^0(1 \times M_v \rho^2)$) and $\sigma(f) = \text{Av}_{x \in \pi_x(\tilde{Q}^{**})} f \leqslant 1$.

Theorem 5.3 gives then (taking care of the fact that now $|\xi^0| \lessapprox M_v \rho^2$, so we have to consider the function χ in (26) defined now by means of $|\xi^0|/CM_v \rho^2$, C being a universal constant such that $|\xi^0| \leqslant CM_v \rho^2$)

$$\begin{aligned} &B_{\rho^2(p \circ \Phi \circ \psi^{-1})}((x^0, \xi^0), 1) \\ &\qquad \approx \{(x, \xi) \in \mathbf{R} \times \mathbf{R};\, |x - x^0| \lessapprox 1,\, |\xi - \xi^0| \lessapprox |\xi^0| + M_v \rho^2 \sigma(f)^{1/4}\}. \end{aligned}$$

Here $(x^0, \xi^0) = \psi(y^0, \eta^0)$.

Therefore we get

$$\begin{aligned}
&B_{p\circ\Phi}((y^0,\eta^0),\rho)\\
&\qquad\approx \psi(B_{p\circ\Phi\circ\psi^{-1}}((x^0,\xi^0),\rho))\\
&\qquad\approx \{(y,\eta)\in \mathbf{R}\times\mathbf{R};\ |y-y^0|\lessapprox\rho,\ |\eta-\eta^0|\lessapprox|\eta^0|+M_\nu\rho\sigma(f)^{1/4}\}.
\end{aligned}$$

We have hence proved

THEOREM 5.5. *Let $p(x,\xi)$, satisfying* (A1) *through* (A4), *be in the form*

$$p(x,\xi)=e(x,\xi)(\xi-\theta(x))^2+M_\nu^2\tilde{V}(x)\sim(\xi-\theta(x))^2+M_\nu^2V(x)$$

(*almost in the sense specified above*) *on a* C.Z. *block Q_ν centered at* $(0,0)\in\mathbf{R}\times\mathbf{R}$, *of sizes $1\times M_\nu$. Let $(x^0,\xi^0)\in Q_\nu$. Then*:

(i) *If $\Phi^{-1}(x^0,\xi^0)=(y^0,\eta^0)\in Q$, an ellipticity* C.Z. *block of sizes $\delta\times M_\nu\delta$ for $\rho^2(p\circ\Phi)(y,\eta)\sim\rho^2(\eta^2+M_\nu^2V(y))$, $1\gtrapprox\delta\gtrapprox\rho$, we have*

$$B_p((x^0,\xi^0),\rho)\approx\Phi(\{(y,\eta)\in\mathbf{R}\times\mathbf{R};\ |y-y^0|\lessapprox\delta,\ |\eta-\eta^0|\lessapprox M_\nu\delta\}).$$

(ii) *If $(y^0,\eta^0)\in Q$, a nonellipticity–nondegeneracy* C.Z. *block of sizes $\sim\rho\times M_\nu\rho$ for $\rho^2(p\circ\Phi)(y,\eta)\sim\rho^2(\eta^2+M_\nu^2V(y))$, we have*

$$\begin{aligned}
&B_p((x^0,\xi^0),\rho)\\
&\qquad\approx\Phi(\{(y,\eta)\in\mathbf{R}\times\mathbf{R};\ |y-y^0|\lessapprox\rho,\ |\eta-\eta^0|\lessapprox|\eta^0|+M_\nu\rho\sigma(f)^{1/4}\})\\
&\qquad=\{(x,\xi);\ |x-x^0|\lessapprox 1,\ |\xi-G(x)|\lessapprox|\xi^0-\theta(x^0)|+M_\nu\rho\sigma(f)^{1/4}\},
\end{aligned}$$

where $\sigma(f):=$ "size" of the $(1/\rho^2)$ d-Taylor polynomial of V defined above, and $G(x):=\theta(x)-\theta(x^0)+\xi^0$.

Remark 5.6. Equation (30) implies max $V\sim\sigma(\tilde{f})=\rho^2\sigma(f)$, hence

$$M_\nu\rho\sigma(f)^{1/4}=M_\nu\rho^{1/2}\sigma(\tilde{f})^{1/4}\sim M_\nu\rho^{1/2}(\max_{y\in\pi_y(Q')}V)^{1/4}.$$

This is a natural order of magnitude (recall (29)).

In fact, suppose that, on Q_ν as above ($Q_\nu\subset\mathbf{R}\times\mathbf{R}$),

$$p(x,\xi)=\xi^2+M_\nu^2\delta,\qquad\text{where}\quad 0<\delta\ll 1,\quad \delta\leqslant\rho^2$$

(but not "too" small).

Consider $\rho^2 p(x, \xi) = \rho^2\xi^2 + M_\nu^2 \rho^2\delta$ and, as above, suppose

$$(x^0, \xi^0) \in Q, \qquad \pi_\xi(\text{center}(Q)) = \bar{\xi}, \quad |\bar{\xi}| \sim M_\nu \rho \quad \text{or} \quad |\bar{\xi}| \lessapprox M_\nu \rho,$$

Q the nonellipticity–nondegeneracy C.Z. block for $\rho^2 p$.

Then we can directly construct subunit symbols *subordinate* to $M_\nu^2 \rho^2\delta$, having "strength" $((\rho^2\delta)^{1/4}, M_\nu(\rho^2\delta)^{1/4}$:

$$(\rho^2\delta)^{1/4} \frac{\partial}{\partial x} \qquad \text{and} \qquad M_\nu(\rho^2\delta)^{1/4} \frac{\partial}{\partial \xi}.$$

Note that $(\rho^2\delta)^{1/4} \leqslant \rho$, so that we have the right order of magnitude associated with size$(Q) \sim \rho \times M_\nu \rho$. Since $\rho^2\xi^2$ allows us to consider the subunit vector field

$$\rho \frac{\partial}{\partial x},$$

we conclude that (noting that $|\xi^0| \leqslant CM_\nu \rho$)

$$B_p((x^0, \xi^0), \rho) \approx \{(x, \xi); |x - x^0| \leqslant \rho, |\xi - \xi^0| \leqslant |\xi^0| + M_\nu \rho^{1/2}\delta^{1/4}\}.$$

But $M_\nu \rho^{1/2}\delta^{1/4} = M_\nu \rho(\delta/\rho^2)$.

Here $\delta/\rho^2 \leqslant 1$ and δ plays the role of V (or $\tilde{f}$), and δ/ρ^2 that of f.

Note that a subunit symbol for $\rho^2 \, |\xi^0|^2$ is:

$$q(x, \xi) = \rho \, |\xi^0| \left(\frac{x - x^0}{\rho}\right) \chi(x, \xi)$$

($\chi \in C_0^\infty$ is the function (26), with M replaced by M_ν, and $|\xi^0|/M$ by $|\xi^0|/CM_\nu$. Note that $|\xi^0|/CM_\nu \leqslant \rho$).

We have $|x - x^0| \leqslant \rho$ on supp χ and

$$H_q(x, \xi) \sim c \, |\xi^0| \frac{\partial}{\partial \xi}$$

where $\chi \equiv 1$. We shall again use this construction in the next subsections.

5.3. *The* $(2+2)$-*Dimensional, Nonelliptic–Nondegenerate Case*

First of all, we show that Remark 5.6 may be generalized in $n + n$ dimensions to the following

Proposition 5.7. *Suppose $p(x, \xi)$ has the form $p(x, \xi) = \xi_1^2 + M^2\delta$ on a C.Z. block Q centered at $(0, 0) \in \mathbf{R}^n \times \mathbf{R}^n$, of sizes $1 \times M$. Suppose $p_{|Q}$ is nonelliptic–nondegenerate. Set $(x, \xi) = (x_1, x', \xi_1, \xi') \in \mathbf{R} \times \mathbf{R}^{n-1} \times \mathbf{R} \times \mathbf{R}^{n-1}$. Then, for $(x^0, \xi^0) \in Q$,*

$$B_p((x^0, \xi^0), 1) \approx \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x_1 - x_1^0| \leqslant 1,$$
$$M\,|x' - x^{0\prime}| + |\xi - \xi^0| \leqslant |\xi_1^0| + M\delta^{1/4}\}.$$

Proof (Part 1). We follow the proof of Theorem 5.5 and Remark 5.6. We construct, using an $(n+n)$-dimensional analogue of the function (26) (with $|\xi^0|/M$ now replaced by $|\xi_1^0|/M$), subunit symbols (c is always a positive universal constant $\leqslant 1$)

$$q_i(x, \xi) = c\,|\xi_1^0|\,(x_i - x_i^0)\,\chi(x, \xi)$$
$$q_{i+n}(x, \xi) = c\,\frac{|\xi_1^0|}{M}\,(\xi_i - \xi_i^0)\,\chi(x, \xi),$$

$i = 1, 2, ..., n$, giving rise to the vector fields

$$H_{q_i}(x, \xi) \sim -c\,|\xi_1^0|\,\frac{\partial}{\partial \xi_i}$$
$$H_{q_{i+n}}(x, \xi) \sim c\,\frac{|\xi_1^0|}{M}\,\frac{\partial}{\partial x_i},$$

$i = 1, 2, ..., n$, on the region on which $\chi \equiv 1$. Using ξ_1^2 we get also the usual subunit symbol

$$q_0(x, \xi) = c\xi_1 \chi_0(x, \xi), \tag{31}$$

where $\chi_0 \in C_0^\infty$, $0 \leqslant \chi_0 \leqslant 1$, $\chi_0 \equiv 1$ for $\operatorname{dist}((x, \xi), (x^0, \xi^0)) \leqslant 1$, 0 for $\operatorname{dist}((x, \xi), (x^0, \xi^0)) \geqslant 2$, and the associated vector field

$$H_{q_0}(x, \xi) \sim \frac{\partial}{\partial x_1}$$

on the region on which $\chi_0 \equiv 1$. Using [18, Corollary 4.4], we can consider the subunit symbols (25)

$$\tilde{q}_i(x, \xi) = cM\varphi_i\left(x - \bar{x}, \frac{\xi - \bar{\xi}}{M}\right),$$

for $(\bar{x}, \bar{\xi}) \in \{(x, \xi);\ |x - x^0| \leqslant 1,\ |\xi - \xi^0| \leqslant M\}$, $i = 1, 2, ..., 2n$. These are subunits for the "potential part" of p: $p_{|\xi_1 = 0} = M^2\delta$.

Thus we have also the vector fields:

$$H_{\tilde{q}_i}(x, \xi) \sim -cM\delta^{1/4} \frac{\partial}{\partial \xi_i}, \qquad i=1, ..., n,$$

$$H_{\tilde{q}_i}(x, \xi) \sim c\delta^{1/4} \frac{\partial}{\partial x_{x_{i-n}}}, \qquad i=n+1, ..., 2n,$$

when $|x-\bar{x}| \leqslant \delta^{1/4}$, $|\xi-\bar{\xi}| \leqslant M\delta^{1/4}$.

From this, we conclude that

$$\{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\, |x_1-x_1^0| \lessapprox 1,\, M\,|x'-x^{0\prime}|+|\xi-\xi^0| \lessapprox |\xi_1^0|+M\delta^{1/4}\}$$
$$\subset B_p((x^0, \xi^0), 1).$$

(Part 2) We prove now the other inclusion.

Clearly, the best displacement at time 1 for x_1 is $|x_1-x_1^0| \leqslant 1$. Consider $q \in \mathscr{S}(p, Q)$. From Lemma 3.7 it follows that

$$q(x, \xi)=q_1(x, \xi)+q_2(x, \xi)$$

with $cq_1 \in \mathscr{S}(\xi_1^2, Q^1)$, $cq_2 \in \mathscr{S}(p_{|\xi_1=0}, Q^1)$, where $Q \subset Q^1 \subset Q^{1**}=Q^{***}$, center$(Q^1)=$center$(Q)$.

For cq_1 we have

$$|\nabla_x cq_1(x, \xi)| \lessapprox |\xi_1^0|+|\xi_1-\xi_1^0|, \tag{32}$$

$$|\nabla_{\xi'} cq_1(x, \xi)| \lessapprox \frac{|\xi_1^0|+|\xi_1-\xi_1^0|}{M}, \tag{33}$$

in fact, $q_1(x, 0, \xi') \equiv 0$ so that $\nabla_x q_1(x, 0, \xi') \equiv 0$ and $\nabla_{\xi'} q_1(x, 0, \xi') \equiv 0$. Therefore (32) and (33) follow.

Now consider a C.Z. decomposition of Q (i.e. Q''') relative to $p_{|\xi_1=0}$. Q is then cut up into subblocks Q_ν with sizes $\delta_\nu \times M\delta_\nu$. Since $p_{|\xi_1=0} \equiv M^2\delta$, $\forall \nu$, $\delta_\nu \sim \delta^{1/4}$ then.

Let $\{q_k\}_{k=0, ..., L-1}$ be subunit symbols giving rise to Γ, a subunit broken path starting at (x^0, ξ^0). Then $q_k=q_{1k}+q_{2k}$ as above, and (32) and (33) still hold for all the q_{1k}.

For q_{2k} we have:

$$q_{2k}(x, \xi)=\sum_{\nu=1}^{N} q_{2k\nu}(x, \xi),$$

where $q_{2k\nu} \in \mathscr{S}(p_{|\xi_1=0}, Q_\nu)$, so that, by [18, Lemma 4.1],

$$|\partial_x q_{2k}(\Gamma(t; x^0, \xi^0))| \\ \lessapprox |\xi_1^0| + \max_\nu \max_{|t| \leqslant 1} |\partial_x q_{2k\nu}(\Gamma_1(t; x^0, \xi^0), 0, \xi^{0\prime})| + |\Gamma_2(t; x^0, \xi^0) - \xi^0|, \tag{34}$$

and, for $i \geqslant 2$,

$$|\partial_{\xi_i} q_{2k}(\Gamma(t; x^0, \xi^0))| \\ \lessapprox \frac{|\xi_1^0|}{M} + \max_\nu \max_{|t| \leqslant 1} |\partial_{\xi_i} q_{2k\nu}(\Gamma_1(t; x^0, \xi^0), 0, \xi^{0\prime})| + \frac{|\Gamma_2(t; x^0, \xi^0) - \xi^0|}{M}. \tag{35}$$

Estimate (27) now reads

$$|\partial_x q_{2k}(\Gamma(t; x^0, \xi^0))| \lessapprox (|\xi_1^0| + M\delta^{1/4}) + |\Gamma_2(t; x^0, \xi^0) - \xi^0|,$$
$$M\, |\partial_{\xi_i} q_{2k}(\Gamma(t; x^0, \xi^0))| \lessapprox (|\xi_1^0| + M\delta^{1/4}) + |\Gamma_2(t; x^0, \xi^0) - \xi^0|$$

($i \geqslant 2$). Using also estimates (32) and (33), we conclude, for $t \neq t_0, t_1, ..., t_L$, as in Lemma 5.4, that

$$M\, |\dot{\Gamma}_1'(t; x^0, \xi^0)| + |\dot{\Gamma}_2(t; x^0, \xi^0)| \\ \lessapprox (|\xi_1^0| + M\delta^{1/4}) + M\, |\Gamma_1'(t; x^0, \xi^0) - x^{0\prime}| + |\Gamma_2(t; x^0, \xi^0) - \xi^0|.$$

By Gronwall's inequality, it follows that

$$|\Gamma_1'(t; x^0, \xi^0) - x^{0\prime}| \lessapprox \frac{|\xi_1^0|}{M} + \delta^{1/4},$$
$$|\Gamma_2(t; x^0, \xi^0) - \xi^0| \lessapprox |\xi_1^0| + M\delta^{1/4}.$$

Thus

$$B_p((x^0, \xi^0), 1) \subset \{(x, \xi) \in \mathbf{R}^n \times \mathbf{R}^n;\ |x_1 - x_1^0| \lessapprox 1, \\ M\, |x' - x^{0\prime}| + |\xi - \xi^0| \lessapprox |\xi_1^0| + M\delta^{1/4}\}. \quad \blacksquare$$

We finally consider the $(2+2)$-dimensional case. Hence let $Q \subset \mathbf{R}^2 \times \mathbf{R}^2$ be a C.Z. block of sizes $1 \times M$, centered at $(0, 0)$, such that on (a large dilate of) Q the symbol $p \geqslant 0$ (satisfying the assumptions of Section 2) has (after the tame canonical transformation Φ of Lemma 2.8) the form

$$p(x, \xi) = \xi_1^2 + p_1(x_1, x_2, \xi_2), \qquad p_1 \in S^2(1 \times M), \quad p_1 \geqslant 0.$$

Recall that, in view of our normalizations (see Section 2), we have

$$|\partial_{x_1}^{\alpha}\,\partial_{x_2}^{\beta}\,\partial_{\xi_2}^{\gamma}\,p_1(x_1, x_2, \xi_2)| \leqslant M^{2-\gamma}, \qquad \alpha+\beta+\gamma=4.$$

Let $(x^0, \xi^0) \cong (x_1^0, \xi_1^0, x_2^0, \xi_2^0)$ be the center of our ball, and ρ (satisfying the above hypotheses; see Section 2) its radius.

Given a block $Q \subset \mathbf{R}^2 \times \mathbf{R}^2$, we shall occasionally write it as $Q = Q^1 \times Q^2$, with $Q^1 = \pi_{(x_1, \xi_1)}(Q)$, $Q^2 = \pi_{(x_2, \xi_2)}(Q)$.

Now let $I_\rho = [x_1^0 - c_0\,\rho, x_1^0 + c_0\,\rho]$, where $0 < c_0 < \frac{1}{4}$ is an *a priori* fixed constant (note that $|I_\rho| = 2$ times the best displacement given by $\rho^2 \xi_1^2$ at time c_0) and let

$$\bar{p}_\rho(x_2, \xi_2) = \frac{1}{2c_0\,\rho} \int_{x_1 \in I_\rho} p_1(x_1, x_2, \xi_2)\, dx_1.$$

Note that, by assumption on ρ, $p_{1\,|\,x_1 \in I_\rho}$ may be Taylor expanded in x_1 in such a way that (as in Section 2) we can suppose $x_1 \mapsto p_1(x_1, \cdot, \cdot)$, a non-negative polynomial of *a priori* bounded degree, still satisfying all our assumptions (possibly replacing the universal constants with other universal constants).

Moreover, since for a non-negative polynomial the average is equivalent to the maximum, and since $\rho^2 \bar{p}_\rho$ satisfies a (s.e.) condition,

$$\text{(CZ1)(iii) does not occur in the C.Z. decomposition relative to } \bar{p}_\rho(x_2, \xi_2). \tag{36}$$

Now apply a C.Z. decomposition of Q^2 associated with $\bar{p}_\rho$ (note that we now have the further freedom of *a priori* choosing the dilation factor λ_1, relative to $p_1, \bar{p}_\rho$).

Hence let Q_ν^2, of sizes $\delta_\nu \times M\delta_\nu$, be one of these blocks. Thus, $\bar{p}_\rho \in S^2(Q_\nu^2)$. Since p_1 is supposed to be a polynomial in x_1, we also have

$$p_1(x_1, \cdot, \cdot) \in S^2(Q_\nu^2) \tag{37}$$

with bounds uniform in $x_1 \in I_\rho$.

In fact,

$$p_1(x_1, \cdot, \cdot) \leqslant C(M\delta_\nu^2)^2, \qquad \text{on} \quad I_\rho \times Q_\nu^2,$$

and $p_1 \in S^2(1 \times M)$ implies

$$|\partial_{x_2}^{\alpha}\,\partial_{\xi_2}^{\beta}\,p_1(x_1, \cdot, \cdot)| \leqslant C_{0,\alpha,\beta}(M\delta_\nu^2)^2\,\delta_\nu^{-\alpha}(M\delta_\nu)^{-\beta} \qquad \text{on} \quad I_\rho \times Q_\nu^2$$

for $\alpha + \beta \geqslant 4$.

By interpolation, the remaining estimates ($1 \leqslant \alpha + \beta \leqslant 3$) follow. Since p_1 is a polynomial in x_1, we have

$$|\partial^{\alpha}_{x_1} \partial^{\gamma}_{x_2} \partial^{\beta}_{\xi_2} p_1(x_1, x_2, \xi_2)| \leqslant C_{\alpha\beta\gamma} \rho^{-\alpha} (M\delta^2_\nu)^2 \delta_\nu^{-\gamma} (M\delta_\nu)^{-\beta} \tag{38}$$

$\forall (x_1, x_2, \xi_2) \in I_\rho \times (Q^2_\nu)'''$, $\forall \alpha, \beta, \gamma$ (and $C_{\alpha\beta\gamma} \leqslant 1$, for $\alpha + \beta + \gamma = 4$). (Note that, p_1 being a polynomial in x_1, (38) holds also for $x_1 \in [x^0_1 - \rho, x^0_1 + \rho]$). By (36), $\bar{p}_{\rho \mid Q^2_\nu}$ is either elliptic or nonelliptic–nondegenerate. We shall refer to these cases as Case 1 and Case 2, respectively. Suppose we are in Case 2. Let $(x^0_2, \xi^0_2) \in Q^2_{\nu_0}$. We can suppose

$$\partial^2_{\xi_2} \bar{p}_\rho(x_2, \xi_2) \geqslant c\delta^2_{\nu_0}, \qquad \text{on} \quad Q^2_{\nu_0},$$

c being a large positive constant. (See the assertion after Lemma 3.3 in Fefferman and Phong [4].)

