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A computer search for finite projective planes of order 9

C.W.H. Lam*, G. Kolesova and L. Thiel
Computer Science Department, Concordia University, Montreal, Que., Canada H3G 1M8

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Dedicated to Professor R.G. Stanton on the occasion of his 68th birthday.

Abstract

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There are four known finite projective planes of order 9. This paper reports the result of a computer search which shows that this list is complete. The computer search starts by generating all 283,657 non-isomorphic latin squares of order 8. Each latin square gives 27 columns of the incidence matrix. Another program attempts to complete each of these incidence matrices to 40 columns. Only 21 of them can be so completed, giving rise to 326 matrices of 40 columns. A third computer program attempts to complete the rest of the matrices. One of the 326 does not complete. The rest complete each to a unique matrix. An isomorphism testing program is then applied to the 325 complete matrices, creating a certificate for each matrix, as well as its collineation group. The certificates are then compared with the known planes and no new ones found. As a further evidence of the correctness of the computer programs, this paper also shows that the computer results are consistent with those expected by using information about the known planes and their associated latin squares.

1. Introduction

A *finite projective plane of order n* is a collection of n^2+n+1 lines and n^2+n+1 points such that:

(1) every line contains $n+1$ points,
(2) every point is on $n+1$ lines,
(3) any two distinct lines intersect at exactly one point, and
(4) any two distinct points lie on exactly one line.

There are four known planes of order 9, namely the Desarguesian Plane, the Left Nearfield Plane, the Right Nearfield Pane and the Hughes Plane. For a detailed description of these planes, as well as their collineation groups, see [11].

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It was an unsettled question whether there were any more planes of order 9. In several special cases, [3, 5, 10], computer searches have discovered no new planes. In this paper, we report the result of an exhaustive computer search which shows that there cannot be any new planes of order 9. We also show that the computer results are consistent with those one would expect by applying the collineation group to the known planes.

In Section 2, we introduce the mathematical preliminaries which are required in the rest of the paper. Section 3 gives the methodology of the computer search and the computer results. Finally, Section 4 addresses the question of correctness of these results.

2. Mathematical preliminaries

The subject of projective planes is vast and we will introduce only the concepts that we need in this paper. For more details, one can refer to the excellent books [4, 11].

Another way to represent a projective plane is to use an *incidence matrix* A of size n^2+n+1 by n^2+n+1. The columns represent the points and the rows represent the lines. The entry A_{ij} is 1 if point j is on line i; otherwise, it is 0. In terms of an incidence matrix, the property of being a projective plane is translated into:

(1) A has constant row sum $n+1$,

(2) A has contant column sum $n+1$,

(3) the inner product of any two distinct rows of A is 1, and

(4) the inner product of any two distinct columns of A is 1.

The property of being a projective plane is preserved under arbitrary and independent relabelling of lines and points. In terms of the incidence matrix, these operations translate to arbitrary and independent permutations of rows and columns. Two incidence matrices that can be transformed one into another by row and column permutations are said to be *isomorphic*. Since we are interested only in non-isomorphic planes, we shall use these operations to reduce our search space. Given a particular column permutation, we shall use the row permutations to sort the rows into a standard form. Thus, the symmetry group acting on the incidence matrix is the symmetric group on n^2+n+1 letters. The subgroup which fixes the incidence matrix is called its *collineation group*.

A *latin square* of order n is an $n \times n$ matrix satisfying the following properties:

(1) all the entries are from a set of n distinct symbols,

(2) in every row, no entry is repeated, and

(3) in every column, no entry is repeated.

Usually, the entries are chosen from the set $\{1, \ldots, n\}$.

While it is well known that the existence of a projective plane of order n is equivalent to the existence of a complete set of mutually orthogonal latin squares of order n, it is also possible to characterize the property of being a projective plane in terms of latin squares of order $n-1$. We do not need the complete characterization. We shall only describe how to extract a latin square of order 8 from a projective plane of order 9.

