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Pseudo-Dedekind Domains and Divisorial Ideals in $R[X]_T$

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I. INTRODUCTION

Throughout this paper R will be a commutative ring with identity, usually an integral domain. Bourbaki [8] has defined an integral domain R to be pseudo-principal if every divisorial ideal of R is principal. We define an integral domain R to be pseudo-Dedekind if every divisorial ideal of R is invertible. In the second section of this paper, we give several alternative characterizations of pseudo-Dedekind domains. For example, for an integral domain R the following conditions are equivalent: (1) R is pseudo-Dedekind, (2) $(AB)^{-1} = A^{-1}B^{-1}$ for all nonzero fractional ideals A and B of R, (3) is completely integrally closed and the product of divisorial ideals is divisorial, and (4) $(\bigcap (a_\alpha))(\bigcap (b_\beta)) = \bigcap (a_\alpha b_\beta)$ where $\{a_\alpha\}, \{b_\beta\} \subseteq K$, the quotient field of R.

In the third section, we investigate divisorial ideals in $R[X]_T$ where $T \subseteq N_v = \{f \in R[X] \mid (A_f)_v = R\}$ is a multiplicatively closed subset of $R[X]$. The main result of this section is that if R is an essential domain and J is a divisorial ideal of $R[X]_T$, then $J \cap R[X]$ is a divisorial ideal of $R[X]$. This result is then used in the fourth section to show that R is pseudo-Dedekind if and only if $R(X)$ is pseudo-principal.

In general, our terminology and notation will follow that given in Gilmer [9]. The reader is referred there for terms and notation not defined in this paper.

II. PSEUDO-DEDEKIND DOMAINS

If R is a Dedekind domain, then the set of nonzero fractional ideals of R forms a group; so $(AB)^{-1} = A^{-1}B^{-1}$ for all nonzero fractional ideals A and B of R. This paper began with the following question. If R is an integral

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domain, when is $(AB)^{-1} = A^{-1}B^{-1}$ for all nonzero fractional ideals A and B of R? We begin by considering this question in a slightly more general context.

Thus let R be a commutative ring with identity having total quotient ring $T(R)$. Let $\mathscr{F}_r(R)$ be the set of *regular* fractional ideals of R. As usual, for $A \in \mathscr{F}_r(R)$, $A^{-1} = \{x \in T(R) \mid xA \subseteq R\} \in \mathscr{F}_r(R)$. We denote $(A^{-1})^{-1}$ by A_v and call A divisorial if $A = A_v$. If $A \in \mathscr{F}_r(R)$, then $A \subseteq A_v \subseteq \bigcap \{Rx \mid Rx \supseteq A \text{ for } Rx \in \mathscr{F}_r(R)\}$. If R is an integral domain (more generally a Marot ring), then $A_v = \bigcap \{Rx \mid Rx \supseteq A \text{ for } Rx \in \mathscr{F}_r(R)\}$, but in general $\subseteq$ may be proper [7].

$A \in \mathscr{F}_r(R)$ is said to be v-invertible if A is a unit in the divisor monoid $\mathscr{D}(R)$ of divisorial ideals with the v-product $A * B = (AB)_v$; i.e., there exists $B \in \mathscr{F}_r(R)$ with $(AB)_v = R$. But then $AB \subseteq (AB)_v = R$ so $B \subseteq A^{-1}$. Hence $R = (AB)_v \subseteq (AA^{-1})_v \subseteq R$, so $R = (AA^{-1})_v$. It is well known that every element of $\mathscr{F}_r(R)$ is v-invertible $\Leftrightarrow \mathscr{D}(R)$ is a group $\Leftrightarrow R$ is completely integrally closed.

Let $A, B \in \mathscr{F}_r(R)$. Now $(A^{-1}B^{-1})AB = (A^{-1}A)(B^{-1}B) \subseteq R \cdot R = R$, so we have $A^{-1}B^{-1} \subseteq (AB)^{-1}$. Since $(AB)^{-1}$ is divisorial, we even have $A^{-1}B^{-1} \subseteq (A^{-1}B^{-1})_v \subseteq (AB)^{-1}$. We begin by considering when either of these two containments may be replaced by equality.

THEOREM 2.1. *For $A \in \mathscr{F}_r(R)$, the following statements are equivalent.*

(1) $(AB)^{-1} = (A^{-1}B^{-1})_v$ *for all* $B \in \mathscr{F}_r(R)$.

(2) $(AB)^{-1} = (A^{-1}B^{-1})_v$ *for all divisorial ideals* B.

(3) $(AA^{-1})^{-1} = (A^{-1}A_v)_v$ $(= (AA^{-1})_v)$.

(4) *A is v-invertible.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3). Clear. (3)$\Rightarrow$(4). Now $AA^{-1} \subseteq R$, so $(AA^{-1})^{-1} \supseteq R$. However, $(AA^{-1})^{-1} = (A^{-1}A_v)_v = (A^{-1}A)_v \subseteq R$, so $(AA^{-1})^{-1} = R$ and hence $(AA^{-1})_v = R$. Therefore A is v-invertible. (4)$\Rightarrow$(1). Suppose that A is v-invertible. We always have $(AB)^{-1} \supseteq (A^{-1}B^{-1})_v$. Now $(A(AB)^{-1})B \subseteq R$, so $A(AB)^{-1} \subseteq B^{-1}$. Hence $A^{-1}A(AB)^{-1} \subseteq A^{-1}B^{-1}$, so $(AB)^{-1} = (AB)_v^{-1} = (A^{-1}A(AB)^{-1})_v \subseteq (A^{-1}B^{-1})_v$. Therefore $(AB)^{-1} = (A^{-1}B^{-1})_v$.

COROLLARY 2.2. *For a ring R, the following statements are equivalent.*

(1) $(A^{-1}B^{-1})_v = (AB)^{-1}$ *for* $A, B \in \mathscr{F}_r(R)$.

(2) *R is completely integrally closed.*

LEMMA 2.3. *For a ring R, the following statements are equivalent.*

(1) $A^{-1}B^{-1} = (A^{-1}B^{-1})_v$ *for* $A, B \in \mathscr{F}_r(R)$.

(2) *The product of divisorial ideals is divisorial.*

Proof. (2)$\Rightarrow$(1). This is clear since A^{-1} and B^{-1} are divisorial. (1)$\Rightarrow$(2). Let C and D be divisorial ideals. Then $CD=((C^{-1})^{-1}(D^{-1})^{-1})_v$ is divisorial.

