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On graphs with a local hereditary property

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Abstract

Let $\mathcal{P}$ be an induced hereditary property and $L(\mathcal{P})$ denote the class of all graphs that satisfy the property $\mathcal{P}$ locally. The purpose of the present paper is to describe the minimal forbidden subgraphs of $L(\mathcal{P})$ and the structure of local properties. Moreover, we prove that $L(\mathcal{P})$ is irreducible for any hereditary property $\mathcal{P}$. © 2001 Elsevier Science B.V. All rights reserved.

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1. Introduction

We consider finite undirected graphs without loops or multiple edges. A graph G has vertex-set $V(G)$ and edge-set $E(G)$, and $H \leqslant G$ means that H is an induced subgraph of G. We say that G *contains* H whenever G contains an induced subgraph isomorphic to H.

In general, we follow the notation and terminology of [3].

Let $\mathcal{I}$ denote the class of all graphs. If $\mathcal{P}$ is a proper nonempty isomorphism closed subclass of $\mathcal{I}$, then $\mathcal{P}$ will also denote the property of being in $\mathcal{P}$. We shall use the terms *class of graphs* and *property of graphs* interchangeably.

A property $\mathcal{P}$ is called (*induced*) *hereditary*, if every (induced) subgraph of any graph with property $\mathcal{P}$ also has property $\mathcal{P}$ and *additive*, if the disjoint union $H \cup G \in \mathcal{P}$ whenever $G \in \mathcal{P}$ and $H \in \mathcal{P}$. Obviously, any hereditary property is induced hereditary, too.

Let us denote by $\mathbb{L}$ ($\mathbb{M}$, resp.) the set of all hereditary (induced hereditary, resp.) properties of graphs. Corresponding sets of additive properties are denoted by $\mathbb{L}^a$ and $\mathbb{M}^a$, respectively. The sets $\mathbb{L}$, $\mathbb{L}^a$, $\mathbb{M}$ and $\mathbb{M}^a$, partially ordered by the set inclusion, form

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complete distributive lattices with the set intersection as the meet operation. Obviously, $(\mathbb{L}, \subseteq)$ is a proper sublattice of $(\mathbb{M}, \subseteq)$, for more details see [1,2].

We list some properties to introduce the necessary notions which will be used in the paper. Let k be a nonnegative integer.

$\mathcal{O} = \{G \in \mathcal{I}$: G is totally disconnected$\}$,
$\mathcal{O}_k = \{G \in \mathcal{I}$: each component of G has at most $k+1$ vertices$\}$,
$\mathcal{I}_k = \{G \in \mathcal{I}$: G contains no subgraph isomorphic to K_{k+2}.

Let $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be properties of graphs. We say that a graph G has property $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$, if the vertex set of G can be partitioned into n sets $V_1, \ldots, V_n$ such that $G[V_i]$, the subgraph of G induced by V_i, has the property $\mathcal{P}_i$ for $i = 1, \ldots, n$. It is easy to see that $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ is (induced) hereditary and additive whenever $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ are (induced) hereditary and additive, respectively. An (induced) hereditary property $\mathcal{R}$ is said to be *reducible* if there exist two (induced) hereditary properties $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{R} = \mathcal{P}_1 \circ \mathcal{P}_2$ and *irreducible*, otherwise.

For a given irreducible property $\mathcal{P} \in \mathbb{M}$, a reducible property $\mathcal{R} \in \mathbb{M}$ is called a *minimal reducible bound* for $\mathcal{P}$ if $\mathcal{P} \subset \mathcal{R}$ and for each reducible property $\mathcal{R}' \subset \mathcal{R}$, $\mathcal{P} \nsubseteq \mathcal{R}'$. The family of all minimal reducible bounds for $\mathcal{P}$ will be denoted by $\mathbf{B}(\mathcal{P})$.

Let $\mathcal{P}$ be a class of graphs. A graph $G = (V, E)$ is said to satisfy a property $\mathcal{P}$ *locally* if $G[N(v)] \in \mathcal{P}$ for every $v \in V(G)$. The class of graphs that satisfy the property $\mathcal{P}$ locally will be denoted by $L(\mathcal{P})$ and we shall call such a class a *local property*.

