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ORIGINAL ARTICLE

Regularity of Po-Γ-semigroups in Terms of Fuzzy Subsemigroups and Fuzzy Bi-ideals

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Abstract In this paper, the notions of fuzzy subsemigroups and fuzzy bi-ideals of a po-Γ-semigroup are introduced with some of their important properties investigated. We obtain some characterizations of regular, intra-regular po-Γ-semigroups in terms of fuzzy bi-ideals. We also give a pointwise characterization of fuzzy regular subsemigroups in a po-Γ-semigroup.

Keywords Po-Γ-semigroup · Fuzzy subsemigroup · Fuzzy bi-ideal · Intra-regular po-Γ-semigroup

1. Introduction

A semigroup is an algebraic structure consisting of a non-empty set S together with an associative binary operation [14]. The formal study of semigroups began in the early 20th century. A. Rosenfeld [18] used the idea of fuzzy set previously introduced by L. A. Zadeh [31] to give the notions of fuzzy subgroups. N. Kuroki [15-17] is the pioneer of fuzzy ideal theory of semigroups. The notion of a Γ-semigroup

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was introduced by Sen [24] as a generalization of semigroups as well as of ternary semigroups. Γ-semigroups have a close connection with Morita equivalence of semigroups. Γ-semigroup have been analyzed by a lot of mathematicians, for instance by Chattopadhyay [1, 2], Dutta and Adhikari [5-7], Hila [12, 13], Chinram [3], Sen et al. [24-28], Seth [29], N. K. Saha [19]. S. K. Sardar and S. K. Majumder [10, 11, 20-22] have introduced the notion of fuzzification of ideals, prime ideals, semiprime ones and ideal extensions of Γ-semigroups and studied them via its operator semigroups. Dutta and Adhikari introduced and studied the notion of po-Γ-semigroup, i.e., partially ordered Γ-semigroup. For some works on fuzzification of Γ-structures we may refer to [8, 9]. Z. W. Mo and X. P. Wang [30] introduced the notion of a fuzzy regular point and gave a pointwise depiction of fuzzy regular subsemigroups, fuzzy completely regular subsemigroups in a semigroup. S. K. Sardar et al. [23] studied properties of fuzzy ideals of po-Γ-semigroups. In this paper, we investigate some properties of fuzzy ideals, fuzzy bi-ideals and fuzzy $(1,2)$-ideals and characterize a po-Γ-semigroup which is left (right) simple, left (right) duo, left (right) regular, intra-regular, regular in terms of fuzzy ideals and fuzzy bi-ideals. Here we will also give a pointwise characterization of fuzzy regular subsemigroups in a po-Γ-semigroup.

2. Preliminaries

In this section, we present some elementary definitions that we will use later in this paper.

Definition 2.1 [27] *Let* $S = \{x, y, z, \cdots\}$ *and* $\Gamma = \{\alpha, \beta, \gamma, \cdots\}$ *be two non-empty sets. Then,* S *is called a* Γ*-semigroup if there exists a mapping* $S \times \Gamma \times S \to S$ *(images to be denoted by* $a\alpha b$*) satisfying*

(1) $x\alpha y \in S$,

(2) $(x\beta y)\gamma z = x\beta(y\gamma z)$

for all $x, y, z \in S$, $\alpha, \beta, \gamma \in \Gamma$.

Definition 2.2 [4] *A* Γ*-semigroup* S *is said to be po-*Γ*-semigroup (partially ordered* Γ*-semigroup) if*

(1) S *and* Γ *are posets,*

(2) $a \leq b$ *in* S *implies that* $a\alpha c \leq b\alpha c$, $c\alpha a \leq c\alpha b$ *in* S *for all* $c \in S$ *and for all* $\alpha \in \Gamma$,

(3) $\alpha \leq \beta$ *in* Γ *implies that* $a\alpha b \leq a\beta b$ *for all* $a, b \in S$.