Because p_1 is a polynomial in x_1, it follows that there exists $J_\rho \subset I_\rho$, such that

$$|J_\rho| \sim |I_\rho| \qquad \text{and} \qquad \partial^2_{\xi_2} p_1(x_1, x^0_2, \xi^0_2) \geqslant c\delta^2_{\nu_0} \qquad \forall x_1 \in J_\rho. \tag{39}$$

By (38), it follows that

$$c\delta^2_{\nu_0} \leqslant \partial^2_{\xi_2} p_1(x_1, x_2, \xi_2) \leqslant C\delta^2_{\nu_0} \tag{40}$$

$\forall (x_1, x_2, \xi_2) \in R_0 = J_\rho \times \{|x_2 - x^0_2| < c'\delta_{\nu_0}\} \times \{|\xi_2 - \xi^0_2| < c'M\delta_{\nu_0}\} := J_\rho \times Q(x^0_2, \xi^0_2, \delta_{\nu_0})$. (We remark that it cannot be either $\partial^2_{x_1} p_1(x, \xi_2) \geqslant CM^2\delta^4_{\nu_0}\rho^{-2}$ or $|\partial_{x_1} \partial^{\alpha}_{x_2} \partial^{\beta}_{\xi_2} p_1(x, \xi_2)| \geqslant CM^{2-\beta}\delta^{4-(\alpha+\beta)}_{\nu_0}\rho^{-1}$, $\alpha + \beta = 1$, for otherwise the same assertion after Lemma 3.3 in [4] would imply the ellipticity of $\bar{p}_\rho$.) As in Lemma 2.5, the Implicit Function Theorem yields

$$\partial_{\xi_2} p_1(x_1, x_2, \xi_2) = 0 \qquad \text{for} \quad (x_1, x_2, \xi_2) \in R_0 \Leftrightarrow \xi_2 = \theta(x_1, x_2) + \xi^0_2$$

with, $\forall \alpha, \beta$,

$$|\partial^{\alpha}_{x_1} \partial^{\beta}_{x_2} \theta(x_1, x_2)| \leqslant C_{\alpha\beta} \rho^{-\alpha} M\delta_{\nu_0} \delta^{-\beta}_{\nu_0}. \tag{41}$$

Since p_1 is a polynomial in x_1, $x_1 \mapsto \theta(x_1, \cdot)$ is an algebraic function for any fixed x_2. Hence we have

LEMMA 5.8. *There exists a region $R_0 \subset \mathbf{R} \times \mathbf{R} \times \mathbf{R}$, $(x^0_2, \xi^0_2) \in Q(x^0_2, \xi^0_2, \delta_{\nu_0}) = \pi_{(x_2, \xi_2)}(R_0)$, of sizes $\rho \times \delta_{\nu_0} \times M\delta_{\nu_0}$ such that, on R_0,*

$$p_1(x_1, x_2, \xi_2) = \delta^2_{\nu_0} e(x, \xi_2)(\xi_2 - \xi^0_2 - \theta(x_1, x_2))^2 + \widetilde{V}(x_1, x_2),$$

with $c \leqslant e \leqslant C$ *satisfying*, $\forall \alpha, \beta, \gamma$, *the estimates*

$$|\partial_{x_1}^{\alpha} \partial_{x_2}^{\beta} \partial_{\xi_2}^{\gamma} e(x, \xi_2)| \leqslant C_{\alpha\beta\gamma} \rho^{-\alpha} \delta_{\nu_0}^{-\beta} (M\delta_{\nu_0})^{-\gamma};$$

θ *satisfying estimates* (41), *and* $\tilde{V}$ *satisfying*, $\forall \alpha, \beta$, *the estimates*

$$|\partial_{x_1}^{\alpha} \partial_{x_2}^{\beta} \tilde{V}(x_1, x_2)| \leqslant C_{\alpha\beta} \rho^{-\alpha} (M\delta_{\nu_0}^2)^2 \delta_{\nu_0}^{-\beta}.$$

Remark 5.8′. Let I_ρ^1 be the interval in $x_1[x_1^0 - \rho, x_1^0 + \rho]$. Then $I_\rho \subset I_\rho^1$ and $|I_\rho^1| \sim |I_\rho|$. Let

$$\bar{p}_\rho^1(x_2, \xi_2) := (\mathrm{Av}_{x_1 \in I_\rho^1} p_1)(x_2, \xi_2).$$

Then $\bar{p}_\rho$ and $\bar{p}_\rho^1$ have equivalent behavior.

In fact, let $\{\tilde{Q}_\nu^2\}$ be a C.Z. decomposition of Q relative to $\bar{p}_\rho^1$ (same parameters A and λ of the C.Z. decomposition relative to $\bar{p}_\rho$). Denote by δ_1 the x_2-size of the $\tilde{Q}_\nu^2$ containing (x_2^0, ξ_2^0), and by δ the x_2-size of the Q_ν^2 containing (x_2^0, ξ_2^0). Since $p_1(x_1, \cdot, \cdot)$ is a polynomial of *a priori* bounded degree, it follows that $\bar{p}_\rho^1 \sim \bar{p}_\rho$ and also that $\bar{p}_\rho^1$ satisfies (s.e.). $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}}$ can be either elliptic or nonelliptic–nondegenerate, and analogously for $\bar{p}_{\rho \mid Q^2_\delta}$.

Suppose $\bar{p}_{\rho \mid Q^2_\delta}$ is elliptic, then $\bar{p}_\rho^1(x_2^0, \xi_2^0) \sim M^2\delta^4$. On the other hand, $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}} \lessapprox M^2\delta_1^4$, hence $\delta \lessapprox \delta_1$.

If $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}}$ is elliptic, it follows that $\delta \sim \delta_1$ (i.e., $\tilde{Q}^2_{\delta_1} = Q^2_\delta$).

If $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}}$ is nonelliptic–nondegenerate, then $\delta \ll \delta_1$ and $Q^2_\delta \subset \tilde{Q}^2_{\delta_1}$ and $\partial^2_{x_2} \bar{p}_\rho^1(x_2^0, \xi_2^0) \sim M^2\delta_1^2$ or $\partial^2_{\xi_2} \bar{p}_\rho^1(x_2^0, \xi_2^0) \sim \delta_1^2$. On the other hand, $0 \leqslant \bar{p}^1_{\rho \mid Q^2_\delta} \lessapprox M^2\delta^2$, $\bar{p}_\rho^1 \in S^2(1 \times M) \Rightarrow \bar{p}^1_{\rho \mid Q^2_\delta} \in S^2(\delta \times M\delta) \Rightarrow$, for $\alpha + \beta = 2$,

$$|\partial_{x_2}^{\alpha} \partial_{\xi_2}^{\beta} \bar{p}_\rho^1(x_2^0, \xi_2^0)| \lessapprox (M\delta^2)^2 \delta^{-\alpha} (M\delta)^{-\beta} = M^{2-\beta}\delta^2 \ll M^{2-\beta}\delta_1^2,$$

contradicting the nondegeneracy. Hence $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}}$ must be elliptic. In particular, $\delta \sim \delta_1$.

Suppose now $\bar{p}_{\rho \mid Q^2_\delta}$ is nonelliptic–nondegenerate. If $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}}$ were elliptic, by a reasoning similar to the one above, we would contradict the nondegeneracy of $\bar{p}_{\rho \mid Q^2_\delta}$. Hence also $\bar{p}^1_{\rho \mid \tilde{Q}^2_{\delta_1}}$ is nonelliptic–nondegenerate and $\delta_1 \sim \delta$. In particular, we must have at least one of the estimates

$$\partial^2_{x_2} \bar{p}_\rho^1(x_2^0, \xi_2^0) \sim M^2\delta^2, \qquad \partial^2_{\xi_2} \bar{p}_\rho^1(x_2^0, \xi_2^0) \sim \delta^2. \quad \blacksquare$$

Consider now

$$\rho^2 p(x, \xi) = \rho^2 \xi_1^2 + \rho^2 p_1(x_1, x_2, \xi_2)$$

and

$$\rho^2 \bar{p}_\rho(x_2, \xi_2).$$

We make a C.Z. decomposition relative to $\rho^2 \bar{p}_\rho$ of Q^2. Let $\hat{Q}^2_{\mu_0}$ be the C.Z. block containing (x_2^0, ξ_2^0). Since also $\rho^2 \bar{p}_\rho(x_2, \xi_2)$ satisfies (s.e.), either $\rho^2 \bar{p}_{\rho \mid \hat{Q}^2_{\mu_0}}$ is elliptic (Case 2.1), or it is nonelliptic–nondegenerate (Case 2.2).

It is clear from (40) and (38) that $\hat{Q}^2_{\mu_0}$ will have sizes $\rho\delta_{\nu_0} \times M\rho\delta_{\nu_0}$.

Moreover, since $0 < \rho < 1$, $\rho \not\sim 1$, it follows that an *a priori* large dilate of $\hat{Q}^2_{\mu_0}$ is completely contained in $Q(x_2^0, \xi_2^0, \delta_{\nu_0})$.

Remark 5.8″. Suppose $\rho^2 \bar{p}_{\rho \mid \hat{Q}^2_{\rho\delta}}$ is nonelliptic–nondegenerate because of $\partial^2_{\xi_2} \bar{p}_\rho(x_2^0, \xi_2^0) \sim \delta^2$. (See the notations above. We denoted $Q^2_{\nu_0}$ by Q^2_δ, and $\hat{Q}^2_{\mu_0}$ by $\hat{Q}^2_{\rho\delta}$.) Then also $\rho^2 \bar{p}^1_{\rho \mid \hat{Q}^2_{\rho\delta}}$ is nonelliptic–nondegenerate with $\partial^2_{\xi_2} \bar{p}^1_\rho(x_2^0, \xi_2^0) \sim \delta^2$.

This is trivial in case $x_1 \mapsto \partial^2_{\xi_2} p_1(x_1, x_2^0, \xi_2^0)$ is a non-negative polynomial. Otherwise, there must be $\bar{x}_1 \in I^1_\rho$ at which $\partial^2_{\xi_2} p_1(\bar{x}_1, x_2^0, \xi_2^0) \sim -\delta^2$. Estimate (38) (still valid with I^1_ρ replacing I_σ) and the assertion after Lemma 3.3 in [3] would then imply the existence of a region R_0 of size $\rho \times \delta \times M\delta$, $(\bar{x}_1, x_2^0, \xi_2^0) \in R_0$, on which $p_1(x_1, x_2, \xi_2) \sim (M\delta^2)^2$. Hence $\rho^2 \bar{p}^1_{\rho \mid \hat{Q}^2_{\rho\delta}}(x_2, \xi_2)$ would then be elliptic and the same would hold for $\rho^2 \bar{p}_{\rho \mid \hat{Q}^2_{\rho\delta}}$.

From Remark 5.8′ and Remark 5.8″ it follows that it is no restriction to consider the above I_ρ for the *a priori* choice of c_0. c_0 is chosen so that we can move x_1, in the construction of the subunit ball, to fill in a full-dimensional region contained in the ball.

On $J_\rho \times \hat{Q}^2_{\mu_0}$ (i.e., on $J_\rho \times (\hat{Q}^2_{\mu_0})''''$), by estimates (41), θ can be Taylor expanded in x_2. We summarize all of this in the following

LEMMA 5.9. *Under the above hypotheses and Case* 2, *Case* 2.2,

$$\theta(x_1, x_2)_{\mid J_\rho \times \hat{Q}^2_{\mu_0}}$$

is essentially a polynomial in x_2, *algebraic function in* x_1. (*By this, we mean that we can replace* θ *by an a priori suitable high-degree Taylor polynomial of* θ *making an error which can be absorbed by using assumption A*4 *of Section* 2.)

Next, we make a C.Z. decomposition of Q relative to $\rho^2 p$. We suppose $(x^0, \xi^0) \in Q_\rho$, a nonellipticity–nondegeneracy C.Z. block, so that, as we have already seen, size$(Q_\rho) \sim \rho \times M\rho$ and $\pi_{\xi_1}(\text{center}(Q_\rho)) = \bar{\xi}_1$ is such that either $|\bar{\xi}_1| \sim M\rho$ or $|\bar{\xi}_1| \lesssim M\rho$.

Remark than on Q_ρ we can suppose p_1 a polynomial of *a priori* bounded degree in (x_1, x_2, ξ_2).

We have the following proposition:

PROPOSITION 5.10. *Suppose $(x^0, \xi^0) \in Q_\rho$ and $\rho^2 p_{|Q_\rho}$ is nonelliptic–nondegenerate. In Case 1 we have:*

$$B_p((x^0, \xi^0), \rho) \approx \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2;\ |x - x^0| \lessapprox \rho,\ |\xi - \xi^0| \lessapprox M\rho\}.$$

In Case 2.1 consider the derived symbol

$$p^*_\rho(x_2, \xi_2) = \left(\frac{|\xi_1^0|}{C_0 M}\right)^4 M^2 + \rho^2 \bar{p}_\rho(x_2, \xi_2). \tag{42}$$

Then there exists a block $\bar{Q}^2$, containing (x_2^0, ξ_2^0), on which $p^*_{\rho\,|\bar{Q}^2}$ is elliptic of size $\sim M^2\delta^4$. We have then

$$B_p((x^0, \xi^0), \rho) \approx \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2;\ |x_1 - x_1^0| \lessapprox \rho,$$
$$M\,|x_2 - x_2^0| + |\xi - \xi^0| \lessapprox M\Delta\},$$

where

(i) $\Delta = |\xi_1^0|\, M^{-1}$ *in case*

$$\delta \sim \frac{|\xi_1^0|}{C_0 M} \gg \rho\delta_{\nu_0} \qquad \text{or} \qquad \delta \sim \frac{|\xi_1^0|}{C_0 M} \sim \rho\delta_{\nu_0};$$

(ii) $\Delta = \rho\delta_{\nu_0} M^{-1}$ *in case*

$$\frac{|\xi_1^0|}{C_0 M} \ll \rho\delta_{\nu_0}.$$

(Here C_0 is a positive universal constant such that $|\xi_1^0|/(C_0 M) \leqslant \rho$.)

Proof (*Case* 1). If $\bar{p}_\rho(x_2, \xi_2)$ is elliptic on Q_ν^2, containing (x_2^0, ξ_2^0), then $\bar{p}_\rho(x_2, \xi_2) \sim (M\delta_\nu^2)^2$.

We localize $\rho^2 \bar{p}_\rho(x_2, \xi_2)$ to subblocks $Q^2_{\mu\nu}$ of Q^2_ν, on which

$$\rho^2 \bar{p}_\rho(x_2, \xi_2) \sim (M(\rho^{1/2}\delta_\nu)^2)^2,$$

with sizes of $Q^2_{\mu\nu} \sim \rho^{1/2}\delta_\nu \times M\rho^{1/2}\delta_\nu$. Let $Q^2_{\mu_0\nu}$ be the one containing (x_2^0, ξ_2^0).

Hence $\rho^2 \bar{p}_{\rho\,|Q^2_{\mu\nu}}$ is elliptic on $Q^2_{\mu\nu}$.

p_1 being a polynomial in x_1, it follows that

$$\exists \bar{x}_1 \in I_\rho \qquad \text{such that} \quad \rho^2 p_1(\bar{x}_1, x_2^0, \xi_2^0) \geqslant c(M(\rho^{1/2}\delta_\nu)^2)^2.$$

On the other hand, we suppose $\rho^2 p(x,\xi)$ is nonelliptic–nondegenerate at $(x^0,\xi^0)\in Q_\rho$, a C.Z. block for $\rho^2 p$. Since $|\bar{x}_1 - x_1^0|\leqslant \rho$, it follows that

$$c(M\rho\delta_\nu^2)^2 \leqslant \rho^2 p(\bar{x}_1, x_2^0, \xi_1^0, \xi_2^0)\leqslant C(M\rho^2)^2,$$

i.e., $\rho\delta_\nu^2 \lessapprox \rho^2$, i.e., $\delta_\nu \lessapprox \rho^{1/2}$.

Hence $\pi_{(x_2,\xi_2)}(Q_\rho)\approx Q^2_{\mu_0\nu}$ (since both contain (x_2^0,ξ_2^0) and sizes $(Q_\rho)\sim$ sizes$(Q^2_{\mu_0\nu})\sim \rho\times M\rho$).

Since at this scale $\rho^2 p_1$ is a polynomial in (x_1, x_2, ξ_2), we apply the Fact in Section 4 to conclude that $\exists I^1_\rho \subset I_\rho$, $|I^1_\rho|\sim I_\rho$, $\exists \tilde{Q}^2_\rho\subset \pi_{(x_2,\xi_2)}(Q_\rho)$ of size $\rho\times M\rho$, such that $\rho^2 p_{1\,|\,I^1_\rho\times\tilde{Q}^2_\rho}\sim M^2\rho^4$.

Hence

$$\rho^2 p(x,\xi)\sim M^2\rho^4$$

$\forall (x,\xi)\in I^1_\rho\times \pi_{x_2}(\tilde{Q}^2_\rho)\times\pi_{\xi_1}(Q_\rho)\times\pi_{\xi_2}(\tilde{Q}^2_\rho) := \tilde{R}_\rho$, and $\rho^2 p(x,\xi)\lessapprox M^2\rho^4$, $\forall (x,\xi)\in Q_p$. Using the subunit vector field $\rho\partial/\partial x_1$ (arising from $\rho^2\xi_1^2$), which allows us to move from (x^0,ξ^0) to the region $\tilde{R}_\rho$, we apply the methods of Part 1 and Part 2 of the Proof of Proposition 5.7 to conclude that (note that $|\xi_1^0|\lessapprox M\rho$, so that $|\xi_1^0| + M\rho\sim M\rho$)

$$B := \{(x,\xi)\in \mathbf{R}^2\times\mathbf{R}^2;\ |x-x^0|\lessapprox\rho,\ |\xi-\xi^0|\lessapprox M\rho\}\approx B_p((x^0,\xi^0),\rho).$$

(Note that in this case $B_p\subset B$ is a trivial consequence of the estimates on subunit symbols at scale $\rho\times M\rho$.)

We now pass to Case 2.1. In this case we consider the derived symbol

$$p_p^*(x_2,\xi_2) := \left(\frac{|\xi_1^0|}{C_0 M}\right)^4 M^2 + \rho^2\bar{p}_\rho(x_2,\xi_2).$$

(Here C_0 is a universal constant such that $|\xi_1^0|\leqslant C_0 M\rho$. We then have $|\xi_1^0|/(C_0 M\rho)\leqslant 1$.) Note that $p_\rho^*\in S^2(\rho\times M\rho)$.

We know that $\rho^2\bar{p}_{\rho\,|\,Q^2_{\mu_0}}$ is elliptic $\sim (M\rho^2\delta^2_{\nu_0})^2$.

Consider a C.Z. decomposition relative to p_ρ^* (note that $p_\rho^*(x_2,\xi_2)$ satisfies (s.e.), since $\rho^2\bar{p}_\rho$ does). The procedure will stop at Q^2_δ containing (x_2^0,ξ_2^0), either because $(|\xi_1^0|/C_0 M\rho)^4\,(M\rho^2)^2$ is elliptic or because $\rho^2\bar{p}_\rho$ is elliptic or because of both conditions. This corresponds respectively to:

$$\text{(i)}\ \ \frac{|\xi_1^0|}{C_0 M}\sim\delta\gg\rho\delta_{\nu_0},\qquad \text{(ii)}\ \ \frac{|\xi_1^0|}{C_0 M}\ll\rho\delta_{\nu_0}\sim\delta,\qquad \text{(iii)}\ \ \frac{|\xi_1^0|}{C_0 M}\sim\rho\delta_{\nu_0}\sim\delta.$$

For (i), we consider the subunit symbols

$$q_i(x,\xi)=c\,|\xi_1^0|\,(x_i-x_i^0)\,\chi(x,\xi), \tag{43}$$

$$q_{i+n}(x,\xi)=c\,\frac{|\xi_1^0|}{M}\,(\xi_i-\xi_i^0)\,\chi(x,\xi), \tag{44}$$

$i=1,2,\ldots,n$, where $\chi\in C_0^\infty$, $0\leqslant\chi\leqslant 1$, $\chi\equiv 1$ when $\max\{|x-x^0|, M^{-1}|\xi-\xi^0|\}\leqslant\frac{1}{2}(|\xi_1^0|/C_0M)$, $\chi\equiv 0$ when $\max\{|x-x^0|, M^{-1}|\xi-\xi^0|\}\geqslant\frac{2}{3}(|\xi_1^0|/C_0M)$.

Let us check the estimates for q_i and q_{i+n}:

$$q_i(x,\xi)^2\leqslant c^2\rho^2\,|\xi_1^0|^2\,\frac{|\xi_1^0|^2}{C_0^2M^2\rho^2}\,\chi(x,\xi)^2\leqslant\rho^2\xi_1^2$$

on supp χ, and

$$q_{i+n}(x,\xi)^2\leqslant c^2\,\frac{|\xi_1^0|^2}{C_0^2M^2}\,|\xi_1^0|^2\,\chi(x,\xi)^2\leqslant c^2\rho^2\,|\xi_1^0|^2\,\chi(x,\xi)^2\leqslant\rho^2\xi_1^2$$

on supp χ.

For $|\alpha|=2$,

$$\begin{aligned}|\partial_x^\alpha q_i(x,\xi)|&\leqslant c\,|\xi_1^0|\,(|\partial_x\chi(x,\xi)|+|x_i-x_i^0|\,|\partial_x^\alpha\chi(x,\xi)|)\\&\lessapprox c\,|\xi_1^0|\left(\frac{M}{|\xi_1^0|}+\frac{|\xi_1^0|}{M}\,\frac{M^2}{|\xi_1^0|^2}\right)\lessapprox M;\end{aligned}$$

$$|\partial_x^\alpha q_{i+n}(x,\xi)|\leqslant c\,\frac{|\xi_1^0|}{M}\,|\xi_i-\xi_i^0|\,|\partial_x^\alpha\chi(x,\xi)|\lessapprox c\,\frac{|\xi_1^0|^2}{M}\,\frac{M^2}{|\xi_1^0|^2}\lessapprox M.$$

For $|\beta|=2$,

$$|\partial_\xi^\beta q_i(x,\xi)|\leqslant c\,|\xi_1^0|\,|x_i-x_i^0|\,|\partial_\xi^\beta\chi(x,\xi)|\lessapprox c\,\frac{|\xi_1^0|^2}{C_0M}\,|\xi_1^0|^{-2}\lessapprox\frac{1}{M};$$

$$\begin{aligned}|\partial_\xi^\beta q_{i+n}(x,\xi)|&\leqslant c\,\frac{|\xi_1^0|}{M}\,(|\xi_i-\xi_i^0|\,|\partial_\xi^\beta\chi(x,\xi)|+|\partial_\xi\chi(x,\xi)|)\\&\lessapprox c\,\frac{|\xi_1^0|}{M}\,(|\xi_1^0|\,|\xi_1^0|^{-2}+|\xi_1^0|^{-1})\sim\frac{1}{M}.\end{aligned}$$

For $|\alpha|=|\beta|=1$,

$$|\partial_x^\alpha \partial_\xi^\beta q_i(x,\xi)| \leqslant c\,\frac{|\xi_1^0|}{M}\,(|\partial_\xi^\beta \chi(x,\xi)| + |x_i - x_i^0|\,|\partial_x^\alpha \partial_\xi^\beta \chi(x,\xi)|)$$

$$\leqslant c\,\frac{|\xi_1^0|}{M}\left(|\xi_1^0|^{-1} + \frac{|\xi_1^0|}{M}\,\frac{M}{|\xi_1^0|^2}\right) \sim 1;$$

$$|\partial_x^\alpha \partial_\xi^\beta q_{i+n}(x,\xi)| \leqslant c\,\frac{|\xi_1^0|}{M}\,(|\partial_x^\alpha \chi(x,\xi)| + |\xi_i - \xi_i^0|\,|\partial_x^\alpha \partial_\xi^\beta \chi(x,\xi)|)$$

$$\lessapprox c\,\frac{|\xi_1^0|}{M}\left(M\,|\xi_1^0|^{-1} + |\xi_1^0|\,\frac{M}{|\xi_1^0|^2}\right) \sim 1.$$

Hence $q_i, q_{i+n} \in \mathscr{S}(\rho^2 p, Q_\rho)$, for $1 \leqslant i \leqslant n$.