THEOREM 2.4. *For a ring R and $A\in\mathscr{F}_r(R)$, the following statements are equivalent.*

(1) $(AB)^{-1}=A^{-1}B^{-1}$ *for all* $B\in\mathscr{F}_r(R)$.

(2) $(AB)^{-1}=A^{-1}B^{-1}$ *for all divisorial ideals B of R.*

(3) $(AA^{-1})^{-1}=A^{-1}(A^{-1})^{-1}$.

(4) A_v *is invertible.*

Proof. (1)$\Rightarrow$(2)$\Rightarrow$(3). Clear. (3)$\Rightarrow$(4). Now $AA^{-1}\subseteq R$ so $R\subseteq(AA^{-1})^{-1}=A^{-1}A_v$. But $A^{-1}A_v=A_v^{-1}A_v\subseteq R$, so $A^{-1}A_v=R$. Hence A_v is invertible. (4)$\Rightarrow$(1). We always have $(AB)^{-1}\supseteq A^{-1}B^{-1}$. Now $A(AB)^{-1}B\subseteq R$ so $A(AB)^{-1}\subseteq B^{-1}$. Hence $A_v(AB)^{-1}\subseteq(A(AB)^{-1})_v\subseteq B^{-1}$. But A_v is invertible, so $(AB)^{-1}\subseteq A^{-1}B^{-1}$.

COROLLARY 2.5. *For a commutative ring R, the following statements are equivalent.*

(1) $(AB)^{-1}=A^{-1}B^{-1}$ *for all* $A, B\in\mathscr{F}_r(R)$.

(2) *For each* $A\in\mathscr{F}_r(R)$, A_v *is invertible.*

(3) *R is completely integrally closed and the product of divisorial ideals is divisorial.*

(4) $\mathscr{D}(R)$ *is a group under the usual ideal product.*

Proof. (1)$\Leftrightarrow$(2). Theorem 2.4. (2)$\Rightarrow$(4). Clear. (4)$\Rightarrow$(3). $\mathscr{D}(R)$ is a group under the usual product implies that $A*B=(AB)_v=AB$, so the product of divisorial ideals is divisorial. Since $\mathscr{D}(R)$ is a group, R is completely integrally closed. (3)$\Rightarrow$(2). $R=(AA^{-1})_v=(A_vA^{-1})_v=A_vA^{-1}$ since A_v and A^{-1} are divisorial. Hence A_v is invertible.

R can be completely integrally closed without having the property that the product of divisorial ideals is divisorial. For example, a Krull domain is completely integrally closed but the product of divisorial ideals is divisorial if and only if R is locally factorial. Also, the product of divisorial ideals can be divisorial without R being completly integrally closed. For example, let R be a one-dimensional Gorenstein domain that is not regular. Then every nonzero ideal of R is divisorial so certainly the product of divisorial ideals is divisorial.

Let us now restrict ourselves to integral domains. A domain R is *pseudo-Dedekind* if every divisorial ideal is invertible. If actually every divisorial ideal is principal, R is called *pseudo-principal* [8]. So by Corollary 2.5, $(AB)^{-1}=A^{-1}B^{-1}$ for all nonzero fractional ideals A and B of R if and only if R is pseudo-Dedekind. Another characterization of pseudo-Dedekind domains may be obtained by considering products of intersections of principal fractional ideals.

Let R be an integral domain with quotient field K. Let $\{a_\alpha\}, \{b_\beta\} \subseteq K$. Then certainly $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) \subseteq \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$. A natural question is when do we actually have equality. Observe that if $\bigcap_\alpha (a_\alpha) = 0$, then we always have equality. (Now $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) = 0$. Let $x \in \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$. Fix $b_0 \in \{b_\beta\}$. Then $x \in \bigcap_\alpha (a_\alpha b_0) = b_0(\bigcap_\alpha (a_\alpha)) = 0$ so $x = 0$. Hence $\bigcap_{\alpha,\beta} (a_\alpha b_\beta) = 0$ also.) Thus we can assume that $\bigcap_\alpha (a_\alpha) \neq 0$ and $\bigcap_\beta (b_\beta) \neq 0$; in particular, $\{a_\alpha\}, \{b_\beta\} \subseteq K - \{0\}$.

Suppose that $\{a_\alpha\}, \{b_\beta\} \subseteq K - \{0\}$. Let $A = (\{a_\alpha^{-1}\})$ and $B = (\{b_\beta^{-1}\})$. Then $A^{-1} = \bigcap_\alpha (a_\alpha)$ and $B^{-1} = \bigcap_\beta (b_\beta)$. Since $AB = \{\{a_\alpha^{-1} b_\beta^{-1}\}\}$, we also have that $(AB)^{-1} = \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$. Hence $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) = \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$ translates to the statement that $A^{-1}B^{-1} = (AB)^{-1}$.

THEOREM 2.6. *For an integral domain R, the following statements are equivalent.*

(1) *R is pseudo-Dedekind.*

(2) $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) = \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$ *for all* $\{a_\alpha\}, \{b_\beta\} \subseteq K$.

(3) $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) = \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$ *for all* $\{a_\alpha\}, \{b_\beta\} \subseteq K - \{0\}$.

Proof. We have already noted that (2) $\Leftrightarrow$ (3). (1) $\Rightarrow$ (3). If R is pseudo-Dedekind, then $A^{-1}B^{-1} = (AB)^{-1}$ for all fractional ideals A and B of R. Hence by the above paragraph, $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) = \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$ for $\bigcap_\alpha (a_\alpha) \neq 0$ and $\bigcap_\beta (b_\beta) \neq 0$. But as previously noted, if $\bigcap_\alpha (a_\alpha) = 0$ or $\bigcap_\beta (b_\beta) = 0$, the result is trivial. (2) $\Rightarrow$ (1). This again follows from the above paragraph.

Note that we always have $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) \subseteq ((\bigcap_\alpha (a_\alpha))(\bigcap (b_\beta)))_v \subseteq \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$. What is somewhat surprising is that $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta))$ can be divisorial wthout being equal to $\bigcap_{\alpha,\beta} (a_\alpha b_\beta)$. Let R be a one-dimensional Gorenstein domain that is not regular. Since every nonzero ideal of R is divisorial, $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta))$ is divisorial. Since R is not pseudo-Dedekind, we must have some $(\bigcap_\alpha (a_\alpha))(\bigcap_\beta (b_\beta)) \neq \bigcap_{\alpha,\beta} (a_\alpha b_\beta)$.

The following interesting lemma gives an alternative proof of Theorem 2.6.