The method is based on choosing a *triangle*, which is defined to be a set of 3 noncollinear points, and then normalizing the incidence matrix into the form shown in Fig. 1. There exist 3 lines each containing two of these 3 points. We permute the rows and columns so that they come first and as a result, the principal submatrix on the top left is as shown. Next, we rearrange the columns to place the remaining 8 points of line 1 at column positions 4 to 11. The remaining points for line 2 are placed at columns 12 to 19 and those for line 3 at columns 20 to 27. Similarly, rows 4 to 11 are the remaining lines through point 1; rows 12 to 19 are the lines through point 2 and rows 20 to 27 are the lines through point 3. Lines 4 to 11 have intersected with lines 2 and 3. Hence, they cannot contain any of the points 12 to 27. They have not intersected with line 1 and so they must each contain one of the points from 4 to 11. However, each of the points 4 to 11 can only be incident on at most one of the lines 4 to 11. Thus, each of the lines from 4 to 11 contains a distinct point from 4 to 11. By permuting the columns 4 to 11, we can ensure that the submatrix formed by rows 4 to 11 and columns 4 to 11

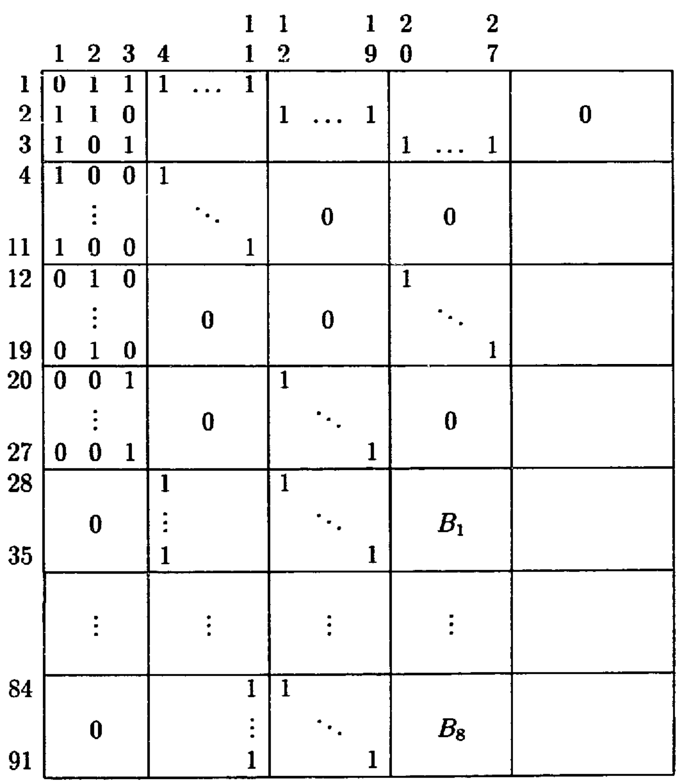

Fig. 1. A normalized incidence matrix for a plane of order 9.

is an identity matrix. The incidence structure of rows 12 to 27 with points 1 to 27 follows from a similar reasoning.

The remaining rows are sorted in a lexicographical order. Naturally, all the remaining lines containing point 4 comes first, followed by those containing point 5, and so on. Thus, the rows 28 to 91 are divided into eight *row blocks*. Those containing point 4 (rows 28 to 35) form the row block 1. Those from rows 36 to 42 form the row block 2 and so on. In each row block, a row has to intersect row 2 exactly once. Hence, the row ordering within a row block can be further refined by the incidence of points 12 to 19. We label the submatrix formed by columns 20 to 27 with the various row blocks as B_1 to B_8. They will give rise to a latin square of order 8.

First, we claim that each of the B_i is a permutation matrix. Since each of its rows must intersect row 3, B_i contains one 1 in each row. Since each column of B_i intersects column $i+3$ of the incidence matrix at most once, there is at most one 1 in each column of B_i. Hence, B_i must be a permutation matrix.

There are several ways to associate the submatrices B_i with a latin square. We have chosen to use points 4 to 11 to define the rows 1 to 8 of the latin square, points 12 to 19 to define the labels 1 to 8 of the entries, and points 20 to 27 to define the columns 1 to 8 of the latin square. Thus, if there is a 1 in row k, column j of B_i, there is a k in row i and column j of the latin square. We next prove that the result is indeed a latin square. Since B_i is a permutation matrix, row i of the resulting square must have distinct entries. In the incidence matrix, a column from the range 20 to 27 must intersect each of the columns in the range 12 to 19 exactly once. Hence, a column of the resulting square must have distinct entries.

