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A posteriori error estimations for mixed finite-element approximations to the Navier–Stokes equations

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ABSTRACT

A posteriori estimates for mixed finite element discretizations of the Navier–Stokes equations are derived. We show that the task of estimating the error in the evolutionary Navier–Stokes equations can be reduced to the estimation of the error in a steady Stokes problem. As a consequence, any available procedure to estimate the error in a Stokes problem can be used to estimate the error in the nonlinear evolutionary problem. A practical procedure to estimate the error based on the so-called postprocessed approximation is also considered. Both the semidiscrete (in space) and the fully discrete cases are analyzed. Some numerical experiments are provided.

1. Introduction

We consider the incompressible Navier–Stokes equations

$$\begin{aligned} &u_t - \Delta u + (u\cdot\nabla)u + \nabla p = f, \\ &\mathrm{div}(u) = 0, \end{aligned} \tag{1}$$

in a bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) with a smooth boundary, subject to homogeneous Dirichlet boundary conditions $u = 0$ on $\partial\Omega$. In (1), u is the velocity field, p the pressure, and f a given force field. For simplicity in the exposition we assume, as in [1–5], that the fluid density and viscosity have been normalized by an adequate change of scale in space and time.

Let u_h and p_h be the semidiscrete (in space) mixed finite element (MFE) approximations to the velocity u and pressure p, respectively, solution of (1) corresponding to a given initial condition

$$u(\cdot, 0) = u_0. \tag{2}$$

We study the a posteriori error estimation of these approximations in the L^2 and H^1 norm for the velocity and in the $L^2/\mathbb{R}$ norm for the pressure. To do this, for a given time $t^* > 0$, we consider the solution $(\tilde u, \tilde p)$ of the Stokes problem

$$\left.\begin{aligned} &-\Delta\tilde u + \nabla\tilde p = f - \frac{\mathrm{d}}{\mathrm{d}t}u_h(t^*) - (u_h(t^*)\cdot\nabla)u_h(t^*) \\ &\mathrm{div}(\tilde u) = 0 \end{aligned}\right\} \quad \text{in } \Omega, \qquad \tilde u = 0, \quad \text{on } \partial\Omega. \tag{3}$$

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In this paper we prove that $\tilde{u}$ and $\tilde{p}$ are approximations to u and p whose errors decay by a factor of $h|\log(h)|$ faster than those of u_h and p_h (h being the mesh size). As a consequence, the quantities $\tilde{u} - u_h$ and $\tilde{p} - p_h$, are asymptotically exact indicators of the errors $u - u_h$ and $p - p_h$ in the Navier–Stokes problem (1)–(2).

Furthermore, the key observation in the present paper is that (u_h, p_h) is also the MFE approximation to the solution $(\tilde{u}, \tilde{p})$ of the Stokes problem (3). Consequently, any available procedure to a posteriori estimate the errors in a Stokes problem can be used to estimate the errors $\tilde{u} - u_h$ and $\tilde{p} - p_h$ which, as mentioned above, coincide asymptotically with the errors $u - u_h$ and $p - p_h$ in the evolutionary NS equations. Many references address the question of estimating the error in a Stokes problem, see for example [6–12] and the references therein. In this paper we prove that any efficient or asymptotically exact estimator of the error in the MFE approximation (u_h, p_h) to the solution of the *steady* Stokes problem (3) is also an efficient or asymptotically exact estimator, respectively, of the error in the MFE approximation (u_h, p_h) to the solution of the *evolutionary* Navier–Stokes equations (1)–(2).

The analysis of the errors $u-\tilde{u}$ and $p-\tilde{p}$ is new and appears in this paper for the first time, although it follows closely [13], where MFE approximations to the Stokes problem (3) (the so-called postprocessed approximations) are considered with the aim of getting improved approximations to the solution of (1)–(2) at any fixed time $t^* > 0$. In [13], most of the results concern only quadratic and cubic elements. For this reason, in the present paper, some new results concerning first order finite elements that had not appeared before have also been included.

In this paper we will refer to $(\tilde{u}, \tilde{p})$ as infinite-dimensional postprocessed approximations (ID-postprocessed approximations). Of course, they are not computable in practice and they are only considered for the analysis of a posteriori error estimators. We remark that the Stokes reconstruction of [5] is exactly the ID-postprocessing approximation $(\tilde{u}, \tilde{p})$ in the particular case of a linear model. We prefer the term ID-postprocessed approximation for historical reasons and consistency with our previous published papers. In [5], the Stokes reconstruction is used to a posteriori estimate the errors of spatially semidiscrete approximations to a linear time-dependent Stokes problem.

The postprocessed approximations to the Navier–Stokes equations were first developed for spectral methods in [14–17], and also developed for MFE methods for the Navier–Stokes equations in [18,19,13].

For the sake of completeness, in the present paper we also analyze the use of the (computable) postprocessed approximations of [13] for a posteriori error estimation. The use of this kind of postprocessing technique to get a posteriori error estimations has been previously studied in [20–23] for nonlinear parabolic equations excluding the Navier–Stokes equations. For the analysis in the present paper we do not assume that the solution u of (1)–(2) possesses more than second-order spatial derivatives bounded in $L^2(\Omega)^d$ up to initial time $t = 0$, since demanding further regularity requires the data to satisfy nonlocal compatibility conditions unlikely to be fulfilled in practical situations [2,3].

In the second part of the paper we consider a posteriori error estimations for the fully discrete MFE approximations $U_h^n \approx u_h(t_n)$ and $P_h^n \approx p_h(t_n)$, ($t_n = t_{n-1} + \Delta t_{n-1}$ for $n = 1, 2, \ldots, N$) obtained by integrating in time with either the backward Euler method or the two-step backward differentiation formula (BDF). For this purpose, we define a Stokes problem similar to (3) but with the right-hand-side depending now on the fully discrete MFE approximation U_h^n (problem (70)–(71) in Section 4 below). We will call infinite-dimensional time-discrete postprocessed approximation (IDTD-postprocessed approximation) to the solution $(\widetilde{U}^n, \widetilde{P}^n)$ of this new Stokes problem. As before, $(\widetilde{U}^n, \widetilde{P}^n)$ is not computable in practice and it is only considered for the analysis of a posteriori error estimation. Again, the analysis of the errors $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ is new and appears for the first time in this paper, although following closely the analysis of [24].

Observe that in the fully discrete case (which is the case in actual computations) the task of estimating the error $u(t_n)-U_h^n$ of the MFE approximation becomes more difficult due to the presence of time discretization errors $e_h^n = u_h(t_n) - U_h^n$, which are added to the spatial discretization errors $u(t_n) - u_h(t_n)$. However we show in Section 4 that if temporal and spatial errors are not very different in size, the quantity $\widetilde{U}^n - U_h^n$ correctly estimates the spatial error because the leading terms of the temporal errors in $\widetilde{U}^n$ and U_h^n cancel out when subtracting $\widetilde{U}^n - U_h^n$, leaving only the spatial component of the error. This is a very convenient property that allows to use independent procedures for the tasks of estimating the errors of the spatial and temporal discretizations. More precisely, we mean that we can choose the tolerance for the temporal error and the tolerance for the spatial error approximately of the same size, in order to control both temporal and spatial errors in an adaptive way. We remark that the temporal error can be routinely controlled by resorting to well-known ordinary differential equations techniques. We refer the reader to [22], where analogous results were obtained for fully discrete finite element approximations to evolutionary convection–reaction–diffusion equations using the backward Euler method, and where an adaptive algorithm is proposed. The performance of an adaptive algorithm in time and space for the Navier–Stokes equations will be the subject of future research.

As in the semidiscrete case, a key point in our results is again the fact that the fully discrete MFE approximation (U_h^n, P_h^n) to the Navier–Stokes problem (1)–(2) is also the MFE approximation to the solution $(\widetilde{U}^n, \widetilde{P}^n)$ of the Stokes problem (70)–(71). As a consequence, we can use again any available error estimator for the Stokes problem to estimate the spatial error of the fully discrete MFE approximations (U_h^n, P_h^n) to the Navier–Stokes problem (1)–(2).

Computable mixed finite element approximations to $(\widetilde{U}^n, \widetilde{P}^n)$, the so-called fully discrete postprocessed approximations, were studied and analyzed in [24] where we proved that the fully discrete postprocessed approximations maintain the increased spatial accuracy of the semidiscrete approximations. The analysis in the second part of the present paper borrows in part from [24]. We also include error bounds for the L^2 norm of the difference between the temporal errors of the Galerkin and postprocessed approximations to the pressure, that had not been proved before. Finally, we propose a computable error estimator based on the fully discrete postprocessed approximation of [24] and show that it also has the excellent property of separating spatial and temporal errors, both for the velocity and the pressure.

The rest of the paper is as follows. In Section 2 we introduce some preliminaries and notation. In Section 3 we study the a posteriori error estimation of semidiscrete in space MFE approximations. In Section 4 we study a posteriori error estimates for fully discrete approximations. Finally, some numerical experiments are shown in Section 5.

2. Preliminaries and notations

We will assume that Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, of class $\mathcal{C}^m$, for $m \geq 2$. When dealing with linear elements ($r = 2$ below) Ω may also be a convex polygonal or polyhedral domain. We consider the Hilbert spaces

$$H = \{u \in L^2(\Omega)^d \mid \operatorname{div}(u) = 0,\ u \cdot n_{|\partial\Omega} = 0\},$$
$$V = \{u \in H_0^1(\Omega)^d \mid \operatorname{div}(u) = 0\},$$

endowed with the inner product of $L^2(\Omega)^d$ and $H_0^1(\Omega)^d$, respectively. For $l \geq 0$ integer and $1 \leq q \leq \infty$, we consider the standard spaces, $W^{l,q}(\Omega)^d$, of functions with derivatives up to order l in $L^q(\Omega)$, and $H^l(\Omega)^d = W^{l,2}(\Omega)^d$. We will denote by $\|\cdot\|_l$ the norm in $H^l(\Omega)^d$, and $\|\cdot\|_{-l}$ will represent the norm of its dual space. We consider also the quotient spaces $H^l(\Omega)/\mathbb{R}$ with norm $\|p\|_{H^l/\mathbb{R}} = \inf\{\|p + c\|_l \mid c \in \mathbb{R}\}$.

We recall the following Sobolev's imbeddings [25]: For $q \in [1, \infty)$, there exists a constant $C = C(\Omega, q)$ such that

$$\|v\|_{L^{q'}} \leq C\|v\|_{W^{s,q}}, \qquad \frac{1}{q'} \geq \frac{1}{q} - \frac{s}{d} > 0, \quad q < \infty,\ v \in W^{s,q}(\Omega)^d. \tag{4}$$

For $q' = \infty$, (4) holds with $\frac{1}{q} < \frac{s}{d}$.

The following inf-sup condition is satisfied (see [26]): there exists a constant $\beta > 0$ such that

$$\inf_{q\in L^2(\Omega)/\mathbb{R}} \sup_{v\in H_0^1(\Omega)^d} \frac{(q, \nabla\cdot v)}{\|v\|_1 \|q\|_{L^2/\mathbb{R}}} \geq \beta, \tag{5}$$

where, here and in the sequel, $(\cdot, \cdot)$ denotes the standard inner product in $L^2(\Omega)$ or in $L^2(\Omega)^d$.

Let $\Pi : L^2(\Omega)^d \longrightarrow H$ be the $L^2(\Omega)^d$ projector onto H. We denote by A the Stokes operator on Ω:

$$A : \mathcal{D}(A) \subset H \longrightarrow H, \quad A = -\Pi\Delta,\ \mathcal{D}(A) = H^2(\Omega)^d \cap V.$$

Applying Leray's projector Π to (1), the equations can be written in the form

$$u_t + Au + B(u, u) = \Pi f \quad \text{in } \Omega,$$

where $B(u, v) = \Pi(u\cdot\nabla)v$ for u, v in $H_0^1(\Omega)^d$.

We shall use the trilinear form $b(\cdot,\cdot,\cdot)$ defined by

$$b(u, v, w) = (F(u, v), w) \quad \forall u, v, w \in H_0^1(\Omega)^d,$$

where

$$F(u, v) = (u\cdot\nabla)v + \frac{1}{2}(\nabla\cdot u)v \quad \forall u, v \in H_0^1(\Omega)^d.$$

It is straightforward to verify that b enjoys skew-symmetry:

$$b(u, v, w) = -b(u, w, v) \quad \forall u, v, w \in H_0^1(\Omega)^d. \tag{6}$$

Let us observe that $B(u, v) = \Pi F(u, v)$ for $u \in V$, $v \in H_0^1(\Omega)^d$.

For $\alpha \in \mathbb{R}$ and $t > 0$, let us consider the operators A^α and e^{-tA}, which are defined by means of the spectral properties of A (see, e.g., [27, p. 33], [28]). Notice that A is a positive self-adjoint operator with compact resolvent in H. An easy calculation shows that

$$\|A^\alpha e^{-tA}\|_0 \leq (\alpha e^{-1})^\alpha t^{-\alpha}, \quad \alpha \geq 0,\ t > 0, \tag{7}$$

where, here and in what follows, $\|\cdot\|_0$ when applied to an operator denotes the operator norm associated with $\|\cdot\|_0$.

