J. Symbolic Computation (1997) **24**, 51–58

On the Stability of Gröbner Bases Under Specializations

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Let R be a Noetherian commutative ring with identity, K a field and π a ring homomorphism from R to K. We investigate for which ideals in $R[x_1, \ldots, x_n]$ and admissible orders the formation of leading monomial ideals commutes with the homomorphism π.

1. Introduction

Let R, R' be Noetherian commutative rings with identity and $\pi : R \to R'$ a ring homomorphism. When does a Gröbner basis of the ideal $I \subseteq R[x_1, \ldots, x_n]$ map to a Gröbner basis of the ideal $I\, R'[x_1, \ldots, x_n]$ generated by the image of I under the natural extension $\pi : R[x_1, \ldots, x_n] \to R'[x_1, \ldots, x_n]$? Obviously it suffices to have

$$\mathrm{lm}(I)\, R'[x_1, \ldots, x_n] = \mathrm{lm}(I\, R'[x_1, \ldots, x_n]), \tag{1.1}$$

where $\mathrm{lm}(I)$ denotes the ideal generated by the leading monomials of the elements of I. This condition has already been studied in Bayer *et al.* (1991) and it has been shown that (1.1) holds for any ideal and any term order if and only if π is flat.

In this paper we study condition (1.1) under the additional assumption that R' is not a general Noetherian commutative ring with identity but a field. First we prove the following necessary and sufficient condition for (1.1). Let $\{g_1, \ldots, g_s\}$ be a Gröbner basis of an ideal $I \subseteq R[x_1, \ldots, x_n]$ with respect to an order $\prec$ and assume that the g_is are ordered in such a way that the leading coefficients of precisely the first r polynomials are not in the kernel $\ker(\pi)$. Then (1.1) holds for I and $\prec$ if and only if the polynomials $\pi(g_{r+1}), \ldots, \pi(g_s)$ can be reduced to 0 modulo $\{\pi(g_1), \ldots, \pi(g_r)\}$. Sufficient but not necessary conditions that (1.1) holds for an ideal and an order can be found in Bayer *et al.* (1991), Pauer (1992), Gräbe (1993) and Assi (1994).

If R' is a field $\ker(\pi)$ is a prime ideal. Let J be a subideal of $\ker(\pi)$. We show that the following two conditions are equivalent.

(*a*) $\ker(\pi)$ is an isolated prime ideal of J.
(*b*) For any ideal I in the univariate polynomial ring $R[x]$ with $I \cap R = J$, (1.1) holds.

Furthermore we use the concept of independence complexes of ideals to give two other

0747–7171/97/010051 + 08 \$25.00/0 sy970113

conditions equivalent to (a) and (b). Note that the implication $(a) \Rightarrow (b)$ is a generalization of the main result in Gianni (1987) and Kalkbrener (1987).

For ideals in multivariate polynomial rings over R we prove the equivalence of the following two conditions.

(c) $\ker(\pi)$ is an isolated prime ideal of J which equals the corresponding primary component.
(d) For any number of variables n, any ideal I in $R[x_1, \ldots, x_n]$ with $I \cap R = J$ and any term order, (1.1) holds.

As a consequence of this result and the already mentioned theorem in Bayer *et al.* (1991) we obtain that π is flat if and only if no proper subideal of $\ker(\pi)$ is primary.

2. Definitions

Throughout this paper let R be a Noetherian commutative ring with identity and K a field. The ideal generated by a subset F of R is denoted by $\langle F \rangle$ and the set of power products in the variables $x_1, \ldots, x_n$ by $PP(x_1, \ldots, x_n)$. Let $\prec$ be an arbitrary admissible order on $PP(x_1, \ldots, x_n)$. For any non-zero polynomial $f \in R[x_1, \ldots, x_n]$ write $f = cX + f'$, where $c \in R \setminus \{0\}$ and $X \in PP(x_1, \ldots, x_n)$ with $X \succ X'$ for every power product X' in f'. With this notation we set

$$\begin{aligned} \mathrm{lc}(f) &:= c, && \text{the leading coefficient of } f, \\ \mathrm{lpp}(f) &:= X, && \text{the leading power product of } f, \\ \mathrm{lm}(f) &:= cX, && \text{the leading monomial of } f. \end{aligned}$$