LEMMA 2.7. *Let R be an integral domain. Then* $\bigcap_\alpha (x_\alpha) \neq 0$ *is invertible if and only if* $(\bigcap_\alpha (x_\alpha))(\bigcap_\beta (y_\beta)) = \bigcap (x_\alpha y_\beta)$ *for all* $\{y_\beta\} \subseteq K$ *(or* $K - \{0\}$*).*

Proof. ($\Rightarrow$). Suppose that $\bigcap_\alpha (x_\alpha) \neq 0$ is invertible. Then $(\bigcap_\alpha (x_\alpha))(\bigcap_\beta (y_\beta)) = \bigcap_\beta (\bigcap_\alpha (x_\alpha) y_\beta) = \bigcap_\beta (\bigcap_\alpha (x_\alpha y_\beta)) = \bigcap_{\alpha,\beta} (x_\alpha y_\beta)$. (Here we have used the fact that if I is an invertible ideal, then $I(\bigcap_\alpha A_\alpha) = \bigcap_\alpha IA_\alpha$ for any collection $\{A_\alpha\}$ of fractional ideals of R [9, p. 80].)

($\Leftarrow$). Now $(\bigcap_\alpha (x_\alpha))^{-1}$ is divisorial, so $(\bigcap_\alpha (x_\alpha))^{-1} = \bigcap_\beta (y_\beta)$ for some $\{y_\beta\} \subseteq K - \{0\}$. Now $R \supseteq (\bigcap_\alpha (x_\alpha))(\bigcap_\alpha (x_\alpha))^{-1} = (\bigcap_\alpha (x_\alpha))(\bigcap_\beta (y_\beta)) = \bigcap_{\alpha,\beta} (x_\alpha y_\beta)$. Now for each α_0, $(x_{\alpha_0}) \supseteq \bigcap_\alpha (x_\alpha) \Rightarrow (x_{\alpha_0}^{-1}) \subseteq (\bigcap_\alpha (x_\alpha))^{-1} = \bigcap_\beta (y_\beta) \subseteq (y_\beta)$. So $R = (\bigcap_\alpha (x_\alpha))(\bigcap_\beta (y_\beta)) = (\bigcap_\alpha (x_\alpha))(\bigcap_\alpha (x_\alpha))^{-1}$ and hence $\bigcap_\alpha (x_\alpha)$ is invertible.

In view of Theorem 2.6, the following question is natural. Let R be an integral domain that satisfies $(\bigcap (a_\alpha))(\bigcap (b_\beta)) = \bigcap (a_\alpha b_\beta)$ for all subsets $\{a_\alpha\}, \{b_\beta\} \subseteq R$. Is R pseudo-Dedekind? While we have not been able to answer this question in general, we remark that the answer is affirmative under additional hypothesis such as R having ACC on divisorial ideals or R being a GCD domain.

Recall that one of the many conditions equivalent to R being a GCD domain is that every finite-type divisorial ideal is principal. R is said to be a G-GCD domain (generalized GCD domain) if every finite type divisorial ideal of R is invertible. For results on G-GCD domains, the reader is referred to [3].

Suppose that R is pseudo-Dedekind. Then R is a G-GCD domain and R is completely integrally closed. Since R is a G-GCD domain, $\mathrm{Inv}(R)$, the group of invertible ideals of R, is a lattice ordered group where as usual $I \leqslant J \Leftrightarrow J \subseteq I$, $I \vee J = I \cap J$ and $I \wedge J = (I+J)_v$. Since every divisorial ideal is invertible, $\mathrm{Inv}(R)$ is even complete. For let $\{I_\alpha\} \subseteq \mathrm{Inv}(R)$, with $I_\alpha \leqslant B$. Then $B \subseteq \bigcap I_\alpha$, so $\bigcap I_\alpha \neq 0$ and since each I_α is invertible, $\bigcap I_\alpha$ is a divisorial ideal and hence invertible since R is pseudo-Dedekind. Hence $\bigvee I_\alpha = \bigcap I_\alpha$.

The converse is also true. Suppose that $\mathrm{Inv}(R)$ is a complete lattice ordered group. Let I be a divisorial ideal, so $I = \bigcap_\alpha Rr_\alpha$. Now $I \neq 0$, so for $0 \neq x \in I$, $xR \subseteq Rr_\alpha$ so $Rr_\alpha \leqslant xR$. Hence $\{Rr_\alpha\}$ is a collection of invertible ideals that is bounded above, so $\bigvee Rr_\alpha = J$, say, where J is an invertible ideal. But $Rr_\alpha \leqslant \bigvee Rr_\alpha = J$, so $J \subseteq Rr_\alpha$. Hence $J \subseteq \bigcap Rr_\alpha = I$. But, if $0 \neq y \in I$, then $yR \subseteq I \subseteq Rr_\alpha$. Thus $Rr_\alpha \leqslant yR$, so $J = \bigvee Rr_\alpha \leqslant yR$, and hence $yR \subseteq J$. Hence $I = J$ is invertible. So R is pseudo-Dedekind.

Now if R is a G-GCD domain, $\mathrm{Inv}(R)$ is order isomorphic to $\mathrm{G}(R^v)$, the group of divisibility of R^v (see [3, p. 219]). So if R is pseudo-Dedekind, $\mathrm{Inv}(R^v)$ is a complete lattice ordered group (every invertible ideal of R^v is principal), so R^v is pseudo-principal So R pseudo-Dedekind implies that R^v is pseudo-principal. The converse is false. If R is a Krull domain, R^v is always a PID and hence pseudo-principal. However, for R a G-GCD domain, it is easily seen that R is pseudo-Dedekind ($\mathrm{Inv}(R)$ is complete) if and only if R^v is pseudo-principal ($G(R^v)$ is complete). In particular, a valuation domain is pseudo-principal if and only if its value group is isomorphic to a complete subgroup of the real numbers.

Another characterization of pseudo-Dedekind domains is as follows. R is pseudo-Dedekind if and only if $\mathrm{Cl}(R) = \mathrm{Pic}(R)$. Here $\mathrm{Cl}(R)$ is the divisor class monoid of R, that is, $\mathrm{Cl}(R) = \mathscr{D}(R)/\mathrm{Princ}(R)$ where $\mathrm{Princ}(R)$ is the subgroup of $\mathrm{Inv}(R)$ consisting of principal fractional ideals. Also, $\mathrm{Pic}(R) = \mathrm{Inv}(R)/\mathrm{Princ}(R)$ is the Picard group of R. Indeed, if R is pseudo-Dedekind, R is completely integrally closed, so $\mathscr{D}(R)$ is a group and hence $\mathrm{Cl}(R)$ is a group. Since every divisorial ideal of R is invertible, clearly $\mathrm{Cl}(R) = \mathrm{Pic}(R)$. Conversely, suppose that the divisor class monoid $\mathscr{D}(R)/\mathrm{Princ}(R) = \mathrm{Pic}(R)$. This says that every divisorial ideal is invertible and hence R is pseudo-Dedekind.