Remark 2.1 Definitions 2.1 and 2.2 are the definitions of one sided Γ-semigroup and one sided po-Γ-semigroup respectively. It may be noted here that in 1981 Sen [24] introduced the notion of both sided Γ-semigroups, later T. K. Dutta and N. C. Adhikari [7] introduced the notion of both sided po-Γ-semigroup and also introduced the notions of operator semigroups of a both sided Γ-semigroup. Throughout this paper unless or otherwise mentioned S stands for one sided po-Γ-semigroup.

Remark 2.2 The partial order relations on S and Γ are denoted by the same symbol $\leq$.

Example 2.1 [7] Let S be the set of all 2×3 matrices over the set of positive integers and Γ be the set of all 3×2 matrices over the same set. Then, S is a Γ-semigroup with respect to the usual matrix multiplication. Also S and Γ are posets with respect to $\leq$ defined by $(a_{ik}) \leq (b_{ik})$ if and only if $a_{ik} \leq b_{ik}$ for all i, k. Then, S is a po-Γ-semigroup.

Definition 2.3 [31] *A fuzzy subset μ of a non-empty set X is a function $\mu : X \to [0, 1]$.*

Definition 2.4 [20] *Let μ be a fuzzy subset of a non-empty set X. Then, the set $\mu_t = \{x \in X \mid \mu(x) \geq t\}$ for $t \in [0, 1]$, is called the level subset or t-level subset of μ.*

3. Fuzzy Subsemigroup and Fuzzy Bi-idea

Definition 3.1 *A non-empty subset I of a po-Γ-semigroup S is said to be a subsemigroup of S if $I\Gamma I \subseteq I$.*

Definition 3.2 *Let S be a po-Γ-semigroup. A non-empty subset I of S is said to be a right ideal (left ideal) of S if*

(1) $I\Gamma S \subseteq I$ (*resp.* $S\Gamma I \subseteq I$),

(2) *for all $a, b \in S$, $a \in I$ and $b \leq a$ imply $b \in I$.*

I is said to be an ideal of S if it is a right ideal as well as a left ideal of S.

Definition 3.3 *A subsemigroup I of a po-Γ-semigroup S is called a bi-ideal of S if*

(1) $I\Gamma S\Gamma I \subseteq I$,

(2) *for all $a, b \in S$, $a \in I$ and $b \leq a$ imply $b \in I$.*

Definition 3.4 *A subsemigroup I of a po-Γ-semigroup S is called a $(1, 2)$-ideal of S if*

(1) $I\Gamma S\Gamma I\Gamma I \subseteq I$,

(2) *for all $a, b \in S$, $a \in I$ and $b \leq a$ imply $b \in I$.*

Definition 3.5 *Let μ and σ be two fuzzy subsets of a po-Γ-semigroup S. Then, we define the following:*

(1) $(\mu \cap \sigma)(x) = \min\{\mu(x), \sigma(x)\}$ *for all* $x \in S$,

(2) $(\mu \cup \sigma)(x) = \max\{\mu(x), \sigma(x)\}$ *for all* $x \in S$,

(3) *for all* $x \in S$,

$$(\mu \circ \sigma)(x) = \begin{cases} \sup\limits_{x \leq y\gamma z} \{\min\{\mu(y), \sigma(z)\}\}, & \textit{if } \exists y, z \in S, \gamma \in \Gamma \textit{ with } x \leq y\gamma z, \\ 0, & \textit{otherwise}. \end{cases}$$

Definition 3.6 *A non-empty fuzzy subset μ of a po-Γ-semigroup S is called a fuzzy subsemigroup of S if $\mu(x\gamma y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, y \in S$ and for all $\gamma \in \Gamma$.*

Definition 3.7 *A non-empty fuzzy subset μ of a po-Γ-semigroup S is called a fuzzy left ideal (right ideal) of S if*

(1) *$x \leq y$ implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$,*

(2) *$\mu(x\alpha y) \geq \mu(y)(\mu(x\alpha y) \geq \mu(x))$ for all $x, y \in S$, for all $\alpha \in \Gamma$.*

μ is said to be a fuzzy ideal of S if it is a fuzzy right ideal as well as a fuzzy left ideal of S.