The total degree of f in $x_1, \ldots, x_n$ is denoted by $\deg(f)$. Furthermore, we define $\mathrm{lc}(0) := \mathrm{lpp}(0) := \mathrm{lm}(0) := 0$ and $\deg(0) := -1$. For an ideal I in $R[x_1, \ldots, x_n]$ we denote the ideal $\langle \{\mathrm{lm}(f) \mid f \in I\} \rangle$ by $\mathrm{lm}(I)$. A finite subset G of an ideal $I \subseteq R[x_1, \ldots, x_n]$ is a Gröbner basis of I w.r.t. $\prec$ if

$$\langle \{\mathrm{lm}(g) \mid g \in G\} \rangle = \mathrm{lm}(I).$$

We will often use the characterization of Gröbner bases in Theorem 2.1 (see Möller, 1988). Let $F = \{f_1, \ldots, f_r\}$ be a subset of $R[x_1, \ldots, x_n]$ and $M := (\mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_r))$. A syzygy w.r.t. M is an r-tuple of polynomials $S = (h_1, \ldots, h_r)$ in $R[x_1, \ldots, x_n]^r$ such that

$$\sum_{i=1}^{r} h_i \cdot \mathrm{lm}(f_i) = 0.$$

The set $S(M)$ of all syzygies w.r.t. M forms an $R[x_1, \ldots, x_n]$-module. An element $S \in S(M)$ is homogeneous of degree X, where $X \in PP(x_1, \ldots, x_n)$, provided that

$$S = (c_1 Y_1, \ldots, c_r Y_r),$$

where $c_i \in R$, $Y_i \in P(x_1, \ldots, x_n)$ and $Y_i \cdot \mathrm{lpp}(f_i) = X$ whenever $c_i \neq 0$. Obviously, $S(M)$ has a finite homogeneous basis.

Theorem 2.1. *Let $F = \{f_1, \ldots, f_r\}$ be a subset of $R[x_1, \ldots, x_n]$ and $M := (\mathrm{lm}(f_1), \ldots, \mathrm{lm}(f_r))$. The following two conditions are equivalent.*

(*a*) *F is a Gröbner basis of* $\langle F\rangle$.
(*b*) *Let* $S_1,\dots,S_m$ *be a basis of* $S(M)$, $S_i=(h_{i1},\dots,h_{ir})$ *homogeneous for every* $i\in\{1,\dots,m\}$. *Then any polynomial* $p_i=\sum_{j=1}^r h_{ij}f_j$ *can be written in the form* $p_i=\sum_{j=1}^r g_{ij}f_j$, *where the* g_{ij} *are in* $R[x_1,\dots,x_n]$ *and* $\mathrm{lpp}(p_i)=\max_{j=1}^r \mathrm{lpp}(g_{ij})\,\mathrm{lpp}(f_j)$.

Let R' be a Noetherian commutative ring with identity. Every ring homomorphism $\pi: R\to R'$ extends naturally to a homomorphism $\pi: R[x_1,\dots,x_n]\to R'[x_1,\dots,x_n]$. The image under π of an ideal $I\subseteq R[x_1,\dots,x_n]$ generates the extension ideal $I\,R'[x_1,\dots,x_n]$. We want to study under which conditions on π and $\prec$ a Gröbner basis of I maps to a Gröbner basis of $I\,R'[x_1,\dots,x_n]$. Note that it suffices to have

$$\mathrm{lm}(I)\,R'[x_1,\dots,x_n]=\mathrm{lm}(I\,R'[x_1,\dots,x_n]). \tag{2.1}$$

We call I stable under π and $\prec$ if it satisfies (2.1) and we will focus on this condition.

