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Weyl submodules in restrictions of simple modules

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ABSTRACT

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 0$. Suppose that $\mathrm{SL}_{n-1}(\mathbb{F})$ is naturally embedded into $\mathrm{SL}_n(\mathbb{F})$ (either in the top left corner or in the bottom right corner). We prove that certain Weyl modules over $\mathrm{SL}_{n-1}(\mathbb{F})$ can be embedded into the restriction $L(\omega){\downarrow}_{\mathrm{SL}_{n-1}(\mathbb{F})}$, where $L(\omega)$ is a simple $\mathrm{SL}_n(\mathbb{F})$-module. This allows us to construct new primitive vectors in $L(\omega){\downarrow}_{\mathrm{SL}_{n-1}(\mathbb{F})}$ from any primitive vectors in the corresponding Weyl modules. Some examples are given to show that this result actually works.

1. Introduction

Let $G = \mathrm{SL}_n(\mathbb{F})$, where $\mathbb{F}$ is an algebraically closed field of characteristic $p > 0$ and $n \geqslant 3$. Consider the subgroup $G^{(q)}$ of G generated by the root elements $x_\alpha(t)$, $x_{-\alpha}(t)$, where α is a simple root distinct from a fixed terminal (simple) root α_q. It is a classical problem to describe the structure of the restriction $L{\downarrow}_{G^{(q)}}$, where L is a simple rational G-module.

In this paper, we focus on primitive (with respect to $G^{(q)}$) vectors of $L{\downarrow}_{G^{(q)}}$. The complete combinatorial description of these vectors is an open problem (stated in [BK1]), although lately there has been some progress in this direction [K,BKS,Sh2].

Another problem of equal importance is the description of primitive vectors in Weyl modules. Known methods of constructing such vectors [CL,CP] and methods of constructing primitive vectors in restrictions $L{\downarrow}_{G^{(q)}}$ [K,BKS,Sh1,Sh2] bear some similarity (e.g. similar lowering operators), which is still not fully understood.

The present paper contains a combinatorial condition under which all primitive vectors (regardless of their nature) of certain Weyl modules over $G^{(q)}$ become primitive vectors of $L{\downarrow}_{G^{(q)}}$. This result is proved by embedding the corresponding Weyl modules into $L{\downarrow}_{G^{(q)}}$ (Theorem A). Examples I and II show that our result actually works, that is, produces nonzero primitive vectors of $L{\downarrow}_{G^{(q)}}$.

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We also hope that Theorem A will be useful for finding new composition factors of $L{\downarrow}_{G^{(q)}}$ and lower estimates of the dimensions of the weight spaces of L.

We order the simple roots $\alpha_1,\dots,\alpha_{n-1}$ so that $x_{\alpha_i}(t)=E+te_{i,i+1}$. Then $x_{\alpha_i+\dots+\alpha_{j-1}}(t)=E+te_{i,j}$ and $x_{-\alpha_i-\dots-\alpha_{j-1}}(t)=E+te_{j,i}$, where $1\leqslant i<j\leqslant n$. Here and in what follows E is the identity $n\times n$ matrix and $e_{i,j}$ is the $n\times n$ matrix having 1 in the ijth position and 0 elsewhere. The root system Φ of G consists of the roots $\pm(\alpha_i+\dots+\alpha_{j-1})$ and the positive root system Φ^+ consists of the roots $\alpha_i+\dots+\alpha_{j-1}$, where $1\leqslant i<j\leqslant n$. Let $\omega_1,\dots,\omega_{n-1}$ denote the fundamental weights corresponding to the roots $\alpha_1,\dots,\alpha_{n-1}$.

In G, we fix the maximal torus T consisting of diagonal matrices and the Borel subgroup B consisting of upper triangular matrices.

The *hyperalgebra* of G is constructed as follows. Consider the following elements of $\mathfrak{sl}_n(\mathbb{C})$: $X_{\alpha_i+\dots+\alpha_{j-1}}=e_{i,j}$, $X_{-\alpha_i-\dots-\alpha_{j-1}}=e_{j,i}$, where $1\leqslant i<j\leqslant n$, and $H_{\alpha_i}=e_{i,i}-e_{i+1,i+1}$, where $1\leqslant i<n$. Following [St, Theorem 2], we denote by $\mathcal{U}_{\mathbb{Z}}$ the subring of the universal enveloping algebra of $\mathfrak{sl}_n(\mathbb{C})$ generated by divided powers $X_\alpha^m/m!$, where $\alpha\in\Phi$ and $m\in\mathbb{Z}^+$ (the set of nonnegative integers). The hyperalgebra of G is the tensor product $\mathcal{U}:=\mathcal{U}_{\mathbb{Z}}\otimes_{\mathbb{Z}}\mathbb{F}$. Elements $X_{\alpha,m}:=(X_\alpha^m/m!)\otimes 1_{\mathbb{F}}$ generate $\mathcal{U}$ as an $\mathbb{F}$-algebra.

Every rational G-module V can be made into a $\mathcal{U}$-module by the rule

$$x_\alpha(t)v=\sum_{m=0}^{+\infty}t^mX_{\alpha,m}v. \tag{1}$$

We also need the elements $H_{\alpha_i,m}=\binom{H_{\alpha_i}}{m}\otimes 1_{\mathbb{F}}$. It is easy to show that these elements actually belong to $\mathcal{U}$ (e.g., [St, Corollary to Lemma 5]). We shall often abbreviate $X_\alpha:=X_{\alpha,1}$ and $H_{\alpha_i}:=H_{\alpha_i,1}$ if this notation does not cause confusion.

For any integers $q_1,\dots,q_m\in\{1,\dots,n-1\}$, we denote by $G^{(q_1,\dots,q_m)}$ the subgroup of G generated by the root elements $x_{\alpha_i}(t)$, $x_{-\alpha_i}(t)$ with $i\in\{1,\dots,n-1\}\setminus\{q_1,\dots,q_m\}$. Note that $G^{(q_1,\dots,q_m)}$ is the universal Chevalley group with root system $\Phi\cap\sum_{i\in\{1,\dots,n-1\}\setminus\{q_1,\dots,q_m\}}\mathbb{Z}\alpha_i$ [H, Theorem 27.3].