The following theorem gives a summary of our various characterizations of pseudo-Dedekind domains.

THEOREM 2.8. *For an integral domain R, the following conditions are equivalent.*

(1) *R is a pseudo-Dedekind domain; i.e., every divisorial ideal of R is invertible.*

(2) *$(AB)^{-1} = A^{-1}B^{-1}$ for all nonzero fractional ideals A and B of R.*

(3) $(\bigcap (a_\alpha))(\bigcap (b_\beta)) = \bigcap (a_\alpha b_\beta)$ *for* $\{a_\alpha\}, \{b_\beta\} \subseteq K$.

(4) *R is completely integrally closed and the product of divisorial ideals is divisorial.*

(5) *The divisor monoid $\mathscr{D}(R)$ is a group under the usual ideal product.*

(6) $\mathrm{Cl}(R) = \mathrm{Pic}(R)$.

(7) $\mathrm{Inv}(R)$, *the group of invertible ideals of R, is a complete lattice ordered group.*

Zafrullah [12] has defined an integral domain R to be a $*$-domain if $(\bigcap_{i=1}^{n} (a_i))(\bigcap_{j=1}^{m} (b_j)) = \bigcap_{i,j} (a_i b_j)$ for all $a_1, ..., a_n, b_1, ..., b_m \in R$ (or equivalently, $\in K$). We observed that R is a $*$-domain if and only if $(AB)^{-1} = A^{-1}B^{-1}$ for all finitely generated nonzero fractional ideals A and B of R. This lead us to investigate domains satisfying $(AB)^{-1} = A^{-1}B^{-1}$ for all nonzero fractional ideals. After our research was completed, we learned that Zafrullah [13] had independently obtained the equivalence of (1), (2), (4), and (5) of Theorem 2.8 and the result (proved in the final section) that R pseudo-Dedekind implies that $R[X]$ is pseudo-Dedekind.

We end this section with some examples of pseudo-Dedekind domains. Clearly a Dedekind domain is a pseudo-Dedekind domain. More generally, a locally factorial Krull domain (sometimes called a π-domain) is pseudo-Dedekind [2]. Under certain finiteness conditions, the converse is true. Suppose, for example, that R is a pseudo-Dedekind domain with ACC on divisorial ideals. Then since R is also completely integrally closed, R is a Krull domain. It then follows from [2] that R is a locally factorial Krull domain. In Section 4 we show that the following statements are equivalent: (1) R is pseudo-Dedekind, (2) $R[X]$ is pseudo-Dedekind, (3) $R(X)$ is pseudo-Dedekind, (4) $R(X)$ is pseudo-principal. We also give an example to show that R being pseudo-Dedekind does not imply that R_S is pseudo-Dedekind.

III. Divisorial Ideals in $R[X]_T$

Throughout this section, R will denote an integral domain with quotient field K. For $f = a_0 + a_1X + \cdots + a_nX^n \in K[X]$, the content A_f of f is the fractional ideal of R generated by $a_0, a_1, ..., a_n$. N_v will denote the multiplicatively closed subset $\{f \in R[X] \mid (A_f)_v = R\}$ of $R[X]$. T will usually denote a multiplicatively closed subset of N_v. An integral domain R is said to be essential if there is a set of prime ideals $\{P_\alpha\}$ of R with $R = \bigcap R_{P_\alpha}$ and each R_{P_α} is a valuation domain. The main result of this section is Theorem 3.10 which states that if R is an essential domain and J is a divisorial ideal of $R[X]_T$, then $J \cap R[X]$ is a divisorial ideal of $R[X]$. However, our first result gives an important case when the extension of a divisorial ideal remains divisorial.

Proposition 3.1. *Let R be an integral domain and T a multiplicatively closed subset of $R[X]$ contained in N_v. Let I be a nonzero fractional ideal of R. Then*

(1) $(I[X]_T)^{-1} = I^{-1}[X]_T$, *and*

(2) $(I[X]_T)_v = I_v[X]_T$.

Proof. (1) It is clear that $I^{-1}[X]_T \subseteq (I[X]_T)^{-1}$. Let $0 \neq a \in I$. Then $aR[X]_T \subseteq I[X]_T$, so $(I[X]_T)^{-1} \subseteq a^{-1}R[X]_T \subseteq K[X]_T$. Hence an element of $(I[X]_T)^{-1}$ has the form f/h where $f \in K[X]$ and $h \in T$. Moreover, $f \in (I[X]_T)^{-1}$. Hence $fI \subseteq fI[X]_T \subseteq R[X]_T$. So for $b \in I$, $bf \in R[X]_T$. Hence $bfg \in R[X]$ for some $g \in N_v$. Then $bA_f = A_{bf} \subseteq (A_{bf})_v = (A_{bf}(A_g)_v)_v = (A_{bf}A_g)_v = (A_{bfg})_v \subseteq R$. (Here $(A_{bf}A_g)_v = (A_{bfg})_v$ via the Dedekind–Mertens Theorem [9, Theorem 28.1] and the fact that $(A_g)_v = R$.) Since $bA_f \subseteq R$ for all $b \in I$, we have $A_f \subseteq I^{-1}$. Hence $f \in I^{-1}[X]$, so $f/h \in I^{-1}[X]_T$. Therefore $(I[X]_T)^{-1} = I^{-1}[X]_T$. (2) $(I[X]_T)_v = ((I[X]_T)^{-1})^{-1} = (I^{-1}[X]_T)^{-1} = (I^{-1})^{-1}[X]_T = I_v[X]_T$.

Let $A \subseteq B$ be integral domains. Let T be a multiplicatively closed subset of B. T is said to be weakly saturated in B with respect to A, if $\overline{T \cap A} = \bar{T} \cap A$ where $\overline{T \cap A}$ is the saturation of $T \cap A$ in A and $\bar{T}$ is the saturation of T in B. T is said to be locally weakly saturated in B with respect to A if T, as a multiplicatively closed subset of B_P, is weakly saturated in B_P with respect to A_P for each prime ideal P of A such that $P \cap (T \cap A) = \emptyset$. Lemma 3.2 gives two conditions equivalent to T being weakly saturated in B with respect to A.