The stability of ideals has been already studied by Bayer *et al.* (1991). They proved the following interesting relation between flat morphisms and the stability of ideals (Bayer *et al.*, 1991, Theorem 3.6). Recall that an R-module N is called flat if the functor $T_N: M\to M\otimes_R N$ on the category of R-modules is exact and the ring homomorphism $\pi: R\to R'$ is called flat if π makes R' a flat R-module.

THEOREM 2.2. *Let* $\pi: R\to R'$ *be a ring homomorphism. Then the following two conditions are equivalent.*

(*a*) *For any natural number* n, *any ideal* I *in* $R[x_1,\dots,x_n]$ *and any admissible order* $\prec$ *on* $PP(x_1,\dots,x_n)$, I *is stable under* π *and* $\prec$.
(*b*) π *is flat.*

In this paper we will concentrate on a special case: we assume that π is a ring homomorphism from R to the field K. Hence the image of R is a subring of K and therefore an integral domain. Thus the kernel, $\ker(\pi)$, is a prime ideal and the quotient field $\bar K$ of $R/\ker(\pi)$ is a subfield of K. Furthermore, it is easy to see that

$$\text{the ideal } \mathrm{lm}(I\,K[x_1,\dots,x_n]) \text{ is generated by the set } \{\mathrm{lm}(\pi(f))\mid f\in I\}. \tag{2.2}$$

A subset $\{x_{i_1},\dots,x_{i_m}\}\subseteq\{x_1,\dots,x_n\}$ is called independent modulo an ideal $J\subseteq K[x_1,\dots,x_n]$ if $J\cap K[x_{i_1},\dots,x_{i_m}]=\{0\}$. The independence complex of J is the set

$$\Delta(J):=\{\{x_{i_1},\dots,x_{i_m}\}\subseteq\{x_1,\dots,x_n\}\mid\{x_{i_1},\dots,x_{i_m}\}\text{ is independent modulo }J\}.$$

Additionally to stability we will consider the following weaker property. We call an ideal $I\subseteq R[x_1,\dots,x_n]$ semi-stable under π and $\prec$ if

$$\Delta(\mathrm{lm}(I)\,K[x_1,\dots,x_n])=\Delta(\mathrm{lm}(I\,K[x_1,\dots,x_n])). \tag{2.3}$$

3. Stability Criteria

First of all we show that the stability of an ideal I can be easily checked if a Gröbner basis of I is known.

THEOREM 3.1. *Let* π *be a ring homomorphism from* R *to* K, I *an ideal in* $R[x_1,\dots,x_n]$ *and* $G=\{g_1,\dots,g_s\}$ *a Gröbner basis of* I *with respect to an admissible order* $\prec$. *We* *assume that the g_is are ordered in such a way that there exists an $r \in \{0, \dots, s\}$ with $\pi(\mathrm{lc}(g_i)) \neq 0$ for $i \in \{1, \dots, r\}$ and $\pi(\mathrm{lc}(g_i)) = 0$ for $i \in \{r+1, \dots, s\}$. Then the following three conditions are equivalent.*

(*a*) *I is stable under π and $\prec$.*
(*b*) *$\{\pi(g_1), \dots, \pi(g_r)\}$ is a Gröbner basis of $I\,K[x_1, \dots, x_n]$ w.r.t. $\prec$.*
(*c*) *For every $i \in \{r+1, \dots, s\}$ the polynomial $\pi(g_i)$ is reducible to 0 modulo $\{\pi(g_1), \dots, \pi(g_r)\}$.*

PROOF. Obviously $\{\pi(g_1), \dots, \pi(g_r)\}$ is a Gröbner basis of $I\,K[x_1, \dots, x_n]$ if and only if

$$\langle \{\pi(lm(g)) \mid g \in G\} \rangle = \mathrm{lm}(I\,K[x_1, \dots, x_n]).$$

Since

$$\langle \{\pi(\mathrm{lm}(g)) \mid g \in G\} \rangle = \mathrm{lm}(I)\,K[x_1, \dots, x_n]$$

(a) and (b) are equivalent.