In $G^{(q_1,\dots,q_m)}$, we fix the maximal torus $T^{(q_1,\dots,q_m)}$ generated by the elements $h_{\alpha_i}(t)=\mathrm{diag}(1,\dots,1,t,t^{-1},1,\dots,1)$, where $t\in\mathbb{F}^*$ is at the ith position and $i\in\{1,\dots,n-1\}\setminus\{q_1,\dots,q_m\}$, and the Borel subgroup generated by $T^{(q_1,\dots,q_m)}$ and the root elements $x_\alpha(t)$ with $\alpha\in\Phi^{(q_1,\dots,q_m)}\cap\Phi^+$.

We denote by $X(T)$ the set of T-weights and by $X^+(T)$ the set of dominant T-weights. For any $\omega\in X^+(T)$, we denote by $L(\omega)$ and $\Delta(\omega)$ the simple rational G-module with highest weight ω and the Weyl G-module with highest weight ω respectively. We fix nonzero vectors v_ω^+ and e_ω^+ of $L(\omega)$ and $\Delta(\omega)$ respectively having weight ω. Similar notations will be used for subtori $T^{(q_1,\dots,q_m)}$. We shall often omit the prefix before the word "weight" if it is clear which torus we mean.

The terminal roots of Φ are α_1 and α_{n-1}. Thus $q=1$ or $q=n-1$. For any weight $\varkappa\in X(T)$, we denote by $\bar\varkappa$ and $\bar{\bar\varkappa}$ the restrictions of $\varkappa$ to $T^{(1)}$ and $T^{(n-1)}$ respectively. The main results of the present paper are as follows.

Theorem A. *Let $G=\mathrm{SL}_n(\mathbb{F})$, $\omega\in X^+(T)$ and $k=0,\dots,p-1$.*

(i) *If $\langle\omega,\alpha_1\rangle-l\not\equiv 0 \pmod p$ for any $l=0,\dots,k-1$ and there is $m=0,\dots,k$ such that $\Delta(\bar\omega+m\bar\omega_2)$ is simple, then the $G^{(1)}$-submodule of $L(\omega)$ generated by $X_{-\alpha_1,k}v_\omega^+$ is isomorphic to $\Delta(\bar\omega+k\bar\omega_2)$.*

(ii) *If $\langle\omega,\alpha_{n-1}\rangle-l\not\equiv 0 \pmod p$ for any $l=0,\dots,k-1$ and there is $m=0,\dots,k$ such that $\Delta(\bar{\bar\omega}+m\bar{\bar\omega}_{n-2})$ is simple, then the $G^{(n-1)}$-submodule of $L(\omega)$ generated by $X_{-\alpha_{n-1},k}v_\omega^+$ is isomorphic to $\Delta(\bar{\bar\omega}+k\bar{\bar\omega}_{n-2})$.*

Theorem B. *Let $G=\mathrm{SL}_n(\mathbb{F})$, $\omega\in X^+(T)$, $k=0,\dots,p-1$ and $q=1$ or $q=n-1$. The $G^{(q)}$-submodule of $L(\omega)$ generated by $X_{-\alpha_q,k}v_\omega^+$ is isomorphic to a Weyl module if and only if $\langle\omega,\alpha_q\rangle-l\not\equiv 0 \pmod p$ for any $l=0,\dots,k-1$ and*

(i) *any nonzero primitive vector of $\Delta(\bar\omega+k\bar\omega_2)$ has weight $\bar\omega+k\bar\omega_2-b_2\bar\alpha_2-\dots-b_{n-1}\bar\alpha_{n-1}$ with $k\geqslant b_2\geqslant\dots\geqslant b_{n-1}\geqslant 0$ in the case $q=1$;*

(ii) *any nonzero primitive vector of* $\Delta(\bar{\bar{\omega}} + k\bar{\bar{\omega}}_{n-2})$ *has weight* $\bar{\bar{\omega}} + k\bar{\bar{\omega}}_{n-2} - b_1\bar{\bar{\alpha}}_1 - \cdots - b_{n-2}\bar{\bar{\alpha}}_{n-2}$ *with* $0 \leqslant b_1 \leqslant \cdots \leqslant b_{n-2} \leqslant k$ *in the case* $q = n-1$.

More precisely, we have $KG^{(1)}X_{-\alpha_1,k}v_\omega^+ \cong \Delta(\bar{\omega} + k\bar{\omega}_2)$ *in case* (i) *and* $KG^{(n-1)}X_{-\alpha_{n-1},k}v_\omega^+ \cong \Delta(\bar{\bar{\omega}} + k\bar{\bar{\omega}}_{n-2})$ *in case* (ii).

Theorem A can be viewed as a special case of the following more general problem (valid for an arbitrary semisimple group G) stated by Irina Suprunenko:

Problem 1. Let α_q be a terminal root of the Dynkin diagram of Φ and $k = 0, \dots, p-1$. Describe the weights $\omega \in X^+(T)$ such that the $G^{(q)}$-submodule of the simple module $L(\omega)$ generated by $X_{-\alpha_q,k}v_\omega^+$ is isomorphic to a Weyl module.

Theorem B solves this problem for $G = \mathrm{SL}_n(\mathbb{F})$ in terms of the Hom-spaces between Weyl modules and is a more refined version of Theorem A giving a necessary and sufficient condition for $X_{-\alpha_q,k}v_\omega^+$ to generate a Weyl module.

Theorem A can easily be used in practice by virtue of the following irreducibility criterion of Weyl modules over groups of type A_{n-1} proved by J.C. Jantzen.

Proposition 2. *(See [J, II.8.21].) The Weyl module* $\Delta(\omega)$ *is simple if and only if for each* $\alpha \in \Phi^+$ *the following is satisfied: Write* $\langle \omega + \rho, \alpha \rangle = ap^s + bp^{s+1}$, *where* $a, b, s \in \mathbb{Z}^+$, $0 < a < p$ *and* ρ *is half the sum of the positive roots of* Φ. *Then there have to be* $\beta_0, \beta_1, \dots, \beta_b \in \Phi^+$ *with* $\langle \omega + \rho, \beta_i \rangle = p^{s+1}$ *for* $1 \leqslant i \leqslant b$ *and* $\langle \omega + \rho, \beta_0 \rangle = ap^s$, *with* $\alpha = \sum_{i=0}^{b} \beta_i$ *and with* $\alpha - \beta_0 \in \Phi \cup \{0\}$.

