Generalized coordinate for warping of naturally curved and twisted beams with general cross-sectional shapes

A.M. Yu *, X.G. Yang, G.H. Nie

School of Aerospace Engineering and Applied Mechanics, Tongji University, Shanghai 200092, China

Received 25 February 2005; received in revised form 25 February 2005
Available online 20 July 2005

Abstract

This paper presents an efficient procedure for analyzing naturally curved and twisted beams with general cross-sectional shapes. The hypothesis concerning the cross-sectional shapes of the beams is abandoned in this analysis, and relatively general equations are derived for the analysis of such a structure. Solving directly such equations for various boundary conditions, which take into account the effects of torsion-related warping as well as transverse shear deformations, can yield the solutions to the problem. The solutions can be used to calculate various internal forces, stresses, strains and displacements of the beams. The present theory will be used to investigate the stresses and displacements of a cantilevered curved beam subjected to action of arbitrary load. The numerical results are very close to the FEM results.

© 2005 Elsevier Ltd. All rights reserved.

Keywords: Naturally curved and twisted beam; St. Venant torsional warping function; Generalized coordinate for warping; The minimum potential energy principle

1. Introduction

Static and dynamic analysis of naturally curved and twisted beams has many important applications in mechanical and civil engineering. The problem is urgently needed to be thoroughly studied in engineering structures, especially in bridge structures associated with curved beams. Washizu (1964) presented an approximate theory of the beams and derived a system of governing equations. The unknowns in these equations are the displacement components \( (u_x, u_y, u_z, \varphi_x, \varphi_y, \varphi_z, \varphi_q) \) together with a generalized warping
coordinate \( z \). The solutions of these equations, however, were not obtained. The work of Aimin et al. (2002), developed such a theory and provided such an example where the beams are considered to have a double symmetric cross-section, a fact which clearly restricts the applicability of these equations.

This paper aims to derive a differential equation for generalized warping coordinate \( z \) which involves the St. Venant torsional warping function, the equation applicable to the case of the beams with general cross-sectional shapes subjected to arbitrary load. Obviously, such a system of equations is of practical use for a variety of engineering applications.

Let the locus of the cross-sectional centroid of a beam be a continuum curve in space, the tangential, normal and bi-normal unit vectors of the curve \( l \) are \( t, n \) and \( b \), respectively. The Frenet–Serret formulae, for a smooth curve, is:

\[
\mathbf{t}' = k_1 \mathbf{n}, \quad \mathbf{n}' = -k_1 \mathbf{t} + k_2 \mathbf{b}, \quad \mathbf{b}' = -k_2 \mathbf{n},
\]

where \( (t)' = \frac{d(t)}{ds} \), \( s \), \( k_1 \) and \( k_2 \) are arc coordinate, curvature and torsion respectively of the curve.

In the cross-section of the beam we introduce \( \zeta \)- and \( \eta \)-directions in coincidence with the principal axes through the centroid \( O_1 \), as shown in Fig. 1. The angle between the \( \zeta \)-axis and normal \( n \) is represented as \( \theta \), which is generally a function of \( s \). If the unit vectors of \( O_1 \zeta \) and \( O_1 \eta \) are represented by \( i_\zeta \) and \( i_\eta \), then

\[
i_\zeta = n \cos \theta + b \sin \theta, \\
i_\eta = n \sin \theta + b \cos \theta.
\]

From Eqs. (1) the following expressions are obtained:

\[
i'_\zeta = k_\eta i_\zeta - k_\zeta i_\eta, \\
i'_\eta = -k_\zeta t + k_\zeta i_\eta, \\
i'_\eta = k_\zeta t - k_\zeta i_\zeta,
\]

in which \( k_\zeta = k_1 \sin \theta, k_\eta = k_1 \cos \theta, k_\zeta = k_2 + \theta' \).

The stress–strain relations of the material are given in the local rectangular coordinates in the form (Washizu, 1964):

\[
\sigma^{\lambda \mu} = \sigma^{\lambda \mu}(e_{\alpha \beta}), \quad \lambda, \mu, \alpha, \beta = 1, 2, 3,
\]

where \( \sigma^{\lambda \mu} \) and \( e_{\alpha \beta} \) are the stress and strain tensors defined with respect to the local rectangular coordinates. These tensors are related with those defined with respect to the curvilinear coordinates by the following relationships:
\[
\sigma^{\mu\nu} = \frac{\partial y^\lambda}{\partial x^\sigma} \frac{\partial y^{\mu\nu}}{\partial x^\rho} x^{\lambda\rho},
\]

(5)

\[
e_{\mu\nu} = \frac{\partial x^\lambda}{\partial y^\rho} \frac{\partial x^{\mu\nu}}{\partial y^\sigma} f_{\lambda\rho\sigma},
\]

(6)

We shall denote the components of \( \sigma^{\mu\nu} \) and \( e_{\mu\nu} \) by \((\sigma_s, \sigma_n, \sigma_g, \tau_{s\xi}, \tau_{s\eta}, \tau_{g\xi}, \tau_{g\eta})\) and \((e_{ss}, e_{sn}, e_{sg}, e_{s\xi}, e_{s\eta}, e_{g\xi}, e_{g\eta})\), respectively, and the assumptions based on the slenderness of the beam will be employed. The stress components \( \sigma_s, \sigma_n \) and \( \tau_{g\xi} \) and \( \tau_{g\eta} \) are assumed to be small compared with those remaining we may put (Washizu, 1964)

\[
\sigma_s = \sigma_n = \tau_{g\xi} = \tau_{g\eta} = 0,
\]

(7)
in the stress–strain relations. If the material of the beam is assumed to be isotropic, Eqs. (4) and (7) yield:

\[
\sigma_s = E e_{ss}, \quad \tau_{s\xi} = 2 G e_{s\xi}, \quad \tau_{s\eta} = 2 G e_{s\eta},
\]

(8)
in which \( E \) is Young’s modulus of elasticity and \( G \) is the shear modulus of the material.