Definition 3.8 *A fuzzy subsemigroup μ of a po-Γ-semigroup S is called a fuzzy bi-ideal of S if*

(1) *$\mu(x\beta s\gamma y) \geq \min\{\mu(x), \mu(y)\}$ for all $x, s, y \in S$, for all $\beta, \gamma \in \Gamma$,*

(2) *$x \leq y$ implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$.*

Example 3.1 Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$ and the operation mapping.

α	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	c

β	0	a	b	c
0	0	0	0	0
a	0	0	0	b
b	0	0	0	b
c	b	b	b	b

γ	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

Now, S and Γ are posets with respect to the relation $\leq$ given by $0 \leq a \leq b \leq c$ and $\gamma \leq \beta \leq \alpha$ respectively. Clearly, S is a po-Γ-semigroup. Moreover the fuzzy set $\mu : S \to [0, 1]$ defined by $\mu(0) = 0.9$, $\mu(a) = 0.8$, $\mu(b) = 0.7$, $\mu(c) = 0.5$ is a fuzzy bi-ideal of S. It can be checked easily that μ is neither a fuzzy left ideal nor a fuzzy right ideal of S.

Definition 3.9 *A fuzzy subsemigroup μ of a po-Γ-semigroup S is called a fuzzy $(1, 2)$-ideal of S if*

(1) *$\mu(x\alpha w\beta(y\gamma z)) \geq \min\{\mu(x), \mu(y), \mu(z)\}$ for all $x, w, y, z \in S$, for all $\alpha, \beta, \gamma \in \Gamma$,*

(2) *$x \leq y$ implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$.*

Definition 3.10 *Let μ be a fuzzy subset of a set X, $\alpha \in [0, 1 - \sup\{\mu(x) \mid x \in X\}]$ and $\beta \in (0, 1]$. Then, a mapping $\mu^C_{\beta\alpha} \mid X \to [0, 1]$ is called a fuzzy magnified translation of μ if $\mu^C_{\beta\alpha}(x) = \beta\mu(x) + \alpha$ for all $x \in X$.*

It can be checked easily that μ is a fuzzy bi-ideal of S if and only if the fuzzy magnified translation $\mu^C_{\beta\alpha}$ of μ is a fuzzy bi-ideal of S.

Definition 3.11 *Let R and S be two po-Γ-semigroups. Then, a mapping $f : R \to S$ is said to be a homomorphism of po-Γ-semigroups if it satisfies the following properties:*

(1) *$f(x\gamma y) = f(x)\gamma f(y)$ for all $x, y \in R$ and for all $\gamma \in \Gamma$,*

(2) $x \leq y \Rightarrow f(x) \leq f(y)$ *for all* $x, y \in R$.

It can be checked that if λ is a fuzzy bi-ideal of S, then $f^{-1}(\lambda)$ is also a fuzzy bi-ideal of R where $(f^{-1}(\lambda))(r\gamma s) := \lambda(f(r\gamma s))$ for all $r, s \in R, \gamma \in \Gamma$, provided $f^{-1}(\lambda)$ non-empty.

Now, suppose that μ and λ be two fuzzy bi-ideals of a po-Γ-semigroup S. Then, $\mu \cap \lambda$ is a fuzzy bi-ideal of S provided $\mu \cap \lambda$ non-empty. Again, if θ is an endomorphism and μ a fuzzy bi-ideal of S, then $\mu[\theta]$ is a fuzzy bi-ideal of S, where $\mu[\theta](x) := \mu(\theta(x))$ for all $x \in S$. Now, let μ be a fuzzy bi-ideal of S. Then, μ^{α} is a fuzzy bi-ideal, for every real number $\alpha \geq 0$ where μ^{α} is defined by $\mu^{\alpha}(x) = (\mu(x))^{\alpha}$ for all $x \in S$.