If $\{\pi(g_1), \dots, \pi(g_r)\}$ is a Gröbner basis of $I\,K[x_1, \dots, x_n]$ then (c) holds. It remains to show that (c) implies (a). Let $f \in I$ with $\pi(f) \neq 0$. By (2.2), it suffices to show that

$$\text{there exists a } g \in I \text{ such that } \mathrm{lpp}(g) \text{ divides } \mathrm{lpp}(\pi(f)) \text{ and } \pi(\mathrm{lc}(g)) \neq 0. \tag{3.1}$$

We do the proof by induction on $\prec$.

Induction basis: If $\mathrm{lpp}(f) = 1$ then $\pi(\mathrm{lc}(f)) \neq 0$ and $\mathrm{lpp}(f) = \mathrm{lpp}(\pi(f))$. Hence, (3.1) holds.

Induction step: Since (3.1) holds if $\pi(\mathrm{lc}(f)) \neq 0$ we assume that $\pi(\mathrm{lc}(f)) = 0$. If there exists an $i \in \{1, \dots, r\}$ such that $\mathrm{lpp}(g_i)$ divides $\mathrm{lpp}(f)$ we define

$$f' := \mathrm{lc}(g_i) \cdot f - \mathrm{lc}(f) \cdot (\mathrm{lpp}(f)/\,\mathrm{lpp}(g_i)) \cdot g_i.$$

Obviously, $\mathrm{lpp}(\pi(f')) = \mathrm{lpp}(\pi(f))$ and $\mathrm{lpp}(f') \prec \mathrm{lpp}(f)$. Thus, (3.1) follows from the induction hypothesis. Otherwise, there exist $j_1, \dots, j_k \in \{r+1, \dots, s\}$ and $c_{j_1}, \dots, c_{j_k} \in R$ such that $\mathrm{lpp}(g_{j_l})$ divides $\mathrm{lpp}(f)$ for $l \in \{1, \dots, k\}$ and

$$\mathrm{lm}(f) = \sum_{l=1}^{k} c_{j_l} \cdot (\mathrm{lpp}(f)/\,\mathrm{lpp}(g_{j_l})) \cdot \mathrm{lm}(g_{j_l}).$$

Let $i \in \{r+1, \dots, s\}$. Since $\pi(g_i)$ is reducible to 0 modulo $\{\pi(g_1), \dots, \pi(g_r)\}$ there exist an $h_i \in I$ and a $b_i \in R \setminus \ker(\pi)$ with $\pi(b_i) \cdot \pi(g_i) = \pi(h_i)$ and $\mathrm{lpp}(g_i) \succ \mathrm{lpp}(\pi(g_i)) = \mathrm{lpp}(h_i)$. Define

$$f' := b \cdot f - \sum_{l=1}^{k} (b/b_{j_l}) \cdot c_{j_l} \cdot (\mathrm{lpp}(f)/\,\mathrm{lpp}(g_{j_l})) \cdot (b_{j_l} \cdot g_{j_l} - h_{j_l}),$$

where $b := \prod_{l=1}^{k} b_{j_l}$. Obviously, $\mathrm{lpp}(\pi(f')) = \mathrm{lpp}(\pi(f))$ and $\mathrm{lpp}(f') \prec \mathrm{lpp}(f)$. Again, (3.1) follows from the induction hypothesis. □

Sufficient but not necessary criteria for the stability of I under π and $\prec$ can be found in Bayer *et al.* (1991), Pauer (1992), Gräbe (1993) and Assi (1994).

Let J be an ideal in R with $J \subseteq \ker(\pi)$. We will now show that every ideal I in the univariate polynomial ring $R[x_1]$ with $I \cap R = J$ is stable (resp. semi-stable) under π if and only if

$$\ker(\pi) \text{ is an isolated prime ideal of } J. \tag{3.2}$$

Another condition equivalent to (3.2) is semi-stability of every ideal I in a multivariate polynomial ring over R with $I \cap R = J$.