2. Internal forces, equilibrium equations and geometry equations

Simplifying stress vectors to the centroid \( O_1 \) on the cross-section \( A \), the principal vector \( Q \) and principal moment \( M \) can be obtained, of which components are respectively denoted by \( Q_s, Q_\xi, Q_\eta \) and \( M_s, M_\xi, M_\eta \), so:

\[
Q = Q_s t + Q_\xi i_\xi + Q_\eta i_\eta, \quad M = M_s t + M_\xi i_\xi + M_\eta i_\eta,
\]

where \( Q_s \) is axial force, \( Q_\xi \) and \( Q_\eta \) are shear forces, \( M_s \) is torque, \( M_\xi \) and \( M_\eta \) are bending moments, as shown in Fig. 2, the external forces and moments per unit length along the axis of the beam are indicated by \( p \) and \( m \) as

\[
p = p_s t + p_\xi i_\xi + p_\eta i_\eta, \quad m = m_s t + m_\xi i_\xi + m_\eta i_\eta.
\]

The equilibrium equations are:

\[
\frac{d}{ds} \{Q\} - [K] \cdot \{Q\} + \{p\} = \{0\},
\]

\[
\frac{d}{ds} \{M\} - [K] \cdot \{M\} - [H] \cdot \{Q\} + \{m\} = \{0\},
\]

(9)

Fig. 2. Stress resultants developed on a typical beam element.
where
\[ \{ Q \} = [Q_s, Q_z, Q_n]^T, \quad \{ M \} = [M_s, M_z, M_n]^T, \]
\[ \{ p \} = [p_s, p_z, p_n]^T, \quad \{ m \} = [m_s, m_z, m_n]^T, \]
\[ \{ K \} = \begin{bmatrix} 0 & k_q & -k_z \\ -k_q & 0 & k_s \\ k_z & -k_s & 0 \end{bmatrix}, \quad [H] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}. \]

The general solutions are Hanwei and Peiyuan (1997) and Yuchun et al. (1999):
\[ \{ Q \} = [A] \cdot \left( \{ Q_0 \} - \int_0^s [A]^T \cdot \{ p \} \, ds \right), \]
\[ \{ M \} = [A] \cdot \left\{ \{ M_0 \} + \int_0^s [A]^T \cdot ([H] \cdot [A] \cdot (\{ Q_0 \} + \{ Q^* \}) - \{ m \}) \, ds \right\}, \]

where \( \{ Q_0 \} \) and \( \{ M_0 \} \) are integration constants, \( \{ Q^* \} = -\int_0^s [A]^T \cdot \{ p \} \, ds \).

If the base vectors of special fixed right-handed rectangular coordinate system are \( i_s, i_x, i_z \), then:
\[ [A] = \begin{bmatrix} t \cdot i_x & t \cdot i_y & t \cdot i_z \\ i_z \cdot i_x & i_z \cdot i_y & i_z \cdot i_z \\ i_n \cdot i_x & i_n \cdot i_y & i_n \cdot i_z \end{bmatrix}. \]

The geometry equations are:
\[ \varepsilon_x = u'_s - k_q u_z + k_z u_n, \quad \varepsilon_z = u'_z + k_s u_x - k_z u_n - \varphi_q, \]
\[ \varepsilon_n = u'_n - k_x u_z + k_z u_x + \varphi_z, \quad \omega_s = \varphi'_s - k_q \varphi_z + k_z \varphi_n, \]
\[ \omega_z = \varphi'_z + k_s \varphi_s - k_x \varphi_n, \quad \omega_n = \varphi'_n - k_z \varphi_s + k_x \varphi_z, \]

where, \( \varepsilon_x, \varepsilon_z, \varepsilon_n, \omega_s, \omega_z, \omega_n \) are respectively generalized strains corresponding to generalized stresses \( Q_s, Q_z, Q_n, M_s, M_z, M_n \) and \( u_s, u_z, u_n, \varphi_s, \varphi_z, \varphi_n \) are generalized displacements corresponding to loads \( p_s, p_z, p_n, m_s, m_z, m_n \). The boundary conditions should be given by prescribed the following qualities
\[ Q_s \text{ or } u_s, \quad Q_z \text{ or } u_z, \quad Q_n \text{ or } u_n, \quad M_s \text{ or } \varphi_s, \quad M_z \text{ or } \varphi_z, \quad M_n \text{ or } \varphi_n \]

Eqs. (12) can be rewritten as:
\[ \frac{d}{ds} \{ \varphi \} - [K] \cdot \{ \varphi \} - \{ \omega \} = \{ 0 \}, \]
\[ \frac{d}{ds} \{ u \} - [K] \cdot \{ u \} - [H] \cdot \{ \varphi \} - \{ \varepsilon \} = \{ 0 \}, \]

where
\[ \{ \varphi \} = [\varphi_s, \varphi_z, \varphi_n]^T, \quad \{ u \} = [u_s, u_z, u_n]^T, \]
\[ \{ \omega \} = [\omega_s, \omega_z, \omega_n]^T, \quad \{ \varepsilon \} = [\varepsilon_x, \varepsilon_z, \varepsilon_n]^T, \]

so the general solutions to the geometry equations are:
\[ \{ \varphi \} = [A] \cdot (\{ \varphi_0 \} + \{ \varphi^* \}), \]
\[ \{ u \} = [A] \cdot \left\{ \{ U_0 \} + \int_0^s [A]^T \cdot (\{ \varepsilon \} + [H] \cdot [A] (\{ \varphi_0 \} + \{ \varphi^* \})) \, ds \right\}, \]
in which \( \{ \varphi_0 \} \) and \( \{ U_0 \} \) are integration constants, \( \{ \varphi^* \} = \int_0^s [A]^T \cdot \{ \omega \} \, ds \).
3. The St. Venant torsional warping function and equivalent constitutive equations