Now, here we state some significant theorems and propositions on fuzzy bi-ideal of a po-Γ-semigroup. As it is very easy to prove the following theorems and propositions, we omit the proofs.

Theorem 3.1 *Let I be a non-empty subset of a po-Γ-semigroup S and χ_I be the characteristic function of I. Then, I is a bi-ideal of S if and only if χ_I is a fuzzy bi-ideal of S.*

Theorem 3.2 *Let S be a po-Γ-semigroup and μ be a non-empty fuzzy subset of S. Then, μ is a fuzzy bi-ideal of S if and only if μ_t is a bi-ideal of S for all $t \in Im\, \mu$, where $\mu_t = \{x \in S \mid \mu(x) \geq t\}$.*

Theorem 3.3 *Let μ be a non-empty fuzzy subset of a po-Γ-semigroup S. Then, the following conditions are equivalent:*

(1) *μ is a fuzzy bi-ideal of S.*

(2) *$\mu \circ \mu \subseteq \mu$, $\mu \circ \chi \circ \mu \subseteq \mu$, and $x \leq y$ implies $\mu(x) \geq \mu(y)$ for all $x, y \in S$, where χ is the characteristic function of S.*

Proof (1) $\Rightarrow$ (2) Let μ be a bi-ideal of S and $x \in S$. Then, two cases may arise namely Case (i): $\exists\, y, z \in S$ and $\gamma \in \Gamma$ such that $x \leq y\gamma z$ and Case (ii): $\nexists\, y, z \in S$ and $\gamma \in \Gamma$ such that $x \leq y\gamma z$. For Case (i): $(\mu \circ \mu)(x) = \sup\limits_{x \leq y\gamma z} \min\{\mu(y), \mu(z)\} \leq \sup\limits_{x \leq y\gamma z} \mu(y\gamma z)$ (since μ is a fuzzy subsemigroup) $\leq \mu(x)$. For Case (ii): $(\mu \circ \mu)(x) = 0 \leq \mu(x)$. Hence, $\mu \circ \mu \subseteq \mu$. Again to prove $\mu \circ \chi \circ \mu \subseteq \mu$, let $a \in S$. Suppose that $\exists$ $x, y, p, q \in S$ and $\gamma, \beta \in \Gamma$ such that $a \leq x\gamma y$ and $x \leq p\beta q$. Then,

$$\begin{aligned}
(\mu \circ \chi \circ \mu)(a) &= \sup_{a \leq x\gamma y} \{\min\{(\mu \circ \chi)(x), \mu(y)\}\} \\
&= \sup_{a \leq x\gamma y} \{\min\{\sup_{x \leq p\beta q} \{\min\{\mu(p), \chi(q)\}\}, \mu(y)\}\} \\
&= \sup_{a \leq x\gamma y} \{\min\{\sup_{x \leq p\beta q} \{\min\{\mu(p), 1\}\}, \mu(y)\}\} \\
&= \sup_{a \leq p\beta q\gamma y} \{\min\{\mu(p), \mu(y)\}\} \\
&\leq \sup_{a \leq p\beta q\gamma y} \mu(p\beta q\gamma y) \leq \mu(a) \text{ (cf.Definition 3.8).}
\end{aligned}$$

Otherwise, $(\mu \circ \chi \circ \mu)(a) = 0 \leq \mu(a)$. Thus, $\mu \circ \chi \circ \mu \subseteq \mu$.

Again from the definition of fuzzy bi-ideal, $x \leq y$ implies that $\mu(x) \geq \mu(y)$ for all $x, y \in S$.