In particular, the best displacement given by subunit symbols belonging to

$$\mathscr{S}\left(\left(\frac{|\xi_1^0|}{C_0 M\rho}\right)^4 (M\rho^2)^2, Q_\rho\right)$$

coincides with the displacement given by the q_i, q_{i+n}, $i = 1, 2, ..., n$.

We want to use the estimates (34) and (35) of Part 2 of the Proof of Proposition 5.7. To this aim we consider

$$W(\xi_1^0) = \left\{(x,\xi);\, |x - x^0| \leqslant \frac{|\xi_1^0|}{C_0 M},\, |\xi - \xi^0| \leqslant \frac{|\xi_1^0|}{C_0}\right\}$$

(so $|\xi_1^0|/C_0 \sim |\xi_1^0|$).

Recall that the $\hat{Q}_\mu^2$ were the C.Z. blocks in $\mathbf{R}\times\mathbf{R}$ relative to $\rho^2\bar{p}_\rho$. We hence partition Q_ρ''' into "completions in $\mathbf{R}^2\times\mathbf{R}^2$" of the $\hat{Q}_\mu^2$, i.e., into blocks $\hat{Q}_\mu = \hat{Q}_\mu^1 \times \hat{Q}_\mu^2$ with $\text{sizes}(\hat{Q}_\mu^1) = \text{sizes}(\hat{Q}_\mu^2)$. Therefore $\text{sizes}(\hat{Q}_\mu) = \text{sizes}(\hat{Q}_\mu^2) = \text{sizes}(\hat{Q}_\mu^1) := \Delta_\mu \times M\Delta_\mu$. Note that, by construction, $(\hat{Q}_{\mu_1})''' \cap (\hat{Q}_{\mu_2})''' \neq \varnothing \Rightarrow \Delta_{\mu_1} \sim \Delta_{\mu_2}$.

Let

$$\mathscr{C} = \{\hat{Q}_\mu;\, \hat{Q}_\mu \cap W(\xi_1^0) \neq \varnothing\}.$$

Then

$$\forall \hat{Q}_\mu \in \mathscr{C}, \qquad \Delta_\mu \lessapprox \frac{|\xi_1^0|}{C_0 M} \quad \text{or} \quad \Delta_\mu \sim \frac{|\xi_1^0|}{C_0 M}.$$

Otherwise, if $|\xi_1^0|/C_0 M \ll \Delta_\mu$, we would have

$$(\hat{Q}_{\mu_0})''' \cap (\hat{Q}_\mu)''' \neq \varnothing \Rightarrow \Delta_\mu \sim \Delta_{\mu_0} = \rho\delta_{\nu_0} \qquad \text{and} \qquad \frac{|\xi_1^0|}{C_0 M} \ll \rho\delta_{\nu_0},$$

a contradiction, since we are considering case (i).

Therefore, using the same notation as (34) and (35),

$$\begin{aligned} &|\partial_x q_{2k}(\Gamma(t; x^0, \xi^0))| \\ &\qquad \lessapprox |\xi_1^0| + \max_{\nu} \max_{|t| \leqslant 1} |\partial_x q_{2k\nu}(\Gamma_1(t; x^0, \xi^0), 0, \xi_2^0)| + |\Gamma_2(t; x^0, \xi^0) - \xi^0| \\ &\qquad \lessapprox |\xi_1^0| + |\Gamma_2(t; x^0, \xi^0) - \xi^0|; \end{aligned}$$

and, $\forall i \geqslant 2$,

$$|\partial_{\xi_i} q_{2k}(\Gamma(t; x^0, \xi^0))| \lessapprox \frac{|\xi_1^0|}{M} + \frac{|\Gamma_2(t; x^0, \xi^0) - \xi^0|}{M},$$

since

$$q \in \mathscr{S}(\rho^2 p_{|\xi_1 = 0}, \hat{Q}_\mu) \Rightarrow |\partial_x q_{|\xi_1 = 0}| \lessapprox |\xi_1^0|, \qquad |\partial_{\xi_i} q_{|\xi_1 = 0}| \lessapprox \frac{|\xi_1^0|}{M}, \quad \forall i \geqslant 2.$$

Hence,

$$M\, |\Gamma_1^2(t; x^0, \xi^0) - x_2^0| + |\Gamma_2(t; x^0, \xi^0) - \xi^0| \lessapprox |\xi_1^0|.$$

Thus case (i) gives

$$\begin{aligned} B_p((x^0, \xi^0), \rho) \approx \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2;\ &|x_1 - x_1^0| \lessapprox \rho, \\ &M\, |x_2 - x_2^0| + |\xi - \xi^0| \lessapprox |\xi_1^0|\}. \end{aligned}$$

For case (ii), we use case (i) to conclude immediately that

$$B_p((x^0, \xi^0), \rho) \approx \{(x, \xi);\ |x_1 - x_1^0| \lessapprox \rho,\ M\, |x_2 - x_2^0| + |\xi - \xi^0| \lessapprox |\xi_1^0|\},$$

where now $|\xi_1^0| \sim M\rho\delta_{\nu_0}$.

For case (iii), by the Fact in Section 4, we have that $\exists I_0^1 \subset I_\rho$, $|I_0^1| \sim \rho\delta_{\nu_0}$, and $Q^2(x_2^0, \xi_2^0, \rho\delta_{\nu_0}) \subset \hat{Q}_{\mu_0}^2$, such that

$$\rho^2 p_1(x_1, x_2, \xi_2) \geqslant (M(\rho\delta_{\nu_0})^2)^2, \qquad \forall (x_1, x_2, \xi_2) \in I_0^1 \times Q^2(x_2^0, \xi_2^0, \rho\delta_{\nu_0}).$$

Since $|\xi_1^0|/C_0 M \ll \rho\delta_{\nu_0}$, we can reason as in Case 1 to conclude that

$$\begin{aligned} B_p((x^0, \xi^0), \rho) \approx \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2;\ &|x_1 - x_1^0| \lessapprox \rho, \\ &M\, |x_2 - x_2^0| + |\xi - \xi^0| \lessapprox M\rho\, \delta_{\nu_0}\}. \quad \blacksquare \end{aligned}$$

5.4. *The* $(2+2)$*-Dimensional Case: An Intermediate Result*

We now study an intermediate step toward the general $(2+2)$-dimensional, nonelliptic–nondegenerate case. In order to do that, we need to make some considerations and assumptions (to be justified in the general case).

DEFINITION. We say that a symbol $p = p(x_1, x', \xi')$ belongs to the class $S^m(\rho \times \delta \times M\delta)$ if it satisfies the mth order estimates

$$|\partial^\alpha_{x_1} \partial^\beta_{x'} \partial^\gamma_{\xi'} p(x_1, x', \xi')| \leqslant C_{\alpha\beta\gamma}(M\delta^2)^m \rho^{-\alpha}\delta^{-|\beta|}(M\delta)^{-|\gamma|}, \qquad \forall\alpha, \beta, \gamma.$$

We hence consider $\rho^2 p(x, \xi) = \rho^2\xi_1^2 + \rho^2 p_1(x, \xi_2)$ on a C.Z. block Q_ρ, centered at $(\bar{x}, \bar{\xi})$, $|\bar{\xi}_1| \lesssim M\rho$, of size $\rho \times M\rho$.

Given $q \in \mathscr{S}(\rho^2 p, Q_\rho)$, we know from Lemma 3.7 that $q = q_1 + q_2$, $cq_1 \in \mathscr{S}(\rho^2\xi_1^2, Q_\rho)$, $cq_2 \in \mathscr{S}(\rho^2 p_1, Q_\rho)$, for a universal constant $c > 0$.

We now make the assumption that the derived symbol $p^*_\rho(x_2, \xi_2)$ (see (42)) is nonelliptic–nondegenerate on a block $Q^2_{\rho\delta} \subset \mathbf{R} \times \mathbf{R}$, centered at (x^*_2, ξ^*_2), containing (x^0_2, ξ^0_2). In particular, it follows that

$$\left(\frac{|\xi^0_1|}{C_0 M}\right)^4 M^2 \lesssim M^2(\rho\delta)^4. \tag{45}$$

Now, $\rho^2\bar{p}_\rho \in S^2(Q^2_{\rho\delta})$, $p_1 \in S^2(1 \times M)$, and the fact that $p_1(x_1, \cdot, \cdot)$ is a polynomial in x_1, at scale ρ, yield that

$$|\partial^\alpha_{x_1} \partial^\beta_{x_2} \partial^\gamma_{\xi_2}(\rho^2 p_1)(x_1, x_2, \xi_2)| \lesssim C_{\alpha\beta}\,\rho^{-\alpha}(M\rho^2\delta^2)^2\,(\rho\delta)^{-\beta}\,(M\rho\delta)^{-\gamma}, \tag{46}$$

$\forall (x_1, x_2, \xi_2) \in (\pi_{x_1}(Q_\rho) \times Q^2_{\rho\,\delta})'''' = (\pi_{(x_1, x_2, \xi_2)}(\tilde{R}_\delta))''''$, where

$$\tilde{R}_\delta = \pi_{x_1}(Q_\rho) \times \pi_{x_2}(Q^2_{\rho\delta}) \times \pi_{\xi_1}(Q_\rho) \times \pi_{\xi_2}(Q^2_{\rho\delta}).$$

From (46) and $\rho^2 p_1 \in S^2(1 \times M)$, it follows that $\rho^2 p_1$ can be *localized*[6] on any subblock of $(\tilde{R}_\delta)''''$ of sizes $\rho\delta \times M\rho\delta$.

We may suppose $\rho^2 p_1$ is a polynomial in (x_1, x_2, ξ_2) on (a large dilate of) Q_ρ. We also suppose that $\rho^2 p_1$ can be written, on $\tilde{R}''''_\delta$, as

$$\rho^2 p_1(x_1, x_2, \xi_2) = \rho^2\delta^2 e(x, \xi_2)(\xi_2 - \xi^v_2 - \theta(x_1, x_2))^2 + M^2\rho^2\delta^4\tilde{V}(x_1, x_2) \tag{47}$$

[6] By this we mean the following: Suppose, on a $1 \times M$ block Q, we are given $p \in S^m(1 \times M)$, and let $Q_\delta \subset Q$ be a smaller block of sizes $\delta \times M\delta$. We say that p can be *localized* to Q_δ if $p_{|Q_\delta} \in S^m(\delta \times M\delta)$. By interpolation one has that

$$p \in S^m(1 \times M), \qquad |p_{|Q_\delta}| \lesssim (M\delta^2)^m \Rightarrow p_{|Q_\delta} \in S^m(\delta \times M\delta).$$

where $|\xi_2^\nu - \xi_2^*| \leqslant M\rho\delta$ and (by (46)):

(i) θ is an algebraic function in x_1, a polynomial of *a priori* bounded degree d in x_2, satisfying the estimates:

$$|\partial_{x_1}^\alpha \partial_{x_2}^\beta \theta(x_1, x_2)| \leqslant C_{\alpha\beta} M\rho\delta\rho^{-\alpha}(\rho\delta)^{-\beta},$$

i.e., $(M\rho\delta)^{-1}\,\theta \in S^0(\rho \times \rho\delta \times M\rho\delta)$, whence it follows that, for x_1 varying at scale $\rho\delta$, $\rho\delta\theta \in S^1(\rho\delta \times M\rho\delta)$: In fact, with $0 < \delta \leqslant 1$,

$$|\partial_{x_1}^\alpha \partial_{x_2}^\beta \theta(x_1, x_2)| \leqslant C_{\alpha\beta} M\rho\delta\rho^{-\alpha}(\rho\delta)^{-\beta} \leqslant C_{\alpha\beta} M\rho\delta(\rho\delta)^{-\alpha}(\rho\delta)^{-\beta}. \tag{48}$$

(ii) $0 \leqslant \tilde{V}$ is the polynomial $\rho^2 p_1$ restricted to the graph of

$$\xi_2 = \theta(x_1, x_2) + \xi_2^\nu$$

and such that

$$\rho^{-2}\tilde{V} \in S^0(\rho \times \rho\delta \times M\rho\delta), \tag{49}$$

whence $M^2(\rho\delta)^4\,\tilde{V}$ can be *localized* when x_1 is ranging at scale $\rho\delta$, to an element of $S^2(\rho\delta \times M\rho\delta)$.

(iii) e is positive, elliptic, and $e \in S^0(\rho \times \rho\delta \times M\rho\delta)$, so it can be *localized* to an element of $S^0(\rho\delta \times M\rho\delta)$.

We now use the symplectic dilation

$$s\colon (x_1, x_2, \xi_1, \xi_2) \mapsto \left(\frac{x_1 - \bar{x}_1}{\rho}, \frac{x_2 - x_2^*}{\rho}, \rho\xi_1, \rho(\xi_2 - \xi_2^*)\right) = (y, \eta),$$

taking Q_ρ''' to be a block of sizes $1 \times M\rho^2$ (hereafter we shall use M in place of $M\rho^2$) and $\tilde{R}_\delta$ to a "band" R_δ of sizes $1 \times \delta \times M \times M\delta$,

$$R_\delta := I \times J_\delta \times I_M \times J_{M\delta}, \tag{50}$$

center$(R_\delta) = (0, 0, \bar{\eta}_1, 0)$, with $|\bar{\eta}_1| \leqslant CM\delta$.

In these new coordinates, writing p_1 for $\rho^2 p_1 \circ s^{-1}$, the symbol $\rho^2 p$ goes over into

$$p(y, \eta) = \eta_1^2 + \delta^2\tilde{e}(y, \eta_2)(\eta_2 - \eta_2^\nu - \tilde{\theta}(y))^2 + M^2\delta^4 V(y),$$

with $\delta\tilde{\theta} \in S^1(1 \times \delta \times M\delta)$, $0 \leqslant V$, $M^2\delta^4 V \in S^2(1 \times \delta \times M\delta)$, $0 < \tilde{e}$ elliptic belonging to $S^0(1 \times \delta \times M\delta)$ (and, when size of $y_1 \sim \delta$, $\delta\tilde{\theta} \in S^1(\delta \times M\delta)$, $M^2\delta^4 V \in S^2(\delta \times M\delta)$, $\tilde{e} \in S^0(\delta \times M\delta)$).

We call (y, η) (x, ξ) again. Hence,

$$p(x, \xi) = \xi_1^2 + \delta^2 e(x, \xi_2)(\xi_2 - \xi_2^\nu - \theta(x))^2 + M^2\delta^4 V(x) = \xi_1^2 + p_1(x, \xi_2) \tag{51}$$

on $(R'''_\delta)^{**}$.

Since e is a harmless, localizable elliptic factor, we drop it in the following. Note that now (45) reads as

$$|\xi_1^0| \leqslant C_0' M\delta, \tag{52}$$

and $p_1^*(x_2, \xi_2)$ is nonelliptic–nondegenerate on the new Q_δ^2.

Since p_1 can be localized to sizes $\delta \times M\delta$, we write

$$R_\delta''' = \bigcup_{k_1, k_2} (I_\delta^{k_1} \times J_\delta \times I_{M\delta}^{k_2} \times J_{M\delta}) = \bigcup_{k_1, k_2} Q_\delta^{k_1 k_2},$$

where $|I_\delta^{k_1}| \sim \delta$, $|I_{M\delta}^{k_2}| \sim M\delta$ (with an *a priori* bounded number of overlappings for their $(\ )^{**}$ dilates).

Let $I_{M\delta}$ be the interval in the ξ_1-axis containing ξ_1^0. Let

$$\bar{R}_\delta = \bigcup_k (I_\delta^k \times J_\delta \times I_{M\delta} \times J_{M\delta}) = \bigcup_k Q_\delta^k \subset R_\delta,$$

with center$(\bar{R}_\delta) = (0, 0, \xi_1^*, 0)$ with $|\xi_1^*| \lessapprox M\delta$.

Moreover, we suppose $(x^0, \xi^0) \in \bar{R}_\delta$.

LEMMA 5.11. *Suppose $p_1(x, \xi_2) \in S^2(1 \times \delta \times M\delta)$ on R_δ''', $|\xi_1^0| \leqslant C_0' M\delta$, $(x^0, \xi^0) \in \bar{R}_\delta$ (in the above notations). Then, for any subunit broken path Γ starting at (x^0, ξ^0),*

$$\Gamma(t; x^0, \xi^0) = (x_1(t), x_2(t), \xi_1(t), \xi_2)),$$

we have

$$\frac{|x_2(t) - x_2^0|}{\delta} + \frac{|\xi(t) - \xi_0|}{M\delta} \leqslant 4C_*,$$

where $0 < C_*$ is an a priori constant.

Proof. Any $q_k \in \mathscr{S}(p, Q)$, giving rise to Γ, can be written as $q_k = q_{1k} + q_{2k}$, where, for a *universal* $0 < c \leqslant 1$ (depending, see Remark 3.3, on an *a priori* cut-off function),

$$cq_{1k} \in \mathscr{S}(\xi_1^2, Q), \qquad cq_{2k} \in \mathscr{S}(p_1, Q).$$

It follows that

$$q_{1k\,|\,\xi_1=0}\equiv 0 \Rightarrow \partial_{x_i} q_{1k\,|\,\xi_1=0}\equiv \partial_{\xi_2} q_{1k\,|\,\xi_1=0}\equiv 0, \qquad i=1,2.$$

Thus, for an *a priori* constant $C>0$,

$$|\partial_{x_i} q_{1k}(x_1(t), x_2^0, \xi_1^0, \xi_2^0)| \leqslant C\,|\xi_1^0| \leqslant C' M\delta$$

and

$$|\partial_{\xi_2} q_{1k}(x_1(t), x_2^0, \xi_1^0, \xi_2^0)| \leqslant C\frac{|\xi_1^0|}{M} \leqslant C'\delta. \tag{53}$$

p_1 can be localized to subblocks of size $\delta\times M\delta$, then the same is true for q_{2k}. By Proposition 3.4 it follows that

$$q_{2k}(x,\xi)^2 \leqslant p_1(x,\xi_2)$$

on R_δ, and $cq_{2k}\in\mathscr{S}(p_1, Q)$ implies (by interpolation we get the needed estimates for $|\alpha|+|\beta|=1$) that $q_{2k}\in\mathscr{S}(p_1,\delta\times M\delta)$.

Since $R_\delta=\bigcup_{v_1,v_2} Q_\delta^{v_1 v_2}$, we write (this is analogous to what has been done in Proposition 5.10)

$$q_{2k}(x,\xi)=\sum_{v_1,v_2} q_{2kv_2v_2}(x,\xi),$$

where $cq_{2kv_1v_2}\in\mathscr{S}(p_1, Q_\delta^{v_1v_2})$ for a universal constant $c>0$, $\operatorname{supp} q_{2kv_1v_2}\subset (Q_\delta^{v_1v_2})^{**}$. Consider Hamilton's equations for the kth segment of Γ. By Taylor expansion we have

$$\begin{aligned}
\dot{x}_2^k &= \partial_{\xi_2} q_k(x,\xi)\\
&= (\partial_{\xi_2} q_k)(x_1(t), x_2^0, \xi_1^0, \xi_2^0) + Q_{1k}(x,\xi)(x_2-x_2^0)\\
&\quad + \frac{\langle Q_{2k}(x,\xi), (\xi(t)-\xi^0)\rangle}{M},\\
\dot{\xi}_i^k &= -\partial_{x_i} q_k(x,\xi)\\
&= -\{(\partial_{x_i} q_k)(x_1(t), x_2^0, \xi_2^0, \xi_2^0) + MQ_{1k}^i(x,\xi)(x_2-x_2^0)\\
&\quad + \langle Q_{2k}^i(x,\xi), (\xi(t)-\xi^0)\rangle\},
\end{aligned}$$

where $|Q_{jk}|\leqslant 1$, $|Q_{jk}^i|\leqslant 1$, $i, j=1,2$, $\forall k=0,1,\ldots,L-1$. Consider

$$\begin{aligned}
(\partial_{\xi_2} q_k)(x_1(t), x_2^0;\xi^0) &= (\partial_{\xi_2} q_{1k})(x_2(t), x_2^0;\xi^0)\\
&\quad + \sum_{v_1,v_2} (\partial_{\xi_2} q_{2kv_1v_2})(x_1(t), x_2^0;\xi^0)
\end{aligned}$$

and

$$(\partial_x q_k)(x_1(t), x_2^0; \xi^0) = (\partial_x q_{1k})(x_1(t), x_2^0; \xi^0) + \sum_{\nu_1, \nu_2} (\partial_x q_{2k\nu_1\nu_2})(x_1(t), x_2^0; \xi^0).$$

We have that $|\partial_x q_{2k\nu_1\nu_2}(x_1(t), x_2^0; \xi^0)| \lessapprox M\delta$ and $|\partial_{\xi_2} q_{2k\nu_1\nu_2}(x_1(t), x_2^0; \xi^0)| \lessapprox \delta$.