Assuming that the deformations of the beam consist of stretching, bending and torsion, then the displacement field can be written in the following form:

\[ u = Wt + Ui_\zeta + Vi_\eta, \]

in which:

\[ W = u_s(s) + \eta \phi_\zeta(s) - \xi \phi_\eta(s) + \sigma(s) \phi(\xi, \eta), \]

\[ U = u_\zeta(s) - \eta \phi_\zeta(s), \quad V = u_\eta(s) + \xi \phi_\eta(s), \]

here, \( \phi(\xi, \eta) \) is the warping function of the St. Venant torsion of a cylindrical shaft which has the same cross-section as the beam under consideration (Timoshenko and Goodier, 1974). Eq. (16) takes into account the effects of torsion-related warping as well as that of transverse shear deformations. The distributions of strain \( e_s, e_n, e_g \), on the cross-section, are:

\[ \sqrt{g} e_{s} = e_s + \eta \omega_\zeta + \sqrt{g} \phi_\eta + k_s \left[ \frac{\partial \phi}{\partial \xi} \right] \eta - \left[ \frac{\partial \phi}{\partial \eta} \right] \xi, \]

\[ 2 \sqrt{g} e_{\xi} = e_\xi - \eta \omega_\zeta + \left[ \sqrt{g} \left( \frac{\partial \phi}{\partial \zeta} \right) + k_\eta \phi \right] \xi, \]

\[ 2 \sqrt{g} e_{\eta} = e_\eta + \xi \omega_\zeta + \left[ \sqrt{g} \left( \frac{\partial \phi}{\partial \eta} \right) - k_\zeta \phi \right] \eta, \]

\[ e_{\xi \eta} = e_{\eta \xi} = 0, \]

where \( e_s, e_\zeta, e_\eta, \omega_\zeta, \omega_\eta \) are the same as Eqs. (12), and \( \zeta \) is a generalized coordinate for warping. Assume that the curvature is small enough to assure that (Washizu, 1964):

\[ \sqrt{g} \approx 1. \]

Introducing stress resultants and moments defined by:

\[ Q_s = \int \int \sigma_s d\zeta d\eta, \quad M_s = \int \int (\tau_{s\eta} \zeta - \tau_{s\zeta} \eta) d\zeta d\eta, \]

\[ Q_\zeta = \int \int \tau_{s\zeta} d\zeta d\eta, \quad M_\zeta = \int \int \sigma_s \eta d\zeta d\eta, \]

\[ Q_\eta = \int \int \tau_{s\eta} d\zeta d\eta, \quad M_\eta = - \int \int \sigma_s \zeta d\zeta d\eta, \]

lead to the equivalent constitutive equations described with generalized strains and generalized coordinate for warping.

The minimum potential energy principle of the entire beam can be written as

\[ \delta U - \delta \int_0^l (\{p\}^T \cdot \{u\} + \{m\}^T \cdot \{\phi\}) ds = 0. \]

Equilibrium Eqs. (9) described with generalized displacements can be derived and another equation which involves the St. Venant torsional warping function is:

\[ \left( \int \int \phi \sigma_s d\zeta d\eta \right)' - k_s \int \int \left[ \frac{\partial \phi}{\partial \zeta} \eta - \left( \frac{\partial \phi}{\partial \eta} \right) \xi \right] \sigma_s d\zeta d\eta \]

\[ - \int \int \left\{ \left( \frac{\partial \phi}{\partial \zeta} \right) + k_\eta \phi \right\} \tau_{s\zeta} + \left[ \frac{\partial \phi}{\partial \eta} - k_\zeta \phi \right] \tau_{s\eta} \right\} d\zeta d\eta = 0. \]
Besides, boundary conditions (13) and the following boundary condition can also be obtained

\[
\left[ \int \int \varphi \sigma, \eta \, d \xi \, d \eta \right]^l_0 = 0.
\]  

(21)

4. The solving process of the generalized coordinate for warping

Equations for the static problem of the beam include: variational Eq. (19), geometry Eqs. (12), and equivalent constitutive Eqs. (18) as well as Eq. (20). Substituting Eqs. (17) into Eqs. (8), then substituting the results obtained into Eq. (20), we have

\[
E \int \int \varphi \{ \dot{\varepsilon}_n + \eta \omega_n - \xi \omega_n' + \varphi \eta'' + k_n D \varphi + k_s \varphi' \} \, d \xi \, d \eta - k_s E \int \int D \{ \varepsilon_n + \eta \omega_n - \xi \omega_n' + \varphi \eta'' + k_s \varphi' \} \, d \xi \, d \eta - G \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) \right] \{ \varepsilon_n - \eta \omega_n' + \left( \frac{\partial \varphi}{\partial \xi} \right) \} \, d \xi \, d \eta - G \int \int \left[ \left( \frac{\partial \varphi}{\partial \eta} \right) \right] \{ \varepsilon_n - \eta \omega_n' + \left( \frac{\partial \varphi}{\partial \eta} \right) \} \, d \xi \, d \eta = 0;
\]

(22)

where

\[
D = \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) \eta - \left( \frac{\partial \varphi}{\partial \eta} \right) \xi \right] \, d \xi \, d \eta
\]

Substituting Eqs. (17) into Eqs. (8), then substituting the results obtained into Eqs. (18), one obtains

\[
Q_s = E A \varepsilon_n + E k_s D \varphi + E S_z \omega_n - E S_n \omega_n + E \int \int \varphi \, d \xi \, d \eta',
\]

\[
Q_\xi = G A \varepsilon_n + G S_z \omega_n + G \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) \right] \, d \xi \, d \eta,
\]

\[
Q_\eta = G A \varepsilon_n + G S_n \omega_n + G \int \int \left[ \left( \frac{\partial \varphi}{\partial \eta} \right) \right] \, d \xi \, d \eta,
\]

\[
M_s = G S_z \omega_n + G S_n \omega_n + G I_p \omega_n - G \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) \eta - \left( \frac{\partial \varphi}{\partial \eta} \right) \xi \right] \, d \xi \, d \eta,
\]

\[
M_\xi = E S_z \varepsilon_n + E I_p \varepsilon_n - E I_\eta \varepsilon_n + E k_s \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) \eta - \left( \frac{\partial \varphi}{\partial \eta} \right) \xi \right] \, d \xi \, d \eta + E \int \int \eta \varphi \, d \xi \, d \eta',
\]