We shall assume, as in [2], that $u_0 \in V \cap H^2(\Omega)^d$, that there exists a constant $\tilde{M}_1$ such that $\|f\|_0 + \|f_t\|_0 \leq \tilde{M}_1$, for $t \in [0, T]$, and that the solution u of (1)–(2) exists on an interval $[0, T]$ and satisfies

$$\|u(t)\|_1 \leq M_1, \quad 0 \leq t \leq T, \tag{8}$$

for some constant M_1. Then, following Theorem 2.3 in [2] we get

$$\|u(t)\|_2 + \|u_t(t)\|_0 + \|p(t)\|_{H^1/\mathbb{R}} \leq M_2, \quad 0 \leq t \leq T.$$

Moreover, assuming that there exists a constant $\tilde{M}_2$ such that

$$\|f\|_1 + \|f_t\|_1 + \|f_{tt}\|_1 \le \tilde{M}_2, \quad 0 \le t \le T, \tag{9}$$

and that for some $k \ge 2$

$$\sup_{0\le t\le T} \|\partial_t^{\lfloor k/2\rfloor} f\|_{k-1-2\lfloor k/2\rfloor} + \sum_{j=0}^{\lfloor (k-2)/2\rfloor} \sup_{0\le t\le T} \|\partial_t^j f\|_{k-2j-2} < +\infty,$$

according to Theorems 2.4 and 2.5 in [2], there exist positive constants M_k and K_k such that the following bounds hold:

$$\|u(t)\|_k + \|u_t(t)\|_{k-2} + \|p(t)\|_{H^{k-1}/\mathbb{R}} \le M_k \tau(t)^{1-k/2}, \tag{10}$$

$$\int_0^t \sigma_{k-3}(s)(\|u(s)\|_k^2 + \|u_s(s)\|_{k-2}^2 + \|p(s)\|_{H^{k-1}/\mathbb{R}}^2 + \|p_s(s)\|_{H^{k-3}/\mathbb{R}}^2)\,\mathrm{d}s \le K_k^2, \tag{11}$$

where $\tau(t) = \min(t, 1)$ and $\sigma_n = \mathrm{e}^{-\alpha(t-s)}\tau^n(s)$ for some $\alpha > 0$. Observe that for $t \le T < \infty$, we can take $\tau(t) = t$ and $\sigma_n(s) = s^n$. For simplicity, we will take these values of τ and σ_n.

Let $\mathcal{T}_h = (\tau_i^h, \phi_i^h)_{i\in I_h}$, $h > 0$, be a family of partitions of suitable domains Ω_h, where h is the maximum diameter of the elements $\tau_i^h \in \mathcal{T}_h$, and ϕ_i^h are the mappings of the reference simplex τ_0 onto τ_i^h.

For $r \ge 2$, we consider the finite-element spaces

$$S_{h,r} = \{\chi_h \in \mathcal{C}(\overline{\Omega}_h) \mid \chi_h|_{\tau_i^h} \circ \phi_i^h \in P^{r-1}(\tau_0)\} \subset H^1(\Omega_h),$$
$$S_{h,r}^0 = S_{h,r} \cap H_0^1(\Omega_h),$$

where $P^{r-1}(\tau_0)$ denotes the space of polynomials of degree at most $r-1$ on τ_0. As it is customary in the analysis of finite-element methods for the Navier–Stokes equations (see e.g., [1–4,29]) we restrict ourselves to quasiuniform and regular meshes $\mathcal{T}_h$, so that, as a consequence of [30, Theorem 3.2.6], the following inverse inequality holds for each $v_h \in (S_{h,r}^0)^d$:

$$\|v_h\|_{W^{m,q}(\Omega_h)^d} \le Ch^{l-m-d(\frac{1}{q'}-\frac{1}{q})}\|v_h\|_{W^{l,q'}(\Omega_h)^d}, \tag{12}$$

where $0 \le l \le m \le 1$, and $1 \le q' \le q \le \infty$.

We shall denote by $(X_{h,r}, Q_{h,r-1})$ the mixed finite-elements spaces that we consider, which are, when $r \ge 3$, the so-called Hood–Taylor element [31,32], given by

$$X_{h,r} = (S_{h,r}^0)^d, \qquad Q_{h,r-1} = S_{h,r-1} \cap L^2(\Omega_h)/\mathbb{R}, \quad r \ge 3,$$

and, when $r = 2$, the so-called mini-element [33], for which $Q_{h,1} = S_{h,2} \cap L^2(\Omega_h)/\mathbb{R}$, and $X_{h,2} = (S_{h,2}^0)^d \oplus \mathbb{B}_h$. Here, $\mathbb{B}_h$ is spanned by the bubble functions b_τ, $\tau \in \mathcal{T}_h$, defined by $b_\tau(x) = (d+1)^{d+1}\lambda_1(x)\cdots\lambda_{d+1}(x)$, if $x \in \tau$ and 0 elsewhere, where $\lambda_1(x), \ldots, \lambda_{d+1}(x)$ denote the barycentric coordinates of x. For these elements a uniform inf-sup condition is satisfied, that is, there exists a constant $\beta > 0$ independent of the mesh grid size h such that

$$\inf_{q_h\in Q_{h,r-1}} \sup_{v_h\in X_{h,r}} \frac{(q_h, \nabla\cdot v_h)}{\|v_h\|_1 \|q_h\|_{L^2/\mathbb{R}}} \ge \beta, \tag{13}$$

see [31,33]. We remark that our analysis can also be applied to other pairs of LBB-stable mixed finite elements (see [13, Remark 2.1]).

The approximate velocity belongs to the discrete divergence-free space

$$V_{h,r} = X_{h,r} \cap \{\chi_h \in H_0^1(\Omega_h)^d \mid (q_h, \nabla\cdot\chi_h) = 0\ \forall q_h \in Q_{h,r-1}\},$$

which is not a subspace of V. We shall frequently write V_h instead of $V_{h,r}$ whenever the value of r plays no particular role.

Let $\Pi_h : L^2(\Omega)^d \longrightarrow V_{h,r}$ be the discrete Leray's projection defined by

$$(\Pi_h u, \chi_h) = (u, \chi_h) \quad \forall \chi_h \in V_{h,r}.$$

We will use the following well-known bounds for $u \in V \cap H^l(\Omega)^d$.

$$\|(I - \Pi_h)u\|_j \le Ch^{l-j}\|u\|_l, \quad 1 \le l \le 2,\ j = 0, 1. \tag{14}$$

These are a consequence of similar bounds for the Stokes projection [3], (12) and the fact that u is divergence-free. We will denote by $A_h : V_h \to V_h$ the discrete Stokes operator defined by

$$(\nabla v_h, \nabla\phi_h) = (A_h v_h, \phi_h) = (A_h^{1/2} v_h, A_h^{1/2}\phi_h) \quad \forall v_h, \phi_h \in V_h.$$

Let $(u, p) \in (H^2(\Omega)^d \cap V) \times (H^1(\Omega)/\mathbb{R})$ be the solution of a Stokes problem with right-hand side g, we will denote by $s_h = S_h(u) \in V_h$ the so-called Stokes projection (see [3]) defined as the velocity component of solution of the following Stokes problem: find $(s_h, q_h) \in (X_{h,r}, Q_{h,r-1})$ such that

$$(\nabla s_h, \nabla \phi_h) + (\nabla q_h, \phi_h) = (g, \phi_h) \quad \forall \phi_h \in X_{h,r}, \tag{15}$$

$$(\nabla \cdot s_h, \psi_h) = 0 \quad \forall \psi_h \in Q_{h,r-1}. \tag{16}$$

The following bound holds for $2 \le l \le r$:

$$\|u - s_h\|_0 + h\|u - s_h\|_1 \le Ch^l(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}). \tag{17}$$

The proof of (17) for $\Omega = \Omega_h$ can be found in [3]. For the general case, Ω_h must be such that the value of $\delta(h) = \max_{x\in\partial\Omega_h} \mathrm{dist}(x, \partial\Omega)$ satisfies $\delta(h) = O(h^{2(r-1)})$. This can be achieved if, for example, $\partial\Omega$ is piecewise of class $\mathcal{C}^{2(r-1)}$, and superparametric approximation at the boundary is used [34]. Under the same conditions, the bound for the pressure is [26]

$$\|p - q_h\|_{L^2/\mathbb{R}} \le C_\beta h^{l-1}(\|u\|_l + \|p\|_{H^{l-1}/\mathbb{R}}), \tag{18}$$

where the constant C_β depends on the constant β in the inf-sup condition (13). We will assume that the domain Ω is of class $\mathcal{C}^m$, with $m \ge r$ so that standard bounds for the Stokes problem [34,35] imply that

$$\|A^{-1}\Pi g\|_{2+j} \le \|g\|_j, \quad -1 \le j \le m-2. \tag{19}$$

For a domain Ω of class $\mathcal{C}^2$ we also have the bound (see [36])

$$\|p\|_{H^1/\mathbb{R}} \le c\|g\|_0. \tag{20}$$

In what follows we will apply the above estimates to the particular case in which (u, p) is the solution of the Navier–Stokes problem (1)–(2). In that case $s_h = S_h(u)$ is the discrete velocity in problem (15)–(16) with $g = f - u_t - (u \cdot \nabla u)$. Note that the temporal variable t appears here merely as a parameter, and then, taking the time derivative, the error bound (17) can also be applied to the time derivative of s_h changing u, p by u_t, p_t.

Since we are assuming that Ω is of class $\mathcal{C}^m$ and $m \ge 2$, from (17) and standard bounds for the Stokes problem [34,35], we deduce that

$$\|(A^{-1}\Pi - A_h^{-1}\Pi_h)f\|_j \le Ch^{2-j}\|f\|_0 \quad \forall f \in L^2(\Omega)^d,\ j = 0, 1. \tag{21}$$

We consider the semidiscrete finite-element approximation (u_h, p_h) to (u, p), solution of (1)–(2). That is, given $u_h(0) = \Pi_h u_0$, we compute $u_h(t) \in X_{h,r}$ and $p_h(t) \in Q_{h,r-1}$, $t \in (0, T]$, satisfying

$$(\dot{u}_h, \phi_h) + (\nabla u_h, \nabla \phi_h) + b(u_h, u_h, \phi_h) + (\nabla p_h, \phi_h) = (f, \phi_h) \quad \forall \phi_h \in X_{h,r}, \tag{22}$$

$$(\nabla \cdot u_h, \psi_h) = 0 \quad \forall \psi_h \in Q_{h,r-1}. \tag{23}$$

For $2 \le r \le 5$, provided that (17)–(18) hold for $l \le r$, and (10)–(11) hold for $k = r$, we have

$$\|u(t) - u_h(t)\|_0 + h\|u(t) - u_h(t)\|_1 \le C\frac{h^r}{t^{(r-2)/2}}, \quad 0 \le t \le T, \tag{24}$$

(see, e.g., [13,2,3]), and also,

$$\|p(t) - p_h(t)\|_{L^2/\mathbb{R}} \le C\frac{h^{r-1}}{t^{(r'-2)/2}}, \quad 0 \le t \le T, \tag{25}$$

where $r' = r$ if $r \le 4$ and $r' = r + 1$ if $r = 5$.

3. A posteriori error estimations. Semidiscrete case

Let us consider the MFE approximation (u_h, p_h) to $(u(t^*), p(t^*))$ at any time $t^* \in (0, T]$, obtained by solving (22)–(23). We consider the ID-postprocessed approximation $(\tilde{u}(t^*), \tilde{p}(t^*))$ in $(V, L^2(\Omega)/\mathbb{R})$ which is the solution of the following Stokes problem written in weak form

$$(\nabla \tilde{u}(t^*), \nabla \phi) + (\nabla \tilde{p}(t^*), \phi) = (f, \phi) - b(u_h(t^*), u_h(t^*), \phi) - (\dot{u}_h(t^*), \phi), \tag{26}$$

$$(\nabla \cdot \tilde{u}(t^*), \psi) = 0, \tag{27}$$

for all $\phi \in H_0^1(\Omega)^d$ and $\psi \in L^2(\Omega)/\mathbb{R}$. We remark that the MFE approximation $(u_h(t^*), p_h(t^*))$ to $(u(t^*), p(t^*))$ is also the MFE approximation to the solution $(\tilde{u}(t^*), \tilde{p}(t^*))$ of the Stokes problem (26)–(27). In Theorems 1 and 2 below we prove that the ID-postprocessed approximation $(\tilde{u}(t^*), \tilde{p}(t^*))$ is an improved approximation to the solution (u, p) of the evolutionary Navier–Stokes equations (1)–(2) at time t^*. Although, as it is obvious, $(\tilde{u}(t^*), \tilde{p}(t^*))$ is not computable in practice, it is however a useful tool to provide a posteriori error estimates for the MFE approximation (u_h, p_h) at any desired time $t^* > 0$.