These inequalities, together with (53), give (using Lemma 4.1)

$$|\partial_{\xi_2} q_k(\Gamma(t; x^0, \xi^0))| \leqslant C_* \delta + |x_2(t) - x_2^0| + \frac{|\xi(t) - \xi^0|}{M}$$

and

$$|\partial_{x_i} q_k(\Gamma(t; x^0, \xi^0))| \leqslant C_* M\delta + M\,|x_2(t) - x_2^0| + |\xi(t) - \xi^0|,$$

so that Lemma 5.4 (adapted to the present situation as in Proposition 5.10) yields

$$M\,|x_2(t) - x_2^0| + |\xi(t) - \xi^0| \leqslant 4C_* M\delta. \quad \blacksquare$$

Write now $\theta(x)$ in (51) as $M\delta b(x_1, x_2)$.

Denote $\bar{b}(x_2) = (\mathrm{Av}_{x_1 \in I} b)(x_2)$ and $b_0(x_1, x_2) := b(x_1, x_2) - \bar{b}(x_2)$. Then $b_0(x_1, x_2)$ is an algebraic function in x_1, a polynomial of *a priori* bounded degree in x_2. We now make the requirement that

$$\max_{x_2 \in J_\delta} |\bar{b}(x_2)| \leqslant C,$$

$0 < C$ a universal constant so that, with

$$\max_{x_2 \in J_\delta^\#} |\bar{b}(x_2)| \leqslant C_d$$

(since $\bar{b}$ is a polynomial, C_d is a universal constant depending on d, C_*, C; $J_\delta^\#$ is the dilate of J_δ by the factor $4(C_* + 1) = C_\#$ (in view of Lemma 5.11), we have

$$\eta_2 = \xi_2 = \xi_2^\nu - M\delta\bar{b}(x_2), \qquad \xi_2 \in (J_{M\delta})', \quad x_2 \in J_\delta^\# \Rightarrow \eta_2 \in (J_{M\delta})''. \tag{54}$$

Define the following canonical transformation

$$\Psi\colon (x_1, x_2, \xi_1, \xi_2) \mapsto (x_1, x_2, \xi_1, \xi_2 - \xi_2^\nu - M\delta\bar{b}(x_2)) = (y, \eta). \tag{55}$$

Ψ is globally defined, and it is *tame* whenever (x, ξ) are ranging at scale $\delta \times M\delta$.

We shall refer to this fact by saying that Ψ is δ-*locally tame*. It follows from (54) that

$$\Psi(R_\delta^\#) \subset R''_\delta,$$

thus $\Psi(R_\delta)$ is of sizes $1 \times \delta \times M \times M\delta$.

We use the new coordinates defined by Ψ (calling them (x, ξ) again). Note that now

$$\Psi(x^0, \xi^0) = (x_1^0, x_2^0, \xi_1^0, \xi_2^0 - \xi_2^\nu - M\delta\bar{b}(x_2^0)) = (y^0, \eta^0) := (x^0_{\text{new}}, \xi^0_{\text{new}}). \quad (56)$$

We have the following important facts:

(F1) $\Psi_* \dfrac{\partial}{\partial x_1} = \dfrac{\partial}{\partial x_1}$;

(F2) p_1 can be localized at scale $\delta \times M\delta$ on R_δ, hence any $q_2 \in \mathscr{S}(p_1, Q)$ can be localized at scale $\delta \times M\delta$ on R_δ.

Thus, subunit symbols for p_1 on blocks of sizes $\delta \times M\delta$ can be pushed forward through Ψ to equivalent subunit symbols for $((\Psi^{-1})^* p_1)$ on equivalent blocks of sizes $\delta \times M\delta$ and vice versa (in view of Proposition 3.5).

(F3) Since, by Lemma 5.11, ξ_1 doesn't leave $I^\#_{M\delta}$ through subunit paths, it follows that Ψ transports the geometry localized at sizes $\delta \times M\delta$. We can pass from one $\delta \times M\delta$-localization to another $\delta \times M\delta$-localization using (F1). Moreover, since we have the subunit symbol

$$q_0(x, \xi) = c\xi_1 \chi(x, \xi)$$

(see (31)), which allows us to move according to the flow of $\partial/\partial x_1$, we also have subunit symbols (relative to p)

$$q_{0\delta}(x, \xi) = c\delta\xi_1 \chi_\delta(x, \xi),$$

where χ_δ is analogous at sizes $\delta \times M\delta$ to the above χ. The $q_{0\delta}$ allow us to move according to $\delta\partial/\partial x_1$. Let us check that $q_{0\delta}$ are indeed subunit symbols for $p_{|\bar{Q}_\delta}$, provided $\xi_1 \in I^\#_{M\delta}$ (for $|\xi_1^0| \lesssim M\delta$, $\bar{Q}_\delta$ being now a generic block of sizes $\delta \times M\delta$ in $\mathbf{R}^2 \times \mathbf{R}^2$, $\pi_{\xi_1}(\bar{Q}_\delta) \subset I^\#_{M\delta}$). We have $|\partial_x^\alpha \partial_\xi^\beta \chi_\delta(x, \xi)| \leqslant C_{\alpha\beta}(M\delta)^{-|\beta|}\delta^{-|\alpha|}$.

$$q_{0\delta}(x, \xi)^2 \leqslant c^2\delta^2\xi_1^2 \leqslant \xi_1^2 \qquad (0 < \delta \leqslant 1);$$

for $|\alpha| = 2$,

$$|\partial_x^\alpha q_{0\delta}(x, \xi)| = \delta\, |\xi_1|\, |\partial_x^\alpha \chi_\delta(x, \xi)| \lesssim \delta M\delta\delta^{-2} = M;$$

for $|\alpha| = |\beta| = 1$,

$$|\partial_x^\alpha \partial_\xi^\beta q_{0\delta}(x, \xi)| \leqslant c\delta(|\partial_x^\alpha \chi_\delta(x, \xi)| + |\xi_1| \, |\partial_x^\alpha \partial_\xi^\beta \chi_\delta(x, \xi)|)$$
$$\lessapprox \delta(\delta^{-1} + M\delta(M\delta^2)^{-1}) = 2;$$

for $|\beta| = 2$,

$$|\partial_\xi q_{0\delta}(x, \xi)| \leqslant c\delta(|\partial_\xi \chi_\delta(x, \xi)| + |\xi_1| \, |\partial_\xi^\beta \chi_\delta(x, \xi)|)$$
$$\lessapprox \delta((M\delta)^{-1} + M\delta(M\delta)^{-2}) = \frac{2}{M}.$$

We can hence move according to vector fields $\sim \delta\partial/\partial x_1$.

Fact. The transformation Ψ allows us to suppose that

$$(\mathrm{Av}_{x_1 \in I} b)(x_2) \equiv 0,$$

and to construct the equivalent subunit ball in the Ψ-coordinates.

This results in a "clustering" of the ξ_2-component of the subunit ball around the graph of the polynomial $\bar{b}(x_2)$.

We can now state the following theorem.

THEOREM 5.12. *Under the above assumptions, we suppose, on an a priori large dilate of R_δ, in Ψ-coordinates,*

$$p(x, \xi) = \xi_1^2 + \delta^2(\xi_2 - M\delta b_0(x_1, x_2))^2 + (M\delta^2)^2 \, V(x_1, x_2),$$

where $\bar{b}_0(x_2) \equiv 0$. Define

$$\sigma(b_0^2) := \max_{x \in I \times J_\delta} (b_0(x_1, x_2))^2 \qquad \text{and} \qquad \sigma(V) := \max_{x \in I \times J_\delta} V(x_1, x_2).$$

Then, in these Ψ-coordinates

$$B_p((x^0, \xi^0), 1) \approx \{(x, \xi); |x_1 - x_1^0| \lessapprox 1, |x_2 - x_2^0| \lessapprox \delta,$$
$$|\xi - \xi^0| \lessapprox M\delta\Delta_0 + M\delta\sigma(V)^{1/4}\},$$

where

$$\Delta_0 := \frac{|\xi_1^0| + |\xi_2^0|}{M\delta} + \sigma(b_0^2)^{1/2}.$$

Proof. Define

$$\ell(x_1, x_2, \xi_2)^2 := c^2\delta^2(\xi_2 - M\delta b_0(x_1, x_2))^2$$

($c > 0$ is a universal normalizing constant).

LEMMA 5.13. *(x^0, ξ^0) can be joined through a subunit broken path to*

$$(x_1^0, x_2^0 \pm c\delta t_3, \xi_1^0, \xi_2^0),$$

where c is the above universal constant, $0 < t_3 \sim 1$. The same holds true $\forall (x^0, \bar{\xi})$ with $|\bar{\xi} - \xi^0| \leqslant M\delta$.

Proof of the Lemma. Consider the subunit Hamiltonian vector field

$$H_\ell(x, \xi) = c\left(\delta \frac{\partial}{\partial x_2} + M\delta^2(\partial_{x_1} b_0(x)) \frac{\partial}{\partial \xi_1} + M\delta^2(\partial_{x_2} b_0(x)) \frac{\partial}{\partial \xi_2}\right).$$

Denote

$$\gamma_0(t; \bar{x}, \bar{\xi}) = \exp(tH_{q_0})(\bar{x}, \bar{\xi}),$$
$$\gamma_\ell(t; \bar{x}, \bar{\xi}) = \exp(tH_\ell)(\bar{x}, \bar{\xi}).$$

We flow along γ_ℓ to the point

$$\gamma_\ell(t_1; x^0, \xi^0) := (x_1^0, x_2^{(1)}, \xi_1^{(1)}, \xi_2^{(1)})$$

where $|t_1| \leqslant 1$, $t_1 \sim 1$, and we can suppose $t_1 > 0$ (see Remark 3.10). Here t_1 is chosen so that

$$(\mathrm{Av}_{x_1 \in I} b_0^2)(x_2^{(1)}) \sim \max_{x_2 \in J_\delta} (\mathrm{Av}_{x_1 \in I} b_0^2)(x_2).$$

This is possible with $t_1 \sim 1$ since $(\mathrm{Av}_{x_1 \in I} b_0^2)(x_2)$ is a non-negative polynomial of *a priori* bounded degree.

We consider also, for $0 < t_3 \sim 1$, $t_3 \leqslant t_1$, t_3 to be determined (depending on universal constants), the point

$$(x_1^0, x_2^0 - c\delta t_3, \xi_1^0, \xi_2^0).$$

We evolve it through $\gamma_\ell(t; x_1^0, x_2^0 - c\delta t_3; \xi^0)$ to reach the point, at time $t_1 + t_3$,

$$(x^{(2)}, \xi^{(2)}) := (x_1^0, x_2^0 + c\delta t_1, \xi_1^{(2)}, \xi_2^{(2)}).$$

We can hence move $(x_1^0, x_2^0 - c\delta t_3; \xi^0)$ to $(x_1^0, x_2^0 + c\delta t_1; \xi^{(2)}) = (x_1^0, x_2^{(1)}; \xi^{(2)})$.

Consider $\ell(x_1, x_2^{(1)}, \xi_2^{(1)})^2$ and flow along $\gamma_0(t; x^{(1)}, \xi^{(1)})$ (note that (x_2, ξ_1, ξ_2) remains in this way *fixed*) to reach, at time $t_2 \sim 1$, a maximum for $x_1 \mapsto \ell(x_1, x_2^{(1)}, \xi_2^{(1)})^2$ (say $\tilde{x}_1$). This is possible in view of the properties of algebraic functions (see Section 4 of [18] and in particular Lemma 4.7).

By Lemma 4.3, we hence have

$$\max_{x_1 \in I} \ell(x_1, x_2^{(1)}, \xi_2^{(1)})^2 \sim \mathrm{Av}_{x_1 \in I} \ell(x_1, x_2^{(1)}, \xi_2^{(1)})^2$$
$$\sim \delta^2 |\xi_2^{(1)}|^2 + (M\delta^2)^2 (\mathrm{Av}_{x_1 \in I} b_0^2)(x_2^{(1)}). \tag{57}$$

On the other hand,

$$(\mathrm{Av}_{x_1 \in I} b_0^2)(x_2^{(1)}) \sim \max_{x_2 \in J_\delta} (\mathrm{Av}_{x_1 \in I} b_0^2)(x_2)$$
$$\sim (\mathrm{Av}_{x_1 \in I} \mathrm{Av}_{x_2 \in J_\delta} b_0(x_1, x_2)^2) \sim \max_{I \times J_\delta} b_0^2 = \sigma(b_0^2), \tag{58}$$

for b_0 is an algebraic function in x_1, polynomial in x_2. Recall also that

$$\max_{I \times J_\delta} b_0^2 \sim \max_{(I \times J_\delta)'} b_0^2.$$

We now have to estimate $|\xi^{(1)} - \xi^{(2)}|$.

Note that $\xi^{(1)}$ and $\xi^{(2)}$ arise from "parallel" trajectories having different initial conditions. Define

$$\gamma_\ell^1(\tau) := \gamma_\ell(\tau; x^0, \xi^0) \qquad \text{and} \qquad \gamma_\ell^2(\tau) := \gamma_\ell(\tau; x_1^0, x_2^0 - c\delta t_3; \xi^0).$$

Then

$$\pi_x \gamma_\ell^2(\tau) = (x_1^0, x_2^0 - c\delta t_3 + c\delta\tau) \qquad \text{and} \qquad \pi_x \gamma_\ell^1(\tau) = (x_1^0, x_2^0 + c\delta\tau).$$

By Taylor expansion, we have

$$(\nabla b_0)(\pi_x \gamma_\ell^2(\tau)) = (\nabla b_0)(\pi_x \gamma_\ell^1(\tau)) + \delta B_0(x^0, t_3, \tau)(-ct_3),$$

where

$$B_0(x^0, t_3, \tau) = \int_0^1 (\partial_{x_2} \nabla b_0)(x_1^0, x_2^0 + c\delta\tau + c\delta s(-t_3))\, ds.$$

Since, by the properties of algebraic functions (see [18, Lemma 4.8]),

$$\max_{x \in I \times J_\delta} |\partial_{x_2} \nabla b_0(x)| \leqslant \frac{C}{\delta} \max_{x \in I \times J_\delta} |\nabla b_0(x)| \leqslant \frac{C'}{\delta^2} \max_{x \in I \times J_\delta} |b_0(x)| \sim \frac{C''}{\delta^2} \sigma(b_0^2)^{1/2},$$

we therefore have

$$\begin{aligned}|\xi^{(1)}-\xi^{(2)}| &= M\delta^2 \left| \int_0^{t_1+t_3} (\nabla b_0)(\pi_x \gamma_\ell^2(\tau))\, d\tau - \int_0^{t_1} (\nabla b_0)(\pi_x \gamma_\ell^1(\tau))\, d\tau \right| \\ &= M\delta^2 \left| \int_{t_1}^{t_1+t_3} (\nabla b_0)(\pi_x \gamma_\ell^2(\tau))\, d\tau \right. \\ &\quad \left. + \int_0^{t_1} \langle B_0(x^0, t_3, \tau), (\pi_x \gamma_\ell^2(\tau) - \pi_x \gamma_\ell^1(\tau)) \rangle \, d\tau \right| \\ &\leqslant CM\delta^2 \frac{(\delta t_3)}{\delta^2} \sigma(b_0^2)^{1/2}\end{aligned}$$

(in fact, $\pi_x \gamma_\ell^2(\tau) - \pi_x \gamma_\ell^1(\tau) = (0, -c\delta t_3)$). Hence

$$|\xi^{(1)}-\xi^{(2)}| \leqslant CM\delta t_3 \sigma(b_0^2)^{1/2}. \tag{59}$$

Now consider $(\tilde{x}_1, x_2^{(1)}, \xi_1^{(1)}, \xi_2^{(1)})$ and $(\tilde{x}_1, x_2^{(1)}, \xi_1^{(2)}, \xi_2^{(2)})$ $(x_2^{(1)} = x_2^{(2)})$ (which belong to R''_δ, in view of our *a priori* normalizations). Since

$$\ell(\tilde{x}_1, x_2^{(1)}, \xi_2^{(1)})^2 \gtrsim M^2 \delta^4 \sigma(b_0^2),$$

it follows that we can find a neighborhood of $(\tilde{x}_1, x_2^{(1)}, \xi_1^{(1)}, \xi_2^{(1)})$ of sizes $\delta\sigma(b_0^2)^{1/2} \times M\delta\sigma(b_0^2)^{1/2}$ on which $\ell(x_1, x_2, \xi_2)^2 \sim \ell(\tilde{x}_1, x_2^{(1)}, \xi_2^{(1)})^2$.

Recalling that $\operatorname{dist}((x, \xi), (\bar{x}, \bar{\xi})) := \max\{|x-\bar{x}|, M^{-1}|\xi - \bar{\xi}|\}$, we call that neighborhood

$$U_1 = \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2; \operatorname{dist}((x, \xi), (\tilde{x}_1, x_2^{(1)}; \xi^{(1)})) \leqslant c\sigma(b_0^2)^{1/2}\delta\}.$$

Consider $\chi_1 \in C_0^\infty(\mathbf{R}^2 \times \mathbf{R}^2)$, $\chi_1 \equiv 1$ on $\frac{1}{2}U_1$, $\operatorname{supp} \chi_1 \subset U_1$, $0 \leqslant \chi_1 \leqslant 1$. Then

$$|\partial_x^\alpha \partial_\xi^\beta \chi_1(x, \xi)| \leqslant C_{\alpha\beta}(M\delta\sigma(b_0^2)^{1/2})^{-|\beta|} (\delta\sigma(b_0^2)^{1/2})^{-|\alpha|}.$$

(Note that $U_1 \subset R''_\delta$, and $U_1 \subset (Q_\delta^{k_0})^{**}$, the one containing $(\tilde{x}_1, x_2^{(1)}; \xi^{(1)})$.)

Hence we can consider the symbols

$$q_1(x, \xi) = c\sigma(b_0^2)^{1/2} M\delta(x_2 - x_2^{(1)})\, \chi_1(x, \xi), \tag{60}$$

$$q_2(x, \xi) = c\sigma(b_0^2)^{1/2} M\delta(x_1 - \tilde{x}_1)\, \chi_1(x, \xi). \tag{61}$$

By normalization (by an *a priori* $c > 0$), $q_1, q_2 \in \mathscr{S}(p, Q)$ (in fact, $q_1, q_2 \in \mathscr{S}(p_1, Q_\delta^{k_0})$). Let us check that they are indeed subunit symbols. We make the check for q_1 since that for q_2 is analogous:

$$q_1(x, \xi)^2 \lesssim M\delta^2\sigma(b_0^2)\delta^2\chi_1(x, \xi) \lesssim \ell(x; \xi_2)^2 \qquad \text{on} \quad \operatorname{supp} \chi_1:$$

for $|\alpha|=2$,

$$|\partial_x^\alpha q_1(x,\xi)| \lessapprox M\delta\sigma(b_0^2)^{1/2}\,(|\partial_x \chi_1(x,\xi)| + |x_2 - x_2^{(1)}|\;|\partial_x^\alpha \chi_1(x,\xi)|)$$

$$\lessapprox M\delta\left(\frac{1}{\delta\sigma(b_0^2)^{1/2}} + \frac{\delta\sigma(b_0^2)^{1/2}}{\delta^2\sigma(b_0^2)}\right)\sigma(b_0^2)^{1/2} = 2M;$$

for $|\beta|=2$,

$$|\partial_\xi^\beta q_1(x,\xi)| \lessapprox M\delta\;|x_2 - x_2^{(1)}|\;|\partial_\xi^\beta \chi_1(x,\xi)|\;\sigma(b_0^2)^{1/2}$$

$$\lessapprox M\delta\sigma(b_0^2)^{1/2}\delta\sigma(b_0^2)^{1/2}\,(M\delta\sigma(b_0^2)^{1/2})^{-2} = M^{-1};$$

for $|\alpha| = |\beta| = 1$,

$$|\partial_x^\alpha\,\partial_\xi^\beta q_1(x,\xi)| \lessapprox M\delta\sigma(b_0^2)^{1/2}\,(|\partial_\xi^\beta \chi_1(x,\xi)| + |x_2 - x_2^{(1)}|\;|\partial_x^\alpha\,\partial_\xi^\beta \chi_1(x,\xi)|)$$

$$\lessapprox M\delta\sigma(b_0^2)^{1/2}\left(\frac{1}{M\delta\sigma(b_0^2)^{1/2}} + \frac{\delta\sigma(b_0^2)^{1/2}}{M\delta^2\sigma(b_0^2)}\right) = 2.$$

Hence $q_1, q_2 \in \mathscr{S}(p_1, Q_\delta^{k_0})$ and $q_1, q_2 \in \mathscr{S}(p, Q)$. Consider then

$$H_1 = H_{q_1}(x,\xi) \sim M\delta\sigma(b_0^2)^{1/2}\,\frac{\partial}{\partial \xi_2},$$

and

$$H_2 = H_{q_2}(x,\xi) \sim M\delta\sigma(b_0^2)^{1/2}\,\frac{\partial}{\partial \xi_1},$$

in $\frac{1}{2}U_1$.

Through the associated $\gamma_1(t;\tilde{x}_1, x_2^{(1)};\xi^{(1)})$ and $\gamma_2(t;\tilde{x}_1, x_2^{(2)};\xi^{(2)})$ we can thus join (in $\frac{1}{2}U_1$)

$$(\tilde{x}_1, x_2^{(1)}, \xi_1^{(1)}, \xi_2^{(1)}) \quad \text{to} \quad (\tilde{x}_1, x_2^{(2)}, \xi_1^{(2)}, \xi_2^{(2)})$$

(recall that $x_2^{(2)} = x_2^{(1)}$), provided $t_3 = \tilde{c}t_1$, where $\tilde{c} > 0$ is a universal constant. This proves that $(x_1^0, x_2^0 - ct_3\delta, \xi_1^0, \xi_2^0)$ can be joined to $(x_1^0, x_2^0, \xi_1^0, \xi_2^0)$. The same kind of argument applies for $(x_1^0, x_2^0 + ct_3\delta, \xi_1^0, \xi_1^0)$ and for points of the kind $(x_1^0, x_2^0 \pm ct_3\delta, \bar{\xi}_1, \bar{\xi}_2)$, with $|\bar{\xi} - \xi^0| \leqslant M\delta$. This proves the lemma. ∎

The lemma immediately yields that the slices

$$\{x;\, |x_1 - x_1^0| \lessapprox 1,\, |x_2 - x_2^0| \lessapprox \delta\} \times \{\bar{\xi}\} \subset B_p((x^0, \xi^0), 1)$$

for $|\bar{\xi} - \xi^0| \lessapprox M\delta$.