Early investigations dealt mostly with the case $|\mathcal{P}| = 1$; i.e., when all neighborhoods are isomorphic. Summaries of results of this type can be found in the survey papers of Hell [5] and Sedlaček [7]. The major question is the existence of an appropriate G. More recently, the cases when $\mathcal{P}$ consists of all cycles, all paths, all matchings, or all forests were investigated. Also, results concerning some extremal problems on such classes of graphs have been obtained, see [4,8].

Assume that $\mathcal{P}$ is an induced hereditary property. The purpose of the present paper is to describe the minimal-forbidden subgraphs of $L(\mathcal{P})$ and the structure of local properties. Moreover, we prove that $L(\mathcal{P})$ is irreducible for any $\mathcal{P} \in \mathbb{L}$.

2. Forbidden subgraphs

Any induced hereditary property $\mathcal{P}$ can be characterized by the set of minimal-forbidden-induced subgraphs:

$$\mathcal{C}(\mathcal{P}) = \{H \in \mathcal{I} : H \notin \mathcal{P} \text{ but } (H - v) \in \mathcal{P} \text{ for any } v \in V(H)\}.$$

It is easy to prove that a property $\mathcal{P} \in \mathbb{M}$ is additive if and only if all minimal-forbidden-induced subgraphs of $\mathcal{P}$ are connected.

Lemma 1. *If $H \in \mathcal{C}(L(\mathcal{P}))$, then H has a universal vertex.*

Proof. Let $F \in \mathscr{C}(L(\mathscr{P}))$ and suppose F has no universal vertex. Let $v \in V(F)$. Then, there is a vertex $x \in V(F)$ such that $x \notin N_F(v)$. Since $F - x \in L(\mathscr{P})$, it follows that $F[N_{F-x}(v)] \in \mathscr{P}$. But $N_{F-x}(v) = N_F(v)$, hence $F[N_F(v)] \in \mathscr{P}$. This proves that $F \in L(\mathscr{P})$, a contradiction. □

Lemma 2. $\mathscr{C}(L(\mathscr{P})) = \{K_1 + H: H \in \mathscr{C}(\mathscr{P})\}$.

Proof. Let $F \in \mathscr{C}(L(\mathscr{P}))$. By Lemma 1, F has a universal vertex v. Let $H = F - v$. We shall prove that $H \in \mathscr{C}(\mathscr{P})$, i.e., $H \notin \mathscr{P}$ and $H - u \in \mathscr{P}$ for every $u \in V(H)$.

Suppose that $H \in \mathscr{P}$. This implies that $H[N_H(u)] \in \mathscr{P}$ for every $u \in V(H)$. Moreover, $F[N_F(u)] \in \mathscr{P}$, because of

(1) If u is a universal vertex of F, too, then $F - u$ is isomorphic to $F - v$ which is equal to $H \in \mathscr{P}$.
(2) If u is not a universal vertex of F, let w be a vertex in F that is nonadjacent to u. Then $F[N_F(u)] = F[N_{F-w}(u)] \in \mathscr{P}$, since $F - w \in \mathscr{P}$.

Thus, by (1) and (2) we have $F[N(x)] \in \mathscr{P}$ for every $x \in V(F)$, i.e., $F \in L(\mathscr{P})$, a contradiction. Thus, $H \notin \mathscr{P}$.

Now suppose that H is not a minimal-forbidden subgraph of $\mathscr{P}$, i.e., there is a vertex $u \in V(H)$ such that $H - u \notin \mathscr{P}$. Hence, $H' = (H - u) + K_1 \notin L(\mathscr{P})$. But H' is isomorphic to $F - u \in L(\mathscr{P})$, a contradiction. Thus, $H \in \mathscr{C}(\mathscr{P})$.