THEOREM 3.2. *Let π be a ring homomorphism from R to K and J an ideal in R with $J \subseteq \ker(\pi)$. Then the following four conditions are equivalent.*

- (a) $\ker(\pi)$ *is an isolated prime ideal of J.*
- (b) *For any ideal I in $R[x_1]$ with $I \cap R = J$, I is stable under π and the uniquely determined admissible order $\prec$ on $PP(x_1)$.*
- (c) *For any natural number n, any ideal I in $R[x_1, \dots, x_n]$ with $I \cap R = J$ and any admissible order $\prec$ on $PP(x_1, \dots, x_n)$, I is semi-stable under π and $\prec$.*
- (d) *For any ideal I in $R[x_1]$ with $I \cap R = J$, I is semi-stable under π and the uniquely determined admissible order $\prec$ on $PP(x_1)$.*

PROOF. Denote the kernel of π by P.

$(a) \Rightarrow (c)$: Let I be an ideal in $R[x_1, \dots, x_n]$ with $I \cap R = J$ and $\prec$ an admissible order on $PP(x_1, \dots, x_n)$. Assume that P is an isolated prime ideal of J and $f \in I$ with $\pi(f) \neq 0$. We first show that

$$\text{there exists a natural number } l \text{ with } \operatorname{lm}(\pi(f))^l \in \operatorname{lm}(I)\, K[x_1, \dots, x_n]. \tag{3.3}$$

Write f in the form $f = a_1 X_1 + \cdots + a_t X_t$, where $a_1, \dots, a_t \in R \setminus \{0\}$ and $X_1, \dots, X_t \in PP(x_1, \dots, x_n)$ with $X_1 \succ \cdots \succ X_t$. Choose $k \in \{1, \dots, t\}$ with $a_1, \dots, a_{k-1} \in P$ and $a_k \notin P$ and define $p := a_1 X_1 + \cdots + a_{k-1} X_{k-1}$ and $h := a_k X_k + \cdots + a_t X_t$. Let $I = Q_1 \cap \ldots \cap Q_m$ be an irredundant primary decomposition of I and denote the radical of Q_i by P_i. We can assume that the Q_is are ordered in such a way that there exists an $m' \in \{1, \dots, m\}$ with $P = P_j \cap R$ for $j \in \{1, \dots, m'\}$ and $P \neq P_j \cap R$ for $j \in \{m'+1, \dots, m\}$. Obviously, $p, h \in P_j$ for $j \in \{1, \dots, m'\}$. Hence, we can choose a natural number l such that for every $j \in \{1, \dots, m'\}$ we have $h^l \in Q_j$. Since P is an isolated prime ideal of $I \cap R$ we can choose for every $j \in \{m'+1, \dots, m\}$ a $q_j \in (Q_j \cap R) \setminus P$. For $q := \prod_{j=m'+1}^{m} q_j$ we have $qh^l \in I$ and $\pi(\operatorname{lm}(qh^l)) = \pi(q) \cdot \operatorname{lm}(\pi(f))^l$. Hence, (3.3) is proved.

For proving semi-stability it suffices to show that

$$\Delta(\operatorname{lm}(I)\, K[x_1, \dots, x_n]) \subseteq \Delta(\langle \{\operatorname{lm}(\pi(f)) \mid f \in I\} \rangle).$$

Let $\{x_{i_1}, \dots, x_{i_k}\} \notin \Delta(\langle \{\operatorname{lm}(\pi(f)) \mid f \in I\} \rangle)$. Then there exists an $f \in I$ such that $\operatorname{lm}(\pi(f)) \in K[x_{i_1}, \dots, x_{i_k}] \setminus \{0\}$. By (3.3), there exists a natural number l with

$$\operatorname{lm}(\pi(f))^l \in (\operatorname{lm}(I)\, K[x_1, \dots, x_n]) \cap K[x_{i_1}, \dots, x_{i_k}]$$

and therefore $\{x_{i_1}, \dots, x_{i_k}\} \notin \Delta(\operatorname{lm}(I)\, K[x_1, \dots, x_n])$. Thus, I is semi-stable under π and $\prec$.