LEMMA 3.2. *The following statements are equivalent for domains $A \subseteq B$ and for a multiplicatively closed subset T of B.*

(1) *T is weakly saturated in B with repect to A.*

(2) *$Ba \cap T \neq \emptyset$ implies $Aa \cap T \neq \emptyset$ for every $a \in A$.*

(3) *$ab \in T$, $a \in A$, $b \in B$ implies $aa' \in T$ for some $a' \in A$.*

Proof. (1) ⇒ (2). Let $a \in A$. Suppose $Ba \cap T \neq \emptyset$. Then $a \in \bar{T}$. Hence $a \in \bar{T} \cap A = \overline{T \cap A}$. Thus $aA \cap T \neq \emptyset$.

(2) ⇒ (1). $\bar{T} \cap A$ is saturated since it is the contraction of the saturated set $\bar{T}$. Hence $T \cap A \subseteq \bar{T} \cap A$ implies $\overline{T \cap A} \subseteq \bar{T} \cap A$. Conversely, let $a \in \bar{T} \cap A$. Then $aB \cap T \neq \emptyset$. Hence $Aa \cap T \neq \emptyset$ by assumption, and therefore $a \in \overline{T \cap A}$. (2) ⇔ (3). Obvious.

As previously mentioned, we will be mainly interested in the case where $T \subseteq N_v$. The next two lemmas show that in this case T is both weakly saturated and locally weakly saturated in $R[X]$ with respect to R.

LEMMA 3.3. *Let T be a multiplicatively closed subset of $R[X]$ such that $T \subseteq N_v$ and $1 \in T$. Then T is weakly saturated in $R[X]$ with respect to R.*

Proof. Let $af \in T$ where $a \in R$ and $f \in R[X]$. Then $(a) \supseteq (aA_f)_v = R$ since $T \subseteq N_v$. So $(a) = R$, i.e., a is a unit. Thus $1 \in aR \cap T$. Hence by Lemma 3.2, T is weakly saturated in $R[X]$ with respect to R.

LEMMA 3.4. *Let T be a multiplicatively closed subset of $R[X]$ such that $1 \in T$ and $T \subseteq N_v$. Then T is locally weakly saturated in $R[X]$ with respect to R.*

Proof. Let P be a prime ideal of R. First note that $N_v(R) \subseteq N_v(R_P)$. Hence $T \subseteq N_v(R_P)$. By Lemma 3.3, T is weakly saturated in $R_P[X]$ with respect to R_P.

LEMMA 3.5. *Let Λ be a nonempty subset of* Spec(R) *such that $R = \bigcap_{P \in \Lambda} R_P$. Let $T \subseteq N_v$ be a multiplicatively closed subset of $R[X]$. Then $\bigcap_{P \in \Lambda} (R_P[X]_T) = R[X]_T$.*

Proof. Clearly $R[X]_T \subseteq \bigcap_{P \in \Lambda} (R_P[X]_T)$. Conversely, let $f/t \in \bigcap_{P \in \Lambda} (R_P[X]_T)$, where $f \in K[X]$ and $t \in T$. Then $f \in R_P[X]_T$ for each $P \in \Lambda$. So $fg = h$ for some $g \in T$ and $h \in R_P[X]$. Now $(A_f)_v = (A_{fg})_v = (A_h)_v$ by the Dedekind–Mertens Theorem since $(A_g)_v = R$. We will show that

$(A_h)_v \subseteq R_P$ which implies that $A_f \subseteq R_P$. There exists an $s \in R \backslash P$ such that $sh \in R[X]$. Then $s(A_h)_v = (sA_h)_v = (A_{sh})_v \subseteq R$ since $sh \in R[X]$. Hence $(A_h)_v \subseteq s^{-1}R \subseteq R_P$. Therefore $f \in A_f[X] \subseteq (\bigcap_{P \in \Lambda} R_P)[X] = R[X]$. Hence $f/t \in R[X]_T$. Therefore $\bigcap_{P \in \Lambda} (R_P[X]_T) = R[X]_T$.

The next two technical lemmas are used in proving Theorems 3.8 and 3.10 which are the main results of this section.

LEMMA 3.6. *Let R be a domain with quotient field K and let T be a weakly saturated subset of $R[X]$ with respect to R. Let $S = T \cap R$, $R\{X\} = R[X]_T$, and $f, g \in R[X]$. If R_S is a GCD domain, then*

(1) $fR\{X\} \cap gR[X] = f_1 R\{X\} \cap gR[X] = f_1 R_S[X] \cap gR[X]$ *(so that $fR\{X\} \cap gR_S[X] = f_1 R\{X\} \cap gR_S[X] = f_1 R_S[X] \cap gR_S[X]$) for any $f_1 \in R_S[X]$ which has the minimal degree with respect to $f = f_1 t$ for some $t \in \bar{T}$.*

(2) *Moreover, if* $\deg f = \deg f_1$, *then $fR\{X\} \cap gR_S[X] = fR_S[X] \cap gR_S[X]$.*