\[
M_\eta = -E S_n \varepsilon_n + E I_\eta \varepsilon_n - E I_\eta \varepsilon_n - E k_s \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) \xi \eta - \left( \frac{\partial \varphi}{\partial \xi} \right) \xi \eta \right] \, d \xi \, d \eta - E \int \int \xi \varphi \, d \xi \, d \eta',
\]

in which, \( A \) is the area of the cross-section, \( S_z \) and \( S_n \) are the first moments of the cross-sectional area with respect to the \( \xi \)- and \( \eta \)-axes, \( I_\xi \) and \( I_\eta \) are the moments of inertia of the cross-sectional area, computed about the \( \xi \)- and \( \eta \)-axes, \( I_{\xi \eta} \) is the product of inertia, and \( I_p = I_\xi + I_\eta \). Six unknown generalized strains can be now obtained from Eqs. (23), implying
\[ \varepsilon_a = \frac{1}{E} \left( D_{11} \sigma_s + D_{12} M_z + D_{13} M_H \right) + k_s H_1 \alpha + H_2 \alpha', \]
\[ \varepsilon_z = \frac{1}{G} \left( D_{21} \sigma_s + D_{22} \sigma_n + D_{23} M_s \right) + H_3 \alpha, \]
\[ \varepsilon_n = \frac{1}{G} \left( D_{22} \sigma_n + D_{23} M_n + D_{33} M_s \right) + H_4 \alpha, \]
\[ \omega_a = \frac{1}{G} \left( D_{23} Q_z + D_{33} Q_n + D_{43} M_s \right) + H_5 \alpha, \]
\[ \omega_z = \frac{1}{G} \left( D_{12} Q_s + D_{23} M_z + D_{53} M_n \right) + k_s H_6 \alpha + H_7 \alpha', \]
\[ \omega_n = \frac{1}{G} \left( D_{13} Q_s + D_{53} M_z + D_{63} M_s \right) + k_s H_8 \alpha + H_9 \alpha', \]

where

\[ D_{11} = \frac{I_{\eta}^{2} - I_{\zeta}^{2} - I_{\eta} \frac{S_{\eta}^{2} - I_{\eta} \frac{S_{\eta}^{2} - I_{\eta}}{D_{1}}}{D_{1}}}{D_{1}}, \quad D_{12} = \frac{I_{\eta} \frac{S_{\zeta}^{2} - I_{\eta} \frac{S_{\eta}^{2} - I_{\eta}}{D_{1}}}{D_{1}}}{D_{1}}, \]
\[ D_{13} = \frac{I_{\eta} \frac{S_{\zeta}^{2} - I_{\eta} \frac{S_{\eta}^{2} - I_{\eta}}{D_{1}}}{D_{1}}}{D_{1}}, \quad D_{21} = \frac{S_{\eta}^{2} - I_{\eta} A}{D_{2}}, \]
\[ D_{22} = S_{\zeta}^{2} S_{\eta} D_{2}, \quad D_{23} = - \frac{S_{\eta} A}{D_{2}}, \quad D_{32} = \frac{S_{\eta}^{2} - I_{\eta} A}{D_{2}}, \quad D_{33} = \frac{S_{\eta} A}{D_{2}}, \]
\[ D_{43} = - \frac{A^{2}}{D_{2}}, \quad D_{52} = \frac{S_{\eta}^{2} - I_{\eta} A}{D_{1}}, \quad D_{53} = - \frac{I_{\eta} A + S_{\eta} S_{\zeta}}{D_{1}}, \quad D_{63} = \frac{S_{\eta}^{2} - I_{\eta} A}{D_{1}}, \]
\[ D_{1} = (I_{\eta}^{2} - 2 I_{\eta} S_{\eta} S_{\zeta} - I_{\eta} A + S_{\eta}^{2} S_{\zeta} + I_{\eta} S_{\eta}), \]
\[ D_{2} = - A (I_{\eta} A - S_{\eta}^{2} - S_{\zeta}^{2}), \]
\[ H_{1} = [(I_{\eta}^{2} - I_{\eta}^{2}) D + (I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) J_{1} + (I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) J_{2}]/D_{1}, \]
\[ H_{2} = [(I_{\eta}^{2} - I_{\eta}^{2}) J_{1} - (I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) J_{5} + (I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) J_{6}]/D_{1}, \]
\[ H_{3} = [(I_{\eta} A - S_{\eta}^{2}) J_{2} - S_{\eta} S_{\zeta} J_{3} - S_{\eta} A J_{4}]/D_{2}, \]
\[ H_{4} = [-S_{\eta} S_{\zeta} J_{2} + (I_{\eta} A - S_{\eta}^{2}) J_{3} + S_{\eta} A J_{4}]/D_{2}, \]
\[ H_{5} = (S_{\eta} A J_{2} - S_{\eta} A J_{3} - A^{2} J_{4})/D_{2}, \]
\[ H_{6} = [(I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) D + (I_{\eta} A - S_{\eta}^{2}) J_{7} + (S_{\eta} S_{\zeta} - I_{\eta} A) J_{8}]/D_{1}, \]
\[ H_{7} = [(I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) J_{1} + (I_{\eta} A - S_{\eta}^{2}) J_{5} + (S_{\eta} S_{\zeta} - I_{\eta} A) J_{6}]/D_{1}, \]
\[ H_{8} = [(I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) D + (I_{\eta} A - S_{\eta}^{2}) J_{7} + (S_{\eta}^{2} - I_{\eta} A) J_{8}]/D_{1}, \]
\[ H_{9} = [(I_{\eta} S_{\eta} - I_{\eta} S_{\zeta}) J_{1} + (I_{\eta} A - S_{\eta}^{2}) J_{5} + (S_{\eta}^{2} - I_{\eta} A) J_{6}]/D_{1}. \]

where, \( J_{i}(i = 1, 2, \ldots, 8) \), which depend only on the curvature and geometry of the beam, are defined as the following integrations given by