$(c) \Rightarrow (b)$: Let I be an ideal in $R[x_1]$ with $I \cap R = J$ and $\prec$ the uniquely determined admissible order on $PP(x_1)$. If $\operatorname{lm}(I\, K[x_1]) = \{0\}$ then I is obviously stable under π and $\prec$. Hence, we can assume that $\operatorname{lm}(I\, K[x_1])$ is generated by x_1^k for some non-negative integer k. It follows from (c) that $\operatorname{lm}(I)\, K[x_1]$ is generated by x_1^l for some non-negative integer l with $k \leq l$. Assume that I is not stable and therefore $k < l$. By (2.2), there exist f_1 and f_2 in I with $\deg(\pi(f_1)) = k$ and $\deg(f_2) = \deg(\pi(f_2)) = l$. Let f_3 be the pseudo-remainder of $x_1^{l-k-1} f_1$ and f_2. Obviously, $l-1 = \deg(\pi(x_1^{l-k-1} f_1)) = \deg(\pi(f_3))$

and $\deg(f_3) < \deg(f_2)$. Hence, we obtain $\deg(f_3) = \deg(\pi(f_3)) = l - 1$, a contradiction to the definition of l.

Since (b) implies (d) it remains to show $(d) \Rightarrow (a)$:

Assume that P is not an isolated prime ideal of J. Let $J = Q_1 \cap \ldots \cap Q_m$ be an irredundant primary decomposition of J and denote the radical of Q_i by P_i. We can assume that the Q_is are ordered in such a way that there exists an $m' \in \{0, \ldots, m-1\}$ with $P \subseteq P_j$ for $j \in \{1, \ldots, m'\}$ and $P \not\subseteq P_j$ for $j \in \{m'+1, \ldots, m\}$. Thus the prime ideal P is not contained in $\bigcup_{j=m'+1}^{m} P_j$ (see Matsumura, 1970, p. 3). Hence, we can choose an element c of P such that

$$c \in \bigcap_{j=1}^{m'} Q_j \qquad \text{and} \qquad c \notin \bigcup_{j=m'+1}^{m} P_j.$$

Furthermore, let $\{a_1, \ldots, a_r\}$ be a generating set of J, $\{b_1, \ldots, b_k\}$ a generating set of $Q_{m'+1} \cap \ldots \cap Q_m$ and

$$G := \{a_1, \ldots, a_r, b_1x_1, \ldots, b_kx_1, cx_1^2 - x_1\}.$$

Obviously, $\langle G \rangle \cap R = J$. We will show that G is a Gröbner basis of $I := \langle G \rangle$. Let $S = (s_1, \ldots, s_r, s_1, \ldots, s_k, s)$ be a homogeneous syzygy w.r.t. the tuple $(a_1, \ldots, a_r, b_1x_1, \ldots, b_kx_1, cx_1^2)$. Since

$$(Q_{m'+1} \cap \ldots \cap Q_m) : c = Q_{m'+1} \cap \ldots \cap Q_m,$$

the coefficient of s is an element of $Q_{m'+1} \cap \ldots \cap Q_m$. Hence, sx_1 is an element of the monomial ideal $\langle \{a_1, \ldots, a_r, b_1x_1, \ldots, b_kx_1\} \rangle$ and therefore, by Theorem 2.1, G is a Gröbner basis.