Proof. (1) If A is an R-module, then we denote $A[X]_T$ by $A\{X\}$. We may assume that T is saturated in $R[X]$ since $R_{\bar{T} \cap R} = R_{\overline{T \cap R}} = R_{\bar{S}} = R_S$. Hence assume that T is saturated in $R[X]$ and that R_S is a GCD domain. Let $B = \{f_1 \in R_S[X] \mid f = f_1 t \text{ for some } t \in T\}$. Now $B \neq \varnothing$ since $f \in B$. Let f_1 be an element of B which has the minimal degree among the elements of B. Then $fR\{X\} \cap gR[X] = f_1 R\{X\} \cap gR[X]$ since we have $f = f_1 t$ where $t \in T$ is a unit in $R\{X\}$. We will show that $f_1 R\{X\} \cap gR[X] = f_1 R_S[X] \cap gR[X]$. Let $f_1 g'/s' = h \in f_1 R\{X\} \cap gR[X]$ where $g' \in R[X]$, $s' \in T$, and $h \in gR[X]$. Let (g', s') be the GCD of g' and s' in $R_S[X]$. (Note that $R_S[X]$ is a GCD domain since R_S is a GCD domain). Now $g' = (g', s')g_1$ and $s' = (g', s')s_1$ for some $g_1, s_1 \in R_S[X]$. Also $s''(g', s')$, $s''g_1$, and $s''s_1 \in R[X]$ for some $s'' \in S$. And $(s''(g', s'))(s''s_1) = (s'')^2 s' \in S^2 T = T$. Hence $s''s_1 \in T$ since T is saturated in $R[X]$. So $g'/s' = g_1/s_1 = s''g_1/s''s_1$, $(s''g_1, s''s_1) = (g_1, s_1) = 1$ in $R_S[X]$, and $s''s_1 \in T$. Hence we may assume that $(g', s') = 1$ in $R_S[X]$. Now $f_1 g' = s'h \Rightarrow s' \mid f_1 g'$ in $R_S[X]$. Since $(s', g') = 1$ in $R_S[X]$, we have $s' \mid f_1$ in $R_S[X]$. Hence $\deg s' = 0$ by the minimality of $\deg f_1$. So $s' \in T \cap R = S$ and hence $h = f_1 g'/s' \in f_1 R_S[X] \cap gR[X]$. Thus $f_1 R\{X\} \cap gR[X] \subseteq f_1 R_S[X] \cap gR[X]$. Since the reverse containment is obvious, we have $f_1 R\{X\} \cap gR[X] = f_1 R_S[X] \cap gR[X]$. Hence $fR\{X\} \cap gR[X] = f_1 R\{X\} \cap gR[X] = f_1 R_S[X] \cap gR[X]$. Localizing the previous equalities by S, we get $fR\{X\} \cap gR_S[X] = f_1 R\{X\} \cap gR_S[X] = f_1 R_S[X] \cap gR_S[X]$.

(2) Moreover, if $\deg f = \deg f_1$, then $t \in \bar{T} \cap R = \bar{S}$ where $f = f_1 t$. Hence $f = f_1 t$, $t \in \bar{S}$. Then $fR_S[X] = fR_{\bar{S}}[X] = f_1 R_{\bar{S}}[X] = f_1 R_S[X]$ so that $fR\{X\} \cap gR_S[X] = fR_S[X] \cap gR_S[X]$.

LEMMA 3.7. *Let R be a domain with quotient field K. Let T be a locally weakly saturated subset of $R[X]$ with respect to R. Let $S=T\cap R$, $f,g\in K[X]\backslash\{0\}$ and $R\{X\}=R[X]_T$. If R_S is locally a* GCD *domain, then $fR\{X\}\cap gR_S[X]$ is a divisorial ideal of $R_S[X]$.*

Proof. We may assume that $f,g\in R[X]$ and $(f,g)=1$ in $K[X]$. Let $\mathrm{Max}(R_S)=\{M_{\alpha S}\}_{\alpha\in\Lambda}$ where each M_α is a prime ideal of R which is maximal with respect to $M_\alpha\cap S=\emptyset$. If A is an R_S-module, then $A_{M_{\alpha S}}=A_{M_\alpha}$ since $M_\alpha\cap S=\emptyset$. We will denote $A_{M_{\alpha S}}=A_{M_\alpha}$ by A_α. If A is an R-module, then we also denote A_{M_α} by A_α and $A[X]_T$ by $A\{X\}$. Hence $R_\alpha\{X\}=R_\alpha[X]_T=(R\{X\})_\alpha$.

Now $(fR\{X\}\cap gR_S[X])_\alpha = fR_\alpha\{X\}\cap g(R_S)_\alpha[X] = fR_\alpha\{X\}\cap gR_\alpha[X]$. Also, $R_\alpha\{X\}=R_\alpha[X]_T$ and $\bar{T}\cap R_\alpha=\overline{T\cap R_\alpha}=\bar{S}$ (since $T\cap R_\alpha\subseteq R[X]\cap K=R$, we have $T\cap R_\alpha=T\cap R=S$), so that $(R_\alpha)_{\bar{T}\cap R_\alpha}=(R_\alpha)_{\bar{S}}=(R_\alpha)_S=R_\alpha$ where $\bar{S}$ is the saturation of S in R_α. Hence by Lemma 3.6, there exist $f_\alpha\in R_\alpha[X]$ such that $fR_\alpha\{X\}\cap gR_\alpha[X]=f_\alpha R_\alpha\{X\}\cap gR_\alpha[X]=f_\alpha(R_\alpha)_S[X]\cap gR_\alpha[X]=f_\alpha R_\alpha[X]\cap gR_\alpha[X]$ and $f\in f_\alpha R_\alpha[X]$. Hence there exist $s_\alpha\in S_\alpha=R\backslash M_\alpha$ so that $f\in f_\alpha/s_\alpha R[X]$. Hence $fR\{X\}\subseteq f_\alpha/s_\alpha R\{X\}$. So $I\equiv fR\{X\}\cap gR_S[X]\subseteq\bigcap_\alpha(f_\alpha/s_\alpha R\{X\}\cap gR_S[X])\equiv J$. Now $J_\alpha\subseteq(f_\alpha/s_\alpha R_\alpha\{X\}\cap gR_S[X])_\alpha=f_\alpha/s_\alpha R_\alpha\{X\}\cap g(R_S)_\alpha[X]=f_\alpha/s_\alpha R_\alpha\{X\}\cap gR_\alpha[X] = f_\alpha R_\alpha\{X\}\cap gR_\alpha[X] = fR_\alpha\{X\}\cap gR_\alpha[X]=I_\alpha$. Thus $J_\alpha\subseteq I_\alpha$. Hence $J\subseteq I$ as R_S-modules. Hence $I=J$. So $fR\{X\}\cap gR_S[X]=\bigcap_\alpha(f_\alpha/s_\alpha R\{X\}\cap gR_S[X])$. We have the following claim.

Claim. $\deg f_\alpha=\deg f_\beta$ for every $\alpha,\beta\in\Lambda$.

Now $fR_\alpha\{X\}\cap gR_\alpha[X]=f_\alpha R_\alpha[X]\cap gR_\alpha[X]$. Localizing both sides by $R\backslash\{0\}$, we get $fK[X]_T\cap gK[X]=f_\alpha K[X]\cap gK[X]=f_\alpha gK[X]$ since $(f_\alpha,g)=1$ in $K[X]$ because $f_\alpha\mid f$ and $(f,g)=1$ in $K[X]$. So $f_\alpha gK[X]=f_\beta gK[X]$ and therefore $f_\alpha K[X]=f_\beta K[X]$.