\[ J_{1} = \int \int \varphi \, d\zeta \, d\eta, \quad J_{2} = \int \int \left[ \frac{\partial \varphi}{\partial \zeta} + k_{\eta} \varphi \right] \, d\zeta \, d\eta, \quad J_{3} = \int \int \left[ \frac{\partial \varphi}{\partial \eta} - k_{\zeta} \varphi \right] \, d\zeta \, d\eta, \]
\[ J_{4} = \int \int \left[ \left( \frac{\partial \varphi}{\partial \zeta} \right)^{2} \varphi - \left( \frac{\partial \varphi}{\partial \eta} \right)^{2} \varphi \right] \, d\zeta \, d\eta, \quad J_{5} = \int \int \eta \varphi \, d\zeta \, d\eta, \quad J_{6} = \int \int \varphi \, d\zeta \, d\eta, \]
\[ J_{7} = \int \int \left[ \frac{\partial \varphi}{\partial \zeta} \right]^{2} \varphi - \left[ \frac{\partial \varphi}{\partial \eta} \right]^{2} \varphi \right] \, d\zeta \, d\eta, \quad J_{8} = \int \int \left[ \left( \frac{\partial \varphi}{\partial \zeta} \right)^{2} \varphi - \left( \frac{\partial \varphi}{\partial \eta} \right)^{2} \varphi \right] \, d\zeta \, d\eta. \]
Substituting Eqs. (24) and their derivatives \( \varepsilon_s', \omega_s', \alpha_s' \) into Eq. (22), we establish the differential equation of \( \alpha \):

\[
\alpha'' + \frac{k_s(H_1 J_1 + H_6 J_5 - H_9 J_6 - H_2 D - H_7 J_7 + H_9 J_8)}{(H_2 J_1 + H_7 J_5 - H_9 J_6 + \Gamma)} \alpha' + \frac{1}{(H_2 J_1 + H_7 J_5 - H_9 J_6 + \Gamma)} \left\{ k_s^2 H_1 J_1 + k_s^2 H_6 J_5 - k_s^2 H_9 J_6 + k_s^2 \int \int \varphi D d\xi d\eta - k_s^2 H_1 J_1 + k_s^2 H_6 J_5 - k_s^2 H_9 J_6 + k_s^2 \int \int D^2 d\xi d\eta \right\} \\
\times \alpha - \frac{G}{E(H_2 J_1 + H_7 J_5 - H_9 J_6 + \Gamma)} \left\{ H_3 J_2 + H_4 J_3 + H_5 J_4 + \int \int \left[ \left( \frac{\partial \varphi}{\partial \eta} \right) + k_\eta \varphi \right] d\xi d\eta \right\} \\
+ \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right) - k_\xi \varphi \right] d\xi d\eta \right\} + \frac{1}{E(H_2 J_1 + H_7 J_5 - H_9 J_6 + \Gamma)} [D_{11} J_1 + D_{12} J_5 - D_{13} J_6] \varphi_s' \\
+ (D_{12} J_1 + D_{25} J_5 - D_{33} J_6) \alpha' + (D_{13} J_1 + D_{33} J_5 - D_{33} J_6) \alpha' + k_s (D_{11} D + D_{12} J_5 - D_{13} J_6) \varphi_s' \\
- (D_{21} J_2 + D_{22} J_3 - D_{23} J_4) \alpha' - (D_{22} J_2 + D_{32} J_3 + D_{33} J_4) \varphi_s' + (D_{23} J_2 - D_{33} J_3 + D_{33} J_4) \alpha' \\
- k_s (D_{11} D + D_{12} J_5 - D_{13} J_6) \alpha' \right\} = 0, \tag{25}
\]

where

\[
\Gamma = \int \int \varphi^2 d\xi d\eta.
\]

Eq. (25) can be written in the form:

\[
\alpha'' + q_1(s) \alpha' + q_2(s) \alpha = f(s). \tag{26}
\]

This is a second order ordinary non-homogeneous linear differential equation with variable coefficients. The solution to such an equation is the sum of two functions \( \alpha_0 \), the complementary solution to the homogeneous equation,

\[
\alpha_0'' + q_1(s) \alpha_0' + q_2(s) \alpha_0 = 0, \tag{27}
\]

and any particular solution \( \alpha^* \) to

\[
(\alpha^*)'' + q_1(s) (\alpha^*)' + q_2(s) \alpha^* = f(s). \tag{28}
\]

Once the relation among \( k_1, k_2 \) and arc coordinate \( s \) is determined, we may find its solution. The general solving steps are as follows:

Let \( \alpha_1(s) \) and \( \alpha_2(s) \) be any two linear independent solutions to the homogeneous Eq. (27), then the complete solution to Eq. (26) is:

\[
\alpha = c_1 \alpha_1(s) + c_2 \alpha_2(s) + \alpha^*, \tag{29}
\]

in which \( \alpha^* \) is a particular solution to Eq. (26). Thus, \{\( \omega \)\} and \{\( \varepsilon \)\} can be expressed with \{\( Q \)\}, \{\( M \)\}, the generalized coordinate for warping \( \alpha \) and integration constants \( c_1, c_2 \). Now the rest is to determine integration constants \{\( Q_0 \)\}, \{\( M_0 \)\}, \{\( \varphi_0 \)\}, \{\( U_0 \)\} as well as \( c_1 \) and \( c_2 \).

5. The generalized coordinate for warping of a plane curved beam under a concentrated force at the free end of the beam and two uniformly distributed loads

\( k_s = 0 \) and \( \theta = 5\pi/6 \) in Eqs. (3) is just the situation of a plane curved beam (see Fig. 3a), and Fig. 3b illustrates the uniform equilateral triangle cross-section at the free end of the beam. It should be noted that \( \xi \)- and \( \eta \)-axes are a set of principal axes through the centroid on the cross-section. Fix the origin of the rect-
angular coordinate system at the end of the beam \((s = 0)\), the axis of the beam being on the plane \(Oxy\). The loads acting are:

\[
\begin{align*}
\{m\} &= \{0\}, \quad \{p\} = \begin{bmatrix} 0 \\ p_\xi \\ p_\eta \end{bmatrix}^T
\end{align*}
\]

If the axis of the beam is a circle with radius \(a\), one has:

\[
\begin{align*}
\beta &= \frac{s}{a}, \quad k_\xi = k_1 \sin \theta, \quad k_\eta = k_1 \cos \theta, \\
x &= a \sin \beta, \quad y = a(1 - \cos \beta).
\end{align*}
\]