Example I. Let $G = \mathrm{SL}_3(\mathbb{F})$ and $\omega = a_1\omega_1 + a_2\omega_2$ be a dominant weight such that $a_1, a_2 < p$ and $a_1 + a_2 \geqslant p + b$, where $b = 0, \dots, p-2$. We put $k := p + b - a_2$. Note that for any $l = 0, \dots, k-1$, we have $0 < a_1 - l < p$ and thus $\langle \omega, \alpha_1 \rangle - l \not\equiv 0 \pmod p$. Notice also that $0 < k < p$. Indeed, $k \geqslant p$ implies $b \geqslant a_2$ and $a_1 + a_2 \geqslant p + b \geqslant p + a_2$. Hence $a_1 \geqslant p$, which is a contradiction. Since the Weyl module $\Delta(\bar{\omega}) = \Delta(a_2\bar{\omega}_2)$ is simple, Theorem A(i) (where $m = 0$) shows that the $G^{(1)}$-submodule of $L(\omega)$ generated by $X_{-\alpha_1,k}v_\omega^+$ is isomorphic to $\Delta((p+b)\bar{\omega}_2)$. The latter module is already not simple. For example, $X_{-\alpha_2,b+1}e^+_{(p+b)\bar{\omega}_2}$ is a nonzero $G^{(1)}$-primitive vector. Thus $X_{-\alpha_2,b+1}X_{-\alpha_1,k}v_\omega^+$ is a nonzero $G^{(1)}$-primitive vector of $L(\omega)$ of weight $\omega - (p + b - a_2)\alpha_1 - (b+1)\alpha_2$.

There is an interesting connection between this example and [Su, Lemma 2.55], which is extensively used in that paper for calculation of degrees of minimal polynomials. In our notation, [Su, Lemma 2.55] is as follows:

Let M *be an indecomposable* $G^{(1)}$*-module with highest weight* $(p+b)\bar{\omega}_2$ *and* $0 \leqslant b < p-1$. *Suppose that* $X_{-\alpha_2,b+1}v^+ \neq 0$, *where* v^+ *is a highest weight vector of* M. *Then* $M \cong \Delta((p+b)\bar{\omega}_2)$.

Therefore, if we somehow prove that $X_{-\alpha_2,b+1}X_{-\alpha_1,k}v_\omega^+ \neq 0$, then it will follow from this lemma that the $G^{(1)}$-submodule of $L(\omega)$ generated by $X_{-\alpha_1,k}v_\omega^+$ is a Weyl module (without applying Theorem A).

Example II. Let $p = 5$, $G = \mathrm{SL}_5(\mathbb{F})$ and $\omega = 3\omega_1 + 3\omega_2 + \omega_3 + 2\omega_4$. Take any $k = 1, \dots, 4$ and apply Theorem A(i) for this k. The value $k = 4$ does not fit, since $\langle \omega, \alpha_1 \rangle - 3 = 0$.

If we apply Theorem A(i) for $k = 1$, then we obtain that $L(\omega)$ contains a $G^{(1)}$-submodule isomorphic to $\Delta(\bar{\omega} + \bar{\omega}_2)$. However, the last module is simple and we do not get any nonzero $G^{(1)}$-primitive vectors in this way except the trivial $X_{-\alpha_1}v_\omega^+$.

The cases $k = 2$ and $k = 3$ on the contrary give new vectors. In the former case, Theorem A(i) implies that $L(\omega)$ contains a $G^{(1)}$-submodule isomorphic to $\Delta(\bar{\omega} + 2\bar{\omega}_2)$. The last module contains nonzero primitive vectors of weights $\bar{\omega} + 2\bar{\omega}_2 - \bar{\alpha}_2$ and $\bar{\omega} + 2\bar{\omega}_2 - \bar{\alpha}_2 - \bar{\alpha}_3 - \bar{\alpha}_4$ by the Carter–Payne theorem [CP]. In the latter case, Theorem A(i) implies that $L(\omega)$ contains a $G^{(1)}$-submodule isomorphic to $\Delta(\bar{\omega}+3\bar{\omega}_2)$. The last module contains nonzero primitive vectors of weights $\bar{\omega}+3\bar{\omega}_2-2\bar{\alpha}_2$ and $\bar{\omega}+3\bar{\omega}_2-2\bar{\alpha}_2-2\bar{\alpha}_3-2\bar{\alpha}_4$ by the Carter–Payne theorem [CP].

Thus except trivial nonzero $G^{(1)}$-primitive vectors of weights $\omega-i\alpha_1$ with $i=0,\dots,3$, the module $L(\omega)$ (which is not a Weyl module) also contains nonzero $G^{(1)}$-primitive vectors of weights $\omega-2\alpha_1-\alpha_2$, $\omega-2\alpha_1-\alpha_2-\alpha_3-\alpha_4$, $\omega-3\alpha_1-2\alpha_2$ and $\omega-3\alpha_1-2\alpha_2-2\alpha_3-2\alpha_4$.

Computer calculations show that examples similar to Example II are quite abundant. Note that in both Examples I and II, we apply $X_{-\alpha_1,k}$ to v^+_ω only for $k>0$. The reason is that the case $k=0$ corresponds to Smith's theorem [Sm] and the only primitive vectors of $L{\downarrow}_{G^{(q)}}$ produced in this way are those proportional to v^+_ω.

We shall use the following result following directly from [St, Theorem 2].

Proposition 3. *The products $\prod_{\alpha\in\Phi^+}X_{-\alpha,m_{-\alpha}}\cdot\prod_{i=1}^{n-1}H_{\alpha_i,n_i}\cdot\prod_{\alpha\in\Phi^+}X_{\alpha,m_\alpha}$, where $m_{-\alpha},n_i,m_\alpha\in\mathbb{Z}^+$, taken in any fixed order form a basis of $\mathcal{U}$.*

We denote by $\mathcal{U}^+$ the subspace of $\mathcal{U}$ spanned by the above products with unitary first and second factors. Given integers $q_1,\dots,q_m\in\{1,\dots,n-1\}$, we denote by $\mathcal{U}^{(q_1,\dots,q_m)}$ the subspace of $\mathcal{U}$ spanned by all the above products such that $m_\alpha=0$ unless $\alpha\in\Phi^{(q_1,\dots,q_m)}$ and $n_i=0$ unless $i\in\{1,\dots,n-1\}\setminus\{q_1,\dots,q_m\}$. One can easily see that $\mathcal{U}^+$ and $\mathcal{U}^{(q_1,\dots,q_m)}$ are subalgebras of $\mathcal{U}$. We let $\mathcal{U}^{(q_1,\dots,q_m)}$ act on any rational $G^{(q_1,\dots,q_m)}$-module according to (1). In the sequel, we shall mean the $X(T)$-grading of $\mathcal{U}$ in which $X_{\alpha,m}$ has weight $m\alpha$ and $H_{\alpha_i,m}$ has weight 0.