In Theorem 1 we obtain the error bounds for the velocity and, in Theorem 2, the bounds for the pressure. The improvement is achieved in both the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms when $r = 3, 4$, and only in the $H^1(\Omega)^d$ norm when using the mini-element ($r = 2$).

In the sequel we will use that for a forcing term satisfying (9) there exists a constant $\tilde{M}_3 > 0$, depending only on $\tilde{M}_2$, $\|A_h u_h(0)\|_0$ and $\sup_{0\le t\le T} \|u_h(t)\|_1$, such that the following bound holds for $0 \le t \le T$, see [4, Proposition 3.2],

$$\|A_h u_h(t)\|_0^2 \le \tilde{M}_3^2. \tag{28}$$

The following inequalities hold for all $v_h, w_h \in V_h$ and $\phi \in H_0^1(\Omega)^d$, see [4, (3.7)],

$$|b(v_h, v_h, \phi)| \le c\|v_h\|_1^{3/2}\|A_h v_h\|_0^{1/2}\|\phi\|_0, \tag{29}$$

$$|b(v_h, w_h, \phi)| + |b(w_h, v_h, \phi)| \le c\|v_h\|_1\|A_h w_h\|_0\|\phi\|_0. \tag{30}$$

The proof of Theorem 1 requires some previous results which we now state and prove.

We will use the fact that $\|A_h^{1/2} w_h\|_0 = \|\nabla w_h\|_0$ for $w_h \in V_h$. Then, since, for $v_h \in V_h$, we have $(A_h^{-1/2} w_h, v_h) = (w_h, A_h^{-1/2} v_h)$, it follows that

$$C^{-1}\|A_h^{-1/2} w_h\|_0 \le \|w_h\|_{-1} \le C\|A_h^{-1/2} w_h\|_0 \quad \forall w_h \in V_h, \tag{31}$$

where the constant C is independent of h.

Lemma 1. *Let (u, p) be the solution of (1)–(2) and fix $\alpha > 0$. Then there exists a positive constant $C = C(M_2, \alpha)$ such that for $w_h^1, w_h^2 \in V_h$ satisfying the threshold condition*

$$\|w_h^l - u\|_j \le \alpha h^{3/2-j}, \quad j = 0, 1, \; l = 1, 2, \tag{32}$$

the following inequalities hold for $j = 0, 1$:

$$\|A_h^{-j/2}\Pi_h(F(w_h^1, w_h^1) - F(w_h^2, w_h^2))\|_0 \le C\|A_h^{(1-j)/2}(w_h^1 - w_h^2)\|_0, \tag{33}$$

$$\|A_h^{-j/2}\Pi_h(F(w_h^1, w_h^1) - F(u, u))\|_0 \le C\|w_h^1 - u\|_{1-j}. \tag{34}$$

Proof. Due to the equivalence (31), and since $\|\Pi_h f\|_0 \le \|f\|_0$ for $f \in L^2(\Omega)^d$, it is sufficient to prove

$$\|F(w_h^1, w_h^1) - F(w, w)\|_{-j} \le C\|w_h^1 - w\|_{1-j}, \quad j = 0, 1, \tag{35}$$

for $w = w_h^2$ or $w = u$. We follow the proof of [19, Lemma 3.1] where a different threshold assumption is assumed. We do this for $w = u$, since the case $w = w_h^2$ can be proved with arguments similar to those in the proof of [19, Lemma 3.1]. We write

$$F(w_h^1, w_h^1) - F(u, u) = F(w_h^1, e_h) + F(e_h, u), \tag{36}$$

where $e_h = w_h^1 - u$. We first observe that

$$\begin{aligned}\|F(e_h, u)\|_0 &= \sup_{\|\phi\|_0=1}\left|(e_h\cdot\nabla u, \phi) + \frac{1}{2}((\nabla\cdot e_h)u, \phi)\right| \\ &\le C\|e_h\|_{L^{2d}}\|\nabla u\|_{L^{2d/(d-1)}} + C\|e_h\|_1\|u\|_{L^\infty} \\ &\le C(\|\nabla u\|_{L^{2d/(d-1)}} + \|u\|_{L^\infty})\|e_h\|_1,\end{aligned}$$

where, in the last inequality, we have used that thanks to Sobolev's inequality (4) we have $\|e_h\|_{L^{2d}} \le C\|e_h\|_1$. Similarly,

$$\begin{aligned}\|F(w_h^1, e_h)\|_0 &\le C\|w_h^1\|_{L^\infty}\|e_h\|_1 + C\|\nabla w_h^1\|_{L^{2d/(d-1)}}\|e_h\|_{L^{2d}} \\ &\le C(\|w_h^1\|_{L^\infty} + \|\nabla w_h^1\|_{L^{2d/(d-1)}})\|e_h\|_1.\end{aligned}$$

The proof of the case $j = 0$ in (35) is finished if we show that for $v = w_h^1$ and $v = u$, both $\|v\|_{L^\infty}$ and $\|\nabla v\|_{L^{2d/(d-1)}}$ are bounded in terms of M_2 and the value α in the threshold assumption (32). This is a consequence of Sobolev's inequality (4) when $v = u$, and, as mentioned above, the case $v = w_h^1$ can be dealt with as in the proof of [19, Lemma 3.1]. Finally, the proof of the case $j = 1$ in (35) is, with obvious changes, that of the equivalent result in [19, Lemma 3.1]. □

In the sequel we consider the auxiliary function $v_h : [0, T] \to V_h$ solution of

$$\dot{v}_h + A_h v_h + \Pi_h F(u, u) = \Pi_h f, \qquad v_h(0) = \Pi_h u_0. \tag{37}$$

According to [13, Remark 4.2] we have

$$\max_{0\le t\le T}\|v_h(t) - \Pi_h u(t)\|_0 \le C|\log(h)|h^2, \tag{38}$$

for some constant $C = C(M_2)$. The following lemma provides a superconvergence result.

Lemma 2. *Let (u, p) be the solution of (1)–(2). Then, there exists a positive constant C such that the solution v_h of (37) and the Galerkin approximation u_h satisfy the following bound,*

$$\|v_h(t) - u_h(t)\|_1 \le C|\log(h)|^2 h^2, \quad t \in (0, T]. \tag{39}$$

Proof. Since for $y_h = A_h^{1/2}(v_h - u_h)$ we have

$$\dot{y}_h + A_h y_h + A_h^{1/2}\Pi_h(F(v_h, v_h) - F(u_h, u_h)) = A_h^{1/2}\rho_h,$$

where $\rho_h = \Pi_h(F(v_h, v_h) - F(u, u))$, it follows that

$$\|y_h(t)\|_0 \le \int_0^t \|A_h^{1/2}\mathrm{e}^{-(t-s)A_h}\|_0 \|\Pi_h(F(v_h, v_h) - F(u_h, u_h))\|_0 + \int_0^t \|A_h \mathrm{e}^{-(t-s)A_h}(A_h^{-1/2}\rho_h(s))\|_0 \,\mathrm{d}s.$$

Applying (33) we have $\|\Pi_h(F(v_h, v_h) - F(u_h, u_h))\|_0 \le C\|y_h\|_0$, so taking into account that

$$\|A_h^{1/2}\mathrm{e}^{-(t-s)A_h}\|_0 \le (2e(t-s))^{-1/2}, \tag{40}$$

it follows that

$$\|y_h(t)\|_0 \le \frac{1}{\sqrt{2e}}\int_0^t \frac{\|y_h(s)\|_0}{\sqrt{t-s}} + \int_0^t \|A_h \mathrm{e}^{-(t-s)A_h}(A_h^{-1/2}\rho_h(s))\|_0 \,\mathrm{d}s.$$

Since applying [13, Lemma 4.2] we obtain

$$\int_0^t \|A_h \mathrm{e}^{-(t-s)A_h}(A_h^{-1/2}\rho_h(s))\|_0 \,\mathrm{d}s \le C|\log(h)| \max_{0\le s\le t} \|\rho_h(s)\|_0,$$

a generalized Gronwall lemma [37, pp. 188–189], together with (33) allow us to conclude

$$\|v_h - u_h\|_1 \le C|\log(h)|\|v_h - u\|_0.$$

Then by writing $\|v_h - u\|_0 \le \|v_h - \Pi_h u\|_0 + \|\Pi_h u - u\|_0$ and applying (14) and (38), the proof is finished if we check that the threshold condition (32) holds for $w_h^1 = u_h$ and $w_h^2 = v_h$. In view of (38), (14) and the inverse inequality (12) we have indeed that $\|v_h - u\|_j = o(h^{3/2-j})$, for $j = 0, 1$. In the case of u_h the threshold condition holds due to (24). □

Lemma 3. *Let (u, p) be the solution of (1)–(2). Then, there exists a positive constant C such that the solution v_h of (37) and the Galerkin approximation u_h satisfy the following bound*

$$\|\dot{v}_h(t) - \dot{u}_h(t)\|_{-1} \le C|\log(h)|^2 h^2, \quad t \in (0, T], \tag{41}$$

where v_h and u_h are defined by (37) and (22)–(23) respectively.

Proof. The difference $v_h - u_h$ satisfies that $\dot{v}_h - \dot{u}_h = A_h(v_h - u_h) + \Pi_h(F(u, u) - F(u_h, u_h))$, so that multiplying by $A_h^{-1/2}$ and taking norms, thanks to (34), we have

$$\|A_h^{-1/2}(\dot{v}_h - \dot{u}_h)\|_0 \le \|A_h^{1/2}(v_h - u_h)\|_0 + C\|u - u_h\|_0.$$

Now we write

$$\|u - u_h\|_0 \le \|u - \Pi_h u\|_0 + \|\Pi_h u - v_h\|_0 + \|v_h - u_h\|_0,$$

so that (14), (38) and (39) allow us to write,

$$\|A_h^{-1/2}(\dot{v}_h - \dot{u}_h)\|_0 \le C|\log(h)|^2 h^2.$$

Then, applying (31) the proof is finished. □

Lemma 4. *Let (u, p) be the solution of (1)–(2) and let u_h be the Galerkin approximation. Then, there exists a positive constant C such that*

$$\|u_t - \dot{u}_h(t)\|_{-1} \le \frac{C}{t^{(r-1)/2}} h^r |\log(h)|^{r'}, \quad t \in (0, T],\ r = 2, 3, 4, \tag{42}$$

where $r' = 2$ when $r = 2$ and $r' = 1$ otherwise.

Proof. The case $r = 3, 4$ is proved in [13, Lemma 5.1]. For the case $r = 2$ we write

$$u_t - \dot{u}_h = (u_t - \Pi_h u_t) + (\Pi_h u_t - \dot{v}_h) + (\dot{v}_h - \dot{u}_h). \tag{43}$$

A simple duality argument and the fact that $\|u_t - \Pi_h u_t\|_0 \le Ch\|u_t\|_1$, easily show that

$$\|(I - \Pi_h)u_t\|_{-1} \le Ch^2\|u_t\|_1 \le C\frac{M_3}{t^{1/2}}h^2.$$

The bound of the third term on the right-hand side of (43) is given in Lemma 3, so that, thanks to the equivalence (31) we are left with estimating

$$y_h = t^{1/2}A_h^{-1/2}(\Pi_h u_t - \dot{v}_h).$$

We notice that

$$\dot{y}_h + A_h y_h = t^{1/2}A_h^{1/2}\dot{\theta}_h + \frac{1}{2}t^{-1/2}A_h^{-1/2}(\Pi_h u_t - \dot{v}_h),$$

where $\theta_h = (\Pi_h - S_h)u$. Thus,

$$y_h(t) = \int_0^t s^{-1/2}A_h^{1/2}\mathrm{e}^{-(t-s)A_h}(s\dot{\theta}_h)\,\mathrm{d}s + \frac{1}{2}\int_0^t s^{-1/2}A_h^{1/2}\mathrm{e}^{-(t-s)A_h}A_h^{-1}(\Pi_h u_s - \dot{v}_h)\,\mathrm{d}s.$$

Recalling (40), by means of the change of variables $\tau = s/t$, it is easy to show that

$$\int_0^t s^{-1/2}\|A_h^{1/2}\mathrm{e}^{-(t-s)A_h}\|_0\,\mathrm{d}s \le \frac{1}{\sqrt{2e}}B\left(\frac{1}{2}, \frac{1}{2}\right), \tag{44}$$

where B is the Beta function (see e.g., [38]). Thus, we have

$$\|y_h\|_0 \le CB\left(\frac{1}{2}, \frac{1}{2}\right)\max_{0\le s\le t}(s\|\dot{\theta}_h\|_0 + \|A_h^{-1}(\Pi_h u_s - \dot{v}_h)\|_0).$$

The first term on the right-hand side above is bounded by CM_4h^2. For the second one we notice that

$$A_h^{-1}(\Pi_h u_t - \dot{v}_h) = \theta_h - (\Pi_h u - v_h),$$

so that using (14), (17) and (38) it is bounded by $M_2h^2|\log(h)|$. □

Theorem 1. *Let (u, p) be the solution of (1)–(2). Then, there exists a positive constant C such that the ID-postprocessed velocity $\tilde{u}$, defined in (26)–(27), satisfies the following bounds:*

(i) *If $r = 2$ then*

$$\|u(t^*) - \tilde{u}(t^*)\|_1 \le \frac{C}{t^{*(1/2)}}h^2|\log(h)|^2. \tag{45}$$

(ii) *If $r = 3, 4$ then*

$$\|u(t^*) - \tilde{u}(t^*)\|_j \le \frac{C}{t^{*(r-1)/2}}h^{r+1-j}|\log(h)|, \quad j = 0, 1. \tag{46}$$

Proof. The proof follows the same steps as [13, Theorem 5.2]. Subtracting (26) from (1), standard duality arguments show that

$$\|\tilde{u}(t^*) - u(t^*)\|_1 \le C(\|F(u(t^*), u(t^*)) - F(u_h(t^*), u_h(t^*))\|_{-1} + \|u_t(t^*) - \dot{u}_h(t^*)\|_{-1}).$$

To bound the second term on the right-hand side above we apply Lemma 4, whereas for the second we apply (35) to get

$$\|F(u(t^*), u(t^*)) - F(u_h(t^*), u_h(t^*))\|_{-1} \le C\|u(t^*) - u_h(t^*)\|_0. \tag{47}$$

Applying (24) the proof of (45) and the case $j = 1$ of (46) are finished.