Suppose that $\deg f=\deg f_\alpha$ for some α, then by the above claim, $\deg f=\deg f_\alpha$ for every $\alpha\in\Lambda$. Then by Lemma 3.6, $fR_\alpha\{X\}\cap gR_\alpha[X]=f(R_\alpha)_S[X]\cap g(R_\alpha)_S[X]=fR_\alpha[X]\cap gR_\alpha[X]$ for every $\alpha\in\Lambda$. Hence $(fR\{X\}\cap gR_S[X])_\alpha=(fR_S[X]\cap gR_S[X])_\alpha$ as R_S-modules for every $\alpha\in\Lambda$. Hence $fR\{X\}\cap gR_S[X]=fR_S[X]\cap gR_S[X]$, which is a divisorial ideal of $R_S[X]$. Thus we may assume that $\deg f>\deg f_\alpha$ for every $\alpha\in\Lambda$. Then by induction on $\deg f$, $fR\{X\}\cap gR_S[X]=\bigcap(f_\alpha/s_\alpha R\{X\}\cap gR_S[X])$ is a divisorial deal of $R_S[X]$ since it is an intersection of divisorial ideals of $R_S[X]$. (Note that if $\deg f=0$, then $\deg f=\deg f_\alpha$ for every α, hence this case has already been handled.)

THEOREM 3.8. *Let T be a locally weakly saturated subset of $R[X]$ with respect to R. Let $S=T\cap R$. Suppose that R_S is locally a GCD domain. If J is a divisorial ideal of $R[X]_T$, then $J\cap R_S[X]$ is a divisorial ideal of $R_S[X]$.*

Proof. We can assume that J is an integral divisorial ideal of $R[X]_T$. Then $J = \bigcap_{f,g} (f/gR[X]_T)$ where $f/gR[X]_T \supseteq J$ and $f, g \in R[X]$. Now $J \cap R_S[X] = \bigcap (f/gR[X]_T \cap R_S[X]) = \bigcap 1/g(fR[X]_T \cap gR_S[X])$ is a divisorial ideal of $R_S[X]$ since it is an intersection of divisorial ideals by Lemma 3.7.

Recall that an integral domain R is called an essential domain if $R = \bigcap_{P \in \Lambda} R_P$ for some collection Λ of prime ideals and each R_P is a valuation domain.

LEMMA 3.9. *Let R be an essential domain and let $T \subseteq N_v$ be a multiplicatively closed subset of $R[X]$. Let $f, g \in K[X] \backslash \{0\}$. Then $fR[X]_T \cap gR[X]$ is a divisorial ideal of $R[X]$.*

Proof. Let $R = \bigcap_{P_\alpha \in \Lambda} R_{P_\alpha}$, $\Lambda \subseteq \mathrm{Spec}(R)$, where each $R_{P_\alpha} = R_\alpha$ is a valuation domain. For an R-module A, $A\{X\}$ is defined to be $A[X]_T$. Now $T \cap R \subseteq N_v \cap R =$ the units of R and T is a locally weakly saturated multiplicatively closed subset of $R[X]$ with respect to R by Lemma 3.4. (We can assume that $1 \in T$.) Hence by Lemma 3.6 applied to R_α, there exist $f_\alpha \in R_\alpha[X]$ such that $fR_\alpha\{X\} \cap gR_\alpha[X] = f_\alpha R_\alpha\{X\} \cap gR_\alpha[X] = f_\alpha R_\alpha[X] \cap gR_\alpha[X]$, where $f \in f_\alpha R_\alpha[X]$. Now there exist $s_\alpha \in R \backslash P_\alpha$ such that $f \in f_\alpha/s_\alpha R[X]$. Then $fR\{X\} \subseteq f_\alpha/s_\alpha R\{X\}$. Hence $fR\{X\} \cap gR[X] \subseteq \bigcap_\alpha (f_\alpha/s_\alpha R\{X\} \cap gR[X]) \subseteq \bigcap_\alpha (f_\alpha/s_\alpha R_\alpha\{X\} \cap gR_\alpha[X]) \subseteq \bigcap_\alpha (f_\alpha R_\alpha\{X\} \cap gR_\alpha[X]) \subseteq \bigcap_\alpha (fR_\alpha\{X\} \cap gR_\alpha[X]) = f(\bigcap_\alpha R_\alpha\{X\}) \cap g(\bigcap_\alpha R_\alpha[X]) = fR\{X\} \cap gR[X]$ where the last equality follows from Lemma 3.5. Thus $fR\{X\} \cap gR[X] = \bigcap_\alpha (f_\alpha/s_\alpha R\{X\} \cap gR[X])$. Then as in the proof of Lemma 3.7, we conclude that $fR\{X\} \cap gR[X]$ is a divisorial ideal of $R[X]$ by induction on the degree of f under the assumption that $(f, g) = 1$ in $K[X]$ and $f, g \in R[X]$.

THEOREM 3.10. *Let R be an essential domain and let $T \subseteq N_v$ be a multiplicatively closed subset of $R[X]$. If J is a divisorial ideal of $R[X]_T$, then $J \cap R[X]$ is a divisorial ideal of $R[X]$.*

Proof. Similar to the proof of Theorem 3.8.

COROLLARY 3.11. *Suppose that R is an essential domain. Let $R(X) = R[X]_N$ where $N = \{f \in R[X] \mid A_f = R\}$, $R\langle X\rangle = R[X]_T$ where $T = \{f \in R[X] \mid$ the leading coefficient of f is a unit$\}$, and $R\{X\} = R[X]_{N_v}$. If J is a divisorial ideal of $R(X)$, $R\langle X\rangle$, or $R\{X\}$, then $J \cap R[X]$ is a divisorial ideal of $R[X]$.*

Proof. This follows from Theorem 3.10.

IV. Extensions of Pseudo-Dedekind Domains

The purpose of this section is to show that the domains R, $R[X]$, and $R(X)$ are simultaneously pseudo-Dedekind domains. We begin with some slightly more general results.

THEOREM 4.1. *If R is pseudo-Dedekind, then $R[X]_T$ is a pseudo-Dedekind domain for any multiplicatively closed subset $T \subseteq N_v$.*

Proof. Let J be an integral divisorial ideal of $R[X]_T$. Then by Theorem 3.10, $J \cap R[X]$ is a divisorial ideal of $R[X]$. Since R is integrally closed, $J \cap R[X] = fI[X]$ for some $0 \neq f \in K[X]$ and a divisorial ideal I of R [11, Lemma 3.2.]. Since R is pseudo-Dedekind, I is invertible and hence $J \cap R[X] = fI[X]$ is invertible. Therefore $J = (J \cap R[X])_T$ is invertible.