We will use this fact in order to show that I is not semi-stable. We have assumed that $J \subseteq P$ and P is not an isolated prime ideal of J. Hence, by definition of m', there exists a $j \in \{m'+1, \ldots, m\}$ with $Q_j \subseteq P_j \subseteq P$. Thus, $\{a_1, \ldots, a_r, b_1, \ldots, b_k, c\} \subseteq P$ and therefore

$$\Delta(\mathrm{lm}(I)\, K[x_1]) = \{\{x_1\}, \emptyset\} \neq \{\emptyset\} = \Delta(\mathrm{lm}(I\, K[x_1])). \quad \square$$

Note that the implication $(a) \Rightarrow (b)$ in Theorem 3.2 is a generalization of the main result in Gianni (1987) and Kalkbrener (1987).

In Theorem 3.2 we have proved that every ideal I in $R[x_1]$ with $I \cap R = J$ is stable if and only if $\ker(\pi)$ is an isolated prime ideal of J. In the following theorem we will give a similar characterization of the stability of multivariate ideals. Note that the implication $(a) \Rightarrow (b)$ in Theorem 3.3 is similar to Proposition 3.10 in Bayer *et al.* (1991) and a generalization of Theorem 2 in Becker (1994).

Theorem 3.3. *Let π be a ring homomorphism from R to K and J an ideal in R with $J \subseteq \ker(\pi)$. Then the following three conditions are equivalent.*

- (a) $\ker(\pi)$ *is an isolated prime ideal of J which equals the corresponding primary component.*
- (b) *For any natural number n, any ideal I in $R[x_1, \ldots, x_n]$ with $I \cap R = J$ and any admissible order $\prec$ on $PP(x_1, \ldots, x_n)$, I is stable under π and $\prec$.*
- (c) *For any ideal I in $R[x_1, x_2]$ with $I \cap R = J$ and any admissible order $\prec$ on $PP(x_1, x_2)$, I is stable under π and $\prec$.*

Proof. Denote the kernel of π by P.

$(a) \Rightarrow (b)$: If P equals the corresponding primary component then it follows from the proof of the previous theorem that we can choose l as 1 in (3.3).

Since (b) implies (c) it remains to show $(c) \Rightarrow (a)$:

If P is not an isolated prime ideal of J it follows from Theorem 3.2 that there exists an ideal I in $R[x_1, x_2]$ which satisfies $I \cap R = J$ and is not semi-stable. Hence, we assume that P is an isolated prime ideal of J which is unequal to the corresponding primary component Q. Let $c \in P$ and $l > 1$ the smallest natural number with $c^l \in Q$. For every non-negative integer j let $B_j = \{b_{j1}, \ldots, b_{ji_j}\}$ be a finite basis of the ideal quotient $J : c^j$. Since $J \subseteq J : c \subseteq J : c^2 \ldots$ is an ascending chain of ideals there exists a natural number r with $J : c^r = J : c^k$ for every $k \geq r$. Define

$$G := \bigcup_{j=0}^{r} \{bx_1^j \mid b \in B_j\} \cup \{cx_2 - x_1\}$$

and $I := \langle G \rangle$. Obviously, $I \cap R = J$. We will now show that G is a Gröbner basis with respect to every admissible order with $x_1 \prec x_2$. Using Theorem 2.1 it suffices to show that for every homogeneous syzygy $S = (s_{11}, \ldots, s_{ri_r}, s)$ w.r.t. the tuple $(b_{11}, \ldots, b_{ri_r}x_1^r, cx_2)$ the monomial sx_1 is an element of the monomial ideal generated by $\bigcup_{j=0}^{r}\{bx_1^j \mid b \in B_j\}$. Let $x_1^{k_1}x_2^{k_2}$ be the degree of S. Obviously, the coefficient of s is an element of the ideal generated by B_{k_1+1} in R. Hence, sx_1 is an element of $\langle\{bx_1^{k_1+1} \mid b \in B_{k_1+1}\}\rangle$ and therefore an element of the ideal generated by $\bigcup_{j=0}^{r}\{bx_1^j \mid b \in B_j\}$.