We now get the error bounds in the L^2 norm. It is easy to see that

$$A(\tilde{u}(t^*) - u(t^*)) = \Pi(F(u(t^*), u(t^*)) - F(u_h(t^*), u_h(t^*))) + \Pi(u_t(t^*) - \dot{u}_h(t^*)).$$

Then, by applying A^{-1} to both sides of the above equations, we obtain

$$\|\tilde{u}(t^*) - u(t^*)\|_0 \le \|A^{-1}\Pi(F(u(t^*), u(t^*)) - F(u_h(t^*), u_h(t^*)))\|_0 + \|A^{-1}\Pi(u_t(t^*) - \dot{u}_h(t^*))\|_0.$$

As regards the nonlinear term, applying [13, Lemma 4.1] we obtain

$$\|A^{-1}\Pi(F(u(t^*),u(t^*))-F(u_h(t^*),u_h(t^*)))\|_0 \le C(\|u(t^*)-u_h(t^*)\|_{-1}+\|u(t^*)-u_h(t^*)\|_1\|u(t^*)-u_h(t^*)\|_0).$$

To bound the second term on the right-hand side above we apply (24), whereas the first one is bounded in the proof of [13, Theorem 5.2] by

$$\|u(t^*)-u_h(t^*)\|_{-1} \le \frac{C}{t^{*(r-2)/2}}h^{r+1}|\log(h)|.$$

Finally, to bound $\|A^{-1}\Pi(u_t(t^*)-\dot{u}_h(t^*))\|_0$ we apply [13, Lemma 5.1] to obtain

$$\|A^{-1}\Pi(u_t(t^*)-\dot{u}_h(t^*))\|_0 \le \frac{C}{t^{*(r-1)/2}}h^{r+1}|\log(h)|,$$

which concludes the proof. □

In the following theorem we obtain the error bounds for the pressure $\tilde{p}$.

Theorem 2. *Let (u,p) be the solution of (1)–(2). Then, there exists a positive constant C such that the ID-postprocessed pressure, $\tilde{p}$, satisfies the following bounds:*

$$\|p(t^*)-\tilde{p}(t^*)\|_{L^2/\mathbb{R}} \le \frac{C}{t^{*(r-1)/2}}h^{r}|\log(h)|^{r'}, \tag{48}$$

where $r'=2$ if $r=2$ and $r'=1$ if $r=3,4$.

Proof. The proof follows the same steps as [13, Theorem 5.3]. Applying the inf-sup condition (5) it is easy to see that

$$\beta\|p(t^*)-\tilde{p}(t^*)\|_{L^2/\mathbb{R}} \le \|\tilde{u}(t^*)-u(t^*)\|_1+\|u_t(t^*)-\dot{u}_h(t^*)\|_{-1}+\|F(u_h(t^*),u_h(t^*))-F(u(t^*),u(t^*))\|_{-1}.$$

Applying now (45) and (46) to bound the first term and arguing as in the proof of Theorem 1 to bound the other two terms we conclude (48). □

Remark 1. As a consequence of Theorems 1 and 2, in the proof of Theorem 3 we obtain that $(\tilde{u}-u_h)$ is an asymptotically exact estimator of the error $(u-u_h)$, while $(\tilde{p}-p_h)$ is an asymptotically exact estimator of the error $(p-p_h)$. However, as we have already observed $\tilde{u}$ and $\tilde{p}$ are not computable in practice. In Theorems 3, 4 and 6 we present different procedures to get computable error estimators.

As we pointed out before the MFE approximations (u_h,p_h) to the velocity and the pressure of the solution (u,p) of the evolutionary Navier–Stokes equations (1)–(2) at any fixed time t^* are also the approximations to the velocity and pressure of the steady Stokes problem (26)–(27). In Theorem 3 below, we show that any a posteriori error estimator of the error in the steady Stokes problem (26)–(27) is also an a posteriori indicator of the error in the approximations to the evolutionary Navier–Stokes equations.

Theorem 3. *Let (u,p) be the solution of (1)–(2) and fix any positive time $t^*>0$. Assume that the Galerkin approximation (u_h,p_h) satisfies, for h small enough and $r=2,3,4$,*

$$\|u(t^*)-u_h(t^*)\|_j \ge C_r h^{r-j}, \quad j=0,1. \tag{49}$$

for some positive constant $C_r=C_r(t^*)$.

(i) *Let us denote by $\xi(t^*)$ any reliable and efficient a posteriori error estimator of the error in the steady Stokes problem (26)–(27), see for example [6,8,39]. That is, we assume that there exist positive constants C_1 and C_2, that are independent of the mesh size h, such that the following bound holds*

$$C_2\xi(t^*) \le \|\tilde{u}(t^*)-u_h(t^*)\|_1+\|\tilde{p}(t^*)-p_h(t^*)\|_0 \le C_1\xi(t^*). \tag{50}$$

Then, $\xi(t^*)$ is also a reliable and efficient estimator of the error in the evolutionary Navier–Stokes equations, i.e., the following bound holds for h small enough

$$\frac{2}{3}C_2\xi(t^*) \le \|u(t^*)-u_h(t^*)\|_1+\|p(t^*)-p_h(t^*)\|_0 \le 2C_1\xi(t^*). \tag{51}$$

(ii) *If $\xi^j_{\rm vel}(t^*)$, $j=0,1$ is an asymptotically exact estimator of the norm $\|\tilde{u}(t^*)-u_h(t^*)\|_j$ of the error in the velocity in the steady Stokes problem, then, it is also an asymptotically exact estimator of the norm of the error $\|u(t^*)-u_h(t^*)\|_j$ in the velocity in the evolutionary Navier–Stokes equations. The same result holds for the pressure in the L^2 norm.*

Proof. Let us first observe that

$$\|u_h(t^*) - u(t^*)\|_1 \le \|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|\tilde{u}(t^*) - u(t^*)\|_1$$

and

$$\|p_h(t^*) - p(t^*)\|_0 \le \|p_h(t^*) - \tilde{p}(t^*)\|_0 + \|\tilde{p}(t^*) - p(t^*)\|_0,$$

so that adding the two above inequalities we get

$$\begin{aligned}\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0 \le\ & \|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0 \\ & + \|\tilde{u}(t^*) - u(t^*)\|_1 + \|\tilde{p}(t^*) - p(t^*)\|_0.\end{aligned}$$

Dividing by $\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0$, using (49) and applying Theorems 1 and 2 we obtain

$$1 \le \frac{\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} + \frac{C t^{*-((r-1)/2)}}{C_r} h|\log(h)|^{r'},$$

where $r' = 2$ for $r = 2$ and $r' = 1$ for $r = 3, 4$. Now, using (50) we get

$$\frac{\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} \le \frac{C_1 \xi(t^*)}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0}.$$

Taking h small enough so that $\frac{C t^{*-((r-1)/2)}}{C_r} h|\log(h)|^{r'} \le 1/2$, we get

$$\frac{1}{2} \le \frac{C_1 \xi(t^*)}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0}$$

and then

$$\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0 \le 2C_1 \xi(t^*). \tag{52}$$

Now, we use the decompositions

$$\|u_h(t^*) - \tilde{u}(t^*)\|_1 \le \|u_h(t^*) - u(t^*)\|_1 + \|u(t^*) - \tilde{u}(t^*)\|_1,$$

and

$$\|p_h(t^*) - \tilde{p}(t^*)\|_0 \le \|p_h(t^*) - p(t^*)\|_0 + \|p(t^*) - \tilde{p}(t^*)\|_0,$$

and add both inequalities as before to get

$$\begin{aligned}\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0 \le\ & \|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0 \\ & + \|u(t^*) - \tilde{u}(t^*)\|_1 + \|p(t^*) - \tilde{p}(t^*)\|_0.\end{aligned}$$

Reasoning as before we get

$$\frac{\|u_h(t^*) - \tilde{u}(t^*)\|_1 + \|p_h(t^*) - \tilde{p}(t^*)\|_0}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} \le 1 + \frac{C t^{*-((r-1)/2)}}{C_r} h|\log(h)|^{r'} \le \frac{3}{2},$$

for h small enough. Using again (50) we obtain

$$\frac{C_2 \xi(t^*)}{\|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0} \le \frac{3}{2},$$

so that

$$\frac{2}{3} C_2 \xi(t^*) \le \|u_h(t^*) - u(t^*)\|_1 + \|p_h(t^*) - p(t^*)\|_0. \tag{53}$$

From (52) and (53) we conclude (51).

Let us now assume that $\xi_{\mathrm{vel}}^j(t^*)$ is an asymptotically exact error estimator for the velocity. Using

$$\|u_h(t^*) - \tilde{u}(t^*)\|_j \le \|u_h(t^*) - u(t^*)\|_j + \|u(t^*) - \tilde{u}(t^*)\|_j, \quad j = 0, 1,$$

we have

$$\lim_{h\to 0} \frac{\|u_h(t^*) - \tilde{u}(t^*)\|_j}{\|u_h(t^*) - u(t^*)\|_j} = 1 + \lim_{h\to 0} \frac{\|u(t^*) - \tilde{u}(t^*)\|_j}{\|u_h(t^*) - u(t^*)\|_j} = 1,$$

the last equality being a consequence of Theorem 1 and the saturation hypothesis (49). As we pointed out before, this limit implies that $(\tilde{u}-u_h)$ is an asymptotically exact estimator of the error $(u-u_h)$. Then

$$\lim_{h\to 0}\frac{\xi^j_{\rm vel}(t^*)}{\|u_h(t^*)-u(t^*)\|_j}=\lim_{h\to 0}\frac{\xi^j_{\rm vel}(t^*)}{\|u_h(t^*)-\tilde{u}(t^*)\|_j}\,\frac{\|u_h(t^*)-\tilde{u}(t^*)\|_j}{\|u_h(t^*)-u(t^*)\|_j}=1,$$

and $\xi^j_{\rm vel}(t^*)$ is also an asymptotically exact estimator of the error in the approximation to the velocity of the evolutionary Navier–Stokes equations. The proof for the pressure can be obtained arguing exactly in the same way. □

Remark 2. We remark that with hypothesis (49) we are merely assuming that the term of order h^{r-j} is really present in the asymptotic expansion of the Galerkin error. The same assumption is also assumed in [20–22,39]. As argued in [40], this is not a very restrictive condition in practice. Let us observe that the constant C_r in (49) is, in general $O(t^{*-(r-2)/2})$, so that the ratio $t^{*-((r-1)/2)}/C_r$ in the proof of Theorem 3 is, in general, $O(t^{*(-1/2)})$.

Remark 3. For some a posteriori error estimators of the steady Stokes problem one can have instead of (50) the following inequalities (see e.g., [12])

$$C_2\xi(t^*)\le\|\tilde{u}(t^*)-u_h(t^*)\|_1+\|\tilde{p}(t^*)-p_h(t^*)\|_0+OSC(f,\mathcal{T}_h)\le C_1\xi(t^*),\tag{54}$$

where $OSC(f,\mathcal{T}_h)$ is the data oscillation term. If this is the case, then, arguing exactly the same as in the proof of Theorem 3, it is possible to obtain a similar bound to (51) that takes into account the oscillation term. More precisely, under hypothesis (54), the following bound holds,

$$\frac{2}{3}C_2\xi(t^*)\le\|u(t^*)-u_h(t^*)\|_1+\|p(t^*)-p_h(t^*)\|_0+OSC(f,\mathcal{T}_h)\le 2C_1\xi(t^*).$$

In Theorem 4 we extend the results of [5] to the nonlinear Navier–Stokes equations. Using the same notation as in [5], in the sequel we will denote by $\xi_{\rm vel}((u_h,p_h),f,H^j)$, $j=0,1$, any a posteriori error estimator of the error $u_h-\tilde{u}$ in the norm of $H^j(\Omega)^d$ in the approximation to the velocity in the steady Stokes problem (26)–(27). We will denote by $\xi_{\rm pres}((u_h,p_h),f,L^2/\mathbb{R})$ any error estimator of the quantity $\|p_h-\tilde{p}\|_{L^2/\mathbb{R}}$.