For each $\omega\in X^+(T)$, we denote by $\nabla(\omega)$ the module contravariantly dual to the Weyl module $\Delta(\omega)$ and denote by $\pi^\omega:\Delta(\omega)\to L(\omega)$ the G-module epimorphism such that $\pi^\omega(e^+_\omega)=v^+_\omega$. We also denote by V^τ for $\tau\in X(T)$ the τ-weight space of a rational T-module V.

A vector v of a rational G-module is called G-*primitive* if $\mathbb{F}v$ is fixed by the Borel subgroup B. We use similar terminology for $G^{(q_1,\dots,q_m)}$ and omit the prefix when it is clear which group we mean. In view of the universal property of Weyl modules [J, Lemma II.2.13 b], we can speak about primitive vectors of a rational module V instead of homomorphisms from Weyl modules to V (we use this language in Theorem B).

Note that Theorems A and B in the case $q=n-1$ are easy consequences of the theorems in the case $q=1$ by a standard argument involving twisting with the automorphism $g\mapsto w_0(g^{-1})^{\mathrm{t}}w_0^{-1}$, where $^{\mathrm{t}}$ stands for the transposition and w_0 stands for the longest element of the Weyl group. Therefore in the remainder of the article we consider only the case $q=1$.

2. Proof of the main results

We fix a weight $\omega=a_1\omega_1+\cdots+a_{n-1}\omega_{n-1}$ of $X^+(T)$ and an integer $k\in\mathbb{Z}^+$. The restriction of ω to $T^{(1)}$ is $\bar{\omega}=a_2\bar{\omega}_2+\cdots+a_{n-1}\bar{\omega}_{n-1}$. Clearly, $X_{-\alpha_1,k}v^+_\omega$ is a (possibly zero) $G^{(1)}$-primitive vector of $T^{(1)}$-weight $\bar{\omega}+k\bar{\omega}_2$. By the universal property of Weyl modules [J, Lemma II.2.13 b], there exists the homomorphism $\varphi^\omega_k:\Delta(\bar{\omega}+k\bar{\omega}_2)\to L(\omega)$ of $G^{(1)}$-modules that takes $e^+_{\bar{\omega}+k\bar{\omega}_2}$ to $X_{-\alpha_1,k}v^+_\omega$. Obviously,

$$KG^{(1)}X_{-\alpha_1,k}v^+_\omega\cong\Delta(\bar{\omega}+k\bar{\omega}_2)/\ker\varphi^\omega_k. \tag{2}$$

Problem 1 can now be reformulated as follows: *Describe the weights $\omega\in X^+(T)$ such that* $\ker\varphi^\omega_k=0$. The analog of this problem for $\Delta(\omega)$ has a trivial solution.

Lemma 4. *The $G^{(1)}$-submodule of $\Delta(\omega)$ generated by $X_{-\alpha_1,k}e^+_\omega$ is isomorphic to $\Delta(\bar{\omega}+k\bar{\omega}_2)$ if $0\leqslant k\leqslant\langle\omega,\alpha_1\rangle$ and is zero otherwise.*

Proof. Suppose temporarily that $\operatorname{char}\mathbb{F}=0$. Then $\Delta(\omega)$ is irreducible. Since $X_{\alpha_1,k}X_{-\alpha_1,k}e^+_\omega=\binom{a_1}{k}e^+_\omega$, we have (recall that α_1 is simple)

$$\dim\Delta(\omega)^{\omega-k\alpha_1}=\begin{cases}1 & \text{if } 0\leqslant k\leqslant a_1;\\ 0 & \text{otherwise.}\end{cases} \tag{3}$$

Now let us return to the situation where $\operatorname{char}\mathbb{F} = p > 0$. Since the character of a Weyl module does not depend on $\operatorname{char}\mathbb{F}$, (3) holds again. Therefore, $X_{-\alpha_1,k}e^+_\omega = 0$ if $k > a_1$. Thus we assume $0 \leqslant k \leqslant a_1$ for the rest of the proof. Consider the decomposition $\Delta(\omega) = \bigoplus_{b\in\mathbb{Z}^+} V^{(b)}$, where

$$V^{(b)} = \bigoplus_{b_2,\dots,b_{n-1}\in\mathbb{Z}^+} \Delta(\omega)^{\omega-b\alpha_1-b_2\alpha_2-\cdots-b_{n-1}\alpha_{n-1}}$$

(the bth level of $\Delta(\omega)$). Note that each $V^{(b)}$ is a $G^{(1)}$-module. By (3), $X_{-\alpha_1,k}e^+_\omega$ is a nonzero vector of $V^{(k)}$ having $T^{(1)}$-weight $\bar{\omega} + k\bar{\omega}_2$. Moreover, the weight space of $V^{(k)}$ corresponding to this weight is one-dimensional. Any other $T^{(1)}$-weight of $V^{(k)}$ is less than this weight. It follows from [M] (see also [J, Proposition II.4.24]) that $\Delta(\omega){\downarrow_{G^{(1)}}}$ has a Weyl filtration. By [J, Proposition II.4.16(iii)], its direct summand $V^{(k)}$ also has a Weyl filtration (as a $G^{(1)}$-module). Any such filtration contains one factor isomorphic to $\Delta(\bar{\omega} + k\bar{\omega}_2)$ and, possibly, some other factors each isomorphic to $\Delta(\tau)$ with $\tau < \bar{\omega} + k\bar{\omega}_2$. Applying [J, II.4.16, Remark 4] to the dual module $V^{(k)*}$, we obtain that $V^{(k)}$ contains a $G^{(1)}$-submodule isomorphic to $\Delta(\bar{\omega} + k\bar{\omega}_2)$. Clearly, this submodule is generated by $X_{-\alpha_1,k}e^+_\omega$. □

We deliberately did not use a basis of $\Delta(\omega)$ in the proof of the above theorem to make it valid for G of arbitrary type.