THEOREM 4.2. *Let T be a multiplicatively closed subset of $R[X]$ such that $T \subseteq N = \{f \in R[X] \mid A_f = R\}$. Then R is pseudo-Dedekind if and only if $R[X]_T$ is pseudo-Dedekind.*

Proof. ($\Rightarrow$). This follows from Theorem 4.1.

($\Leftarrow$). Suppose that $R[X]_T$ is a pseudo-Dedekind domain. Let I be a divisorial ideal of R. By Proposition 3.1, $I[X]_T$ is a divisorial ideal of $R[X]_T$ and so $I[X]_T(I[X]_T)^{-1} = R[X]_T$. Also by Proposition 3.1, $(I[X]_T)^{-1} = I^{-1}[X]_T$. Hence $II^{-1}[X]_T = I[X]_T I^{-1}[X]_T = I[X]_T(I[X]_T)^{-1} = R[X]_T$. Since $T \subseteq N$, $II^{-1} = II^{-1}[X]_T \cap R = R[X]_T \cap R = R$. Hence I is invertible and therefore R is a pseudo-Dedekind domain.

COROLLARY 4.3. *Let R be an integral domain. Then the following statements are equivalent.*

(1) *R is pseudo-Dedekind.*

(2) *$R[X]$ is pseudo-Dedekind.*

(3) *$R(X)$ is pseudo-principal.*

Proof. Since $\operatorname{Pic}(R(X)) = 0$ [1, Theorem 2], (3) is equivalent to $R(X)$ being pseudo-Dedekind. Then by Theorem 4.2, (1), (2), and (3) are equivalent.

It is not however the case that R pseudo-Dedekind implies that $R[[X]]$ is pseudo-Dedekind. Samuel has given an example of a Noetherian UFD for which $R[[X]]$ is *not* a UFD. Now R being a UFD is certainly pseudo-Dedekind. Now $R[[X]]$ is also still a Krull domain. But if $R[[X]]$ were pseudo-Dedekind, it would have to be locally factorial and hence even factorial since $\operatorname{Pic}(R[[X]]) = \operatorname{Pic}(R) = 0$. Also, the localization of a pseudo-Dedekind need not be a pseudo-Dedekind domain: The ring E of entire functions is pseudo-principal, but $\dim E > 1$ [9, pp. 147–148]. Let Q be a prime ideal of E with $\operatorname{ht} Q > 1$. Then E_Q is a valuation domain with $\dim E_Q > 1$, so E_Q is not completely integrally closed. Hence E_Q is not pseudo-Dedekind by Corollary 2.5.

A domain R is defined to be locally pseudo-Dedekind if R_P is pseudo-Dedekind for every prime ideal P of R. Although R is pseudo-Dedekind does not necessarily imply that R_P is pseudo-Dedekind, we can extend this local property to $R[X]$ and $R(X)$.

THEOREM 4.4. *The following are equivalent for a domain R.*

(1) *R is locally pseudo-Dedekind.*

(2) *$R[X]$ is locally pseudo-Dedekind.*

(3) *$R(X)$ is locally pseudo-Dedekind.*

Proof. (1) $\Rightarrow$ (2). Suppose that R is locally pseudo-Dedekind. Let Q be a prime ideal of $R[X]$ and let $P = Q \cap R$. We will show that $T \subseteq N(R_P) = \{f \in R_P[X] \mid A_f = R_P\}$ where $T = R[X] \backslash Q$. Let $f \in T$. Then $f \notin P[X]$ and hence $A_f \nsubseteq P$. So $(A_f)_P = R_P$ and therefore $A_f = R_P$ if we consider f as a polynomial over R_P. Thus $T \subseteq N(R_P)$. Clearly $T \cap R = (R[X] \backslash Q) \cap R = R \backslash P$. Hence $R[X]_T = (R[X]_P)_T = (R_P[X])_T$ so $R[X]_Q = R_P[X]_T$ is a pseudo-Dedekind domain by Theorem 4.1. (2) $\Rightarrow$ (3). Suppose that $R[X]$ is locally pseudo-Dedekind and let Q_N be a prime ideal of $R(X) = R[X]_N$ where $N = \{f \in R[X] \mid A_f = R\}$ and Q is a prime ideal of $R[X]$ such that $Q \cap N = \varnothing$. Now $R(X)_{Q_N} = (R[X]_N)_{Q_N} = R[X]_Q$ is pseudo-Dedekind by assumption. (3) $\Rightarrow$ (1). Let P be a prime ideal of R. Clearly $P[X] \cap N = \varnothing$. Now $R(X)_{P[X]_N} = (R[X]_N)_{P[X]_N} = R[X]_{P[X]} = R_P(X)$. Hence $R_P(X) = R(X)_{P[X]_N}$ is pseudo-Dedekind by assumption. Hence R_P is a pseudo-Dedekind domain by Corollary 4.3.

In [10], Matsuda proved that the group ring $R[X; G]$ is a pseudo-principal domain if and only if R is a pseudo-principal domain and G has type $(0, 0, \ldots)$. We extend this result to pseudo-Dedekind domains.

THEOREM 4.5. *$R[X; G]$ is a pseudo-Dedekind domain if and only if R is a pseudo-Dedekind domain and G has type* $(0, 0, \ldots)$.

Proof. Suppose R is a pseudo-Dedekind domain and G has type $(0, 0, \ldots)$. Let $N' = \{f \in R[Y] \mid A_f = R\}$ and $N = \{f \in D[Y] \mid A_f = D\}$ where $D = R[X; G]$. Clearly $N' \subseteq N$. Now $D[Y]_{N'} = R[X; G][Y]_{N'} = R[Y][X; G]_{N'} = R[Y]_{N'}[X; G] = R(Y)[X; G]$ is a pseudo-principal domain. Since $N' \subseteq N$, D is a pseudo-Dedekind domain by Theorem 4.2.

Conversely, suppose that $D = R[X; G]$ is pseudo-Dedekind. Then $D[Y]_{N'} = R(Y)[X; G]$ is pseudo-Dedekind by Theorem 4.2. Since

$R(Y)[X; G]$ is integrally closed, $\mathrm{Pic}(R(Y)[X; G]) = \mathrm{Pic}(R(Y)) = 0$ where the first equality follows from [4, Corollary 5.6] and the second equality follows from [1, Theorem 2]. Hence $R(Y)[X; G]$ is pseudo-principal. By [10, Theorem 8], $R(Y)$ is pseudo-principal and G has type $(0, 0, \ldots)$. By Theorem 4.2, R is pseudo-Dedekind.

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