The symmetry operations of permuting columns of the incidence matrix induce symmetry operations on the latin squares. Interchanging two points in the range 4 to 11 induces an interchange of two rows of the latin square. Permuting points in the range 12 to 19 corresponds to relabelling the entries in the latin square. A permutation of points in the range 20 to 27 becomes a column permutation on the latin square. These three types of operations are the *isotopic* operations. Together, they partition the latin squares into *isotopy classes*. Moreover, the first 3 points can be permuted amongst themselves. For example, interchanging points 1 and 3 corresponds to transposing the latin square. These 3! permutations of the first 3 points translate to the *conjugate* operations on the latin square. They merge the isotopy classes into *main classes*. For a detailed description of these symmetry operations on latin squares, see [2].

3. Methodology and results

The search for a projective plane of order 9 is divided into four steps:

(1) generate one representative from each main class of latin squares of order 8,

(2) translate the representative latin square into the first 27 columns of the incidence matrix and attempt all possible ways to extend it to 40 columns,

(3) extend the resulting partial incidence matrix to 91 columns, and

(4) perform isomorphism testing on the complete incidence matrices, deriving a certificate for each as well as its collineation group.

Finally, the certificates are compared with those of the known planes.

There are several reasons for organizing the search into such steps. First of all, there exist published numbers of latin squares of order 8 under the action of different symmetry groups, which provide a check on the starting points of our search. Moreover, it is too expensive to perform isomorph rejection beyond this point. Unfortunately, some of the published numbers seem to be wrong. For more details, see [7]. The reason for completing the incidence matrix first to 40 columns and then to 91 columns is for efficiency purposes. The bulge of the search occurs around column 30, reaching around 10^9 cases. In this first range of columns, we use the method of a compatibility matrix, which is first introduced in [1] and also described in greater detail in [8, 12]. Since there are only 326 survivors at column 40, we can afford to use a slower program to complete it. For this purpose, we use an existing program which we had used mainly for estimation.

Out of the 283,657 latin squares of order 8, only 21 have extensions to 40 columns, giving rise to 326 partial incidence matrices. One of these 326 (arising from latin square 16) cannot be completed. The remaining 325 each completes

Table 1
Distribution of 325 complete incidence matrices arising from 21 latin squares

LS	automorphism					
id	Conjugate	Isotopic	Right	Left	Hughes	Desarguesian
1	6	1	1			
2	6	1		1		
3	2	3			1	
4	2	6			4	
5	2	2			1	
6	2	6	1	1		
7	1	1			1	
8	6	1			1	
9	6	3			1	
10	1	2			1	
11	2	3			1	
12	6	6	3	3		
13	6	1			1	
14	2	2			1	
15	1	12			1	
16	2	48	1	1		
17	2	192			8	
18	2	6			1	
19	2	128			8	
20	6	1536	108	108	64	
21	6	256				2
total			114	114	95	2

exactly once. The distribution of the 325 complete incidence matrices according to the latin squares is given in Table 1. The first column identifies the latin square. The next two columns contain information about the size of the automorphism group of the latin square. The second column gives the number of conjugation operations that fix the latin square. The third column gives the number of isotopic operations that fix the latin square. The next four columns identify the number of times a particular plane occurs as an extension of a given latin square.

4. Consistency of the results

While it is impossible to prove that a computer search is correct, we shall do the next best thing; that is, to explain how one can derive the raw data of the computer search by working backwards from the known planes and using only the information about their collineation groups and the associated latin squares. The fact that the computer results are consistent with those predicted by theoretical methods should give the readers confidence in the correctness of the computer search. Those interested in a discussion of this method of consistency checking

Table 2
Desarguesian Plane

		automorphism		
triangle	Latin S.	triangle	Latin S.	copies
1	21	768	1536	2

Table 3
Right Nearfield Plane

		automorphism		
triangle	Latin S.	triangle	Latin S.	copies
1	1	6	6	1
2	12	12	36	3
3	6	12	12	1
4	16	96	96	1
5	20	96	9216	96
6	20	768	9216	12

Table 4
Left Nearfield Plane

		automorphism		
triangle	Latin S.	triangle	Latin S.	copies
1	2	6	6	1
2	12	12	36	3
3	6	12	12	1
4	16	96	96	1
5	20	96	9216	96
6	20	768	9216	12

Table 5
Hughes Plane

triangle	Latin S.	automorphism triangle	automorphism Latin S.	copies
1	4	12	12	1
2	15	12	12	1
3	10	2	2	1
4	5	4	4	1
5	11	6	6	1
6	19	32	256	8
7	17	48	384	8
8	18	12	12	1
9	14	4	4	1
10	3	6	6	1
11	13	6	6	1
12	7	1	1	1
13	4	4	12	3
14	8	6	6	1
15	9	18	18	1
16	20	144	9216	64

are referred to [9]. Tables 2 to 5 give the intermediate results used in deriving these checks.