Lemma 5. *The modules $KG^{(1)}X_{-\alpha_1,k}v^+_\omega$ and $KG^{(1)}X_{-\alpha_1,k}e^+_\omega$ decompose into direct sums of their T-weight subspaces. These sums are exactly the decompositions into $T^{(1)}$-weight subspaces.*

Proof. The only fact we need to prove is that $\overline{\omega - b_1\alpha_1 - \cdots - b_{n-1}\alpha_{n-1}} = \overline{\omega - c_1\alpha_1 - \cdots - c_{n-1}\alpha_{n-1}}$ and $b_1 = c_1$ imply $b_i = c_i$ for any $i = 1,\dots,n-1$. This is obvious, since the first equality is equivalent to $\bar{\omega} + b_1\bar{\omega}_2 - b_2\bar{\alpha}_2 - \cdots - b_{n-1}\bar{\alpha}_{n-1} = \bar{\omega} + c_1\bar{\omega}_2 - c_2\bar{\alpha}_2 - \cdots - c_{n-1}\bar{\alpha}_{n-1}$. □

Before proving Theorem B, we need to describe the standard bases for Weyl modules over $G^{(1)}$. Let $\varkappa = d_2\bar{\omega}_2 + \cdots + d_{n-1}\bar{\omega}_{n-1}$ be a weight of $X^+(T^{(1)})$. A sequence $\lambda = (\lambda_2,\dots,\lambda_n)$ of nonnegative integers is called *coherent* with $\varkappa$ if $d_i = \lambda_i - \lambda_{i+1}$ for any $i = 2,\dots,n-1$. The *diagram* of λ is the set

$$[\lambda] = \{(i,j)\in\mathbb{Z}^2 \mid 2 \leqslant i \leqslant n \text{ and } 1 \leqslant j \leqslant \lambda_i\}.$$

We shall think of $[\lambda]$ as an array of boxes. For example, if $\lambda = (5,3,2,0)$ then

$$[\lambda] = \begin{array}{|c|c|c|c|c|} \hline \ & \ & \ & \ & \ \\ \hline \ & \ & \ & \multicolumn{2}{c}{} \\ \cline{1-3} \ & \ & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}.$$

Note that in our terminology the top row of this diagram is the second row.

A *λ-tableau* is a function $t : [\lambda] \to \{2,\dots,n\}$, which we regard as the diagram $[\lambda]$ filled with integers in $\{2,\dots,n\}$. A λ-tableau t is called *row standard* if its entries weakly increase along the rows, that is $t(i,j) \leqslant t(i,j')$ if $j < j'$. A λ-tableau t is called *regular row standard* if it is row standard and every entry in row i of t is at least i. Finally, a λ-tableau t is called *standard* if it is row standard and its entries strictly increase down the columns, that is $t(i,j) < t(i',j)$ if $i < i'$. For example,

$$t = \begin{array}{|c|c|c|c|c|} \hline 2 & 3 & 3 & 4 & 5 \\ \hline 3 & 4 & 4 & \multicolumn{2}{c}{} \\ \cline{1-3} 4 & 5 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

is a standard $(5,3,2,0)$-tableau. For any λ-tableau t, we put

$$F_t := \prod_{2\leqslant a<b\leqslant n} X_{-\alpha_a-\cdots-\alpha_{b-1},N_{a,b}},$$

where $N_{a,b}$ is the number of entries b in row a of t, $X_{-\alpha_a-\cdots-\alpha_{b-1},N_{a,b}}$ precedes $X_{-\alpha_c-\cdots-\alpha_{d-1},N_{c,d}}$ if $b<d$ or $b=d$ and $a<c$.

Remark 6. One can easily see that the number of entries greater than 2 in the second (top) row of t is exactly minus the coefficient at α_2 in the weight of F_t.

For t as in the above example, we have

$$F_t = X_{-\alpha_2,2}\, X_{-\alpha_2-\alpha_3}\, X_{-\alpha_3,2}\, X_{-\alpha_2-\alpha_3-\alpha_4}\, X_{-\alpha_4}.$$

Proposition 7. *(See [CL].) Let $\varkappa$ be a weight of $X^+(T^{(1)})$ and $\lambda=(\lambda_2,\dots,\lambda_n)$ be a sequence coherent with $\varkappa$. Then the vectors $F_t e^+_\varkappa$, where t is a standard λ-tableau, form a basis of $\Delta(\varkappa)$.*

Now suppose that $m=3,\dots,n$ and $\lambda_2-\lambda_3=d_2\geqslant 1$. For any regular row standard λ-tableau t, we define $\rho_m(t)$ to be the $(\lambda_2-1,\lambda_3,\dots,\lambda_n)$-tableau obtained from t by removing one entry m from the second row, if such removal is possible, and shifting all elements of the resulting row to the left.

One can easily check that for any $2\leqslant s<m\leqslant n$ and $N\in\mathbb{Z}^+$, there holds

$$[X_{\alpha_1+\cdots+\alpha_{m-1}}, X_{-\alpha_s-\cdots-\alpha_{m-1},N}] = X_{-\alpha_s-\cdots-\alpha_{m-1},N-1}X_{\alpha_1+\cdots+\alpha_{s-1}}. \tag{4}$$

Note that (4) holds for any $N\in\mathbb{Z}$ if we define $X_{\alpha,N}:=0$ for $N<0$. Let I^+ denote the left ideal of $\mathcal{U}$ generated by the elements $X_{\alpha,N}$ with $\alpha\in\Phi^+$ and $N>0$.

Lemma 8. *Let $m=3,\dots,n$, $\lambda_2-\lambda_3=d_2\geqslant 1$, t be a regular row standard λ-tableau and $1\leqslant k$. We have*

$$X_{\alpha_1+\cdots+\alpha_{m-1}}F_tX_{-\alpha_1,k}\equiv F_{\rho_m(t)}X_{-\alpha_1,k-1}(H_{\alpha_1}+1-k) \pmod{I^+}$$

if $\rho_m(t)$ is well-defined and

$$X_{\alpha_1+\cdots+\alpha_{m-1}}F_tX_{-\alpha_1,k}\equiv 0 \pmod{I^+}$$

otherwise.