Eq. (25) can be reduced to:

\[
\begin{align*}
x'' &= \frac{G}{E(\Gamma - \frac{J_3}{J_4})} \left\{ \int \int \left[ \left( \frac{\partial \varphi}{\partial \xi} \right)^2 + k_\eta \varphi \right] d\xi d\eta + \int \int \left[ \left( \frac{\partial \varphi}{\partial \eta} \right)^2 - k_\xi \varphi \right]^2 d\xi d\eta - \frac{f_3}{A} - \frac{J_3^2}{I_p} \right\} x \\
&= f_1 Q_\xi + f_2 M_s,
\end{align*}
\]

where

\[
\begin{align*}
f_1 &= \frac{\left( \frac{J_4}{J_3} - \frac{J_3}{A} \right)}{E(\Gamma - \frac{J_4}{J_3})}, \quad f_2 = \frac{\left( \frac{J_4}{J_3} - k_\eta \frac{J_3}{J_4} \right)}{E(\Gamma - \frac{J_4}{J_3})}
\end{align*}
\]
The solution to Eq. (30) must be:
\[ x = c_1 ch \theta s + c_2 sh \theta s + x^* \]  
(31)

where
\[
\bar{\alpha} = \sqrt{G \left\{ \int \int \left[ \left( \frac{\partial \phi}{\partial \zeta} \right) + k_\eta \phi \right]^2 \, d\xi \, d\eta + \int \int \left[ \left( \frac{\partial \phi}{\partial \eta} \right) - k_\zeta \phi \right]^2 \, d\xi \, d\eta - \frac{f_3^2}{A - \frac{J_4}{I_p}} \right\} / E \left( \Gamma - \frac{J_5^2}{I_\zeta} \right) }
\]

Thus, the complete solution of Eq. (30) becomes:
\[
x = c_1 ch \theta s + c_2 sh \theta s \\
+ \frac{f_1}{(k_1^2 + \bar{\alpha}^2)} \left\{ \frac{1}{2} Q_{0s} \sin \beta - \frac{1}{2} Q_{0s} \cos \beta - \frac{\sqrt{3}}{2 \bar{\alpha}^2} Q_{0o}(k_1^2 + \bar{\alpha}^2) + \frac{\sqrt{3}}{4} \frac{p_y \alpha}{a} \left[ s \frac{s}{a \bar{\alpha}^2} (k_1^2 + \bar{\alpha}^2) \right] - \sin \beta \right\} \\
- \frac{1}{4} p_y a \left[ \frac{3s}{a \bar{\alpha}^2} (k_1^2 + \bar{\alpha}^2) + \sin \beta \right] + \frac{f_2}{(k_1^2 + \bar{\alpha}^2)} \left\{ M_{0s} \cos \beta + M_{0o} \sin \beta - Q_{0s} a \left[ \cos \beta - \frac{1}{\bar{\alpha}^2} (k_1^2 + \bar{\alpha}^2) \right] \right\} \\
+ \frac{1}{2} p_y a^2 \left[ \sin \beta - \frac{s}{a \bar{\alpha}^2} (k_1^2 + \bar{\alpha}^2) \right] \right\} \\
\left\{ \frac{1}{2} M_{0s} \cos \beta + M_{0o} \sin \beta + Q_{0s} \sin \beta + \frac{\sqrt{3}}{2} p_y a^2 (\sin \beta - \beta) - \frac{\sqrt{3}}{2} p_y a^2 (\sin \beta - \beta), \right\} \\
M_{\zeta} = \frac{\sqrt{3}}{2} M_{0s} \sin \beta - \frac{\sqrt{3}}{2} M_{0o} \cos \beta + \frac{1}{2} M_{0o} \sin \beta - \frac{\sqrt{3}}{2} Q_{0s} \cos \beta + \frac{\sqrt{3}}{2} Q_{0o} \cos \beta - \frac{\sqrt{3}}{2} Q_{0o} x - p_y a y, \\
M_y = -\frac{1}{2} M_{0s} \cos \beta - \frac{1}{2} M_{0o} \cos \beta - \frac{3}{2} M_{0o} \sin \beta - \frac{\sqrt{3}}{2} Q_{0s} x + \frac{\sqrt{3}}{2} Q_{0o} x + \frac{1}{2} Q_{0o} x + p_y a x, \\
Q_s = Q_{0s} \cos \beta + Q_{0o} \sin \beta + \frac{\sqrt{3}}{2} p_y a^2 \left[ \sin \beta - \beta \right] + \frac{1}{2} p_y a^2, \\
Q_\zeta = \frac{\sqrt{3}}{2} Q_{0s} \sin \beta - \frac{\sqrt{3}}{2} Q_{0o} \cos \beta + \frac{1}{2} Q_{0o} y - \frac{1}{4} p_y a (3 \sin \beta + \beta) - \frac{\sqrt{3}}{4} p_y a (\sin \beta - \beta), \\
Q_y = \frac{1}{2} Q_{0o} \sin \beta - \frac{1}{2} Q_{0o} \cos \beta - \frac{\sqrt{3}}{2} Q_{0o} y - \frac{\sqrt{3}}{2} Q_{0o} x + \frac{\sqrt{3}}{2} Q_{0o} x - \frac{1}{2} Q_{0o} x + p_y a x, \\
\phi_s = \phi_{0s} \cos \beta + \phi_{0o} \sin \beta + a \cos \beta \int_0^\beta \left( \omega_s \cos \beta + \frac{\sqrt{3}}{2} \omega_\zeta \sin \beta + \frac{1}{2} \omega_\eta \sin \beta \right) \, d\beta \\
+ a \sin \beta \int_0^\beta \left( \omega_s \sin \beta - \frac{\sqrt{3}}{2} \omega_\zeta \cos \beta - \frac{1}{2} \omega_\eta \cos \beta \right) \, d\beta, \\
\phi_\zeta = \frac{\sqrt{3}}{2} \phi_{0s} \sin \beta - \frac{\sqrt{3}}{2} \phi_{0o} \cos \beta + \frac{1}{2} \phi_{0o} y \\
+ \frac{\sqrt{3}}{4} a \sin \beta \int_0^\beta \left( \omega_s \cos \beta + \frac{\sqrt{3}}{2} \omega_\zeta \sin \beta + \frac{1}{2} \omega_\eta \sin \beta \right) \, d\beta
\[-\frac{\sqrt{3}}{2} a \cos \beta \int_{0}^{\beta} \left( \omega_s \sin \beta - \frac{\sqrt{3}}{2} \omega_z \cos \beta - \frac{1}{2} \omega_n \cos \beta \right) \, d\beta + \frac{1}{4} a \int_{0}^{\beta} \left( \omega_z - \sqrt{3} \omega_n \right) \, d\beta,\]