First, we have to determine how many distinct latin squares can be extracted from a given projective plane. A latin square is uniquely determined, up to the symmetries of the latin square, by its defining triangle, and if two triangles are isomorphic under the action of the collineation group, then their associated latin squares are the same. Hence, for each plane, we have to first find out the number of non-isomorphic triangles. The number of entries in Tables 2 to 5 gives the number of such triangles.

Even though two triangles are non-isomorphic, they may still give rise to the same latin square. For example, triangle 5 and 6 of the Right Nearfield plane both give rise to latin square 20. So, we have to generate a representative from each class of non-isomorphic triangles, extract the latin square, and find out which latin square it is. It is comforting to note that, in all cases, the resulting latin squares are in the list of 21 that can be extended to a plane. Moreover, as the tables show, they are all associated with the correct latin squares. For example, Table 1 shows that the Right Nearfield plane arises from extending latin squares 1, 6, 12, 16 and 20; and in Table 3, the latin squares arising from the triangles are exactly these five squares. In [6], Killgrove and Parker have also shown that there are 21 latin squares of order 8 in the known planes, which agrees with our results.

Next, we want to determine, given a latin square, how many distinct ways are there to extend it to a particular plane. For this purpose, we have to know the automorphism groups of both the triangles and the latin squares. The sizes of these automorphism groups are also given in the tables. The automorphisms of the latin square map one completed incidence matrix into another. These images need not be distinct. If the original incidence matrix is equal to one of its images, then the automorphism of the latin square that created this image is also a collineation fixing the triangle. The number of distinct incidence matrices generated by the automorphisms of the latin squares is equal to the size of the automorphism group of the latin square divided by the size of the automorphism group of the triangle. For example, latin square 12 has three distinct completions to the Right Nearfield plane and the quotient of the respective sizes of the automorphism groups gives $36/12=3$. The last column in the tables gives this number of distinct copies arising from a given triangle. In some cases, such as latin square 20 in the Right Nearfield case, the same latin square arises from two triangles. The total number of distinct copies expected is the sum of the two values $96+12=108$, as found by the computer search.

There is one further consistency check to verify that the triangles are generated correctly. In a plane of order 9, there are $91\times 90\times 81/6$ triangles. For each triangle, the size of its orbit is equal to the quotient of the sizes of the collineation group and the automorphism group. For example, for the Right Nearfield case, we find

$$311{,}040\times\left(\frac{1}{6}+\frac{1}{12}+\frac{1}{12}+\frac{1}{96}+\frac{1}{96}+\frac{1}{768}\right)=\frac{91\times 90\times 81}{6}.$$

Let us now speculate about the probability that an undiscovered plane of order 9 exists but is missed by our computer search. If one exists, then it can be constructed as an extension of one or more of the latin squares. A mistake could occur if we do not consider all the latin squares, or if there is an error in our extension programs. As reported in [7], there are some discrepancies in the published values of the number of latin squares of order 8. Our number is internally consistent as well as being consistent with two of the four published results. Because of the consistency checking on the number of latin squares, we are certain that our number is correct. Our extension programs performed faultlessly on the known planes. It is difficult to imagine how it could fail to find a new plane, if one exist, and yet produce no contradictory results on the known ones.

What is the effect of an undetected hardware error? A common error is the random changing of bits in a computer word. The occurrence of such an error during the running of the extension programs will mean possibly the loss of a branch of the search tree. There is a possibility that the undiscovered plane is in this branch. If this is the only branch where the plane occurs, then we are in trouble. Fortunately, the results from the known planes in Table 1 indicate that, typically, a plane can be constructed from several latin squares. Moreover, the collineation group of an undiscovered plane is likely to be small, which implies that there are more non-isomorphic triangles and, consequently, more distinct latin squares embedded in the incidence matrix. Thus, it is highly unlikely that the undiscovered plane is obtainable only as an extension of one latin square. If it is the extension of more than one square, then the problem of a hardware error is less serious, because it is unlikely that the random errors are so selective that they deleted only the branches containing the undiscovered plane and yet left all the known ones untouched.

Because of the agreement of the computer results with those obtained by theoretical means, we are confident that the computer program is correct and that there is no unknown projective plane of order 9.

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