The key point in Theorem 3 comes from the observation that if we decompose

$$u-u_h=(u-\tilde{u})+(\tilde{u}-u_h),\tag{55}$$

the first term on the right hand side of (55), $u-\tilde{u}$, is in general smaller, by a factor of size $O(h\log(h))$, than the second one, $\tilde{u}-u_h$ (Theorem 1). Then, to estimate the error $u-u_h$ we can safely omit the term $u-\tilde{u}$ in (55). Comparing with the analysis of [5] for a nonstationary linear Stokes model problem the main difference is that the two terms in (55) are taken into account. In Theorem 4 we show that this kind of technique can also be applied to the nonlinear Navier–Stokes equations. The advantage of this point of view is that hypothesis (49) is not required for the proof of Theorem 4. Let us finally observe that $(\dot{u}_h,\dot{p}_h)$ are the MFE approximations to the solution $(\tilde{u}_t,\tilde{p}_t)$ of the Stokes problem that we obtain deriving respect to the time variable the Stokes problem (26)–(27). Then, we will denote by $\xi_{\rm vel}((\dot{u}_h,\dot{p}_h),f_t,H^j)$, $j=-1,0,1$, any a posteriori error estimator of the error $\dot{u}_h-\tilde{u}_t$ in the norm of $H^j(\Omega)^d$ in the approximation to the velocity of the corresponding steady Stokes problem. The proof of the following theorem follows the steps of the proof of [21, Theorem 1].

Theorem 4. *Let (u,p) be the solution of (1)–(2) and let (u_h,p_h) be its MFE Galerkin approximation. Then, the following a posteriori error bound holds for $0\le t\le T$ and a constant C independent of h.*

$$\begin{aligned}\|(u-u_h)(t)\|_0\le{}& C\|u_0-u_h(0)\|_0+C\xi_{\rm vel}((u_h(0),p_h(0)),f(0),L^2)+\xi_{\rm vel}((u_h(t),p_h(t)),f(t),L^2)\\&+Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((u_h,p_h),f,L^2)+Ct^{1/2}\max_{0\le s\le t}\xi_{\rm vel}((\dot{u}_h,\dot{p}_h),f_s,H^{-1}).\end{aligned}\tag{56}$$

Proof. Let us denote by $\eta=u-\tilde{u}$. From (26)–(27) it follows that

$$\eta_t+A\eta+\Pi(F(u,u)-F(u_h,u_h))=\Pi(\dot{u}_h-\tilde{u}_t).$$

Then η satisfies the equation

$$\begin{aligned}\eta(t)={}&\mathrm{e}^{-At}\eta(0)+\int_0^t\mathrm{e}^{-A(t-s)}\Pi(F(\tilde{u},\tilde{u})-F(u,u))\,\mathrm{d}s\\&+\int_0^t\mathrm{e}^{-A(t-s)}\Pi(F(u_h,u_h)-F(\tilde{u},\tilde{u})\,\mathrm{d}s)+\int_0^t\mathrm{e}^{-A(t-s)}\Pi(\dot{u}_h-\tilde{u}_t)\,\mathrm{d}s.\end{aligned}$$

Taking into account (7) we get

$$\|\eta(t)\|_0 \le \|\eta(0)\|_0 + C\int_0^t \frac{\|A^{-1/2}\Pi(F(\tilde u,\tilde u)-F(u,u))\|_0}{\sqrt{t-s}}\,ds + C\int_0^t \frac{\|A^{-1/2}\Pi(F(u_h,u_h)-F(\tilde u,\tilde u))\|_0}{\sqrt{t-s}}\,ds + C\int_0^t \frac{\|A^{-1/2}\Pi(\dot u_h-\tilde u_t)\|_0}{\sqrt{t-s}}\,ds.$$

We first observe that for any $v \in L^2(\Omega)^d$ we have $\|A^{-1/2}\Pi v\|_0 \le C\|v\|_{-1}$. Then, taking into account (35) we get

$$\|A^{-1/2}\Pi(F(\tilde u,\tilde u)-F(u,u))\|_0 \le C\|\tilde u - u\|_0,$$
$$\|A^{-1/2}\Pi(F(u_h,u_h)-F(\tilde u,\tilde u))\|_0 \le C\|u_h - \tilde u\|_0.$$

Let us observe that in order to apply (35) we require u_h to satisfy (32), which holds due to (24), and $\|\tilde u\|_\infty$ and $\|\nabla\tilde u\|_{L^{2d/(d-1)}}$ to be bounded. Using (4) both norms are bounded in terms of $\|\tilde u\|_2$. Applying (19) we get

$$\|\tilde u\|_2 \le C(\|\dot u_h\|_0 + \|u_h\cdot\nabla u_h\|_0) \le C(\|A_h u_h\|_0 + \|\Pi_h F(u_h,u_h)\|_0 + \|\Pi_h f\|_0 + \|u_h\cdot\nabla u_h\|_0).$$

Finally, using that $\|A_h u_h\|_0$ is uniformly bounded, see (28), and arguing as in (29) to bound the second and forth terms above we conclude $\|\tilde u\|_2$ is uniformly bounded. Then, we arrive at

$$\|\eta(t)\|_0 \le \|\eta(0)\|_0 + C\int_0^t \frac{\|\eta(s)\|_0}{\sqrt{t-s}}\,ds + C\int_0^t \frac{\|u_h(s)-\tilde u(s)\|_0}{\sqrt{t-s}}\,ds + C\int_0^t \frac{\|\dot u_h(s)-\tilde u_s(s)\|_0}{\sqrt{t-s}}\,ds.$$

And then,

$$\|\eta(t)\|_0 \le \|\eta(0)\|_0 + C\int_0^t \frac{\|\eta(s)\|_0}{\sqrt{t-s}}\,ds + Ct^{1/2}\max_{0\le s\le t}\xi_{\mathrm{vel}}((u_h,p_h),f,L^2) + Ct^{1/2}\max_{0\le s\le t}\xi_{\mathrm{vel}}((\dot u_h,\dot p_h),f_s,L^2).$$

A standard application of a generalized Gronwall lemma [37] gives

$$\|\eta(t)\|_0 \le C\|\eta(0)\|_0 + Ct^{1/2}\max_{0\le s\le t}\xi_{\mathrm{vel}}((u_h,p_h),f,L^2) + Ct^{1/2}\max_{0\le s\le t}\xi_{\mathrm{vel}}((\dot u_h,\dot p_h),f_s,L^2).$$

Now, using decomposition (55) we conclude the proof. □

Remark 4. We observe that using the same proof, a similar bound for the $H^1(\Omega)^d$ norm of the error can be obtained by changing only $\xi_{\mathrm{vel}}((u_h,p_h),f,L^2)$ by $\xi_{\mathrm{vel}}((u_h,p_h),f,H^1)$ and $\xi_{\mathrm{vel}}((\dot u_h,\dot p_h),f_t,H^{-1})$ by $\xi_{\mathrm{vel}}((\dot u_h,\dot p_h),f_t,L^2)$. Let us also remark that Theorem 4 allows to obtain a posteriori upper error bounds for the error in the approximation to the nonlinear Navier–Stokes equations using only upper error bounds for some Stokes problems depending only on the data and the computed approximation. However, the estimation of the error at a time t requires the estimation of the error of a family of Stokes problems with the right hand side depending on τ, for all $\tau \in [0,t]$.

We now propose a simple procedure to estimate the error which is based on computing a MFE approximation to the solution $(\tilde u(t^*),\tilde p(t^*))$ of (26)–(27) on a MFE space with better approximation capabilities than $(X_{h,r},Q_{h,r-1})$ in which the Galerkin approximation (u_h,p_h) is defined. This procedure was applied to the p-version of the finite-element method for evolutionary convection–reaction–diffusion equations in [20]. The main idea here is to use a second approximation of different accuracy than that of the Galerkin approximation of (u,p) and whose computational cost hardly adds to that of the Galerkin approximation itself.

Let us fix any time $t^* \in (0,T]$ and let us approximate the solution $(\tilde u,\tilde p)$ of the Stokes problem (26)–(27) by solving the following discrete Stokes problem: find $(\tilde u_h(t^*),\tilde p_h(t^*)) \in \widetilde X \times \widetilde Q$ satisfying

$$(\nabla\tilde u_h(t^*),\nabla\tilde\phi) + (\nabla\tilde p_h(t^*),\tilde\phi) = (f,\tilde\phi) - (F(u_h(t^*),u_h(t^*)),\tilde\phi) - (\dot u_h(t^*),\tilde\phi) \quad \forall\tilde\phi\in\widetilde X, \tag{57}$$

$$(\nabla\cdot\tilde u_h(t^*),\tilde\psi) = 0 \quad \forall\tilde\psi\in\widetilde Q, \tag{58}$$

where $(\widetilde X,\widetilde Q)$ is either:

(a) The same-order MFE over a finer grid. That is, for $h' < h$, we choose $(\widetilde X,\widetilde Q) = (X_{h',r},Q_{h',r-1})$.
(b) A higher-order MFE over the same grid. In this case we choose $(\widetilde X,\ \widetilde Q) = (X_{h,r+1},Q_{h,r})$.

We now study the errors $u-\tilde u_h$ and $p-\tilde p_h$.

Theorem 5. *Let (u,p) be the solution of (1)–(2) and for $r = 2,3,4$, and let (10)–(11) hold with $k = r+2$ Then, there exists a positive constant C such that the postprocessed MFE approximation to u, $\tilde u_h$ satisfies the following bounds for $r = 2,3,4$ and $t \in (0,T]$:*

(i) *if the postprocessing element is* $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$, *then*

$$\|u(t) - \tilde{u}_h(t)\|_j \leq \frac{C}{t^{(r-2)/2}}(h')^{r-j} + \frac{C}{t^{(r-1)/2}}h^{r+1-j}|\log(h)|^{r'}, \quad j = 0, 1, \tag{59}$$

$$\|p(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{(r-2)/2}}(h')^{r-1} + \frac{C}{t^{(r-1)/2}}h^{r}|\log(h)|^{r'}, \tag{60}$$

(ii) *if the postprocessing element is* $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$, *then*

$$\|u(t) - \tilde{u}_h(t)\|_j \leq \frac{C}{t^{(r-1)/2}}h^{r+1-j}|\log(h)|^{r'}, \quad j = 0, 1, \tag{61}$$

$$\|p(t) - \tilde{p}_h(t)\|_{L^2/\mathbb{R}} \leq \frac{C}{t^{(r-1)/2}}h^{r}|\log(h)|^{r'}. \tag{62}$$

For $r = 2$ *only the case* $j = 1$ *in* (59) *and* (61) *holds. In* (59)–(62), $r' = 2$ *when* $r = 2$ *and* $r' = 1$ *otherwise.*

Proof. The cases $r = 3, 4$ have been proven in Theorems 5.2 and 5.3 in [13]. Following the same arguments, we now prove the results corresponding to $r = 2$ and $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$, the case $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$ being similar, yet easier. We decompose the error $u - \tilde{u}_h = (u - s_{h'}) + (s_{h'} - \tilde{u}_h)$, where $(s_{h'}, q_{h'}) \in X_{h',2} \times Q_{h',1}$ is the solution of

$$(\nabla s_{h'}, \nabla \phi_{h'}) - (q_{h'}, \nabla \cdot \phi_{h'}) = (f - F(u, u) - u_t, \phi_{h'}) \quad \forall \phi_{h'} \in X_{h',2}, \tag{63}$$

$$(\nabla \cdot s_{h'}, \psi_{h'}) = 0 \quad \forall \psi_{h'} \in Q_{h',1}, \tag{64}$$

that is, $s_{h'}$ is the Stokes projection of u onto $V_{h'}$. Since in view of (17)–(18) we have

$$\|u - s_{h'}\|_1 + \|p - q_{h'}\|_{L^2/\mathbb{R}} \leq CM_2 h',$$

we only have to estimate $s_{h'} - \tilde{u}_h$ and $q_{h'} - \tilde{p}_h$. To do this, we subtract (57) from (63), and take inner product with $\tilde{e}_h = s_{h'} - \tilde{u}_h$ to get

$$\|\nabla \tilde{e}_h\|_0^2 \leq (\|u_t - \dot{u}_h\|_{-1} + \|F(u_h, u_h) - F(u, u)\|_{-1})\|\tilde{e}_h\|_1.$$

Now applying Lemma 4, (35) and (24) the proof of (59) is finished.