Conversely, let $H \in \mathscr{C}(\mathscr{P})$ and $F = K_1 + H, V(K_1) = w$. Obviously $F \notin L(\mathscr{P})$ since $H \notin \mathscr{P}$. Suppose that there is a proper induced subgraph $H' < F, H' \in \mathscr{C}(L(\mathscr{P}))$. Since the graph $H \in L(\mathscr{P}), w \in V(H')$ and the graph H' has a universal vertex $u \neq w$ such that $H' - u \in \mathscr{C}(\mathscr{P})$. However $H' - u$ is isomorphic to $H' - w < H$, a contradiction. Thus, $F \in \mathscr{C}(L(\mathscr{P}))$. □

Lemmas 1 and 2 imply the following theorems.

Theorem 1. *A property $\mathscr{Q} = L(\mathscr{P})$ for some $\mathscr{P} \in \mathbb{M}$ if and only if every $H \in \mathscr{C}(\mathscr{Q})$ has a universal vertex.*

Theorem 2. *Let $\mathscr{P}, \mathscr{Q} \in \mathbb{M}$. Then,*

(a) $L(\mathscr{P}) \in \mathbb{M}^a$,
(b) $\mathscr{P} \subset L(\mathscr{P})$,
(c) *If $\mathscr{P} \subseteq \mathscr{Q}$, then $L(\mathscr{P}) \subseteq L(\mathscr{Q})$,*
(d) *$L(\mathscr{P}) = L(\mathscr{Q})$ if and only if $P = Q$.*
(e) *If $G \in \mathscr{P}$, then $G + K_1 \in L(\mathscr{P})$.*

Proof. Conditions (a)–(c) follow immediately from the corresponding definitions. To prove (d), suppose that $\mathscr{P} \neq \mathscr{Q}$. Then there is a graph H such that, say, $H \notin \mathscr{P}$ and $H \in \mathscr{Q}$. From this it follows that: $K_1 + H \notin L(\mathscr{P})$ and $K_1 + H \in L(\mathscr{Q})$. Thus, $L(\mathscr{P}) \neq L(\mathscr{Q})$, a contradiction. Condition (e) follows from the proof of Lemma 2. □

Let $\mathscr{P} \in \mathbb{M}$. Then

$$c(\mathscr{P}) = \sup\{k\colon K_{k+1} \in \mathscr{P}\}$$

is called the *completeness* of $\mathscr{P}$.

From the definition of completeness, Lemma 2 and Theorem 2, we have the following:

Theorem 3. *Let $\mathscr{P} \in \mathbb{M}$. Then $c(L(\mathscr{P})) = c(\mathscr{P}) + 1$.*

Let $\mathbb{L}_k = \{\mathscr{P} \in \mathbb{L}\colon c(\mathscr{P}) = k\}$. The set $\mathbb{M}_k$ is defined analogously. $(\mathbb{M}_k, \subseteq)$ and $(\mathbb{L}_k, \subseteq)$ are distributive sublattices of $(\mathbb{M}, \subseteq)$ and $(\mathbb{L}, \subseteq)$, respectively (see [2]).

By the previous results we have immediately

Theorem 4. *Let $L(\mathbb{L}_k) = \{L(\mathscr{P})\colon \mathscr{P} \in \mathbb{L}_k\}$. Then,*

(a) $L(\mathbb{L}_k) \subset \mathbb{L}_{k+1}$,
(b) $L(\mathbb{L}_k)$ *is isomorphic to* $\mathbb{L}_k$,
(c) $L(\mathbb{L})$ *is isomorphic to* $\mathbb{L}$.

3. Irreducibility

There are different approaches to show that an additive hereditary property $\mathscr{P}$ is irreducible in $\mathbb{L}^{\mathrm{a}}$ (see [1], Chapter 3). The following deep Theorem of Nešetřil and Rödl implies that some properties have exactly one-trivial minimal reducible bound.