Consider now the following function:

$$L(x,\xi)^2 := c^2\delta^2\xi_1^2 + \ell(x;\xi_2)^2. \tag{62}$$

For fixed ξ, L^2 is of the kind considered in Lemma 4.3.

We now move (as we are allowed to) from (x_1^0, x_2^0) to a point $(\bar{x}_1, \bar{x}_2)$ in $I \times J_\delta$ which is maximum for $L(x, \xi^0)^2$.

By the properties of algebraic functions of Section 4 (see Lemma 4.3)

$$L(\bar{x},\xi^0)^2 \sim \mathrm{Av}_{x\in I\times J_\delta} L(x,\xi^0)^2$$
$$\sim \delta^2\,|\xi_1^0|^2 + \delta^2\,|\xi_2^0|^2 + (M\delta^2)^2\,\sigma(b_0^2) := \varDelta_0^2(M\delta^2)^2. \tag{63}$$

Thus

$$\varDelta_0 = \frac{|\xi_1^0| + |\xi_2^0|}{M\delta} + \sigma(b_0^2)^{1/2}.$$

Note that $L(x,\xi^0)^2 \lesssim (M\delta^2)^2$. $L(x,\xi)^2$ being smooth (at scale $\delta\times M\delta$) and $\geqslant 0$, $\exists U_2$, a neighborhood of $(\bar{x};\xi^0)$ of sizes $\delta\varDelta_0 \times M\delta\varDelta_0$ on which $L(x,\xi)^2 \sim \varDelta_0^2(M\delta^2)^2$. Note that $U_2 \subset R''_\delta$ and $U_2 \subset (Q_\delta^{k_1})^{**}$, the one containing $(\bar{x}_1, \bar{x}_2, \xi_1^0, \xi_2^0)$.

Let $\chi_2 \in C_0^\infty(\mathbf{R}^2\times\mathbf{R}^2)$, $0 \leqslant \chi_2 \leqslant 1$, $\chi_2 \equiv 1$ on $\frac{1}{2}U_2$, $\operatorname{supp}\chi_2 \subset U_2$. Hence

$$|\partial_x^\alpha\,\partial_\xi^\beta \chi_2(x,\xi)| \leqslant C_{\alpha\beta}(M\delta\varDelta_0)^{-|\beta|}\,(\delta\varDelta_0)^{-|\alpha|}.$$

Consider the symbols

$$q_3(x,\xi) = cM\delta^2\varDelta_0\left(\frac{x_1-\bar{x}_1}{\delta}\right)\chi_2(x,\xi), \tag{64}$$

$$q_4(x,\xi) = cM\delta^2\varDelta_0\left(\frac{x_2-\bar{x}_2}{\delta}\right)\chi_2(x,\xi). \tag{65}$$

We check that they are subunit symbols for p at scale $1\times M$ (and $\delta\times M\delta$, since they can be localized).

Consider q_3 (the check for q_4 is completely identical):

$$q_3(x,\xi)^2 = c^2(M\delta^2)^2\,\varDelta_0^2\,\frac{(x_1-\bar{x}_1)^2}{\delta^2}\,\chi_2(x,\xi)^2 \leqslant c^2\varDelta_0^2(M\delta^2)^2\,\chi_2(x,\xi)^2$$
$$\lesssim L(x,\xi)^2 \leqslant p(x,\xi) \qquad \text{on} \quad \operatorname{supp}\chi_2.$$

For $|\alpha|=2$,

$$|\partial_x^\alpha q_3(x,\xi)| \lesssim M\delta^2\Delta_0\left(\frac{|\partial_x\chi(x,\xi)|}{\delta}+\frac{|x_1-\bar{x}_1|}{\delta}\,|\partial_x^\alpha\chi_2(x,\xi)|\right)$$

$$\lesssim M\delta^2\Delta_0\left(\frac{1}{\Delta_0\delta^2}+\frac{\Delta_0\delta}{\delta}\frac{1}{(\Delta_0\delta)^2}\right)=2M;$$

for $|\beta|=2$,

$$|\partial_\xi^\beta q_3(x,\xi)| \lesssim M\delta^2\Delta_0\,\frac{|x_1-\bar{x}_1|}{\delta}\,|\partial_\xi^\beta\chi_2(x,\xi)| \lesssim M\delta^2\Delta_0\Delta_0\,\frac{1}{(M\delta\Delta_0)^2}=\frac{1}{M};$$

for $|\alpha|+|\beta|=1$,

$$|\partial_x^\alpha\,\partial_\xi^\beta q_3(x,\xi)| \lesssim M\delta^2\Delta_0(\delta^{-1}\,|\partial_\xi\chi_2(x,\xi)|+\Delta_0\,|\partial_x^\alpha\,\partial_\xi^\beta\chi_2(x,\xi)|)$$

$$\lesssim M\delta^2\Delta_0\left(\frac{1}{M\Delta_0\delta^2}+\frac{\Delta_0}{\Delta_0\delta M\delta\Delta_0}\right)=2.$$

Hence $q_3, q_4\in\mathscr{S}(p,Q)$ (and $q_3, q_4\in\mathscr{S}(p_1^\#, Q_\delta^{k_1})$ as well, with $p_1^\#(x,\xi)=\delta^2(\xi_1-\xi_1^0)^2+p_1(x,\xi_2)$).

Considering

$$H_{q_3}(x,\xi)\sim M\delta\Delta_0\,\frac{\partial}{\partial\xi_1}$$

$$H_{q_4}(x,\xi)\sim M\delta\Delta_0\,\frac{\partial}{\partial\xi_2},$$

we are allowed to move in the ξ-direction by an amount $\sim|\xi_1^0|+|\xi_2^0|+M\delta\sigma(b_0^2)^{1/2}$.

We now move (x_1,x_2) to reach, at time ~ 1, the point $(\bar{x}_1,\bar{x}_2)$, a point at which $V(x_1,x_2)$ is comparable to its maximum. Hence

$$V(\bar{x}_1,\bar{x}_2)\sim\sigma(V)=\max_{x\in I\times J_\delta} V(x_1,x_2)\sim\max_{(x_1,x_2)\in(I\times J_\delta)^\#} V(x_1,x_2)$$

because of Theorem 4.4.

Note that it follows from the above constructions that we can join (x_1,x_2,ξ_1,ξ_2) to $(\bar{x}_1,\bar{x}_2,\xi_1,\xi_2)$ $\forall\xi$ such that $|\xi-\xi^0|\leqslant M\delta$.

$V\geqslant 0$ and Theorem 4.4 yields that there exists a region $R(V)$ of sizes $\sim\delta\sigma(V)^{1/4}\times\delta\sigma(V)^{1/4}$ in (x_1,x_2)-space, containing $(\bar{x}_1,\bar{x}_2)$, on which

$$V(x_1,x_2)\geqslant\tfrac{1}{2}\sigma(V).$$

As we have already seen, by [18, Corollary 4.3], we can construct $\varphi_1, \varphi_2 \in C_0^\infty(\mathbf{R}^2)$ such that $\operatorname{supp} \varphi_i \subset R(V)$, $i=1, 2$,

$$\varphi_i(x)^2 \leqslant c_1 \sigma(V),$$

$$\partial_{x_i} \varphi_i(x) \equiv c_3 \delta^{-1} \sigma(V)^{1/4}, \qquad \partial_{x_j} \varphi_i(x) \equiv 0, \qquad i \neq j, i=1, 2, \qquad \forall x \in \tfrac{1}{2} R(V),$$

and such that for $i=1, 2$,

$$|\partial_x^\alpha \varphi_i(x)| \leqslant C_\alpha \sigma(V)^{1/2-|\alpha|/4} \delta^{-|\alpha|}.$$

We now construct, for a generic $\bar{\xi}$ such that $|\bar{\xi}-\xi^0| \leqslant M\delta$, subunit symbols q_5, q_6. Let $\psi \in C_0^\infty(\mathbf{R}^2)$, $0 \leqslant \psi \leqslant 1$, $\psi \equiv 1$ on $\{\xi; |\xi-\bar{\xi}| \leqslant \frac{2}{3} M\delta\}$, $\psi \equiv 0$ on $\{\xi; |\xi-\bar{\xi}| \geqslant M\delta\}$. Define

$$q_5(x, \xi) = cM\delta^2 \varphi_1(x)\, \psi(\xi) \tag{66}$$

$$q_6(x, \xi) = cM\delta^2 \varphi_2(x)\, \psi(\xi). \tag{67}$$

Consider q_5 (q_6 is similar):

$$q_5(x, \xi)^2 \leqslant c^2 M^2 \delta^4 \sigma(V) \leqslant V(x_1, x_2) \qquad \text{on} \quad R(V) \times \operatorname{supp} \psi;$$

$|\alpha| = 2$,

$$|\partial_x^\alpha q_5(x, \xi)| \lesssim \psi(\xi)\, M\delta^2\, |\partial_x^\alpha \varphi_1(x)| \lesssim M\delta^2 \delta^{-2} \sigma(V)^{1/2-|\alpha|/4} = M;$$

$|\beta| = 2$,

$$|\partial_\xi^\beta q_5(x, \xi)| \lesssim M\delta^2 \sigma(V)^{1/2}\, (M\delta)^{-2} \lesssim \frac{1}{M};$$

$|\alpha| = |\beta| = 1$,

$$\begin{aligned} |\partial_x^\alpha \partial_\xi^\beta q_5(x, \xi)| &\lesssim M\delta^2\, |\partial_x^\alpha \varphi_1(x)|\, |\partial_\xi^\beta \psi(\xi)| \\ &\lesssim M\delta^2 \delta^{-|\alpha|} \sigma(V)^{1/2-|\alpha|/4}\, (M\delta)^{-1} \lesssim 1. \end{aligned}$$

Hence $q_5, q_6 \in \mathscr{S}(p, Q)$ and $q_5, q_6 \in \mathscr{S}(p_1^\#, \delta \times M\delta)$ as well. We can therefore flow along the trajectories γ_5, γ_6, generated by

$$H_{q_5}(x, \xi) \sim M\delta\sigma(V)^{1/4} \frac{\partial}{\partial \xi_1}$$

$$H_{q_6}(x, \xi) \sim M\delta\sigma(V)^{1/4} \frac{\partial}{\partial \xi_2},$$

whence

$$\{(x,\xi);\ |x_1-x_1^0|\lessapprox 1,\ |x_2-x_2^0|\lessapprox \delta,$$
$$|\xi-\xi^0|\lessapprox M\delta\Delta_0+M\delta\sigma(V)^{1/4}\}\subset B_p((x^0,\xi^0),1).$$

We now prove the "opposite inclusion."

Applying Lemma 5.11 in the Ψ-coordinates yields that

$$M\,|x_2(t)-x_2^0|+|\xi(t)-\xi^0|\leqslant 4C_* M\delta$$

for any subunit broken path $(x(t),\xi(t))$ starting at (x^0,ξ^0). Let $(x^k(t), \xi^k(t)):=\gamma_k(t)$ be a segment (generated by the subunit Hamiltonian $q_k(x,\xi)$) of a broken path $\Gamma(t;x^0,\xi^0)=(x(t),\xi(t))$ starting at (x^0,ξ^0).

Consider

$$\dot{\xi}^k(t)=-\partial_x q_k(\Gamma(t;x^0,\xi^0))$$

(for $t\in(t_k,t_{k+1})$). As already noted, for a universal constant $c>0$,

$$q_1=q_{1k}+q_{2k},\qquad cq_{1k}\in\mathscr{S}(\xi_1^2,Q),\quad cq_{2k}\in\mathscr{S}(p_1,Q).$$

As previously done, write $R_\delta'''=\bigcup_\nu Q_\delta^\nu$.

Localize, as we are allowed to, p_1 to such Q_δ^ν. As above,

$$q_{1k}(x,0,\xi_2)\equiv 0\Rightarrow|\nabla_x q_{1k}(x,\xi_1^0,\xi_2)|\lessapprox|\xi_1^0|.$$

Hence from

$$(\nabla_x q_{1k})(\Gamma(t))=(\nabla_x q_{1k})(\Gamma^1(t),\xi_1^0,\xi_2(t))+O(|\xi_1(t)-\xi_1^0|),$$

it follows that

$$|\nabla_x q_{1k}(\Gamma(t))|\lessapprox|\xi_1^0|+|\xi_1(t)-\xi_1^0|.$$

Moreover,

$$|\dot{\xi}^k(t)|\lessapprox|\xi_1^0|+\max_\nu\max_{x\in(I\times J_\delta)^{\#}}|\partial_x q_{2k\nu}(x(t);0,\xi_2(t))|+|\xi_1(t)-\xi_1^0|.\tag{68}$$

Now, $q_{2k\nu}=q_{2k\nu}^1+q_{2k\nu}^2$, where $q_{2k\nu}^1$ is subordinate to $\ell(x,\xi_2)^2$ essentially in Q_δ^ν, and $q_{2k\nu}^2$ is subordinate to $M^2\delta^4V(x_1,x_2)$ essentially in Q_δ^ν. Hence

$$|\partial_x^\alpha\partial_\xi^\beta q_{2k\nu}^j(x,\xi)|\leqslant C_{\alpha\beta}M\delta^2(M\delta)^{-|\beta|}\delta^{-|\alpha|},\qquad|\alpha|+|\beta|\leqslant 2.$$

Let

$$\Sigma=\{(x,\xi)\in R'''_\delta;\ \xi_2=M\delta b_0(x_1,x_2)\}\qquad\text{and}\qquad\Sigma_\nu=\Sigma\cap Q_\delta^\nu.\tag{69}$$

Σ is the zero set of $\partial_{\xi_2} p = \partial_{\xi_2} p_1$. Then

$$\nabla_x (q^1_{2k\nu \,|\, \Sigma_\nu}) = (\nabla_x q^1_{2k\nu})_{|\Sigma_\nu} + M\delta (\partial_{\xi_2} q^1_{2k\nu})_{|\Sigma_\nu} \nabla_x b_0(x) = 0. \tag{70}$$

Since $|\partial_{\xi_2} q^1_{2k\nu}(x, \xi)| \lessapprox M\delta^2 (M\delta)^{-1} = \delta$, by Taylor expanding $\nabla_x q^1_{2k\nu}$ at Σ_ν, we obtain (using $|\xi_2 - M\delta b_0(x)| \leqslant |\xi_2 - \xi^0_2| + |\xi^0_2| + M\delta\, |b_0(x)|$)

$$|\nabla_x q^1_{2k\nu}(x, \xi)| \lessapprox M\delta^2\, |\nabla_x b_0(x_1, x_2)| + |\xi^0| + M\delta \max_{x \in (I \times J_\delta)^{\#}} |b_0(x)| + |\xi - \xi^0| \tag{71}$$

for $(x, \xi) \in Q^\nu_\delta$.

Remark once more that the maxima of $|b_0|$ on rectangles of comparable diameters are comparable, and the same applies to V.

Now,

$$\max_{x \in (I \times J_\delta)^{\#}} \delta\, |\nabla b_0| \lessapprox \max_{x \in (I \times J_\delta)^{\#}} |b_0| = \sigma(b_0^2)^{1/2},$$

whence

$$|\nabla_x q^1_{2k\nu}(x, \xi)| \lessapprox |\xi^0| + M\delta\sigma(b_0^2)^{1/2} + |\xi - \xi^0|, \qquad (x, \xi) \in B, \tag{72}$$

where

$$B = \{(x, \xi);\, |x_1 - x^0_1| \leqslant C_*,\, |x_2 - x^0_2| \leqslant C_*\delta,\, |\xi - \xi^0| \leqslant C_* M\delta\}.$$

Of course, only the Q^ν_δ whose ( )**-dilate intersect B are to be considered.

By Lemma 4.2 we also get

$$|\nabla_x q^2_{2k\nu}(x, \xi)| \lessapprox |\xi - \xi^0| + |\xi^0| + M\delta\sigma(V)^{1/4}. \tag{73}$$

Finally, we obtain, for any $k = 0, 1, ..., L-1$,

$$|\partial_x q_k(\Gamma(t; x^0, \xi^0))| \lessapprox |\xi^0| + M\delta\sigma(b_0^2)^{1/2} + M\delta\sigma(V)^{1/4} + |\Gamma_2(t; x^0, \xi^0) - \xi^0|. \tag{74}$$

Applying Lemma 5.4 gives

$$\begin{aligned} |\Gamma_2(t; x^0, \xi^0) - \xi^0| &\lessapprox |\xi^0| + M\delta\sigma(b_0^2)^{1/2} + M\delta\sigma(V)^{1/4} \\ &= M\delta\Delta_0 + M\delta\sigma(V)^{1/4}. \end{aligned} \tag{75}$$

We have hence proved

$$\begin{aligned} B_p((x^0, \xi^0), 1) \subset \{(x, \xi);\, &|x_1 - x^0_1| \lessapprox 1,\, |x_2 - x^0_2| \lessapprox \delta, \\ &|\xi - \xi^0| \lessapprox M\delta\Delta_0 + M\delta\sigma(V)^{1/4}\}. \end{aligned}$$

Thus

$$B_p((x^0, \xi^0), 1) \approx \{(x, \xi); |x_1 - x_1^0| \leqslant 1, |x_2 - x_2^0| \leqslant \delta,$$
$$|\xi - \xi^0| \leqslant M\delta\Delta_0 + M\delta\sigma(V)^{1/4}\}. \quad \blacksquare$$

Remark 5.14. In the case the ξ_2^ν in formula (47) happens to be ξ_2^0, we have that $(\xi_2^0)_{\text{new}} = -M\delta\bar{b}(x_2^0)$. Hence the Δ_0 in Theorem 5.12 takes the form

$$\Delta_0 = \frac{|\xi_1^0|}{M\delta} + |\bar{b}(x_2^0)| + \sigma(b_0^2)^{1/2}. \tag{76}$$

5.5. *The* (2+2)-*Dimensional Case*: *Conclusion*

We now present the construction of B_p in the general (2+2)-dimensional nonelliptic–nondegenerate case.

We recall the general setup, using the results and notations of the preceding Sections 5.3 and 5.4.

We are considering $\rho^2 p$, nonelliptic–nondegenerate on a C.Z. block Q_ρ centered at $(\bar{x}, \bar{\xi})$, $|\bar{\xi}_1| \lessapprox M\rho$, of sizes $\rho \times M\rho$.

The derived symbol $\bar{p}_\rho^*(x_2, \xi_2)$ is supposed to be nonelliptic–nondegenerate on a block $Q_{\rho\delta}^2 \subset \mathbf{R} \times \mathbf{R}$, centered at (x_2^*, ξ_2^*) containing (x_2^0, ξ_2^0). In particular, $|\xi_1^0| \lessapprox M(\rho\delta)$.

$\rho^2 \bar{p}_\rho \in S^2(Q_{\rho\delta}^2)$, $p_1 \in S^2(1 \times M)$, $x_1 \mapsto p_1(x_1, \cdot, \cdot)$ polynomial in x_1 at scale ρ, yield that estimates (46) are valid for $\rho^2 p_1$ on the region

$$\tilde{R}_\delta := \pi_{x_1}(Q_\rho) \times \pi_{x_2}(Q_{\rho\delta}^2) \times \pi_{\xi_1}(Q_\rho) \times \pi_{\xi_2}(Q_{\rho\delta}^2)$$

(and actually on a large dilate of it; see Section 5.4). Moreover, $\rho^2 p_1$ can then be localized (see the footnote at the beginning of Section 5.4) to sub-blocks of $\tilde{R}_\delta$ of sizes $\rho\delta \times M\rho\delta$. By Lemma 5.8 and Lemma 5.9 it follows that there exists a region $R_\delta \subset \tilde{R}_\delta^*$ of the form

$$R_\delta := I_\rho \times J_{\rho\delta} \times I_{M\rho} \times J_{M\rho\delta}$$

(which we shall refer to as a "good band") with $I_\rho \times I_{M\rho} \subset \pi_{(x_1, \xi_1)}(Q_\rho)$, $J_{\rho\delta} \times J_{M\rho\delta} \subset Q_{\rho\delta}^{2**}$ center$(J_{\rho\delta} \times J_{M\rho\delta}) = (x_2^0, \xi_2^0)$, on which $\rho^2 p_1$ can be written in the form (see (47))

$$\rho^2 p_1(x_2, x_2, \xi_2) = \rho^2\delta^2 e(x, \xi_2)(\xi_2 - \xi_2^0 - \theta(x_1, x_2))^2 + M^2\delta^4\rho^2\tilde{V}(x_1, x_2)$$

(see Lemma 5.8), where θ is an algebraic function in x_1, polynomial of *a priori* bounded degree in x_2; θ, $\tilde{V}$ satisfying the estimates (48), (49) in (i) and (ii) of Section 5.4.

We have hence Taylor expanded first $x_1 \mapsto p_1(x_1, \cdot, \cdot)$ at scale ρ, then $x_2 \mapsto \theta(\cdot, x_2)$ at scale $\rho\delta$, and finally, applying Consequence 1 in Section 2, $\rho^2 p_1$ in all the variables at scale $\rho \times \rho\delta \times M\rho\delta$. We may therefore regard $\rho^2 \tilde{V}$ as the polynomial $(M^2\delta^4)^{-1}\, \rho^2 p_{1\,|\,\tilde{R}_\delta}$ evaluated at the graph $\xi_2 = \xi_2^0 + \theta(x_1, x_2)$. All this can be achieved by choosing, in an *a priori* way, λ (the initial dilation parameter of the C.Z. decomposition) and $M_{\min}$. After the symplectic dilation

$$s\colon (x_1, x_2, \xi_1, \xi_2) \mapsto \left(\frac{x_1 - \bar{x}_1}{\rho}, \frac{x_2 - x_2^*}{\rho}, \rho\xi_1, \rho(\xi_2 - \xi_2^*)\right),$$

using M in place of $M\rho^2$, we can hence suppose that $R_\delta = I \times J_\delta \times I_M \times J_{M\delta}$ is a region such that (note that x_1^0 might not belong to I) $R_\delta'''{}^* \subset Q''''$ and on which $\rho^2 p$ (which we call p again) can be written as

$$p(x, \xi) = \xi_1^2 + \delta^2(\xi_2 - \xi_2^0 - M\delta b(x_2, x_2))^2 + (M\delta^2)^2\, V(x_1, x_2)$$

(see (51)), with

$$0 < e \text{ elliptic}, \quad e \in S^0(1 \times \delta \times M\delta), \qquad M\delta^2 b \in S^1(1 \times \delta \times M\delta),$$

$$0 \leqslant V, \qquad M^2\delta^4 V \in S^2(1 \times \delta \times M\delta)$$

(see Section 5.4).