(2) ⇒ (1) Let (2) hold. Since $\mu \circ \mu \subseteq \mu$, so for $x, y \in S$ and $\gamma \in \Gamma$ we obtain

$$\mu(x\gamma y) \geq (\mu \circ \mu)(x\gamma y) \geq \min\{\mu(x), \mu(y)\}.$$

So, μ is a fuzzy subsemigroup of S. Now, let $x, y, z \in S$ and $\beta, \gamma \in \Gamma$. Since $\mu \circ \chi \circ \mu \subseteq \mu$, it follows that

$$\begin{aligned}\mu(x\beta y\gamma z) &\geq (\mu \circ \chi \circ \mu)(x\beta y\gamma z)\\ &\geq \min\{(\mu \circ \chi)(x\beta y), \mu(z)\}\\ &\geq \min\{\min\{\mu(x), \chi(y)\}, \mu(z)\}\\ &= \min\{\min\{\mu(x), 1\}, \mu(z)\}\\ &= \min\{\mu(x), \mu(z)\}.\end{aligned}$$

Hence, μ is a fuzzy bi-ideal of S.

Definition 3.12 *A po-Γ-semigroup S is called left (right) duo if every left (resp. right) ideal of S is a two sided ideal of S. S is called duo if it is left and right duo.*

Example 3.2 Let $S = -\mathbb{Z}$ and $\Gamma = -2\mathbb{Z}$. So, clearly S is a po-Γ-semigroup with respect to usual multiplication and usual $\leq$ of $\mathbb{Z}$. Then, any ideal (left, right or both-sided) of S is of the form $(-n\mathbb{Z}]$ where for any subset A of S, $(A] = \{x \in S \mid x \leq y$ for some $y \in A\}$. Hence, clearly S is duo.

Definition 3.13 *A po-Γ-semigroup S is called fuzzy left (right) duo if every fuzzy left (resp. right) ideal of S is a fuzzy ideal of S. S is called fuzzy duo if it is fuzzy left and fuzzy right duo.*

Example 3.3 Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \beta, \gamma\}$. Then, S and Γ are posets with respect to the relation $\leq$ given by $0 \leq a \leq b \leq c$ and $\gamma \leq \alpha \leq \beta$ respectively. Now, we define $s\alpha t$ for all $s, t \in S$ and for all $\alpha \in \Gamma$ as follows:

α	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	b

β	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

γ	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

Then, S is a po-Γ-semigroup. Now, any fuzzy left ideal or fuzzy right ideal of S is a fuzzy both sided ideal as $x \leq y \Rightarrow \mu(x) \geq \mu(y)$ for all $x, y \in S$. Hence, S is fuzzy duo po-Γ-semigroup. It is easy to check that S is duo.

Definition 3.14 *A po-Γ-semigroup S is called left (right) regular if for each $a \in S$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq x\alpha a\beta a$ (resp. $a \leq a\beta a\alpha x$).*

Definition 3.15 *A po-Γ-semigroup S is called regular if for each $a \in S$, there exist $x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$.*

Theorem 3.4 *If S is a regular left duo (right duo, duo) po-Γ-semigroup, then the following conditions are equivalent:*

(1) *μ is a fuzzy right ideal (resp. fuzzy left ideal, fuzzy ideal) of S.*

(2) *μ is a fuzzy bi-ideal of S.*

Proof (1) $\Rightarrow$ (2) Let μ be a fuzzy right ideal of S, $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. Then,

$$\mu(x\alpha y) \geq \mu(x) \geq \min\{\mu(x), \mu(y)\} \text{ (cf. Definition 3.7)}$$

and

$$\mu(x\alpha y\beta z) = \mu(x\alpha(y\beta z)) \geq \mu(x) \geq \min\{\mu(x), \mu(z)\}.$$

Now, let $x, y \in S$ such that $x \leq y$. Then, $\mu(x) \geq \mu(y)$ (since μ means a fuzzy right ideal of S). Hence, μ is a fuzzy bi-ideal of S.

Similarly, we can prove the other cases.