To prove (60), again we subtract (57) from (63), rearrange terms and apply the inf-sup condition (5) to get

$$\beta\|q_{h'} - \tilde{p}_h\|_{L^2/\mathbb{R}} \leq \|\nabla \tilde{e}_h\|_0 + \|u_t - \dot{u}_h\|_{-1} + \|F(u_h, u_h) - F(u, u)\|_{-1}$$

and the proof is finished with the same arguments used to prove (59). □

To estimate the error in $(u_h(t^*), p_h(t^*))$ we propose to take the difference between the postprocessed and the Galerkin approximations:

$$\tilde{\eta}_{h,\mathrm{vel}}(t^*) = \tilde{u}_h(t^*) - u_h(t^*), \qquad \tilde{\eta}_{h,\mathrm{pres}}(t^*) = \tilde{p}_h(t^*) - p_h(t^*).$$

In the following theorem we prove that this error estimator is efficient and asymptotically exact both in the $L^2(\Omega)^d$ and $H^1(\Omega)^d$ norms, and it has the advantage of providing an improved approximation when added to the Galerkin MFE approximation.

Theorem 6. *Let* (u, p) *be the solution of* (1)–(2) *and fix any positive time* $t^* > 0$. *Assume that condition* (49) *is satisfied. Then, there exist positive constants* h_0, $\gamma_0 < 1$, *and* C_1, C_2, C_3 *and* C_4 *such that, for* $h < h_0$ *and* $0 < \gamma < \gamma_0$, *the error estimators* $\tilde{\eta}_{h,\mathrm{vel}}(t^*)$ $\tilde{\eta}_{h,\mathrm{pres}}(t^*)$ *satisfy the following bounds when* $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$ *and* $h' < \gamma h$:

$$C_1 \leq \frac{\|\tilde{\eta}_{h,\mathrm{vel}}(t^*)\|_j}{\|(u - u_h)(t^*)\|_j} \leq C_2, \quad j = 0, 1, \qquad C_3 \leq \frac{\|\tilde{\eta}_{h,\mathrm{pres}}(t^*)\|_{L^2/\mathbb{R}}}{\|(p - p_h)(t^*)\|_{L^2/\mathbb{R}}} \leq C_4. \tag{65}$$

Furthermore, if $(\widetilde{X}, \widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$, *with* $h' = h^{1+\epsilon}$, $\epsilon > 0$, *or* $(\widetilde{X}, \widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$ *then*

$$\lim_{h \to 0} \frac{\|\tilde{\eta}_{h,\mathrm{vel}}(t^*)\|_j}{\|(u - u_h)(t^*)\|_j} = 1, \quad j = 0, 1, \qquad \lim_{h \to 0} \frac{\|\tilde{\eta}_{h,\mathrm{pres}}(t^*)\|_{L^2/\mathbb{R}}}{\|(p - p_h)(t^*)\|_{L^2/\mathbb{R}}} = 1. \tag{66}$$

For the mini element, the case $j = 0$ *in* (65) *and* (66) *must be excluded.*

Proof. We will prove the estimates for the velocity in the case $r = 3, 4$, since the estimates for the pressure and the case $r = 2$ are obtained by similar arguments but with obvious changes. Let us observe that for $j = 0, 1$,

$$\begin{aligned}\|u(t^*) - u_h(t^*)\|_j &\leq \|\tilde{\eta}_{h,\mathrm{vel}}(t^*)\|_j + \|\tilde{u}_h(t^*) - u(t^*)\|_j \\ &\leq \|\tilde{\eta}_{h,\mathrm{vel}}(t^*)\|_j + \frac{C}{(t^*)^{(r-2)/2}}(h')^{r-j} + \frac{C}{(t^*)^{(r-1)/2}}h^{r+1-j}|\log(h)|.\end{aligned}$$

On the other hand

$$\|\tilde{\eta}_{h,\mathrm{vel}}(t^*)\|_j \le \|u(t^*)-u_h(t^*)\|_j + \|\tilde{u}_h(t^*)-u(t^*)\|_j$$
$$\le \|u(t^*)-u_h(t^*)\|_j + \frac{C}{(t^*)^{(r-2)/2}}(h')^{r-j} + \frac{C}{(t^*)^{(r-1)/2}} h^{r+1-j}|\log(h)|.$$

Using (49) we get

$$\left|\frac{\|\tilde{\eta}_{h,\mathrm{vel}}(t^*)\|_j}{\|(u-u_h)(t^*)\|_j} - 1\right| \le \frac{C}{C_r}\left((t^*)^{-(r-2)/2}\left(\frac{h'}{h}\right)^{r-j} + (t^*)^{-(r-1)/2}|\log(h)|h\right). \tag{67}$$

Taking $h' \le \gamma h$ and h and γ sufficiently small, the bound (65) is readily obtained. The proof of (66) follows straightforwardly from (67), since in the case when $(\widetilde{X},\widetilde{Q}) = (X_{h',r}, Q_{h',r-1})$ with $h' = h^{1+\epsilon}$, $\epsilon > 0$, the term $(h'/h)^{r-j} \to 0$ when h tends to zero, and in the case when $(\widetilde{X},\widetilde{Q}) = (X_{h,r+1}, Q_{h,r})$ the term containing the parameter h' is not present. □

Remark 5. The value of h_0 for which Theorem 6 holds for $h \le h_0$ is unknown in practice and in particular it depends on the Reynolds number. When using the estimator in a practical problem, if one needs to know if it will be applicable, it is usually enough to look at the postprocessed approximation to see if, in some sense, it smooths out the Galerkin approximation and, as a consequence, it can be used in practice to estimate the error for the current value of h. We also want to remark that, in the case of high Reynolds numbers, it is convenient to use, instead of the method proposed in this paper, the postprocessing procedure presented in [41]. This new postprocessing was presented in [42] for semidiscretizations in space of nonlinear convection–diffusion problems and extended in [23] for full discretizations. In [23] a posteriori error estimates, based on this technique, are obtained, both for the semidiscrete in space and the fully discrete cases. The use of the new postprocessing procedure to get a posteriori error estimations in the case of the Navier–Stokes equations will be the subject of future research.

4. A posteriori error estimations. Fully discrete case

In practice, it is not possible to compute the MFE approximation exactly, and, instead, some time-stepping procedure must be used to approximate the solution of (22)–(23). Hence, for some time levels $0 = t_0 < t_1 < \cdots < t_N = T$, approximations $U_h^n \approx u_h(t_n)$ and $P_h^n \approx p_h(t_n)$ are obtained. In this section we assume that the approximations are obtained with the backward Euler method or the two-step BDF which we now describe. For simplicity, we consider only constant stepsizes, that is, for $N \ge 2$ integer, we fix $k = T/N$, and we denote $t_n = nk$, $n = 0, 1, \ldots, N$. For a sequence $(y^n)_{n=0}^N$ we denote

$$Dy^n = y^n - y^{n-1}, \quad n = 1, 2 \ldots, N.$$

Given $U_h^0 = u_h(0)$, a sequence (U_h^n, P_h^n) of approximations to $(u_h(t_n), p_h(t_n))$, $n = 1, \ldots, N$, is obtained by means of the following recurrence relation:

$$(d_t U_h^n, \phi_h) + (\nabla U_h^n, \nabla\phi_h) + b(U_h^n, U_h^n, \phi_h) - (P_h^n, \nabla\cdot\phi_h) = (f, \phi_h) \quad \forall \phi_h \in X_{h,r}, \tag{68}$$
$$(\nabla\cdot U_h^n, \psi_h) = 0, \quad \forall \psi_h \in Q_{h,r-1}, \tag{69}$$

where $d_t = k^{-1}D$ in the case of the backward Euler method and $d_t = k^{-1}(D + \frac{1}{2}D^2)$ for the two-step BDF. In this last case, a second starting value U_h^1 is needed. Here, we will always assume that U_h^1 is obtained by one step of the backward Euler method. Also, for both the backward Euler and the two-step BDF, we assume that $U_h^0 = u_h(0)$, which is usually the case in practical situations.

We now define the IDTD-postprocessed approximation. Given an approximation $d_t^* U_h^n$ to $\dot{u}_h(t_n)$, the IDTD-postprocessed velocity and pressure $(\widetilde{U}^n, \widetilde{P}^n)$ are defined as the solution of the following Stokes problem:

$$(\nabla\tilde{U}^n, \nabla\phi) + (\nabla\tilde{P}^n, \phi) = (f, \phi) - b(U_h^n, U_h^n, \phi) - (d_t^* U_h^n, \phi), \quad \forall \phi \in H_0^1(\Omega)^d, \tag{70}$$
$$(\nabla\cdot\tilde{U}^n, \psi) = 0, \quad \forall \psi \in L^2(\Omega)/\mathbb{R}. \tag{71}$$

For reasons already analyzed in [22,24] we define

$$d_t^* U_h^n = \Pi_h f - A_h U_h^n - \Pi_h F(U_h^n, U_h^n) \tag{72}$$

as an adequate approximation to the time derivative $\dot{u}_h(t_n)$.

For the analysis of the errors $u(t) - \tilde{U}^n$ and $p(t) - \tilde{P}^n$ we follow [24], where the MFE approximations to the Stokes problem (70)–(71) are analyzed. We start by decomposing the errors $u(t) - \tilde{U}^n$ and $p(t) - \tilde{P}^n$ as follows,

$$u(t_n) - \tilde{U}^n = (u(t) - \tilde{u}(t_n)) + \tilde{e}^n, \tag{73}$$
$$p(t_n) - \tilde{P}^n = (p(t_n) - \tilde{p}(t_n)) + \tilde{\pi}^n, \tag{74}$$

where $\tilde{e}^n = \tilde{u}(t_n) - \tilde{U}^n$ and $\tilde{\pi}^n = \tilde{p}(t_n) - \tilde{P}^n$ are the temporal errors of the IDTD-postprocessed velocity and pressure $(\tilde{U}^n, \tilde{P}^n)$. The first terms on the right-hand sides of (73)–(74) are the errors of the postprocessed approximation that were studied in the previous section.

Let us denote by $e_h^n = u_h(t_n) - U_h^n$, the temporal error of the MFE approximation to the velocity, and by $\pi_h^n = p_h(t_n) - P_h^n$, the temporal error of the MFE approximation to the pressure. In the present section we bound $(\tilde{e}^n - e_h^n)$ and $(\tilde{\pi}^n - \pi_h^n)$ in terms of e_h^n.

The error bounds in the following lemma are similar to those of [24, Proposition 3.1] where error estimates for MFE approximations of the Stokes problem (70)–(71) are obtained.

Lemma 5. *There exists a positive constant $C = C(\max_{0\le t\le T} \|A_h u_h(t)\|_0)$ such that for $1 \le n \le N$ the following bounds hold*

$$\|\tilde{e}^n - e_h^n\|_j \le Ch^{2-j}(\|e_h^n\|_1 + \|e_h^n\|_1^3 + \|A_h e_h^n\|_0), \quad j = 0, 1 \tag{75}$$

$$\|\tilde{\pi}^n - \pi_h^n\|_{L^2/\mathbb{R}} \le Ch(\|e_h^n\|_1 + \|e_h^n\|_1^3 + \|A_h e_h^n\|_0). \tag{76}$$

Proof. Let us first observe that, due to (28), $\max_{0\le t\le T} \|A_h u_h(t)\|_0$ is bounded independently of h. Let us denote by $l = g + (d_t^* U_h^n - \dot{u}_h(t_n))$ where $g = F(U_h^n, U_h^n) - F(u_h(t_n), u_h(t_n))$. Subtracting (70)–(71) from (26)–(27) we have that the temporal errors $(\tilde{e}^n, \tilde{\pi}^n)$ of the IDTD-postprocessed velocity and pressure are the solution of the following Stokes problem

$$(\nabla \tilde{e}^n, \nabla \phi) + (\nabla \tilde{\pi}^n, \phi) = (l, \phi), \quad \forall \phi \in H_0^1(\Omega)^d, \tag{77}$$

$$(\nabla \cdot \tilde{e}^n, \psi) = 0, \quad \forall \psi \in L^2(\Omega)/\mathbb{R}. \tag{78}$$

On the other hand, subtracting (68)–(69) from (22)–(23) and taking into account that, thanks to definition (72), $d_t U_h^n = d_t^* U_h^n$, we get that the temporal errors (e_h^n, π_h^n) of the fully discrete MFE approximation satisfy

$$(\nabla e_h^n, \nabla \phi_h) + (\nabla \pi_h^n, \phi_h) = (l, \phi_h), \quad \forall \phi_h \in X_{h,r},$$
$$(\nabla \cdot e_h^n, \psi_h) = 0, \quad \forall \psi_h \in Q_{h,r-1},$$

and, thus, (e_h^n, π_h^n) is the MFE approximation to the solution $(\tilde{e}^n, \tilde{\pi}^n)$ of (77)–(78). Using then (21) we get

$$\|\tilde{e}^n - e_h^n\|_j \le Ch^{2-j}\|l\|_0.$$

For the pressure we apply (18) and (20) to obtain

$$\|\tilde{\pi}^n - \pi_h^n\|_{L^2/\mathbb{R}} \le Ch\|\tilde{\pi}^n\|_{H^1/\mathbb{R}} \le Ch\|l\|_0.$$