By Lemma 5.11 we have an *a priori* box containing the subunit ball:

$$\pi_{x_1}(Q)^\# \times \tilde{B} = \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2;\ |x_1 - x_1^0| \leqslant 4C_*,$$
$$|x_2 - x_2^0|\ M\delta + |\xi - \xi^0| \leqslant 4C_* M\delta\}.$$

Hence $\pi_{(x_2, \xi_2)}(B_p((x^0, \xi^0), 1)) \subset \tilde{B}$. Consider Ψ (see (55) in Section 5.4 and notations used therein). Then, by picking λ *a priori* large, we have that

$$\xi_2 \in (J_{M\delta})', \qquad x_2 \in J_\delta^\# \Rightarrow \xi_2 - \xi_2^0 - M\delta\bar{b}(x_2) \in (J_{M\delta})''.$$

Thus, we can achieve the situation in which

$$\Psi(R_\delta^\#) \subset R_\delta''.$$

Applying then Facts (F1), (F2), and (F3) of the previous Section 5.4, we have, working in Ψ-coordinates, that the subunit ball is contained in an equivalent, through Ψ, box which we call again $\pi_{x_1}(Q)^\# \times \tilde{B}$, with $\pi_{x_1}(Q)^\# \times \tilde{B} \subset R_\delta''$. It follows that we can work, in Ψ-coordinates, sitting in the region $\tilde{R}_\delta'''{}^*$.

Let us now set, for (x^0, ξ^0) the center of the subunit ball in Ψ-coordinates,

$$\tilde{\Delta}_0 := \frac{|\xi_1^0|}{M\delta} + |\bar{b}(x_2^0)| + \sigma(b_0^2)^{1/2} + \sigma(V)^{1/4}. \tag{77}$$

Consider now a C.Z. decomposition of $\tilde{R}_\delta''*$, relative to $p_1(x_1, x_2, \xi_2)$ in $\mathbf{R}^2 \times \mathbf{R}^2$, into blocks Q_ν of various sizes and centers, which we call $\delta_\nu \times M\delta_\nu$ and (x^ν, ξ^ν) respectively. We write $\tilde{R}_\delta$ here as

$$\tilde{R}_\delta = \bigcup_{k_1, k_2} (I_\delta^{k_1} \times \pi_{x_2}(Q_\delta^2) \times I_{M\delta}^{k_2} \times \pi_{\xi_2}(Q_\delta^2)),$$

with $|I_\delta^{k_1}| \sim \delta$, $|I_{M\delta}^{k_2}| \sim M\delta$ (and *a priori* bound on overlappings for their $(\ )^{**}$ dilates). By picking λ larger than an *a priori* λ_0, we can achieve the situation in which, in Ψ-coordinates, denoting by $Q^\natural$ the $k(\lambda)^{1/4}$-dilate of Q_ν, we have

$$Q_\nu^{\natural\natural} \subset \tilde{R}_\delta'''* \qquad \text{and} \qquad Q_\nu^{\natural\natural} \cap Q_\mu^{\natural\natural} \neq \varnothing \Rightarrow \delta_\nu \sim \delta_\mu.$$

Since $p_1(x_1, x_2, \xi_2)$ need not satisfy condition (s.e.), we have to introduce, recalling Fefferman and Phong's Calderon–Zygmund decomposition (see Section 2 before Lemma 2.3), a stopping condition: we stop cutting when the sizes of the block Q_ν, i.e., $\delta_\nu \times M\delta_\nu$, satisfy

$$\delta_\nu \sim \delta\tilde{\Delta}_0. \tag{78}$$

Note that we are allowed to use C.Z. since p_1 can be localized to sizes $\delta \times M\delta$ (i.e., it defines an element of $S^2(\delta \times M\delta)$ when restricted to a block of sizes $\delta \times M\delta$). Since $(x_2^0, \xi_2^0) \in Q_\delta^2$, a nonellipticity–nondegeneracy C.Z. block for $\bar{p}_1^*(x_2, \xi_2)$ (after the rescaling s), it follows that each δ_ν is such that $0 < \delta_\nu \leqslant \delta$.

Suppose $p_{1\,|\,Q_\nu}$, for some C.Z. block Q_ν, is nonelliptic–nondegenerate. This might be caused either by the x_1, or by the x_2, or by the ξ_2 variable. Corresponding to these cases, by Lemma 2.5 and Remark 2.6, we have the following

Lemma 5.15. *On a large dilate of Q_ν either*

(i) $p_1(x, \xi_2) = \delta_\nu e_\nu(x, \xi_2)(\xi_2 - \xi_2^\nu - \theta_\nu(x_1, x_2))^2 + (M\delta_\nu^2)^2\, V_\nu(x)$,

with $0 < e_\nu$, elliptic, belonging to $S^0(\delta_\nu \times M\delta_\nu)$, $\delta_\nu\theta_\nu \in S^1(\delta_\nu \times M\delta_\nu)$, $0 \leqslant (M\delta_\nu^2)^2\, V_\nu \in S^2(\delta_\nu \times M\delta_\nu)$; or

$$\text{(ii)}\quad p_1(x,\xi_2)=\delta_\nu^2 e_\nu(x,\xi_2)(M(x_2-x_2^\nu) - g_\nu(x_1,\xi_2))^2+M^2\delta_\nu^4 V_\nu(x_1,\xi_2),$$

with e_ν *as above,* $\delta_\nu g_\nu\in S^1(\delta_\nu\times M\delta_\nu)$, $0\leqslant M^2\delta_\nu^4 V_\nu\in S^2(\delta_\nu\times M\delta_\nu)$; *or*

$$\text{(iii)}\quad p_1(x,\xi_2)=\delta_\nu^2 e_\nu(x,\xi_2)(M(x_1-x_1^\nu) - g_\nu(x_2,\xi_2))^2+M^2\delta_\nu^4 V_\nu(x_2,\xi_2),$$

with e_ν, $\delta_\nu g_\nu$, $M^2\delta_\nu^4 V_\nu$ *having the same properties as above. Moreover,* θ_ν, g_ν *are "rescaled" algebraic functions. By Lemma* 2.4, *the other cases left out are*

$$\text{(iv)}\quad p_{1\,|\,Q_\nu}\ \text{elliptic},\qquad p_{1\,|\,Q_\nu}\sim M^2\delta_\nu^4;$$

$$\text{(v)}\quad \delta_\nu\sim\delta\tilde{\Delta}_0.$$

We remark that cases (i), (ii), (iii) of the above lemma are due to the fact (see Remark 2.6) that $p_1\geqslant 0$, 4 th-order derivatives under control and non-elliptic–nondegeneracy, yield that $\partial_{x_1}^2 p_1$ or $\partial_{x_2}^2 p_1$ or $\partial_{\xi_2}^2 p_1$ are "big."

In the case (i) above, define the manifold (at scale $\delta_\nu\times M\delta_\nu$)

$$\Sigma_{2,\nu}=\{(x,\xi);\ \xi_2=\xi_2^\nu+\theta_\nu(x_1,x_2),\ (x,\xi)\in Q_\nu^\natural\}=\{(x,\xi)\in Q_\nu^\natural;\ \partial_{\xi_2}p_1=0\}.$$

Define also

$$\theta_\nu^0(x_1,x_2)=\theta_\nu(x_1,x_2)-\frac{1}{N_{\max}}\sum_{j=1}^{N_{\max}}\theta_\nu(x_1^j,x_2):=\theta_\nu(x_1,x_2)-\theta_\nu^*(x_2),\tag{79}$$

where $N_{\max}$ is an *a priori* chosen number (depending on the subellipticity constants). The $\Sigma_{2,\nu}$ give rise to the stratification mentioned in the Introduction, stratification caused, as we will see in a moment, by the graphs of the functions θ_ν^*.

Remark 5.16. The good band R_δ is not *a priori* unique. There might be other good bands farther away from x_1^0. It follows that some of the δ_ν in case (i) of Lemma 5.15 can be $\sim\delta$. Hence the normal form (i) for p_1 would hold for x_1 in an interval of size 1. (We may think of the example

$$p_1(x_1,x_2,\xi_2)=\delta^2(x_1(\tfrac{1}{2}-x_1)\,\xi_2-Mx_2)^2+M^2\delta^4 V(x_1,x_2)$$

for $|x_1|\leqslant 1$, $|x_2|\leqslant\delta$, $|\xi_2|\leqslant M\delta$.)

We use the good band for moving (x_2, ξ_1, ξ_2). We in fact move from (x^0, ξ^0) to the good band through the trajectory with subunit Hamiltonian ξ_1. Hence, from Theorem 5.12 follows

LEMMA 5.17. *As long as* $(x, \xi) \in R''_\delta$ (*i.e.*, $x_1 \in I_\delta$),

$$R''_\delta \cap B_p((x^0, \xi^0), 1) \approx \{(x, \xi); x_1 \in I_\delta, |x_2 - x_2^0| \leqslant \delta, |\xi - \xi^0| \leqslant M\delta\tilde{\Delta}_0\}.$$

Moreover, our a priori choice of the constant c_0 *when finding* R_δ *was made in such a way that we can, by subunit paths, reach* R_δ *starting from* (x^0, ξ^0), *move there, and go back to conclude that*

$$\{(x, \xi); |x_1 - x_1^0| \lessapprox 1, |x_2 - x_2^0| \lessapprox \delta, |\xi - \xi^0| \lessapprox M\delta\tilde{\Delta}_0\} \subset B_p((x^0, \xi^0), 1).$$

Our purpose is now to describe what happens when we exit the good band, moving x_1 by order ~ 1. In fact, exiting R_δ gives new contributions to the ξ-size of B_p (not to the x_2-size, which is already δ, the biggest allowed by the C.Z. decomposition of $p_1^*(x_2, \xi_2)$). Of course, blocks on which (v) of Lemma 5.15 holds don't give any contributions. Since we have *a priori* information on the size of the region containing the subunit ball (i.e., $\pi_{x_1}(Q)^{\#} \times \tilde{B}$), we shall have to consider just those Q_ν for which $Q_\nu \cap (\pi_{x_1}(Q) \times \tilde{B}) \neq \varnothing$.

LEMMA 5.18. *Suppose the block* Q_ν *can be reached through a subunit path starting at* (x^0, ξ^0) *and passing through* R_δ. *Then cases* (ii), (iii), (iv) *of Lemma* 5.15 *are equivalent, that is, the displacement given by subunit paths is* $\sim \delta_\nu$ *in the* x*-direction,* $\sim M\delta_\nu$ *in the* ξ*-direction.*

Proof. Case (iv) is an immediate consequence of the elliptic case. The Q_ν, on which (iv) holds, contribute to the subunit displacement by an amount $\sim \delta_\nu$ in the x-direction, $\sim M\delta_\nu$ in the ξ-direction.

Let us consider the case (iii):

$$\delta_\nu^2(M(x_1 - x_1^\nu) - g_\nu(x_2, \xi_2))^2 + M^2\delta_\nu^4 V_\nu(x_2, \xi_2) \lessapprox M^2\delta_\nu^4.$$

Hence, for a subunit Hamiltonian for $p_1^{\#} := \delta_\nu^2(\xi_1 - \xi_1^\nu)^2 + p_{1\,|\,Q_\nu}$ on Q_ν, we have $|\partial_x q| \leqslant M\delta_\nu$, $|\partial_\xi q| \leqslant \delta_\nu$. On the other hand, we can always move x_1 according to the subunit (for ξ_1^2) vector field $\partial/\partial x_1$ to reach (at a time ~ 1), with $(x_2, \xi_2) = (\bar{x}_2, \bar{\xi}_2)$ *fixed*, a maximum in Q_ν of the *polynomial* $x_1 \mapsto \ell_{1,\nu}(x_1, \cdot, \cdot)^2$ where $\ell_{1,\nu}^2$ is the "quadratic" part of the normal form (iii). The maximum is therefore of the order of (by virtue of the bounds on g_ν)

$$\mathrm{Av}_{x_1 \in \pi_{x_1}(Q_\nu)} \ell_{1,\nu}^2 \sim M^2\delta_\nu^4$$

(the corresponding point being, say, $\bar{x}_1$). It follows (rescaling, as usual, to the unit cube) that we can find a neighborhood U of sizes $\sim \delta_v \times M\delta_v$ on which

$$\ell_{1,v}(x_1, x_2, \xi_2)^2 \sim M^2\delta_v^4.$$

Hence the elliptic construction applies also in this situation.

Case (ii). As in case (iii) we have that the gradient of subunit Hamiltonians satisfies the above inequalities. As above, $p_{1\,|\,Q_v} \lesssim M^2\delta_v^4$. Since in the good band we can move x_2 by order $\sim \delta$ ($\geqslant \delta_v$), it follows that we can reach (at time ~ 1) a maximum point for the polynomial $x_2 \mapsto \ell_{2,v}(\cdot, x_2, \cdot)^2$ (at x_1, ξ_2 fixed), where $\ell_{2,v}^2$ is the quadratic part of the normal form (ii). In fact, we start with $(x_1, x_2; \bar{\xi}) \in R_\delta \cap B_p$. Let $\bar{x}_1$ be such that $(\bar{x}_1, x_2; \bar{\xi}) \in Q_v$. We then move $(x_1, x_2; \bar{\xi})$ to $(x_1, \bar{x}_2; \bar{\xi})$ where x_2 is such that

$$\ell_{2,v}(\bar{x}_1, \bar{x}_2, \bar{\xi}_2)^2 \sim \mathrm{Av}_{x_2 \in \pi_{x_2}(Q_v)} \ell_{2,v}^2 \sim M^2\delta_v^4.$$

(Again, this is possible because $\ell_{2,v}^2$ is a non-negative polynomial in x_2 and by virtue of the bounds on g_v.) Then $\ell_{2,v}(\bar{x}_1, \bar{x}_2, \bar{\xi}_2)^2 \sim M^2\delta_v^4$ and we conclude as above using the elliptic case. ∎

We now study the bounds for the gradients of subunit Hamiltonians at points at which the normal form (i) of Lemma 5.15 holds. (Hence, in the Q_v with this property, $\Sigma_{2,v}$ is a nonempty manifold.)

PROPOSITION 5.19. *Suppose the block Q_v, on which we have the normal form* (i) *for p_1, can be reached through a subunit path starting at (x^0, ξ^0). Define on Q_v the function θ_v^0 (see* (79)) *and let $(\bar{x}, \bar{\xi}) \in Q_v$ be a reachable point*: $(\bar{x}, \bar{\xi}) = \Gamma(\bar{t}; x^0, \xi^0)$, $\bar{t} \sim 1$. (*Note that we enter and leave Q_v, generically, by means of $\partial/\partial x_1$.*)

Let γ be an arc of subunit path starting at $(\bar{x}, \bar{\xi})$, with subunit Hamiltonian q. Then the following bound for the speed of the ξ-component of γ holds:

$$|\partial_x q(x, \xi)| \lesssim M\delta_v \tilde{\Delta}_v, \qquad (x, \xi) \in \gamma,$$

where, with $I_v^2 := \pi_{x_2}(Q_v)$,

$$\begin{aligned}\tilde{\Delta}_v &= \tilde{\Delta}_v(\bar{x}, \bar{\xi}) \\ &= \frac{|\xi_1^v - \bar{\xi}_1| + |\bar{\xi}_2 - \xi_2^v - \theta_v^*(\bar{x}_2)|}{M\delta_v} + \sigma\left(\left(\frac{\theta_v^0}{M\delta_v}\right)^2\right)^{1/2} \\ &\quad + \sigma(V_v)^{1/4} + \frac{\|\partial_{x_2}\theta_v^*\|_{L^\infty(I_v^2)}}{M},\end{aligned}$$

with $\sigma(f) := \max_{x \in \pi_x(Q_v)} |f(x)|$. *In the case in which the normal form* (i) *holds on a good band* (*i.e., with* x_1 *ranging order* 1), $\sigma(f) := \max_{x \in 1 \times \delta_v} |f(x)|$ *with* $|x_1 - \bar{x}_1| \lesssim 1$ *in the subunit ball. Note that the function*

$$(\bar{x}, \bar{\xi}) \mapsto \tilde{\Delta}_v(\bar{x}, \bar{\xi})$$

is continuous *on* Q_v.

Proof. Define the tame (at scale $\delta_v \times M\delta_v$) canonical transformation

$$\Psi_v: (x_1, x_2, \xi_1, \xi_2) \mapsto (x_1, x_2, \xi_1, \xi_2 - \xi_2^v - \theta_v^*(x_2)).$$

Since we can always move according to the vector field $\partial/\partial x_1$, we apply the proof of Theorem 5.12 to the symbol

$$p_v^{\#}(x, \xi) := \delta_v^2(\xi_1 - \xi_1^v)^2 + (p_1 \circ \Psi_v^{-1})(x_1, x_2, \xi_2),$$

$(x, \xi) \in \Psi_v(Q_v^{\natural\natural})$, the only modification being that we have to substitute $\mathrm{Av}_{x_1 \in \pi_{x_1}(Q_v)}$ with the discrete average

$$\mathrm{Avd}_{x_1 \in \pi_{x_1}(Q_v)} := \frac{1}{N_{\max}} \sum_{k=1}^{N_{\max}} \delta(x_1 - x_1^k),$$

where δ is the Dirac function. This modification allows us to use Theorem 4.4′ of Section 4. In Ψ_v-coordinates we have the bound given by (we write $\Psi_v = (\Psi_{vy}, \Psi_{v\eta})$)

$$|\partial_y(q \circ \Psi_v^{-1})(y, \eta)|$$
$$\lesssim M\delta_v \left(\frac{|\Psi_{v\eta}(x^v, \xi^v) - \Psi_{v\eta}(\bar{x}, \bar{\xi})|}{M\delta_v} + \sigma \left(\left(\frac{\theta_v^0}{M\delta_v} \right)^2 \right)^{1/2} + \sigma(V_v)^{1/4} \right).$$

Pulling things back to Q_v (by means of Ψ_v^{-1}), we have

$$\partial_{x_2} q(x, \xi) = (\partial_{y_2}(q \circ \Psi_v^{-1}))(\Psi_v(x, \xi)) + (\partial_{\eta_2}(q \circ \Psi_v^{-1}))(\Psi_v(x, \xi)) \frac{\partial \eta_2}{\partial x_2}(x, \xi).$$

Noting that

$$\left| \partial_{\eta_2}(q \circ \Psi_v^{-1})(\Psi_v(x, \xi)) \frac{\partial \eta_2}{\partial x_2}(x, \xi) \right| \leqslant C\delta_v \|\partial_{x_2} \theta_v^*\|_{L^\infty(I_v^2)}$$
$$= CM\delta_v \frac{\|\partial_{x_2} \theta_v^*\|_{L^\infty(I_v^2)}}{M}$$

gives the proposition. ∎

Remark that, by Theorem 4.4′,

$$\frac{\|\partial_{x_2}\theta_v^*\|_{L^\infty(I_v^2)}}{M} \lesssim (M\delta_v)^{-1} \max_{x_2 \in I_v^2} |\theta_v^*(x_2)|.$$

Let $\mathcal{N} := \{v; Q_v \cap (\pi_{x_1}(Q)^\# \times \tilde{B}) \neq \emptyset\}$ and let Δ_v be the *optimal subunit displacement* relative to Q_v.

By this we mean:

(1) In cases (ii) and (iii) of Lemma 5.15 we have, by Lemma 5.18, $\Delta_v = \delta_v$, the length of the x-side of Q_v; whereas,

(2) in case (i)

$$\Delta_v := \max_{(\bar{x}, \bar{\xi}) \in Q_v} \tilde{\Delta}_v(\bar{x}, \bar{\xi}).$$

Combining Lemmas 5.15, 5.17 and 5.18 and Proposition 5.19 gives the following structure theorem for the subunit ball of radius ρ:

THEOREM 5.20. *Define* $\Delta_0^+ := \max\{\tilde{\Delta}_0, \max\{\Delta_v; v \in \mathcal{N}\}\}$. *Then, after the symplectic scaling s and the transformation* Ψ (*see* (55)), *calling p the symbol* $\rho^2 p \circ s^{-1} \circ \Psi^{-1}$ (*i.e., setting* $\rho = 1$ *and* $M = M\rho^2$),

$$B_1 \subset B_p((x^0, \xi^0), 1) \subset B_2,$$

where

$$B_1 = \{(x, \xi) \in \mathbf{R} \times \mathbf{R}^2;\ |x_1 - x_1^0| \lesssim 1,\ |x_2 - x_2^0| \lesssim \delta,\ |\xi - \xi^0| \lesssim M\delta\tilde{\Delta}_0\}$$

and

$$B_2 = \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R}^2;\ |x_1 - x_1^0| \lesssim 1,\ |x_2 - x_2^0| \lesssim \delta,\ |\xi - \xi^0| \lesssim M\delta\Delta_0^+\}.$$

Remark 5.21. Suppose $|\xi_1^0|/M \sim \delta_v$ for some v. The use of the good band makes it possible to conclude that for those v, $\Delta_v = |\xi_1^0|/(M\delta_v)$.

This is already contained in the *stopping condition* (78). It is equivalent to taking a C.Z. decomposition relative to the symbol $(|\xi_1^0|/M)^4 M^2 + p_1(x_1, x_2, \xi_2)$.