Proof. Let $N_{a,b}$ denote the number of entries b in row a of t. Consider the representation $F_t=F_3\cdots F_n$, where

$$F_j = X_{-\alpha_2-\cdots-\alpha_{j-1},N_{2,j}}\cdots X_{-\alpha_{j-2}-\alpha_{j-1},N_{j-2,j}}X_{-\alpha_{j-1},N_{j-1,j}}.$$

Clearly, $X_{\alpha_1+\cdots+\alpha_{m-1}}$ commutes with any F_j such that $j\neq m$. Using (4) and the fact that $X_{\alpha_1+\cdots+\alpha_{s-1}}$ commutes with any factor of F_m for $s=2,\dots,m-1$, we obtain

$$X_{\alpha_1+\cdots+\alpha_{m-1}}F_m = F_mX_{\alpha_1+\cdots+\alpha_{m-1}} + \sum_{s=2}^{m-1}\left(\prod_{l=2}^{m-1}X_{-\alpha_l-\cdots-\alpha_{m-1},N_{l,m}-\delta_{l,s}}\right)X_{\alpha_1+\cdots+\alpha_{s-1}}.$$

Here and in what follows $\delta_{l,s}$ equals 1 if $l=s$ and equals 0 otherwise. Since $X_{\alpha_1+\cdots+\alpha_{s-1}}$ commutes with any F_j for $s=2,\dots,m$ and $j=m+1,\dots,n$, we obtain

$$X_{\alpha_1+\cdots+\alpha_{m-1}}F_tX_{-\alpha_1,k}=F_tX_{\alpha_1+\cdots+\alpha_{m-1}}X_{-\alpha_1,k}\\ +\sum_{s=2}^{m-1}F_1\cdots F_{m-1}\left(\prod_{l=2}^{m-1}X_{-\alpha_l-\cdots-\alpha_{m-1},N_{l,m}-\delta_{l,s}}\right)F_{m+1}\cdots F_nX_{\alpha_1+\cdots+\alpha_{s-1}}X_{-\alpha_1,k}.$$

Since $m\geqslant 3$ the first summand and any product under the summation sign for $s>2$ in the right-hand side of the above formula belongs to I^+. Hence

$$X_{\alpha_1+\cdots+\alpha_{m-1}}F_tX_{-\alpha_1,k}\equiv F_1\cdots F_{m-1}\left(\prod_{l=2}^{m-1}X_{-\alpha_l-\cdots-\alpha_{m-1},N_{l,m}-\delta_{l,2}}\right)\\ \times F_{m+1}\cdots F_nX_{-\alpha_1,k-1}(H_{\alpha_1}-k+1)\pmod{I^+}.$$

If $N_{2,m}>0$ then the right-hand side of the above formula equals $F_{\rho_m(t)}X_{-\alpha_1,k-1}(H_{\alpha_1}+1-k)$. Otherwise it equals zero and $\rho_m(t)$ is not well-defined. □

We also need the iterated version of ρ_m. Suppose that $M=(m_1,\dots,m_l)$ is a sequence with entries in $\{3,\dots,n\}$ and $\lambda_2-\lambda_3=d_2\geqslant l$. For any regular row standard λ-tableau t, we define $\rho_M(t)$ to be the $(\lambda_2-l,\lambda_3,\dots,\lambda_n)$-tableau obtained from t by removing the entries $m_1,\dots,m_l$ (taking into account their multiplicities) from the second row, if such removal is possible, and shifting all elements of the resulting row to the left. We clearly have $\rho_M(t)=\rho_{m_1}\circ\cdots\circ\rho_{m_l}(t)$ if the second row of t contains entries $m_1,\dots,m_l$. Hence applying Lemma 8, we obtain the following result.

Corollary 9. *Let $M=(m_1,\dots,m_l)$ be a sequence with entries in $\{3,\dots,n\}$, $\lambda_2-\lambda_3=d_2\geqslant l$, t be a regular row standard λ-tableau and $l\leqslant k$. We have*

$$\left(\prod_{i=1}^{l}X_{\alpha_1+\cdots+\alpha_{m_i-1}}\right)F_tX_{-\alpha_1,k}\equiv F_{\rho_M(t)}X_{-\alpha_1,k-l}\left(\prod_{i=1}^{l}H_{\alpha_1}+i-k\right)\pmod{I^+}$$

if $\rho_M(t)$ is well-defined and

$$\left(\prod_{i=1}^{l}X_{\alpha_1+\cdots+\alpha_{m_i-1}}\right)F_tX_{-\alpha_1,k}\equiv 0\pmod{I^+}$$

otherwise.

In what follows, $\mathrm{coeff}_{\alpha_1}(\beta)$ denotes the coefficient at α_1 of a root $\beta\in\Phi$.

Proof of Theorem B. *"Only if part."* Suppose that the $G^{(1)}$-submodule of $L(\omega)$ generated by $X_{-\alpha_1,k}v_\omega^+$ is isomorphic to a Weyl module. Then $X_{-\alpha_1,k}v_\omega^+\neq 0$ and $X_{\alpha_1,k}X_{-\alpha_1,k}v_\omega^+=\binom{a_1}{k}v_\omega^+\neq 0$. Hence $a_1-l\not\equiv 0\pmod p$ for $l=0,\dots,k-1$, since $k<p$.

Now let v be a nonzero $G^{(1)}$-primitive vector of $KG^{(1)}X_{-\alpha_1,k}v_\omega^+$. By Lemma 5, v is a T-weight vector. It has T-weight $\omega-\delta$, where δ is a sum of positive roots. Clearly, the coefficient at α_1 of δ equals k. We claim that

$$\delta\in E(1,k):=\{\beta_1+\cdots+\beta_l\mid\beta_1,\dots,\beta_l\in\Phi^+,\mathrm{coeff}_{\alpha_1}(\beta_1)>0,\dots,\\ \mathrm{coeff}_{\alpha_1}(\beta_l)>0,\mathrm{coeff}_{\alpha_1}(\beta_1)+\cdots+\mathrm{coeff}_{\alpha_1}(\beta_l)=k\}.\tag{5}$$

Indeed, by Proposition 3, the products $\prod_{\alpha\in\Phi^+}X_{\alpha,m_\alpha}$ taken in any fixed order form a basis of $\mathcal{U}^+$. Let us assume now that this order is such that any factor X_{α,m_α} with $\mathrm{coeff}_{\alpha_1}(\alpha)>0$ is situated to the left of any factor X_{β,m_β} with $\mathrm{coeff}_{\alpha_1}(\beta)=0$. Since $v\neq 0$, we have $(\prod_{\alpha\in\Phi^+}X_{\alpha,m_\alpha})v=cv_\omega^+$ for some $c\in\mathbb{F}^*$ and $m_\alpha\in\mathbb{Z}^+$ such that $\sum_{\alpha\in\Phi^+}m_\alpha\alpha=\delta$. Since v is $G^{(1)}$-primitive, the order of factors we have chosen implies that $m_\alpha=0$ if $\mathrm{coeff}_{\alpha_1}(\alpha)=0$. On the other hand, $\sum_{\alpha\in\Phi^+}m_\alpha\,\mathrm{coeff}_{\alpha_1}(\alpha)=k$. Hence (5) directly follows.