(2) $\Rightarrow$ (1) Let μ be a fuzzy bi-ideal of S, $x, y \in S$ and $\gamma \in \Gamma$. Since S is regular, there exist $\alpha, \beta \in \Gamma$ and $z' \in S$ such that $x \leq x\alpha z'\beta x$. So, $x \in (x\Gamma S\Gamma x] = \{x\gamma_1 s\gamma_2 x \ \gamma_1, \gamma_2 \in \Gamma, s \in S\} \subseteq (x\Gamma S \cap S\Gamma x] \subseteq (x\Gamma S] \cap (S\Gamma x]$ (cf. Proposition 3.12 [23]). Hence, $x\gamma y \in ((x\Gamma S] \cap (S\Gamma x])\Gamma S \subseteq (x\Gamma S]\Gamma S \cap (S\Gamma x]\Gamma S \subseteq (x\Gamma S] \cap (S\Gamma x]$ (since S is left duo, $(S\Gamma x]\Gamma S \subseteq (S\Gamma x]$). Hence, $x\gamma y \in (x\Gamma S]$ and $x\gamma y \in (S\Gamma x]$, i.e., $x\gamma y \leq x\alpha_1 s_1$ and $x\gamma y \leq s_2\alpha_2 x$ for some $s_1, s_2 \in S$ and $\alpha_1, \alpha_2 \in \Gamma$. Again since S is regular, there exist $\bar{\alpha}, \bar{\beta} \in \Gamma$ and $z \in S$ such that $x\gamma y \leq (x\gamma y)\bar{\alpha}z\bar{\beta}(x\gamma y) \leq (x\alpha_1 s_1)\bar{\alpha}z\bar{\beta}(s_2\alpha_2 x)$. Then, by the definition of fuzzy bi-ideal we see that $\mu(x\gamma y) \geq \mu((x\alpha_1 s_1)\bar{\alpha}z\bar{\beta}(s_2\alpha_2 x)) = \mu(x\alpha_1(s_1\bar{\alpha}z)\bar{\beta}s_2\alpha_2 x) = \mu(x\alpha_1 s_3\bar{\beta}s_2\alpha_2 x)$, where $s_3 = s_1\bar{\alpha}z] = \mu(x\alpha_1(s_3\bar{\beta}s_2)\alpha_2 x) \geq \min\{\mu(x), \mu(x)\} = \mu(x)$.

Now, let $x, y \in S$ such that $x \leq y$. Then, $\mu(x) \geq \mu(y)$ (since μ is a fuzzy bi-ideal of S). Hence, μ is a fuzzy right ideal of S. Similarly, we can prove the other cases.

The following example shows that the converse of Theorem 3.4 is not true.

Example 3.4 Let $S = \{0, a, b, c\}$ and $\Gamma = \{\alpha, \gamma\}$. Then, S and Γ are posets with respect to the relation $\leq$ given by $0 \leq a \leq b \leq c$ and $\gamma \leq \alpha$ respectively. Now, we define $s\alpha t$ for all $s, t \in S$ and for all $\alpha \in \Gamma$ as follows:

α	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	b

γ	0	a	b	c
0	0	0	0	0
a	0	0	0	0
b	0	0	0	0
c	0	0	0	0

Then, S is a po-Γ-semigroup. Now, any fuzzy subset μ of S becomes a fuzzy both sided ideal as $x \leq y \Rightarrow \mu(x) \geq \mu(y) \ \forall x, y \in S$. Let μ be any fuzzy bi-ideal of S. Then, μ is fuzzy right ideal of S and we know a fuzzy right ideal is always a fuzzy bi-ideal. But here S is not regular as there exists no $x \in S$ and $\beta, \eta \in \Gamma$ such that $c \leq c\beta x\eta c$.