Remark 5.22. In case (ii) of Lemma 5.15 we define the rescaled algebraic function $F_v(\xi_2) := (M\delta_v)^{-1} (g_v^*(\xi_2) - g_v^*(\bar{\xi}_2)) + \bar{x}_2$, for some

$(\bar{x}_2, \bar{\xi}_2) \in \pi_{(x_2, \xi_2)}(Q_\nu)$, $\nu \in \mathcal{N}$. Then, in Q_ν^*, we can fill in, by subunit trajectories, a box of the kind

$$\{(x, \xi); |x_1 - \bar{x}_1| \lessapprox \delta_\nu, |x_2 - F_\nu(\xi_2)|\, M\delta_\nu + |\xi_1 - \bar{\xi}_1| \lessapprox M\delta_\nu D_\nu,$$
$$|\xi_2 - \bar{\xi}_2| \lessapprox M\delta_\nu\},$$

where now

$$D_\nu = D_\nu(\bar{x}, \bar{\xi}) := \frac{|\bar{\xi}_1 - \xi_1^\nu|}{M\delta_\nu} + |\bar{x}_2 - x_2^\nu - (M\delta_\nu)^{-1}\, g_\nu^*(\bar{\xi}_2)|$$
$$+ (\max_{(x_1, \xi_2) \in \pi_{(x_1, \xi_2)}(Q_\nu)} (g_\nu^0/(M\delta_\nu))^2)^{1/2} + (\max_{(x_1, \xi_2) \in \pi_{(x_1, \xi_2)}(Q_\nu)} V_\nu)^{1/4}.$$

We stress once more that it was the use of the good band which allowed us to get the best possible displacement in this case.

We next show, by studying an example, that Theorem 5.20 is optimal. We shall in fact exhibit a symbol for which the "stratification" we referred to in the Introduction occurs, and for which one is able to compute the "critical radii."

THE EXAMPLE $\xi_1^2 + (x_1\xi_2 - Mb)^2$. Consider on a (large dilate of a) block $Q \subset \mathbf{R}^2 \times \mathbf{R}^2$, centered at (0, 0) and of sizes $1 \times M$, the symbol

$$p(x, \xi) = \xi_1^2 + (x_1\xi_2 - Mb)^2$$

with $1 \gg b \gtrsim M^{\varepsilon - 2}$. We shall study $B_p(\gamma^0, \rho)$ as ρ varies, where $\gamma^0 = (x^0, \xi^0)$. In this example one can prove that the subunit ball $B_p((x^0, \xi^0), 1)$, for a suitable choice of $(x^0, \xi^0) = (\mu, 0, 0, 0)$, $1 \gg \mu > 0$, is not a box in the following sense. One can travel through subunit paths to regions in which the contributions allowed in the ξ-direction are strictly greater than the one given by the "good band" (to which (x^0, ξ^0) belongs). Since the choice of μ may be made in such a way that the time elapsed to reach such regions is of order 1, it is not possible to go back through subunit paths to points of the form $(x^0, \bar{\xi})$, with $|\bar{\xi}| \sim$ displacement strictly greater than the good-band displacement (the constants in $\sim$ being *a priori*). This prevents the ball from being a bow.

We omit the computations.

We now want to prove that, when considering the subunit ball of radius ρ, there exists a "critical radius" ρ_{cr}, determined depending on (x^0, ξ^0), such that, for $\rho \leqslant c\rho_{\mathrm{cr}}$ and $\rho \geqslant C\rho_{\mathrm{cr}}$ ($c, C > 0$ *a priori* constants), $B_p(\gamma^0, \rho)$ is essentially a box. Note then that for any fixed center (x^0, ξ^0), the number of such $\rho_{\rm cr}$ is *a priori* bounded. We shall use the following notations: $I_\rho = I_\rho(x_1^0) = [x_1^0 - \rho, x_1^0 + \rho]$ and

$$\bar{p}_\rho(x_2, \xi_2) := (\mathrm{Av}_{x_1 \in I_\rho} p_1)(x_2, \xi_2).$$

We hence consider, on the above Q of sizes $1 \times M$, centered at $(0, 0)$, the operator $\rho^2 p(x, \xi)$.

We already assume (see Assumption (A2v)) that

$$\rho_{\min} < \rho < \rho_{\max}. \tag{80}$$

On the other hand, considering $\rho^2 p$ on Q, we have that the C.Z. procedure stops at $Q_v \subset Q$ because either $\rho^2 p_{|Q_v}$ is elliptic or $\rho^2 p_{|Q_v}$ is nondegenerate.

Whenever $\gamma^0 \in Q_v$, block on which $\rho^2 p_{|Q_v}$ is elliptic, the ball is a box, hence we shall only deal with the case in which $\rho^2 p_{|Q_v}$ is nonelliptic–non-degenerate.

Since $\rho^2 p(x, \xi) = \rho^2 \xi_1^2 + \rho^2 p_1(x, \xi_2)$, nonelliptic–nondegeneracy will occur in Q_v such that sizes$(Q_v) \sim \rho \times M\rho$. We therefore have the following first condition: suppose $\gamma^0 \in Q_v$ with $\rho^2 p_{|Q_v}$ nonelliptic–nondegenerate, then $\rho^2 p(\gamma^0) \leqslant CM^2 \rho^4$, i.e.,

$$\rho \geqslant \sigma(\gamma^0) := \left(\frac{|\xi_1^0|^2}{M^2} + \left| \mu \frac{\xi_2^0}{M} - b \right|^2 \right)^{1/2}, \tag{81}$$

whence, whenever $\rho \lessapprox \sigma(\gamma^0)$ or $\rho \sim \sigma(\gamma^0)$, the ball is a box. In fact, since $M^2 \sigma(\gamma^0)^2 = p(\gamma^0)$ and $\rho \lessapprox \sigma(\gamma^0)$ (or $\rho \sim \sigma(\gamma^0)$) implies

$$\rho^2 p(\gamma^0) \lessapprox CM^2 \rho^4 = C\rho^2 M^2 \rho^2 \leqslant C' \rho^2 M^2 \sigma(\gamma^0)^2 \sim \rho^2 p(\gamma^0),$$

we have that $\rho^2 p(\gamma^0)$ is as big as the maximum of $\rho^2 p$ on the block Q_v of sizes $\rho \times M\rho$. Since $\rho^2 p$ is a polynomial, it follows that the ball is a box.

For $\rho \geqslant \rho_0 = \sigma(\gamma^0)/C^{1/2}$, we consider now a C.Z. decomposition relative to $p_\rho^*(x_2, \xi_2)$. In this case

$$\begin{aligned} p_\rho^*(x_2, \xi_2) &= \rho^2 \bar{p}_\rho(x_2, \xi_2) + \left(\frac{|\xi_1^0|}{M} \right)^4 M^2 \\ &= \rho^2 (\mu \xi_2 - Mb)^2 + \frac{1}{3} \rho^4 \xi_2^2 + \left(\frac{|\xi_2^0|}{M} \right)^4 M^2 \\ &= \rho^2 \left(\mu^2 + \frac{1}{3} \rho^2 \right) \left(\xi_2 - \frac{Mb\mu}{\mu^2 + (1/3)\,\rho^2} \right)^2 + \frac{M^2 b^2 \rho^4}{3(\mu^2 + (1/3)\,\rho^2)} \\ &\quad + \left(\frac{|\xi_1^0|}{M\rho} \right)^4 M^2 \rho^4. \end{aligned} \tag{82}$$

We look at

$$\partial_{\xi_2}^2 p_\rho^*(x_2, \xi_2) \equiv 2\rho^2\sigma(\mu, \rho), \qquad \partial_{x_2}^2 p_\rho^* \equiv 0, \tag{83}$$

where $\sigma(\mu, \rho) := \mu^2 + \frac{1}{3}\rho^2$.

It follows that

(i) $\partial_{\xi_2}^2 p_\rho^* \sim \rho^4$ in case $|\mu| \leqslant \rho$;

(ii) $\partial_{\xi_2}^2 p_\rho^* \sim \rho^2\mu^2$ in case $\rho \leqslant |\mu|$.

In case (i),

$$p_\rho^*(x_2, \xi_2) \sim \rho^4 \left(\xi_2 - \frac{Mb\mu}{\sigma(\mu, \rho)}\right)^2 + \left\{\frac{b^2}{\rho^2} + \left(\frac{|\xi_1^0|}{M\rho}\right)^4\right\} (M\rho^2)^2.$$

In case (ii),

$$p_\rho^*(x_2, \xi_2) \sim \rho^2\mu^2 \left(\xi_2 - \frac{Mb\mu}{\sigma(\mu, \rho)}\right)^2 + \left\{\frac{b^2}{\mu^2} + \left(\frac{|\xi_1^0|}{M\rho}\right)^4\right\} (M\rho^2)^2.$$

Remark that we are supposing $\rho^2 p$ (and hence p_ρ^*) can be localized to $Q_\nu \ni \gamma^0$, sizes$(Q_\nu) \sim \rho \times M\rho$. It follows that it must be, in case (i),

$$\frac{b^2}{\rho^2} + \left(\frac{|\xi_1^0|}{M\rho}\right)^4 := G_1(\rho) \leqslant C_1;$$

in case (ii),

$$\frac{b^2}{\mu^2} + \left(\frac{|\xi_1^0|}{M\rho}\right)^4 := G_2(\rho) \leqslant C_1,$$

where $C_1 > 0$ is a universal constant.

Hence, if $G_1(\rho) \geqslant C_2$, for an *a priori* constant $C_2 > 0$, we have in case (i) the ellipticity of p_ρ^* and the ball is a box; if $G_2(\rho) \geqslant C_2$, we again have ellipticity of p_ρ^* and the same conclusion holds in case (ii).

(Remark that if $G_1(\rho) \leqslant C_1$, then $b^2/\rho^2 \leqslant C_1/2$ or $(|\xi_1^0|/(M\rho))^4 \leqslant C_1/2$ or both; likewise for $G_2(\rho) \leqslant C_1$. Regarding $G_1(\rho) \geqslant C_2$, we have that at least one of b^2/ρ^2 and $(|\xi_1^0|/(M\rho))^4$ is greater than or equal to $C_2/2$.)

At any rate, the conditions on G_1 and G_2 determine a range of values of ρ. We next *suppose*

$$\tfrac{1}{3}\,|\mu| \geqslant \rho_0$$

and examine the following cases:

$$|\mu| \leqslant \rho, \quad \rho \in \{\rho \in \mathbf{R}_+;\, G_1(\rho) \leqslant C_1\} := S(G_1) \tag{84}$$

(in case $S(G_1)\cap[|\mu|, \rho_{\max}]\neq\varnothing$);

$$\rho\in[\rho_0, |\mu|]\cap S(G_2) \tag{85}$$

(in case the intersection of the two sets is non-empty).

In case (84) p^*_ρ is nondegenerate at scale $\sim\rho^2\times M\rho^2$ on a block $Q^2_{\nu_0}$, $\gamma^0_2\in Q^2_{\nu_0}$. p^*_ρ may be elliptic or nonelliptic–nondegenerate on $Q^2_{\nu_0}$. Since it can be localized to $Q^2_{\nu_0}$, it follows that

$$p^*_\rho(\gamma^0_2)\sim\rho^4\left(\xi^0_2-\frac{Mb\mu}{\sigma(\mu, \rho)}\right)^2+G_1(\rho)(M\rho^2)^2\leqslant C(M\rho^4)^2,$$

i.e.,

$$H_1(\rho):=\frac{1}{\rho^4}\left(\frac{\xi^0_2}{M}-\frac{b\mu}{\sigma(\mu, \rho)}\right)^2+\frac{G_1(\rho)}{\rho^4}\leqslant C. \tag{86}$$

If p^*_ρ is elliptic on $Q^2_{\nu_0}$, the ball is a box; if it is nonelliptic–nondegenerate, then, in any case, this is so for

$$\rho\in C_1:=[|\mu|, \rho_{\max}]\cap S(G_1)\cap S(H_1)$$

(a possibly empty set). Since C_1 is an intersection of level sets of rational functions of ρ, quotients of polynomials of *a priori* bounded degree (independent of γ^0 and b), it follows that C_1 has an *a priori* bounded number of connected components. The same kind of argument applies in case (85), and we get a condition on the corresponding $H_2(\rho)$:

$$p^*_\rho(\gamma^0_2)\sim\rho^2\mu^2\left(\xi^0_2-\frac{Mb\mu}{\sigma(\mu, \rho)}\right)^2+G_2(\rho)(M\rho^2)^2\leqslant C(M(\rho\mu)^2)^2,$$

whence

$$H_2(\rho):=\frac{1}{(\rho\mu)^2}\left(\frac{\xi^0_2}{M}-\frac{b\mu}{\sigma(\mu, \rho)}\right)^2+\frac{G_2(\rho)}{\mu^4}\leqslant C, \tag{87}$$

and the condition

$$\rho\in C_2:=[\rho_0, |\mu|]\cap S(G_2)\cap S(H_2)$$

(a possibly empty set). As for C_1, C_2 consists of an *a priori* bounded number of connected components. (In case $p^*_\rho(\gamma^0_2)\sim M^2(\rho\mu)^4$, it follows that the ball is a box since we would have that at $(\bar{x}_1, x^0_2; \xi^0)$ the polynomial $\rho^2p_1(x, \xi_2)+(|\xi^0_1|/M)^4M^2$ is as big as its maximum on a block of sizes $\rho|\mu|\times M\rho|\mu|$.)

We distinguish now among the following cases:

$$\rho \in \left[\rho_0, \frac{1}{3} |\mu| \right] \cap C_2; \tag{88}$$

$$\rho \in \left(C_2 \cap \left[\frac{1}{3} |\mu|, |\mu| \right] \right) \cup (C_1 \cap [\,|\mu|, 2\,|\mu|\,]); \tag{89}$$

$$\rho \in [3\,|\mu|, \rho_{\max}] \cap C_1 \tag{90}$$

(in case these sets are not empty).

Case (88). Consider a C.Z. decomposition relative to $\rho^2 p_1(x, \xi_2)$. Call Q_ν the C.Z. block for which $\gamma_2^0 \in \pi_{(x_2, \xi_2)}(Q_\nu)$.

Look at $\partial^2_{\xi_2}(\rho^2 p_1) = 2\rho^2 x_1^2$. The "good band" R in this situation has sizes $\sim \rho \times \rho\,|\mu| \times M\rho\,|\mu|$. Call $M\rho\,|\mu|\,\Delta_0$ the ξ-displacement given by R. The stopping condition for $\rho^2 p_1$ now reads: $(\operatorname{diam}_x Q_\nu) \sim \Delta_0\,\rho\,|\mu|$.

Since $\rho^2 p_1(x, \xi_2) = \rho^2 x_1^2(\xi_2 - Mb/x_1)^2$ and $I_\rho \subset [\frac{2}{3}\mu, \frac{4}{3}\mu]$ (we may suppose $\mu > 0$, as we shall from now on), it follows that $\rho^2 p_1(x, \xi_2) \sim \rho^2 \mu^2 (\xi_2 - Mb/x_1)^2$ on R and $\forall x_1 \in I_\rho$, whence

$$M\rho\mu\Delta_0 = |\xi_1^0| + |\xi_2^0 - M\bar{b}| + M(\bar{b}^2 - (\bar{b})^2)^{1/2}, \tag{91}$$

where we have set $\bar{b}^2 := \operatorname{Av}_{x_1 \in I_\rho}(b^2/x_1^2)$ and $\bar{b} := \operatorname{Av}_{x_1 \in I_\rho}(b/x_1)$. Let

$$W = \{(x, \xi);\, |x_1 - x_1^0| \leqslant \rho,\, |x_2 - x_2^0| \leqslant c\rho\mu,\, |\xi - \xi^0| \leqslant c\Delta_0 M\rho\mu\} \tag{92}$$

where $c > 0$ is a universal constant (note that $\pi_{(x_2, \xi_2)}(W) \subset Q^{2**}_{\nu_0}$). Consider

$$N = \{\nu;\, Q_\nu \cap W \neq \varnothing,\, \operatorname{diam}_x Q_\nu \gtrsim \rho\mu\Delta_0\}.$$

(Note that $\pi_{x_1}(Q_\nu^\natural \cap W)$ contains a subinterval of diameter $\sim \delta_\nu$.)

We are going to prove that either

$$B_p(\gamma^0, \rho) \approx \{(x, \xi);\, |x_1 - x_1^0| \leqslant \rho,\, |x_2 - x_2^0| \leqslant \rho|\mu|,\, |\xi - \xi^0| \leqslant M\rho|\mu|\}$$

or

$$B_p(\gamma^0, \rho) \approx W.$$

In either case, the ball is a box.

Suppose, for some $\nu \in N$, $\rho^2 p_{1\,|\,Q_\nu}$ *is elliptic*. It then follows that

$$\partial^2_{\xi_2}(\rho^2 p_{1\,|\,Q_\nu}) = 2\rho^2 x_1^2 \leqslant C\delta_\nu^2,$$

whence, since $|x_1| \sim |\mu|$ on $Q_\nu \cap W$, $\delta_\nu \gtrsim \rho\,|\mu|$.

On the other hand, $p_\rho^*(x_2, \xi_2) \leqslant CM^2\rho^4\mu^4$, whence, since $\bar{p}_\rho(x_2, \xi_2) \gtrsim M^2\delta_\nu^4$ for $(x_2, \xi_2) \in \pi_{(x_2, \xi_2)}(Q_\nu \cap W)$, $\delta_\nu \sim \rho\,|\mu|$.

Hence Q_ν is of sizes $\sim \rho\mu \times M\rho\mu$. Let $W_\nu = Q_\nu^\natural \cap R$. Let $\bar{\xi}_2 \in \pi_{\xi_2}(W_\nu)$. It follows that $p_\rho^*(x_2, \bar{\xi}_2) \sim M^2(\rho\mu)^4$, and that, with $R^2 = \pi_{(x_2, \xi_2)}(R)$,

$$\max_{(x_2, \xi_2) \in R^2} p_\rho^*(x_2, \xi_2) \sim M^2(\rho\mu)^4.$$

By inspection of the form of p_ρ^*, one gets:

$$\begin{aligned}
&\max_{(x_2, \xi_2) \in R^2} p_\rho^*(x_2, \xi_2) \\
&\quad \sim \rho^2\mu^2 \left(\xi_2^0 - \frac{Mb\mu}{\sigma(\mu, \rho)} + c\Delta_0 M\rho\mu\right)^2 + \rho^2\mu^2 \left(\xi_2^0 - \frac{Mb\mu}{\sigma(\mu, \rho)} - c\Delta_0 \rho\mu\right)^2 \\
&\qquad + V(\mu, \rho)\, M^2(\rho\mu)^4 \\
&\quad \sim \rho^2\mu^2 \left\{\left(\xi_2^0 - \frac{Mb\mu}{\sigma(\mu, \rho)}\right)^2 + c^2\Delta_0^2 M^2(\rho\mu)^2\right\} \\
&\qquad + V(\mu, \rho)\, M^2(\rho\mu)^4 \sim M^2(\rho\mu)^4,
\end{aligned} \tag{93}$$

where $V(\mu, \rho) := b^2/(\mu^4(\mu^2 + \frac{1}{3}\rho^2)) + (|\xi_1^0|/(M\rho\mu))^4$.

It follows that at least one of

$$\rho^2\mu^2 \left(\xi_2^0 - \frac{Mb\mu}{\sigma(\mu, \rho)}\right)^2, \qquad c^2\Delta_0^2 M^2(\rho\mu)^4, \qquad V(\mu, \rho)\, M^2(\rho\mu)^4$$
$$\geqslant \frac{1}{3}\,\tilde{c} M^2(\rho\mu)^4,$$

from which it follows that the ball is a box of sizes $\sim \rho \times \rho\,|\mu| \times M\rho\,|\mu|$. (In case $\rho^2\mu^2(\xi_2^0 - M(b\mu/\sigma(\rho, \mu)))^2 \geqslant \tilde{c}M^2(\rho\mu)^4/3$, $p_\rho^*(x_2, \xi_2^0) \sim M^2(\rho\mu)^4$ then.)

Suppose now, for some $\nu \in N$, $\rho^2 p_{1\,|\,Q_\nu}$ *is nonelliptic–nondegenerate because of* $\partial_{x_1}^2(\rho^2 p_1)$. Again, it follows that $\delta_\nu \sim \rho\mu$. (In fact, $\partial_{\xi_2}^2(\rho^2 p_1)_{|\,Q_\nu} = 2\rho^2 x_1^2 \lesssim \delta_\nu^2$, $\partial_{x_1}^2(\rho^2 p_1)_{|\,Q_\nu} = 2\rho^2\xi_2^2 \sim M^2\delta_\nu^2$, and since $\rho^2 p_{1\,|\,Q_\nu} = \rho^2\xi_2^2(x_1 - Mb/\xi_2)^2$, we have that

$$\begin{aligned}
\mathrm{Av}_{x_1 \in \pi_{x_1}(W_\nu)}\, \rho^2 p_{1\,|\,Q_\nu} &:= \bar{p}_\nu \\
&= \rho^2 \mathrm{Av}_{|x_1 - \bar{x}_1| \leqslant c\delta_\nu}\, p_{1\,|\,Q_\nu} \\
&\sim \rho^2 \frac{M^2\delta_\nu^2}{\rho^2} \left(\bar{x}_1 - \frac{Mb}{\xi_2}\right)^2_{|\,Q_\nu} + \frac{\rho^2\delta_\nu^2}{3}\, \frac{M^2\delta_\nu^2}{\rho^2}.
\end{aligned}$$

Hence, $M^2\delta_\nu^4 \lesssim \bar{p}_{\nu\,|\,\pi_{\xi_2}(Q_\nu)} \lesssim p^*_{\rho\,|\,Q_{\nu_0}} \lesssim M^2(\rho\mu)^4$.

Since $x_1 \in \pi_{x_1}(W_\nu) \Rightarrow |x_1| \sim \mu$, from $\partial^2_{\xi_2}(\rho^2 p_1) = 2\rho^2 x_1^2$ we have $\rho^2\mu^2 \lesssim \delta_\nu^2 \lesssim \rho^2\mu^2$.) Now fix $\bar{\xi}_2 \in \pi_{\xi_2}(W_\nu)$ for such a ν. $\rho^2 p_1$ being a polynomial, taking the average with respect to x_1 in $\pi_{x_1}(Q_\nu^\natural \cap R)$ yields $p^*_\rho(x_2, \bar{\xi}_2) \sim M^2(\rho\mu)^4$, whence

$$\max_{(x_2, \xi_2) \in R^2} p^*_\rho(x_2, \xi_2) \sim M^2(\rho\mu)^4.$$

As before, it follows that the ball is a box of sizes $\sim \rho \times \rho\,|\mu| \times M\rho\,|\mu|$.