\[\varphi_\eta = \frac{1}{2} \varphi_{0s} \sin \beta - \frac{1}{2} \varphi_{0z} \cos \beta - \frac{\sqrt{3}}{2} \varphi_{0n} + \frac{1}{2} a \sin \beta \int_{0}^{\beta} \left( \omega_s \cos \beta + \frac{\sqrt{3}}{2} \omega_z \sin \beta + \frac{1}{2} \omega_n \sin \beta \right) \, d\beta\]

\[-\frac{1}{2} a \cos \beta \int_{0}^{\beta} \left( \omega_s \sin \beta - \frac{\sqrt{3}}{2} \omega_z \cos \beta - \frac{1}{2} \omega_n \cos \beta \right) \, d\beta - \frac{\sqrt{3}}{4} a \int_{0}^{\beta} \left( \omega_z - \sqrt{3} \omega_n \right) \, d\beta,\]

\[u_s = U_{0s} \cos \beta + U_{0z} \sin \beta + a \cos \beta \int_{0}^{\beta} \left[ \epsilon_s \cos \beta - \frac{\sqrt{3}}{2} \sin \beta (\epsilon_z + \varphi_\eta) - \frac{1}{2} \sin \beta (\epsilon_\eta - \varphi_\zeta) \right] \, d\beta\]

\[+ a \sin \beta \int_{0}^{\beta} \left[ \epsilon_s \sin \beta - \frac{\sqrt{3}}{2} \cos \beta (\epsilon_z + \varphi_\eta) + \frac{1}{2} \cos \beta (\epsilon_\eta - \varphi_\zeta) \right] \, d\beta,\]

\[u_\xi = \frac{\sqrt{3}}{2} U_{0s} \sin \beta - \frac{\sqrt{3}}{2} U_{0z} \cos \beta + \frac{1}{2} U_{0n} + \frac{\sqrt{3}}{2} a \sin \beta \times \int_{0}^{\beta} \left[ \epsilon_s \cos \beta + \frac{\sqrt{3}}{2} \sin \beta (\epsilon_z + \varphi_\eta) + \frac{1}{2} \sin \beta (\epsilon_\eta - \varphi_\zeta) \right] \, d\beta\]

\[- \frac{\sqrt{3}}{2} a \cos \beta \int_{0}^{\beta} \left[ \epsilon_s \sin \beta - \frac{\sqrt{3}}{2} \cos \beta (\epsilon_z + \varphi_\eta) - \frac{1}{2} \cos \beta (\epsilon_\eta - \varphi_\zeta) \right] \, d\beta\]

\[+ \frac{1}{4} a \int_{0}^{\beta} [(\epsilon_z + \varphi_\eta) - \sqrt{3} (\epsilon_\eta - \varphi_\zeta)] \, d\beta,\]

\[u_\eta = \frac{1}{2} U_{0s} \sin \beta - \frac{1}{2} U_{0z} \cos \beta - \frac{\sqrt{3}}{2} U_{0n}\]

\[+ \frac{1}{2} a \sin \beta \int_{0}^{\beta} \left[ \epsilon_s \cos \beta + \frac{\sqrt{3}}{2} \sin \beta (\epsilon_z + \varphi_\eta) - \frac{1}{2} \sin \beta (\epsilon_\eta - \varphi_\zeta) \right] \, d\beta\]

\[- \frac{1}{2} a \cos \beta \int_{0}^{\beta} \left[ \epsilon_s \sin \beta - \frac{\sqrt{3}}{2} \cos \beta (\epsilon_z + \varphi_\eta) - \frac{1}{2} \cos \beta (\epsilon_\eta - \varphi_\zeta) \right] \, d\beta\]

\[+ \frac{\sqrt{3}}{4} a \int_{0}^{\beta} [(\epsilon_z + \varphi_\eta) - \sqrt{3} (\epsilon_\eta - \varphi_\zeta)] \, d\beta.\]

(33)

If the beam is fixed at \( s = 0 \) and loaded by a concentrated force \( F \) applied at the other end \( (s = l) \) in the vertical direction (see Fig. 3b). The boundary conditions are:

\[s = 0 (\beta = 0), \quad U_{0s} = U_{0z} = U_{0n} = 0, \quad \varphi_{0s} = \varphi_{0z} = \varphi_{0n} = 0, \quad \alpha = 0,\]

\[s = l (\beta = \beta_1), \quad Q_\xi = 0, \quad Q_\zeta = \frac{1}{2} F, \quad Q_\eta = \frac{\sqrt{3}}{2} F, \quad M_\xi = M_\zeta = M_\eta = 0, \quad \alpha' = 0,\]