Now it remains to notice that

$$E(1,k)=\{b_1\alpha_1+\cdots+b_{n-1}\alpha_{n-1}\mid k=b_1\geqslant b_2\geqslant\cdots\geqslant b_{n-1}\geqslant 0\}.$$

"If part." We assume that $a_1-l\not\equiv 0 \pmod p$ for any $l=0,\dots,k-1$ and any nonzero primitive vector of $\Delta(\bar\omega+k\bar\omega_2)$ has weight as in (i). In particular, we have $k\leqslant a_1$. Suppose that $KG^{(1)}X_{-\alpha_1,k}v_\omega^+$ is not isomorphic to a Weyl module. Then by (2), we get $\ker\varphi_k^\omega\neq 0$. Since $\ker\varphi_k^\omega$ is a submodule of $\Delta(\bar\omega+k\bar\omega_2)$, it contains a nonzero primitive vector u. Our assumption implies that u has weight $\bar\omega+k\bar\omega_2-b_2\bar\alpha_2-\cdots-b_{n-1}\bar\alpha_{n-1}$, where $k\geqslant b_2\geqslant\cdots\geqslant b_{n-1}\geqslant 0$.

The universal property of Weyl modules implies the existence of the $G^{(1)}$-module homomorphism $\gamma:\Delta(\bar\omega+k\bar\omega_2)\to KG^{(1)}X_{-\alpha_1,k}e_\omega^+$ such that $\gamma(e^+_{\bar\omega+k\bar\omega_2})=X_{-\alpha_1,k}e_\omega^+$. Lemma 4 shows that γ is an isomorphism. Since $\pi^\omega\circ\gamma=\varphi_k^\omega$ (to prove it, apply both sides to $e^+_{\bar\omega+k\bar\omega_2}$), we have $\gamma(u)\in\operatorname{rad}\Delta(\omega)$.

Take any sequence $\lambda=(\lambda_2,\dots,\lambda_n)$ of nonnegative integers coherent with $\bar\omega+k\bar\omega_2$. In particular, we have $\lambda_2-\lambda_3=\langle\bar\omega+k\bar\omega_2,\bar\alpha_2\rangle=a_2+k\geqslant b_2$. By Proposition 7, we have the representation $u=\sum_{s\in S}c_sF_se^+_{\bar\omega+k\bar\omega_2}$, where $c_s\in\mathbb{F}^*$ and S is a nonempty set consisting of standard λ-tableaux s such that F_s has weight $-b_2\alpha_2-\cdots-b_{n-1}\alpha_{n-1}$. Obviously, any tableau $s\in S$ has exactly b_2 entries greater than 2 in the second row (see Remark 6).

Let us fix some tableau $t\in S$, denote be $m_1,\dots,m_{b_2}$ all the entries greater than 2 in the second row of t (taking into account multiplicities) and put $M:=(m_1,\dots,m_{b_2})$. Clearly, $\rho_M(t)$ is well-defined. Moreover, for any $s\in S$ such that $\rho_M(s)$ is well-defined, $\rho_M(s)$ is a standard $(\lambda_2-b_2,\lambda_3,\dots,\lambda_n)$-tableau whose every entry in the second row is 2 and $F_{\rho_M(s)}$ has weight

$$-b_2\alpha_2-\cdots-b_{n-1}\alpha_{n-1}+\left(\sum_{i=1}^{b_2}\alpha_2+\cdots+\alpha_{m_i-1}\right)=-b_3'\alpha_3-\cdots-b_{n-1}'\alpha_{n-1},$$

where $b_3',\dots,b_{n-1}'$ are nonnegative integers (independent of s). Applying γ to the above representation of u, we obtain

$$\gamma(u)=\sum_{s\in S}c_sF_sX_{-\alpha_1,k}e_\omega^+\in\operatorname{rad}\Delta(\omega).$$

Multiplying this formula by $(\prod_{i=1}^{b_2}X_{\alpha_1+\cdots+\alpha_{m_i-1}})$ on the left, taking into account $b_2\leqslant k$ and applying Corollary 9, we obtain

$$\left(\prod_{i=1}^{b_2}a_1+i-k\right)\sum\{F_{\rho_M(s)}X_{-\alpha_1,k-b_2}e_\omega^+\mid s\in S\text{ and }\rho_M(s)\text{ is well-defined}\}$$
$$\in\operatorname{rad}\Delta(\omega).\tag{6}$$

Since $b_2\leqslant k$ and we assumed $a_1-l\not\equiv 0 \pmod p$ for any $l=0,\dots,k-1$, the first factor of the product in the left-hand side of the above formula is nonzero. Moreover, if s and s' are distinct tableaux of S and both $\rho_M(s)$ and $\rho_M(s')$ are well-defined, then $\rho_M(s)\neq\rho_M(s')$. Notice that the summation in (6) is nonempty, since at least $s=t$ satisfies the restrictions.

By Lemma 4, the $G^{(1)}$-submodule W of $\Delta(\omega)$ generated by $X_{-\alpha_1,k-b_2}e_\omega^+$ is isomorphic to $\Delta(\bar\omega+(k-b_2)\bar\omega_2)$. Note that $(\lambda_2-b_2,\lambda_3,\dots,\lambda_n)$ is coherent with $\bar\omega+(k-b_2)\bar\omega_2$. Therefore by Proposition 7, the left-hand side of (6) is nonzero. It belongs to a proper $G^{(1)}$-submodule $W\cap\operatorname{rad}\Delta(\omega)$ of W and hence to $\operatorname{rad} W$. Note that $X_{\alpha_1,k-b_2}X_{-\alpha_1,k-b_2}e^+_\omega = \binom{a_1}{k-b_2}e^+_\omega \neq 0$, whence $X_{-\alpha_1,k-b_2}e^+_\omega \notin \operatorname{rad}\Delta(\omega)$ and indeed $W \cap \operatorname{rad}\Delta(\omega) \neq W$.