Then, to finish the proof, it only remains to bound $\|l\|_0$. From the definition of $d_t^* U_h^n$ it is easy to see that

$$d_t^* U_h^n - \dot{u}_h(t_n) = A_h e_h^n - \Pi_h(F(U_h^n, U_h^n) - F(u_h(t_n), u_h(t_n))),$$

so that

$$\|d_t^* U_h^n - \dot{u}_h(t_n)\|_0 \le \|A_h e_h^n\|_0 + \|g\|_0.$$

Now, by writing g as

$$g = F(e_h^n, u_h(t_n)) + F(u_h(t_n), e_h^n) - F(e_h^n, e_h^n),$$

and using (29)–(30) we get

$$\|g\|_0 \le (\|A_h u_h(t_n)\|_0 \|e_h^n\|_1 + \|e_h^n\|_1^{3/2} \|A_h e_h^n\|_0^{1/2}),$$

from which we finally conclude (75) and (76). □

Let us consider the quantities $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ as a posteriori indicators of the error in the fully discrete approximations to the velocity and pressure respectively. Then, we obtain the following result:

Theorem 7. *Let (u, p) be the solution of (1)–(2) and let (9) hold. Assume that the fully discrete MFE approximations (U_h^n, P_h^n), $n = 0, \dots, N = T/k$ are obtained by the backward Euler method or the two-step BDF (68)–(69), and let $(\widetilde{U}^n, \widetilde{P}^n)$ be the solution of (70)–(71). Then, for $n = 1, \dots, N$,*

$$\|\widetilde{U}^n - U_h^n\|_j \le \|\tilde{u}(t_n) - u_h(t_n)\|_j + C'_{l_0} h^{2-j} \frac{k^{l_0}}{t_n^{l_0}}, \quad j = 0, 1, \tag{79}$$

$$\|\widetilde{P}^n - P_h^n\|_{L^2/\mathbb{R}} \le \|\tilde{p}(t_n) - p_h(t_n)\|_{L^2/\mathbb{R}} + C'_{l_0} h \frac{k^{l_0}}{t_n^{l_0}}, \tag{80}$$

where C'_{l_0} is the constant in (82)–(83), $l_0 = 1$ for the backward Euler method and $l_0 = 2$ for the two-step BDF.

Proof. In [24, Theorems 5.4 and 5.7] we prove that if (9) and the case $l = 2$ in (17) hold, then, for k small enough, the errors e_h^n of these two time integration procedures satisfy that

$$\|e_h^n\|_0 + t_n\|A_h e_h^n\|_0 \le \mathcal{C}_{l_0}\frac{k^{l_0}}{{t_n}^{l_0-1}}, \quad 1 \le n \le N, \tag{81}$$

for a certain constants $\mathcal{C}_1$ and C_2, where $l_0 = 1$ for the backward Euler method and $l_0 = 2$ for the two-step BDF. Since $\|A_h^{1/2}e_h^n\|_0 \le \|e_h^n\|_0^{1/2}\|A_h e_h^n\|_0^{1/2}$, and then $\|e_h^n\|_1 \le C\|e_h^n\|_0^{1/2}\|A_h e_h^n\|_0^{1/2}$, from (81) and (75)–(76) we finally reach that for k small enough

$$\|\tilde{e}^n - e_h^n\|_j \le C'_{l_0}h^{2-j}\frac{k^{l_0}}{t_n^{l_0}}, \quad j = 0, 1, \ 1 \le n \le N, \tag{82}$$

$$\|\tilde{\pi}^n - \pi_h^n\|_{L^2/\mathbb{R}} \le C'_{l_0}h\frac{k^{l_0}}{t_n^{l_0}}, \quad 1 \le n \le N, \tag{83}$$

where C'_{l_0} is a positive constant.

Let us decompose the estimators as follows:

$$\begin{aligned}\widetilde{U}^n - U_h^n &= (\widetilde{U}^n - \tilde{u}(t_n)) + (\tilde{u}(t_n) - u_h(t_n)) + (u_h(t_n) - U_h^n)\\ &= (\tilde{u}(t_n) - u_h(t_n)) + (e_h^n - \tilde{e}^n),\end{aligned} \tag{84}$$

$$\begin{aligned}\widetilde{P}^n - P_h^n &= (\widetilde{P}^n - \tilde{p}(t_n)) + (\tilde{p}(t_n) - p_h(t_n)) + (p_h(t_n) - P_h^n)\\ &= (\tilde{p}(t_n) - p_h(t_n)) + (\pi_h^n - \tilde{\pi}^n),\end{aligned} \tag{85}$$

which implies

$$\|\widetilde{U}^n - U_h^n\|_j \le \|\tilde{u}(t_n) - u_h(t_n)\|_j + \|e_h^n - \tilde{e}^n\|_j, \quad j = 1, 2,$$
$$\|\widetilde{P}^n - P_h^n\|_{L^2/\mathbb{R}} \le \|\tilde{p}(t_n) - p_h(t_n)\|_{L^2/\mathbb{R}} + \|\pi_h^n - \tilde{\pi}^n\|_{L^2/\mathbb{R}}.$$

Thus, in view of (82)–(83) we obtain (79) and (80). □

Let us comment on the practical implications of this theorem. Observe that from (84) and (85) the fully discrete estimators $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ can be both decomposed as the sum of two terms. The first one is the semidiscrete a posteriori error estimator we have studied in the previous section (see Remark 1) and which we showed it is an asymptotically exact estimator of the spatial error of U_h^n and P_h^n respectively. On the other hand, as shown in (82)–(83), the size of the second term is in asymptotically smaller than the temporal error of U_h^n and P_h^n respectively. We conclude that, as long as the spatial and temporal errors are not too unbalanced (i.e., they are not of very different sizes), the first term in (84) and (85) is dominant and then the quantities $\widetilde{U}^n - U_h^n$ and $\widetilde{P}^n - P_h^n$ are a posteriori error estimators of the spatial error of the fully discrete approximations to the velocity and pressure respectively. The control of the temporal error can be then accomplished by standard and well established techniques in the field of numerical integration of ordinary differential equations.

Now, we remark that $(\widetilde{U}^n, \widetilde{P}^n)$ are obviously not computable. However, we observe that the fully discrete approximation (U_h^n, P_h^n) of the evolutionary Navier–Stokes equation is also the approximation to the Stokes problem (70)–(71) whose solution is $(\widetilde{U}^n, \widetilde{P}^n)$. This is true in the case of the backward Euler method and the two-step BDF since, as we commented in the proof of Lemma 5, $d_t^*U_h^n = d_tU_h^n$. Then, one can use any of the available error estimators for a steady Stokes problem to estimate the quantities $\|\widetilde{U}^n - U_h^n\|_j$ and $\|\widetilde{P}^n - P_h^n\|_{L^2/\mathbb{R}}$, which, as we have already proved, are error indicators of the spatial errors of the fully discrete approximations to the velocity and pressure, respectively.

To conclude, we show a procedure to get computable estimates of the error in the fully discrete approximations. We define the fully discrete postprocessed approximation $(\widetilde{U}_h^n, \widetilde{P}_h^n)$ as the solution of the following Stokes problem (see [24]):

$$(\nabla\widetilde{U}_h^n, \nabla\tilde{\phi}) + (\nabla\widetilde{P}_h^n, \tilde{\phi}) = (f, \tilde{\phi}) - b(U_h^n, U_h^n, \tilde{\phi}) - (d_t^*U_h^n, \tilde{\phi}) \quad \forall\tilde{\phi} \in \widetilde{X}, \tag{86}$$

$$(\nabla\cdot\widetilde{U}_h^n, \tilde{\psi}) = 0 \quad \forall\tilde{\psi} \in \widetilde{Q}, \tag{87}$$

where $(\widetilde{X}, \widetilde{Q})$ is as in (57)–(58).

Let us denote by $\tilde{e}_h^n = \tilde{u}_h(t_n) - \widetilde{U}_h^n$ and $\tilde{\pi}_h^n = \tilde{p}_h(t_n) - \widetilde{P}_h^n$ the temporal errors of the fully discrete postprocessed approximation $(\widetilde{U}_h^n, \widetilde{P}_h^n)$ (observe that the semidiscrete postprocessed approximation $(\tilde{u}_h, \tilde{p}_h)$ is defined in (57)–(58)). Let us denote, as before, by e_h^n the temporal error of the MFE approximation to the velocity. Then, we have the following bounds.

Lemma 6. *There exists a positive constant $C = C(\max_{0\le t\le T}\|A_h u_h(t)\|_0)$ such that for $1 \le n \le N$ the following bounds hold*

$$\|\tilde{e}_h^n - e_h^n\|_j \le Ch^{2-j}(\|e_h^n\|_1 + \|e_h^n\|_1^3 + \|A_h e_h^n\|_0), \quad j = 0, 1, \tag{88}$$

$$\|\tilde{\pi}_h^n - \pi_h^n\|_{L^2(\Omega)/\mathbb{R}} \le Ch(\|e_h^n\|_1 + \|e_h^n\|_1^3 + \|A_h e_h^n\|_0). \tag{89}$$

Proof. The bound (88) is proved in [24, Proposition 3.1]. To prove (89) we decompose

$$\|\tilde{\pi}_h^n - \pi_h^n\|_{L^2(\Omega)/\mathbb{R}} \le \|\tilde{\pi}_h^n - \tilde{\pi}^n\|_{L^2(\Omega)/\mathbb{R}} + \|\tilde{\pi}^n - \pi_h^n\|_{L^2(\Omega)/\mathbb{R}}.$$

The second term above is bounded in (76) of Lemma 5. For the first we observe that $\tilde{\pi}_h^n$ is the MFE approximation in $\widetilde{Q}$ to the pressure $\tilde{\pi}^n$ in (77)–(78) so that the same arguments used in the proof of Lemma 5 allow us to obtain

$$\|\tilde{\pi}_h^n - \tilde{\pi}^n\|_{L^2(\Omega)/\mathbb{R}} \le Ch\|\tilde{\pi}^n\|_{H^1/\mathbb{R}} \le Ch(\|e_h^n\|_1 + \|e_h^n\|_1^3 + \|A_h e_h^n\|_0). \quad \square$$

Using (81) as before, we get the analogous to (82) and (83), i.e., for k small enough the following bound holds

$$\|\tilde{e}_h^n - e_h^n\|_j \le C'_{l_0} h^{2-j}\frac{k^{l_0}}{t_n^{l_0}}, \quad j = 0, 1, 1 \le n \le N, \tag{90}$$

$$\|\tilde{\pi}_h^n - \pi_h^n\|_{L^2/\mathbb{R}} \le C'_{l_0} h\frac{k^{l_0}}{t_n^{l_0}}, \quad 1 \le n \le N \tag{91}$$

where C'_{l_0} is a positive constant.

Similarly to (84)–(85) we write $\widetilde{U}_h^n - U_h^n = (\tilde{u}_h(t_n) - u_h(t_n)) + (e_h^n - \tilde{e}_h^n)$ and $\widetilde{P}_h^n - P_h^n = (\tilde{p}_h(t_n) - p_h(t_n)) + (\pi_h^n - \tilde{\pi}_h^n)$, so that in view of (90)–(91) we have the following result.

Theorem 8. *Let (u, p) be the solution of (1)–(2) and let (9) hold. Assume that the fully discrete MFE approximations (U_h^n, P_h^n), $n = 0, \ldots, N = T/k$ are obtained by the backward Euler method or the two-step BDF (68)–(69), and let $(\widetilde{U}_h^n, \widetilde{P}_h^n)$ be the solution of (86)–(87). Then, for $n = 1, \ldots, N$,*

$$\|\widetilde{U}_h^n - U_h^n\|_j \le \|\tilde{u}_h(t_n) - u_h(t_n)\|_j + C'_{l_0} h^{2-j}\frac{k^{l_0}}{t_n^{l_0}}, \quad j = 0, 1, \tag{92}$$

$$\|\widetilde{P}_h^n - P_h^n\|_{L^2/\mathbb{R}} \le \|\tilde{p}_h(t_n) - p_h(t_n)\|_{L^2/\mathbb{R}} + C'_{l_0} h\frac{k^{l_0}}{t_n^{l_0}}, \tag{93}$$

where C'_{l_0} is the constant in (90)–(91), $l_0 = 1$ for the backward Euler method and $l_0 = 2$ for the two-step BDF.

The practical implications of this result are similar to those of Theorem 7, that is, the first term on the right-hand side of (92) is an error indicator of the spatial error (see Theorem 6) while the second one is asymptotically smaller than the temporal error. As a consequence, the quantity $(\widetilde{U}_h^n - U_h^n)$ is a computable estimator of the spatial error of the fully discrete velocity U_h^n whenever the temporal and spatial errors of U_h^n are more or less of the same size. As before, similar arguments apply for the pressure. We remark that having balanced spatial and temporal errors in the fully discrete approximation is the more common case in practical computations since one usually looks for a final solution with small total error.

As in the semidiscrete case, the advantage of these error estimators is that they produce enhanced (in space) approximations when they are added to the Galerkin MFE approximations.