Theorem 5 (Nešetřil and Rödl [6]). *Let $\mathscr{C}(\mathscr{P})$ be a finite set of 2-connected graphs. Then for every graph G of property $\mathscr{P}$ there exists a graph H of property $\mathscr{P}$ such that for any partition $\{V_1, V_2\}$ of $V(H)$ there is an i, $i = 1$ or 2 for which the subgraph $H[G_i]$ induced by V_i in H contains G.*

Corollary 1 (Borowiecki et al. [1]). *Let $\mathscr{C}(\mathscr{P})$ be a finite set of 2-connected graphs, then the property $\mathscr{P}$ has exactly one minimal-reducible bound $\mathscr{R} = \mathscr{O} \circ \mathscr{P}$.*

Particularly for the property $L(\mathscr{O}_k)$ it follows:

Lemma 3. $\mathbf{B}(L(\mathscr{O}_k)) = \{\mathscr{O} \circ L(\mathscr{O}_k)\}$ *for* $\mathscr{O}_k \in \mathbb{L}^{\mathrm{a}}$.

Proof. The set of minimal-forbidden subgraphs for $\mathscr{O}_k \in \mathbb{L}$ is given by

$$\mathscr{C}(\mathscr{O}_k) = \{G \in \mathscr{I}\colon |V(G)| = k + 2 \text{ and } G \text{ is connected}\}.$$

Thus, if $H \in \mathscr{C}(L(\mathscr{O}_k))$, then H is 2-connected. Since $\mathscr{C}(L(\mathscr{O}_k))$ is finite, then each minimal reducible bound $\mathscr{Q} \circ \mathscr{Q}'$ for $L(\mathscr{O}_k)$, by Corollary 1, has the form $\mathscr{Q} \circ L(\mathscr{O}_k)$, i.e., one of factors has to be $L(\mathscr{O}_k)$. By the minimality we have $\mathscr{Q} = \mathscr{O}$ which proves the lemma. □

From above the following lemma follows.

Lemma 4. *The property $L(\mathcal{O}_k)$ is irreducible for $\mathcal{O}_k \in \mathbb{L}$, $k \geqslant 1$.*

Theorem 6. *If $\mathcal{P} \in \mathbb{L}^{\mathrm{a}}$, then $L(\mathcal{P})$ is irreducible.*

Proof. Suppose that for some property $\mathcal{P} \in \mathbb{L}_k^{\mathrm{a}}$, $L(\mathcal{P})$ is reducible, i.e., $L(\mathcal{P}) = \mathcal{Q}_1 \circ \mathcal{Q}_2$. Then by Theorem 3 $c(\mathcal{Q}_1 \circ \mathcal{Q}_2) = k+1$, $k \geqslant 1$ and $L(\mathcal{O}_k) \subset \mathcal{Q}_1 \circ \mathcal{Q}_2$. By Lemma 3, $L(\mathcal{O}_k) \subset \mathcal{O} \circ L(\mathcal{O}_k) \subset \mathcal{Q}_1 \circ \mathcal{Q}_2$. But $c(\mathcal{O} \circ L(\mathcal{O}_k)) = c(\mathcal{O}) + c(L(\mathcal{O}_k)) + 1 = k+2 \leqslant c(\mathcal{Q}_1 \circ \mathcal{Q}_2) = k+1$, a contradiction. □

4. Concluding remarks

Let $L^r(\mathcal{P}) = L(L^{r-1}(\mathcal{P}))$ and $L^r(\mathbb{L}_k) = L(L^{r-1}(\mathbb{L}_k))$ for $r \geqslant 2$.

By induction on r we can prove the following statements.

Theorem 7. *Let $\mathcal{P} \in \mathbb{M}$ and $r \geqslant 1$. Then,*

(1) $\mathcal{C}(L^r(\mathcal{P})) = \{K_r + H \colon H \in \mathcal{C}(\mathcal{P})\}$.
(2) $c(L^r(\mathcal{P})) = c(\mathcal{P}) + r$.
(3) $L^r(\mathcal{P}) \subseteq L^{r+1}(\mathcal{P})$.
(4) $L^r(\mathcal{P}) \in \mathbb{M}^{\mathrm{a}}$.
(5) $L^r(\mathbb{L}_k) \subseteq \mathbb{L}_{k+r}$.
(6) *$L^r(\mathbb{L}_k)$ is isomorphic to $\mathbb{L}_k$.*
(7) *If $r \geqslant 2$, then $L^r(\mathcal{P})$ is irreducible for any $\mathcal{P} \in \mathbb{L}$.*

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