In other words, we proved that $\operatorname{rad}\Delta(\bar\omega + (k-b_2)\bar\omega_2)$ contains a nonzero vector u' of weight $\bar\omega + (k-b_2)\bar\omega_2 - b'_3\bar\alpha_3 - \cdots - b'_{n-1}\bar\alpha_{n-1}$. As an immediate consequence of this fact, we get $n \geqslant 4$. For any weight $\varkappa \in X(T)$, we denote by $\widetilde{\varkappa}$ its restriction to $T^{(1,2)}$. By Lemma 4, the $G^{(1,2)}$-submodule W' of $\Delta(\bar\omega + (k-b_2)\bar\omega_2)$ generated by $e^+_{\bar\omega+(k-b_2)\bar\omega_2}$ is isomorphic to $\Delta(\widetilde{\omega})$ (the restriction of $\bar\omega + (k-b_2)\bar\omega_2$ to $T^{(1,2)}$ is $\widetilde{\omega}$). Clearly, u' belongs to a proper submodule $W' \cap \operatorname{rad}\Delta(\bar\omega + (k-b_2)\bar\omega_2)$ of W' and thus belongs to $\operatorname{rad} W'$. In this way, we proved that $\Delta(\widetilde{\omega})$ is not simple.

Consider the $G^{(1,2)}$-submodule W'' of $\Delta(\bar\omega + k\bar\omega_2)$ generated by $e^+_{\bar\omega+k\bar\omega_2}$. By Lemma 4, W'' is isomorphic to $\Delta(\widetilde{\omega})$ (the restriction of $\bar\omega + k\bar\omega_2$ to $T^{(1,2)}$ is also $\widetilde{\omega}$). Therefore W'' is not simple and contains a nonzero $G^{(1,2)}$-primitive vector u'' of $T^{(1,2)}$-weight $\widetilde{\omega} - d_3\widetilde{\alpha}_3 - \cdots - d_{n-1}\widetilde{\alpha}_{n-1}$, where $d_3, \dots, d_{n-1}$ are nonnegative integers not equal simultaneously to zero. By Lemma 5, we obtain that u'' has $T^{(1)}$-weight $\bar\omega + k\bar\omega_2 - d_3\bar\alpha_3 - \cdots - d_{n-1}\bar\alpha_{n-1}$. Note that this weight does not have the form described in (i). Since $x_{\alpha_2}(t)$ commutes with any $x_{-\alpha_i}(s)$, where $i = 3, \dots, n-1$, and

$$u'' \in W'' = \mathbb{F}\langle x_{-\alpha_i}(s) \mid i = 3, \dots, n-1,\ s \in \mathbb{F}\rangle e_{\bar\omega+k\bar\omega_2},$$

we obtain that u'' is $G^{(1)}$-primitive. This is a contradiction. □

Proof of Theorem A. Suppose that the hypothesis of (i) holds. The weights of $\Delta(\bar\omega_2)$ are $\varkappa_1, \dots, \varkappa_{n-1}$, where $\varkappa_i = \bar\omega_2 - \bar\alpha_2 - \cdots - \bar\alpha_i$ and each weight space is one-dimensional.

Suppose for a while that $\operatorname{char}\mathbb{F} = 0$. It is well known that for any $\varkappa \in X^+(T^{(1)})$, the module $\Delta(\varkappa) \otimes \Delta(\bar\omega_2)$ is a direct sum of $\Delta(\varkappa + \varkappa_i)$ over $i = 1, \dots, n-1$ such that $\varkappa + \varkappa_i \in X^+(T^{(1)})$ (see, for example, [BK2, Lemma 4.8]). Thus the module $\Delta(\bar\omega + m\bar\omega_2) \otimes \Delta(\bar\omega_2)^{\otimes k-m}$ is a direct sum of several copies of $\Delta(\bar\omega + m\bar\omega_2 + \varkappa_{i_1} + \cdots + \varkappa_{i_{k-m}})$ over sequences $i_1, \dots, i_{k-m}$ of integers in $\{1, \dots, n-1\}$ such that $\bar\omega + m\bar\omega_2 + \varkappa_{i_1} + \cdots + \varkappa_{i_{k-m}} \in X^+(T^{(1)})$. Moreover, the module $\Delta(\bar\omega + k\bar\omega_2)$ enters into this sum with multiplicity one.

Let us return to the case $\operatorname{char}\mathbb{F} = p > 0$. Applying the main result of [M], we obtain that the module $V := \Delta(\bar\omega + m\bar\omega_2) \otimes \Delta(\bar\omega_2)^{\otimes k-m}$ has a filtration with factors $\Delta(\bar\omega + m\bar\omega_2 + \varkappa_{i_1} + \cdots + \varkappa_{i_{k-m}})$ over the same sequences $i_1, \dots, i_{k-m}$ with the same multiplicities. By [J, II.4.16 Remark 4] applied to the dual module V^*, V has a submodule isomorphic to $\Delta(\bar\omega + k\bar\omega_2)$.

Now recall that $\Delta(\bar\omega + m\bar\omega_2) \cong \nabla(\bar\omega + m\bar\omega_2)$ by the hypothesis of the present lemma and $\Delta(\bar\omega_2) \cong \nabla(\bar\omega_2)$. Therefore, V is isomorphic to $\nabla(\bar\omega + m\bar\omega_2) \otimes \nabla(\bar\omega_2)^{\otimes k-m}$ and by the main result of [M] has a filtration with factors $\nabla(\bar\omega + m\bar\omega_2 + \varkappa_{i_1} + \cdots + \varkappa_{i_{k-m}})$ over the same sequences $i_1, \dots, i_{k-m}$ with the same multiplicities. Applying [J, Proposition II.4.13], we obtain that $\operatorname{Hom}_{G^{(1)}}(\Delta(\varkappa), V) = 0$ unless $\varkappa = \bar\omega + m\bar\omega_2 + \varkappa_{i_1} + \cdots + \varkappa_{i_{k-m}}$. Since V has a submodule isomorphic to $\Delta(\bar\omega + k\bar\omega_2)$, any nonzero primitive vector of $\Delta(\bar\omega + k\bar\omega_2)$ has weight $\bar\omega + m\bar\omega_2 + \varkappa_{i_1} + \cdots + \varkappa_{i_{k-m}}$ with $i_1, \dots, i_{k-m}$ as above. It remains to apply Theorem B(i).

Part (ii) can be proved similarly but tensoring with $\Delta(\bar{\bar\omega}_{n-2})$ and applying Theorem B(ii). □

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