5. Numerical experiments

We consider the equations

$$\begin{aligned} &u_t - \nu\Delta u + (u\cdot\nabla)u + \nabla p = f, \\ &\mathrm{div}(u) = 0, \end{aligned} \tag{94}$$

in the domain $\Omega = [0, 1] \times [0, 1]$ subject to homogeneous Dirichlet boundary conditions. For the numerical experiments of this section we approximate the equations using the mini-element [33] over a regular triangulation of Ω induced by the set of nodes $(i/N, j/N)$, $0 \le i, j \le N$, where $N = 1/h$ is an integer. For the time integration we use the two-step BDF method with fixed time step. For the first step we apply the backward Euler method. In the first numerical experiment we study the semidiscrete in space case. To this end in the numerical experiments we integrate in time with a time-step small enough in order to have negligible temporal errors. We take the forcing term $f(t, x)$ such that the solution of (94) with $\nu = 0.05$ is

$$\begin{aligned} u^1(x, y, t) &= 2\pi\varphi(t)\sin^2(\pi x)\sin(\pi y)\cos(\pi y), \\ u^2(x, y, t) &= -2\pi\varphi(t)\sin^2(\pi y)\sin(\pi x)\cos(\pi x), \\ p(x, y, t) &= 20\varphi(t)x^2 y. \end{aligned} \tag{95}$$

We chose $\varphi(t) = t$ in the first numerical experiment.

When using the mini-element it has been observed and reported in the literature (see for instance [12,43,8,44–46]) that the linear part of the approximation to the velocity, u_h^l, is a better approximation to the solution u than u_h itself. The bubble

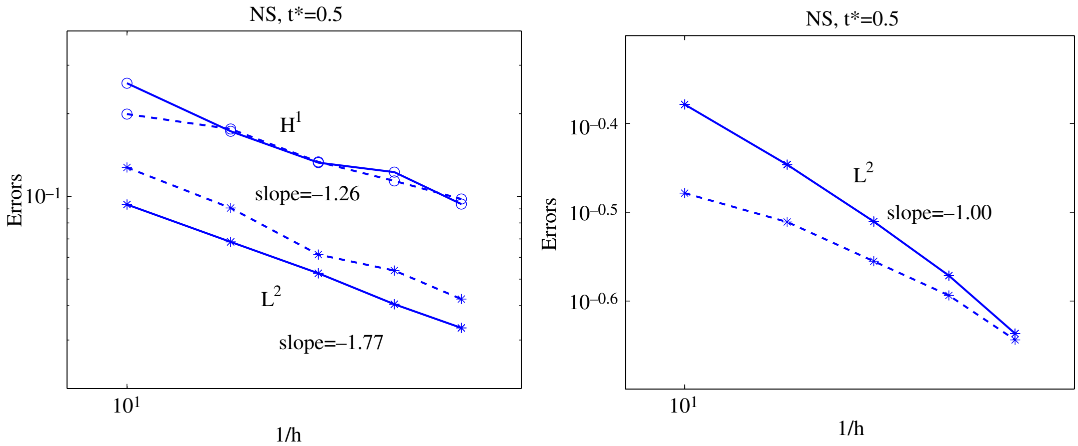

Fig. 1. Errors (solid lines) and estimations (dashed lines) in L^2 (asterisks) and H^1 (circles) for $h = 1/10, 1/12, 1/14, 1/16$ and $1/18$ and $h' = 1/24, 1/30, 1/34, 1/38$ and $1/40$ respectively. On the left, error estimations for the first component of the velocity. On the right, error estimations for the pressure.

Table 1
Efficiency indexes.

h	$\|\theta_{\text{vel}}\|_0$	$\|\theta_{\text{vel}}\|_1$	$\|\theta_{\text{pre}}\|_{L^2/\mathbb{R}}$
1/10	1.3640	0.7721	1.2588
1/12	1.3280	1.0197	1.1602
1/14	1.1695	1.0068	1.1084
1/16	1.3259	0.9290	1.0526
1/18	1.2741	1.0438	1.0167

part of the approximation is only introduced for stability reasons and does not improve the approximation to the velocity and pressure terms. For this reason in the numerical experiments of this section we only consider the errors in the linear approximation to the velocity. Also, following [19], we postprocess only the linear approximation to the velocity, i.e., we solve the Stokes problem (57)–(58) with u_h^l and $\dot{u}_h^l$ on the right-hand-side instead of u_h and $\dot{u}_h$. The finite element space at the postprocessed step is the same mini-element defined over a refined mesh of size h'. We show the Galerkin errors and the a posteriori error estimates obtained at time $t^* = 0.5$ by taking the difference between the postprocessed and the standard approximations to the velocity and the pressure. In Fig. 1, we have represented the errors in the first component of the velocity of the Galerkin approximation in the L^2 and H^1 norms and the errors for the pressure in the L^2 norm using solid lines. We have used dashed lines to represent the error estimations. The results for the second component of the velocity are completely analogous and they are not reported here. The L^2 errors of the pressure, on the right of Fig. 1, are approximately twice as those of the H^1 errors of the velocity, on the left of Fig. 1, in this example. We can observe that with the procedure we propose in this paper we get very accurate estimations of the errors, especially in the H^1 norm of the velocity. The difference between the behavior of the error estimations in the L^2 and H^1 norms of the velocity are due to the fact that for first order approximations the postprocessed procedure increases the rate of convergence of the standard method only in the H^1 norm for the velocity and the L^2 norm for the pressure. However, since the postprocessed method produces smaller errors than the Galerkin method also in the L^2 norm it can also be used to estimate the errors in this norm, as it can be checked in the experiment. On the right of Fig. 1 we can clearly observe the asymptotically exact behavior of the estimator in the L^2 errors in the pressure in agreement with (66) of Theorem 6.

Let us denote by

$$\theta_{\text{vel}} = \frac{\tilde{u}_h^1(t^*) - u_h^1(t^*)}{u^1(t^*) - u_h^1(t^*)}, \qquad \theta_{\text{pre}} = \frac{\tilde{p}_h(t^*) - p(t^*)}{p(t^*) - p_h(t^*)},$$

the efficiency indexes for the first component of the velocity and for the pressure. In Table 1 we have represented the values of the L^2 and H^1 norms of the velocity index and the $L^2/\mathbb{R}$ norm of the pressure index for the experiments in Fig. 1. We deduce again from the values of the efficiency indexes that the a posteriori error estimates are very accurate, all the values are remarkably close to 1, which is the optimal value for the efficiency index. More precisely, we can observe that the values of the efficiency index in the L^2 norm for the velocity in this experiment belong to the interval [1.1695, 1.3640]. The values in the H^1 norm for the velocity lie on the interval [0.7721, 1.0438] and, finally, the values for the pressure are in the interval [1.0167, 1.2588].

We next show that the a posteriori error estimators we propose can also be used to compute indicators of the local errors. The idea is the following. To estimate the error in, for example, the first component of the velocity, on an element of the partition, τ_i^h, or on a patch of elements, $\cup_{i\in I}\tau_i^h$, we propose to compute the quantities $\|\tilde{u}_h^1(t^*) - u_h^1(t^*)\|_{j,\tau_i^h}$, or

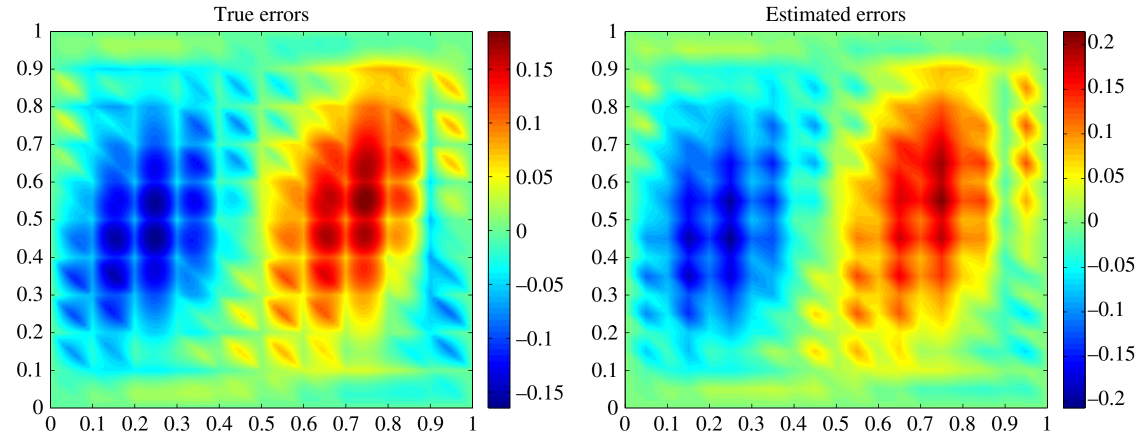

Fig. 2. On the left: true errors for the first component of the velocity for $h = 1/10$. On the right: estimated errors for the first component of the velocity for $h' = 1/20$.

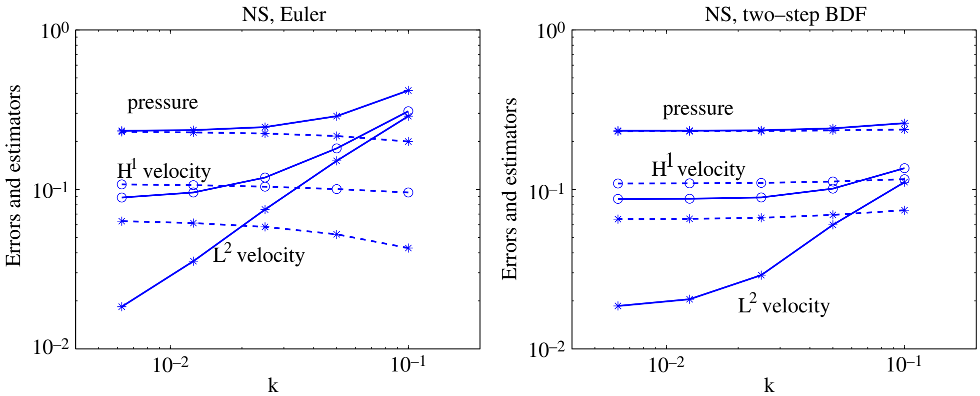

Fig. 3. Errors (solid lines) and estimations (dashed lines) in L^2 (asterisks) and H^1 (circles) for $h = 1/18$. On the left: Euler; on the right: two-step BDF for $k = 1/10$–$k = 1/160$.

$\|\tilde{u}_h^1(t^*) - u_h^1(t^*)\|_{j,\cup_{i\in I}\tau_i^h}$, respectively, for $j = 0, 1$. In Fig. 2 we have represented, on the left, the distribution of the true errors, $u_h^1(0.5) - u(0.5)$, for $h = 1/10$, over the full domain $[0, 1] \times [0, 1]$. On the right, we have represented the distribution of the estimated errors $u_h^1(0.5) - \tilde{u}_h^1(0.5)$, i.e., the difference between the Galerkin approximation computed with $h = 1/10$ and the postprocessed approximation computed with $h' = 1/20$. We can observe in the figure that both distributions are very similar and, as a consequence, our error indicators not only compute very accurate global error estimations but also reproduce very well the local behavior of the errors. A proof for the local error bounds following the lines of [21] will be the subject of future research.

To conclude, we show a numerical experiment to check the behavior of the estimators in the fully discrete case. We choose the forcing term f such that the solution of (94) is (95) with $\varphi(t) = \sin((2\pi + \pi/2)t)$. The value of $\nu = 0.05$ and the final time $t^* = 0.5$ are the same as before. In Fig. 3, on the left, we have represented the errors obtained using the implicit Euler method as a time integrator for different values of the fixed time step k ranging from $k = 1/10$ to $k = 1/160$ halving each time the value of k. For the spacial discretization we use the mini-element with always the same value of $h = 1/18$. We use solid lines for the errors in the Galerkin method and dashed lines for the estimations, as before. The L^2 norm errors are marked with asterisks while the H^1 norm errors are marked with circles. We estimate the errors using the postprocessed method computed with the same mini-element over a refined mesh of size $h' = 1/40$. We observe that the Galerkin errors decrease as k decreases until a value that corresponds to the spatial error of the approximation. On the contrary, the error estimations lie on an almost horizontal line, both for the velocity in the L^2 and H^1 norms and for the pressure. This means, that the error estimations we propose are a measure of the spatial errors, even when the errors in the Galerkin method are polluted by errors coming from the temporal discretization. In this experiment the error estimations are very accurate for the spatial errors of the velocity in the H^1 norm and for the errors in the pressure. As commented above, the fact that postprocessing linear elements does not increase the convergence rate in the L^2 norm is reflected in the precision of the error estimations in the L^2 norm. On the right of Fig. 3 we have represented the errors obtained when we integrate in time with the two-step BDF and fixed time step. The only remarkable difference is that, as we expected from the second order rate of convergence of the method in time, the temporal errors are smaller for the same values of the fixed time step k. Again, the estimations lie on a horizontal line being essentially the same as in the experiment on the left.

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