Since P is an isolated prime ideal of J we have $B_j \subseteq P$ for $j \in \{0, \ldots, l-1\}$ and $B_l \not\subseteq P$. Hence, $\mathrm{lm}(I)\, K[x_1, \ldots, x_n] = \{x_1^l\}$ and $\mathrm{lm}(I\, K[x_1, \ldots, x_n]) = \{x_1\}$. □

Let I be an ideal in $R[x_1, \ldots, x_n]$ such that $\ker(\pi)$ is an isolated prime ideal of $I \cap R$ but unequal to the corresponding primary component. It has been proved in the above theorem that in this case I is not necessarily stable. The next example shows that even the Gröbner basis property may not be preserved for Gröbner bases of I.

Example 3.1. Let $\mathbb{Q}$ denote the rational numbers and define $R := \mathbb{Q}[y]$, $K := \mathbb{Q}$. Let π be the natural map from $\mathbb{Q}[y]$ to $\mathbb{Q}[y]/\langle y \rangle$ and I the ideal in $R[x_1, x_2, x_3, x_4]$ generated by

$$\{y^2,\ yx_1,\ x_1^2,\ yx_2 + x_1,\ x_1x_4 + x_3\}.$$

The set

$$G = \{y^2,\ yx_1,\ x_1^2,\ yx_2 + x_1,\ yx_3,\ x_1x_3,\ x_3^2,\ x_1x_4 + x_3\}$$

is a Gröbner basis of I with respect to the lexicographical order $\prec$ with $x_4 \succ x_3 \succ x_2 \succ x_1$. Thus, $I \cap R = \langle\{y^2\}\rangle$ and $\ker(\pi) = \langle\{y\}\rangle$ is an isolated prime ideal of $I \cap R$. Obviously, I is semi-stable but not stable under π and $\prec$ and the image of G under π is not a Gröbner basis.

As a consequence of Theorems 2.2 and 3.3 we obtain the following characterization of flatness.

Corollary 3.1. *Let π be a ring homomorphism from R to K.*

(a) *The ring homomorphism π is flat iff no proper subideal of the kernel of π is primary.*

(*b*) *If* $\langle 0\rangle \subseteq R$ *is primary but not prime then* π *is not flat.*
(*c*) *If* $\langle 0\rangle \subseteq R$ *is prime then* π *is flat iff the kernel of* π *is* $\langle 0\rangle$.

Proof. Denote the kernel of π by P.

(a) Assume that there exists a proper subideal Q of P which is primary. By Theorem 3.3, there exists an ideal $I \subseteq R[x_1,\ldots,x_n]$ and an admissible order $\prec$ such that I is not stable under π and $\prec$. Hence, by Theorem 2.2, π is not flat.

Assume that no proper subideal Q of P is primary and let I be an ideal in $R[x_1,\ldots,x_n]$ and $\prec$ an admissible order. If $I \cap R \not\subseteq P$ then

$$\operatorname{lm}(I\,K[x_1,\ldots,x_n]) = \langle 1\rangle = \operatorname{lm}(I)\,K[x_1,\ldots,x_n]. \tag{3.4}$$

Otherwise, P is an isolated prime ideal of $I \cap R$ which equals the corresponding primary component. By Theorem 3.3, $\operatorname{lm}(I\,K[x_1,\ldots,x_n]) = \operatorname{lm}(I)\,K[x_1,\ldots,x_n]$. Together with (3.4) and Theorem 2.2, π is flat.

(b) and (c) follow from (a) immediately. □

Example 3.2. Let $R := \mathbb{Q}[x]/\langle x^2(x-1)\rangle$ and consider the following homomorphisms from R to $\mathbb{Q}$: π_1 is the natural map from R to $\mathbb{Q}[x]/\langle x\rangle$ and π_2 is the natural map from R to $\mathbb{Q}[x]/\langle x-1\rangle$. Then π_2 is flat and π_1 is not.

Acknowledgements

I am grateful to Hans-Gert Gräbe and Urs Stammbach for helpful discussions and suggestions.

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Originally received 28 August 1995
Accepted 16 January 1997