where \( l = \pi a/2 \), the integration constants determined by the aforementioned conditions are
Fig. 4. (a) Horizontal displacement $U_P$ of point $P$ at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_x = 20$ N/cm, $p_y$. (b) Horizontal displacement $U_P$ of point $P$ at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_x$, $p_y = 10$ N/cm. (c) Vertical displacement $V_P$ of point $P$ at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_x = 20$ N/cm, $p_y$. (d) Vertical displacement $V_P$ of point $P$ at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_x$, $p_y = 10$ N/cm. (e) Warping displacement $W_P$ of point $P$ at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_x = 20$ N/cm, $p_y$. (f) Warping displacement $W_P$ of point $P$ at the free end of the beam under a concentrated force $F = 25$ N applied at the free end and uniformly distributed loads $p_x$, $p_y = 10$ N/cm.
\[ Q_{0s} = \frac{\sqrt{3}}{2} p_s a + \frac{1}{2} p_y a + \frac{1}{2} F, \quad Q_{0t} = -\frac{\sqrt{3}}{2} p_s a - \frac{1}{2} p_y a, \quad Q_{0g} = \frac{\pi}{4} p_s a - \frac{\sqrt{3} \pi}{4} p_y a + F, \]
\[ M_{0s} = -\frac{1}{4} p_s a^2 \left( 1 - \frac{\pi}{2} \right) + \frac{\sqrt{3}}{2} p_y a^2 \left( 1 - \frac{\pi}{2} \right) + Fa, \quad M_{0t} = -\frac{1}{2} p_s a^2 + \frac{\sqrt{3}}{2} p_y a^2 - Fa, \]
\[ M_{0g} = -\frac{\sqrt{3}}{2} p_s a^2 - \frac{1}{2} p_y a^2, \]
\[ c_1 = \frac{f_1}{(k_1^2 + \bar{\alpha}^2)} \left[ \frac{1}{2} Q_{0t} + \frac{\sqrt{3}}{2\bar{\zeta}} Q_{0g} (k_1^2 + \bar{\alpha}^2) \right] - \frac{f_2}{(k_1^2 + \bar{\alpha}^2)} \left( M_{0s} + \frac{k_1}{2\bar{\zeta}} Q_{0g} \right), \]
\[ c_2 = -c_1 \theta \bar{\alpha} l - \frac{f_1}{\bar{\alpha} c h \bar{\alpha} (k_1^2 + \bar{\alpha}^2)} \left[ \frac{1}{2a} Q_{0t} + \frac{\sqrt{3}}{4\alpha^2} p_s (k_1^2 + \bar{\alpha}^2) - \frac{3}{4\alpha^2} p_y (k_1^2 + \bar{\alpha}^2) \right] \]
\[ \quad + \frac{f_2}{\bar{\alpha} c h \bar{\alpha} (k_1^2 + \bar{\alpha}^2)} \left[ \frac{1}{a} M_{0s} - Q_{0g} + \frac{1}{2\alpha \xi} p_s a (k_1^2 + \bar{\alpha}^2) + \frac{\sqrt{3}}{2\alpha \xi} p_y a (k_1^2 + \bar{\alpha}^2) \right]. \tag{34} \]

So far, the solutions to this problem have been obtained. The beam in Fig. 3 is made of steel with the following properties
\[ E = 2.106 \times 10^5 \text{ MPa}, \quad G = 0.816 \times 10^5 \text{ MPa}, \]
\[ b = 1.732 \text{ cm}, \quad a = 48 \text{ cm}. \]

The St. Venant torsional warping function (Sokolnikoff, 1956)
\[ \varphi = -\frac{1}{6b} (\eta^3 - 3\xi^2 \eta) = -\frac{1}{10.392} (\eta^3 - 3\xi^2 \eta). \]

Let \( U_P \) and \( V_P \) present the displacements in the horizontal and vertical directions of point \( P \) on the cross-section (\( \beta = \frac{\pi}{2} \)) shown in Fig. 3b, respectively, and \( W_P \) present the displacement in the \( s \) direction of point \( P \). Theoretical results for \( U_P, V_P \) and \( W_P \) are obtained using the equations developed in this paper and compared with a 3-D finite element analysis (referred as the FEM results), according to the ANSYS program.

![Fig. 5. Distribution of the axial stress \( \sigma_a \) on \( \xi\)-axis at the root of the beam under uniformly distributed loads \( p_\xi = 20 \text{ N/cm}, p_\eta = 10 \text{ N/cm} \) and a concentrated force \( F = 25 \text{ N} \) applied at the free end.](image-url)
To analyze the beam shown in Fig. 3 by the finite element method (FEM), we partition it into 35299 three-dimensional solid elements (SOLID 92), and the total number of nodal points is 54482. These cases for different values of $p_n$ and $p_g$ are shown in Fig. 4(a)–(f). It is evident that the theoretical results are very close to the FEM results.

It is also interesting to compute the stress distributions at the root ($\beta = 0$) of the beam. Fig. 5 shows the distribution of the axial stress $\sigma_s$ on the $\xi$-axis at the root of the beam under uniformly distributed loads $p_\xi = 20$ N/cm, $p_\eta = 10$ N/cm and a concentrated force $F = 25$ N applied at the free end. Here, the axial stress distribution is in close agreement between theoretical results and the FEM results. While Fig. 6 shows the distributions of the shear stresses $\tau_{s\xi}$ and $\tau_{s\eta}$ on the $\xi$-axis at the root of the beam, respectively.

6. Conclusions

A differential equation of generalized warping coordinate $\alpha$ for naturally curved and twisted beams with general cross-sectional shapes subjected to arbitrary load has been derived. The numerical results obtained by solving the present equations are very close to the FEM results for different values of $p_\xi$ and $p_\eta$. The highlights of this paper are as follows:

(i) From mathematical point of view, the boundary value problem for the theory of the beams is fully defined in Eqs. (12), (13), (18) and (29). The complex structural behavior of the beams is modeled accurately as the following features are included shear deformations and torsion-related warping.

(ii) It should be emphasized that this theory is not limited to the beams with an equilateral triangle cross-section, it can be extended to those with general cross-sectional shapes. Once the generalized coordinate for warping $\alpha$ is found, the solutions (23) and (24) remain valid.

(iii) This theory is not limited to the beams with solid cross section either. In the case of thin-walled special curved beams made of either isotropic or composite materials, it can also be extended as long as several assumptions given by Washizu (1964) remain valid. Of course, the stress–strain (constitutive) relations for the beams with thin-walled cross-sections should be supplied. the constitutive equations, providing an additional information about the material and geometrical properties of the body under consideration, complete the formulation of the boundary value problem for the 1-D theory.
Acknowledgement

The work is supported in part by Program for New Century Excellent talents in University (NCET) by the Ministry of Education (MOE) of China.

References