Finally, suppose, for $\nu \in N$, $\rho^2 p_{1\,|\,Q_\nu}$ *is nonelliptic–nondegenerate because of* $\partial^2_{\xi_2}$. It follows that the ball is a box $\approx W$ (in this case, $\delta_\nu^2 \sim \partial^2_{\xi_2}(\rho^2 p_1)_{|\,Q_\nu} \sim \rho^2\mu^2$).

In fact, define $B = \{(x, \xi) \in Q;\, x_1 \in I_\rho\}$ and $N' = \{\nu;\, Q_\nu \cap B \neq \emptyset, \pi_{(x_2, \xi_2)}(Q_\nu) \cap Q^{2**}_{\nu_0} \neq \emptyset\}$. It follows that, for any $\nu \in N'$, $\delta_\nu \sim \rho\mu$, and that

$$\rho^2 p_{1\,|\,Q_\nu \cap B}(x, \xi) = \rho^2 x_1^2 \left(\xi_2 - \frac{Mb}{x_1}\right)^2.$$

Observe that we *a priori* know that

$$\pi_{x_1}(B_p(\gamma^0, \rho)) \subset I_\rho(x_1^0).$$

A symbol q subordinate to $\rho^2 p$ can be written in the form $q_1 + q_2$, with q_1 subordinate to $\rho^2\xi_1^2$, and q_2 subordinate to $\rho^2 p_1$. Since p^*_ρ can be localized to sizes $\rho\mu \times M\rho\mu$, and $\rho^2 p_1 \leqslant C p^*_\rho$, it follows ($\Gamma$ being a subunit path), that

$$|\partial_{\xi_2} q(\Gamma(t; \gamma^0))| \leqslant C\rho\,|\mu|.$$

Denoting by Γ_2 the ξ-projection of Γ, we have

$$|\partial_x q(\Gamma(t; \gamma^0))| \leqslant C(\Delta_0 M\rho\,|\mu| + |\Gamma_2(t, \gamma^0) - \xi^0|).$$

This follows upon using the transformation Ψ introduced earlier (which, in this case, is written in the form

$$(x, \xi) \mapsto \left(x; \xi_1, \xi_2 - M\,\mathrm{Av}_{x_1 \in I_\rho}\left(\frac{b}{x_1}\right)\right),$$

the Taylor expansion of $\partial_x q_2$ with respect to

$$\Sigma = \left\{(y, \eta);\, \eta_2 = Mb\left(\frac{1}{y_1} - \mathrm{Av}_{y_1 \in I_\rho}\left(\frac{1}{y_1}\right)\right),\, y_1 \in I_\rho\right\},$$

and changing the variables back. This proves that also in this case the ball is a box:

$$B_p((x^0,\xi^0),\rho)\approx\{(x,\xi);\,|x_1-x_1^0|\leqslant\rho,\\ |x_2-x_2^0|\leqslant\rho\,|\mu|,\,|\xi-\xi^0|\leqslant\Delta_0 M\rho\,|\mu|\}.$$

This concludes Case (88).

Case (90). In this case we have that $0\in I_\rho(x_1^0)$, and

$$p_\rho^*(x_2,\xi_2)\sim\rho^4\left(\xi_2-M\frac{b\mu}{\sigma(\mu,\rho)}\right)^2+M^2b^2\rho^2+\left(\frac{|\xi_1^0|}{M}\right)^4M^2.$$

The ξ-displacement given by the "good band" R is now given by

$$M\rho^2\Delta_0\sim|\xi_1^0|+|\xi_2^0-M\bar{b}|+M(\bar{b}^2-(\bar{b})^2)^{1/2},$$

where

$$\bar{b}^2:=\mathrm{Av}_{x_1\in\pi_{x_1}(R)}(b^2/x_1),\qquad \bar{b}:=\mathrm{Av}_{x_1\in\pi_{x_1}(R)}(b/x_1).$$

Note that

$$B_{p_\rho^*}((x_2^0,\xi_2^0),1)\approx\{(x_2,\xi_2);\,|x_2-x_2^0|\leqslant\rho^2,\,|\xi_2-\xi_2^0|\leqslant M\rho^2\Delta_1\},$$

with

$$M\rho^2\Delta_1\sim|\xi_1^0|+\left|\xi_2^0-M\frac{b\mu}{\sigma(\mu,\rho)}\right|+Mb^{1/2}\rho^{1/2}\sim|\xi_1^0|+|\xi_2^0|+Mb^{1/2}\rho^{1/2}$$

because

$$M\frac{b\mu}{\sigma(\mu,\rho)}\sim\frac{Mb\mu}{\rho^2}=M\frac{b^{1/2}}{\rho^{3/2}}b^{1/2}\frac{\mu}{\rho^{1/2}}\leqslant CMb^{1/2}\rho^{1/2}$$

in this case. Also, $M\bar{b}^{2^{1/2}}, M\bar{b}\lesssim Mb^{1/2}\rho^{1/2}$. We have to consider the following two cases (the stopping condition is now given by $\mathrm{diam}_x\,Q_\nu\sim\Delta_0\rho^2$):

(i) $M\rho^2\Delta_0\gtrsim Mb^{1/2}\rho^{1/2}$ (or $M\rho^2\Delta_0\sim Mb^{1/2}\rho^{1/2}$);

(ii) $M\rho^2\Delta_0\lesssim Mb^{1/2}\rho^{1/2}$.

Since $M\rho^2\Delta_0\sim M\rho^2\Delta_0+Mb^{1/2}\rho^{1/2}$ in case (i), we get

$$M\rho^2\Delta_0\sim|\xi_1^0|+|\xi_2^0|+Mb^{1/2}\rho^{1/2}$$

(in fact, $|\xi_1^0-M\bar{b}|+Mb^{1/2}\rho^{1/2}\sim|\xi_2^0|+Mb^{1/2}\rho^{1/2}$), which is the maximum displacement allowed. Hence in case (i) the ball is a box.

In case (ii), we have $|\xi_1^0|, |\xi_2^0-M\bar{b}|\lesssim Mb^{1/2}\rho^{1/2}$, from which it follows that $M\rho^2\Delta_1\sim|\xi_2^0|+M\rho^{1/2}b^{1/2}$.

We now look at the following quantities:

$$\sigma_1(p_\rho^*) := \max_{(x_2, \xi_2) \in R^2} p_\rho^*(x_2, \xi_2) \sim \rho^4 \left(\left| \xi_2^0 - \frac{Mb\mu}{\sigma(\mu, \rho)} \right|^2 + M^2 \rho^4 \Delta_0^2 \right) + M^2 b^2 \rho^2$$

$$\sim \rho^4 \, |\xi_2^0|^2 + M^2 b^2 \rho^2 + \rho^4 \, |\xi_2^0 - M\bar{b}|^2 + M^2 \rho^4 (\bar{b^2} - (\bar{b})^2) + \rho^4 \, |\xi_1^0|^2$$

$$\sim \rho^4 (|\xi_1^0|^2 + |\xi_2^0|^2) + M^2 b^2 \rho^2 = (M\rho^4)^2 \left\{ \left(\frac{|\xi_1^0|}{M\rho^2} \right)^2 + \left(\frac{|\xi_2^0|}{M\rho^2} \right)^2 + \frac{b^2}{\rho^6} \right\}, \tag{94}$$

and

$$\sigma_2(p_\rho^*) := p_\rho^*(x_2, \xi_2^0) \sim \rho^4 \left(\xi_2^0 - \frac{Mb\mu}{\sigma(\mu, \rho)} \right)^2$$

$$+ M^2 b^2 \rho^2 \sim (M\rho^4)^2 \left\{ \left(\frac{|\xi_2^0|}{M\rho^2} \right)^2 + \frac{b^2}{\rho^6} \right\}. \tag{95}$$

Consider a C.Z. decomposition relative to $\rho^2 p_1$, and let $\hat{Q}$ be a C.Z. block such that $(0, 0) \in \hat{Q}$. As a consequence, $\rho^2 p_{1 \mid \hat{Q}}$ must be elliptic (in the present case (ii)) and sizes $(\hat{Q}) \sim (b\rho)^{1/2} \times M(b\rho)^{1/2}$. We may suppose that $\xi_1^0 \in \pi_{\xi_1}(\hat{Q})$ (since otherwise we would be in case (i) above: it would be $|\xi_1^0| \sim M\rho^{1/2} b^{1/2}$ and thus $M\rho^2 \Delta_0 \sim M(b\rho)^{1/2}$).

We distinguish now among the following cases:

(A) $|\xi_2^0| \geqslant CM(b\rho)^{1/2}$;

(B) $|\xi_2^0| \sim M(b\rho)^{1/2}$;

(C) $\xi_2^0 \in \pi_{\xi_2}(\hat{Q})$ (i.e., $|\xi_2^0| \leqslant CM(b\rho)^{1/2}$).

(A) We have that

$$\sigma_1(p_\rho^*) \sim \sigma_2(p_\rho^*) \qquad \text{and} \qquad M\rho^2 \Delta_1 \sim |\xi_2^0|.$$

$\rho^2 p_1$ being a non-negative polynomial, it follows that $\exists \bar{x}_1 \in \frac{1}{8} I_\rho$ (say) such that

$$\rho^2 p_1(\bar{x}_1, x_2^0, \xi_2^0) \sim (M\rho^4)^2 \left(\frac{|\xi_2^0|}{M\rho^2} \right)^2.$$

We can therefore find a neighborhood of $(\bar{x}_1, x_2^0, \xi_1^0, \xi_2^0)$ of sizes

$$\frac{|\xi_2^0|}{M\rho^2} \rho^2 \times M\rho^2 \frac{|\xi_2^0|}{M\rho^2}$$

on which $\rho^2 p_1 \sim (M\rho^4)^2 (|\xi_2^0|/(M\rho^2))^2 = \rho^4 |\xi_2^0|^2$, whence we have the possibility of moving, through subunit paths, by order $|\xi_2^0|$ in the ξ-variables, i.e., the maximum allowed. Hence, the ball is a box:

$$B_p(\gamma^0, \rho) \approx \{(x, \xi);\, |x_1 - x_1^0| \leqslant \rho,\, |x_2 - x_2^0| \leqslant \rho^2,\, |\xi - \xi^0| \leqslant |\xi_2^0|\}.$$

(B) This case is completely analogous to case (A).

(C) In this case $M\rho^2 \Delta_1 \sim M b^{1/2} \rho^{1/2}$ is the maximum ξ-displacement allowed.

Since $|\mu| \leqslant \rho/3$ now, we can reach $x_1 = 0$ at time $\frac{1}{3}$, and using the ellipticity of $\rho^2 p_{1\,|\,\hat{Q}}$, we fill in a region of sizes

$$\sim \frac{b^{1/2}}{\rho^{3/2}} \rho^2 \times M\rho^2 \frac{b^{1/2}}{\rho^{3/2}}.$$

It follows that the ball is a box:

$$B_p(\gamma^0, \rho) \approx \{(x, \xi);\, |x_1 - x_2^0| \leqslant \rho,\, |x_2 - x_2^0| \leqslant \rho^2,\, |\xi - \xi^0| \leqslant M b^{1/2} \rho^{1/2}\}.$$

This concludes Case (90).

Case (89). It gives $|\mu|$ as critical radius (applying the construction at the beginning of this section).

To complete the discussion, we have to consider the following cases we have left out so far (recall that $\rho \geqslant \rho_0$):

$$\tfrac{1}{3} |\mu| \leqslant \rho_0 \leqslant |\mu|, \quad |\mu| \leqslant \rho_0 \leqslant 3\,|\mu|, \qquad \rho_0 \geqslant 3\,|\mu|.$$

In the first case, condition (88), (89), or (90) may hold, whence the conclusions of Case (88), Case (89), and Case (90) follow.

In the second case, condition (88) is empty, while (89) or (90) may hold, whence the conclusions of Case (89) and Case (90) follow.

In the third case, only condition (90) holds, whence the conclusion of Case (90) holds true.

We may summarize the result as follows:

(1) If $\rho \leqslant |\mu|/3$ the ball is a box:

$$B_{\rho^2 p}(\gamma^0, 1) \approx \{(x, \xi);\, |x_1 - x_1^0| \leqslant \rho,\, |x_2 - x_2^0| \leqslant \rho\,|\mu|,\, |\xi - \xi^0| \leqslant \Delta_0 M\rho\,|\mu|\}$$

with Δ_0 given by (91),

(2) If $\rho \geqslant 3\,|\mu|$ the ball is a box:

$$B_{\rho^2 p}(\gamma^0, 1) \approx \{(x, \xi);\, |x_1 - x_1^0| \leqslant \rho,\, |x_2 - x_2^0| \leqslant \rho^2,$$
$$|\xi - \xi^0| \leqslant |\xi_1^0| + |\xi_2^0| + M b^{1/2} \rho^{1/2}\}.$$

We hence have a "transition" of the geometry at the radius:

$$\rho_{\text{cr}} \sim |\mu| = |x_1^0|.$$

We finally want to comment on the resulting geometry of these subunit balls. It follows from Theorem 5.20 that

$$\pi_x(B_p((x^0, \xi^0), 1)) \approx \{x;\, |x_1 - x_1^0| \leqslant 1,\, |x_2 - x_2^0| \leqslant \delta\}.$$

Hence, while the subunit localization in x gives naturally expected results, the ξ-localization presents the above described "stratification," related to the *stability*, as x_1 varies in the interval $[x_1^0 - 1, x_1^0 + 1]$, of normal forms (with respect to ξ_2) of the symbol $p_1(x_1, x_2, \xi_2)$, where x_1 may be viewed as a parameter. These normal forms are in turn related to the degeneration of the algebraic variety (p_1 can be supposed a polynomial)

$$\Sigma_2 = \{(x, \xi) \in \mathbf{R}^2 \times \mathbf{R};\, \partial p_1/\partial \xi_2 = 0\}.$$

Example A. We give here an example of the symbol for which the good band is not unique, but for which the stratification doesn't take place. Consider

$$p(x, \xi) = \xi_1^2 + \delta^2 (\tfrac{1}{2} - x_1)^2 (\tfrac{1}{8} - x_1)^2 (\xi_2 - M x_1 x_2)^2 + M^2 \delta^4 V(x_1, x_2),$$

on Q of sizes $1 \times M$ centered at $(0, 0)$, $0 < \delta \leqslant 1$. For this symbol, in the case $|\xi_1^0|/M \leqslant \delta$,

$$B_p((x^0, \xi^0), 1) \approx \{(x, \xi);\, |x_1 - x_1^0| \leqslant 1,\, |x_2 - x_2^0| \leqslant \delta,\, |\xi - \xi^0| \leqslant M\delta\tilde{\Delta}_0\}.$$

where

$$\tilde{\Delta}_0 = \frac{|\xi_1^0| + |\xi_2^0|}{M\delta} + \sigma(b^2)^{1/2} + \sigma(V)^{1/4},$$

$b(x_1, x_2) = x_1 x_2$.

The absence of the stratification is also due to the "stability" of the function V.

Example B. We now give an example of a 2nd-order differential operator for which the stratification doesn't occur. Consider a 2nd-order differential operator in $\mathbf{R}^2$ with symbol

$$p(x, \xi) = e(x_1, x_2)\, \xi_1^2 + a(x_1, x_2)\, \xi_2^2,$$

where $a \geqslant 0$ is a C^∞-function, $0 < e \sim 1$ is an elliptic factor, and p satisfies the Assumptions of Section 2.[1] As in Section 2, we microlocally reduce p to a symbol belonging to the class $S^2(1 \times M)$ (still denoted by p). Let hence Q be a block $1 \times M$ ($M \gg 1$) centered at $(\bar{x}, \bar{\xi})$, with $|\bar{\xi}| \sim M$. Hence $p_{|Q} \in S^2(1 \times M)$ and $\xi \in \pi_\xi(Q) \Rightarrow |\xi| \sim M$. Now, $|\bar{\xi}| = \max\{|\bar{\xi}_1|, |\bar{\xi}_2|\}$, hence it might well be $|\bar{\xi}_1| \ll M$ and $|\bar{\xi}_2| \sim M$. For simplicity we assume $a(x_1, x_2)$ is a polynomial (otherwise, by subellipticity, this can be achieved by considering the subunit ball of radius ρ), such that $a = a_1(x_1)\, a_2(x_2)$ with a_1, a_2 non-negative polynomials. Assume $a \ll 1$. Let $(x^0, \xi^0) \in Q$ be the center of our subunit ball. Under these assumptions, Q itself is a nonellipticity–nondegeneracy block for p. Since $|\xi_2| \sim M$ on Q, we have on Q

$$p(x, \xi) \sim \xi_1^2 + a(x_1, x_2)\, M^2$$

and $|\xi_1| \lesssim M$, $\xi_1 \in \pi_{\xi_1}(Q)$. Denote

$$\bar{a}(x_2) := (\mathrm{Av}_{|x_1 - x_1^0| \leqslant 1}\, a)(x_2)$$

and consider the derived symbol

$$p_1^*(x_2, \xi_2) = \left(\frac{|\xi_1^0|}{CM}\right)^4 M^2 + \bar{a}(x_2)\, \xi_2^2 \sim \left(\frac{|\xi_1^0|}{CM}\right)^4 M^2 + \bar{a}(x_2)\, M^2.$$

Then $p_1^*(x_2, \xi_2) \lesssim M^2$. We consider a C.Z. decomposition of $\pi_{(x_2, \xi_2)}(Q)$ relative to p_1^*. Let Q_δ^2 be a C.Z. block in $\mathbf{R} \times \mathbf{R}$, of sizes $\delta \times M\delta$, at which the procedure stops. In particular (since $|\xi_2| \sim M$), $\bar{a}(x_2) \lesssim \delta^4$. We have the following cases:

(i) p_1^* is elliptic on Q_δ^2 because $|\xi_1^0|/M \sim \delta$;

(ii) p_1^* is elliptic on Q_δ^2 because $|\xi_1^0|/M \sim \delta$ and $\bar{a}(x_2) \sim \delta^4$;

(iii) p_1^* is elliptic on Q_δ^2 because $|\xi_1^0|/M \ll \delta$ and $\bar{a}(x_2) \sim \delta^4$.

In all these cases we have

$$B_p((x^0, \xi^0), 1) \approx \{(x, \xi);\, |x_1 - x_1^0| \leqslant 1,\, |x_2 - x_2^0|\, M + |\xi - \xi^0| \leqslant M\delta\}.$$

[1] The case $X_1^2 + X_2^2$ for real vector fields X_1, X_2 satisfying a subelliptic condition (say, the well-known Hörmander finite-type condition) can be treated by using the Weyl Pseudodifferential Calculus: if $p_i(x, \xi)$ is the symbol of X_i, the Weyl symbol of $X_1^2 + X_2^2$ is $p_1(x, \xi)^2 + p_2(x, \xi)^2 \geqslant 0$, and we apply the methods so far developed.

(iv) p_1^* is nonelliptic–nondegenerate on Q_δ^2 because $|\xi_1^0|/M \ll \delta$ and $\partial_{x_2}^2 \bar{a}(x_2) \sim \delta^2$; in this case it follows that $\bar{a}(x_2) = \bar{a}_1 a_2(x_2) \sim \delta^2(x_2 - x_2^*)^2 + \delta^4\alpha$, where $x_2^* \in \pi_{x_2}(Q_\delta^2)$ and $0 < \alpha \leqslant 1$. Since

$$p(x, \xi) \sim \xi_1^2 + \frac{a_1(x_1)}{\bar{a}_1} (\delta^2(x_2 - x_2^*)^2 + \delta^4\alpha)\, M^2,$$

moving x_1 to a maximum for a_1 on the interval $[x_1^0 - 1, x_1^0 + 1]$ yields $a_1(x_1)/\bar{a}_1 \sim 1$ so that (using the fact that the subunit ball relative to $\xi_1^2 + \bar{p}_1(x_2, \xi_2)$ contains the one relative to p), we have

$$\begin{aligned} B_p((x^0, \xi^0), 1) \approx \{(x, \xi);\ & |x_1 - x_1^0| \leqslant 1, \\ & |x_2 - x_2^0|\, M + |\xi_1 - \xi_1^0| \leqslant |\xi_1^0| + |x_2^0 - x_2^*|\, M + M\delta\alpha^{1/4}, \\ & |\xi_2 - \xi_2^0| \leqslant M\delta\}. \end{aligned}$$

(The case in which $\bar{a}(x_2) \sim \delta^2(x_2 - \tilde{x}_2)^2$ is ruled out by (s.e.) (relative to $\bar{p}_1(x_2, \xi_2) := \bar{a}(x_2)\, \xi_2^2$; see Section 2.) Here center$(Q_\delta^2) = (\tilde{x}_2, \tilde{\xi}_2)$ and $M\delta^2 \gg 1$. In fact, take the testing box in $\mathbf{R} \times \mathbf{R}$

$$B = \left\{ (x_2, \xi_2) \in \mathbf{R} \times \mathbf{R};\ |x_2 - \tilde{x}_2| \leqslant c_\varepsilon^{1/2} \frac{\delta (M\delta^2)^{-1+\varepsilon/2}}{\sqrt{2C}}, \right.$$

$$\left. |\xi_2 - \tilde{\xi}_2| \leqslant \sqrt{2C}\, \frac{M\delta (M\delta^2)^{-\varepsilon/2}}{c_\varepsilon^{1/2}} \right\}.$$

Then $B \subset (\pi_{x_2}(Q_\delta^2) \times \pi_{\xi_2}(Q))^*$ (note that $(M\delta)^{-1} \leqslant \delta^2/\delta = \delta$). It follows that

$$\max_B \bar{p}_1(x_2, \xi_2) \leqslant C \max_B (\delta^2 (x_2 - \tilde{x}_2)^2 M^2) \leqslant \frac{c_\varepsilon}{2} (M\delta^2)^\varepsilon,$$

and hence that $\bar{p}_1$ doesn't satisfy (s.e.).)

In case $|\bar{\xi}_1| \sim M$ and $|\bar{\xi}_2| \ll M$ or $|\bar{\xi}_2| \sim M$, $p_{|Q}$ is elliptic, hence $B_p((x^0, \xi^0), 1) \approx Q$.

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