Theorem 3.5 *Let S be a regular left duo (right duo, duo) po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *μ is a fuzzy bi-ideal of S.*

(2) *μ is a fuzzy $(1, 2)$-ideal of S.*

Proof (1) ⇒ (2) Let μ be a fuzzy bi-ideal of S and $x, w, y, z \in S$, $\alpha, \beta, \gamma \in \Gamma$. Then,

$$\begin{aligned}\mu(x\alpha w\beta(y\gamma z)) &= \mu((x\alpha w\beta y)\gamma z)\\ &\geq \min\{\mu(x\alpha w\beta y), \mu(z)\} \text{ (cf. Definition 3.6)}\\ &\geq \min\{\min\{\mu(x), \mu(y)\}, \mu(z)\} \text{ (cf. Definition 3.8)}\\ &= \min\{\mu(x), \mu(y), \mu(z)\}.\end{aligned}$$

Again for any $x, y \in S$ with $x \leq y$, $\mu(x) \geq \mu(y)$ (since μ is a fuzzy bi-ideal). Hence, μ is a fuzzy $(1,2)$-ideal of S.

(2) ⇒ (1) Let μ be a fuzzy $(1,2)$-ideal of S and $x, w, y \in S$, $\gamma, \delta \in \Gamma$. Since S is regular, there exist $\alpha, \beta \in \Gamma$ and $z' \in S$ such that $x \leq x\alpha z'\beta x$. So $x \in (x\Gamma S\Gamma x] \subseteq (x\Gamma S \cap S\Gamma x] \subseteq (x\Gamma S] \cap (S\Gamma x]$. Hence, $x\gamma w \in (x\Gamma S] \cap (S\Gamma x]$ (using the second part of the proof of Theorem 3.4). So $x\gamma w \in (x\Gamma S]$ and $x\gamma w \in (S\Gamma x]$, i.e., $x\gamma w \leq x\alpha_1 s_1$ and $x\gamma w \leq s_2\alpha_2 x$ for some $s_1, s_2 \in S$ and $\alpha_1, \alpha_2 \in \Gamma$. Again since S is regular, there exist $\bar{\alpha}, \bar{\beta} \in \Gamma$ and $z \in S$ such that $x\gamma w \leq (x\gamma w)\bar{\alpha}z\bar{\beta}(x\gamma w) \leq (x\alpha_1 s_1)\bar{\alpha}z\bar{\beta}(s_2\alpha_2 x)$. Then, by definition of fuzzy $(1,2)$-ideal we have

$$\begin{aligned}\mu(x\gamma w\delta y) &\geq \mu(((x\alpha_1 s_1)\bar{\alpha}z\bar{\beta}(s_2\alpha_2 x))\delta y)\\ &= \mu((x\alpha_1(s_1\bar{\alpha}z)\bar{\beta}s_2\alpha_2 x)\delta y)\\ &= \mu((x\alpha_1 s_3\bar{\beta}s_2\alpha_2 x)\delta y) \text{ (where } s_3 = s_1\bar{\alpha}z)\\ &= \mu(x\alpha_1(s_3\bar{\beta}s_2)\alpha_2 x\delta y) = \mu(x\alpha_1 s_4\alpha_2(x\delta y)) \text{ (where } s_4 = s_3\bar{\beta}s_2)\\ &\geq \min\{\mu(x), \mu(x), \mu(y)\} = \min\{\mu(x), \mu(y)\}.\end{aligned}$$

Now, let $x, y \in S$ such that $x \leq y$. Then, $\mu(x) \geq \mu(y)$ as μ is a fuzzy $(1,2)$-ideal of S. Hence, μ is a fuzzy bi-ideal of S.

Theorem 3.6 *Let S be a regular po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S left duo (right duo, duo).*

(2) *S fuzzy left duo (fuzzy right duo, fuzzy duo).*

Proof (1) ⇒ (2) Let S be left duo, μ be any fuzzy left ideal of S and $a, b \in S, \gamma \in \Gamma$. Since S is regular and left duo, $a\gamma b \in (a\Gamma S] \cap (S\Gamma a]$ (using the second part of the proof of Theorem 3.4). So there exist $s \in S$, $\alpha \in \Gamma$ such that $a\gamma b \leq s\alpha a$. Then, by definition of fuzzy left ideal we see that $\mu(a\gamma b) \geq \mu(s\alpha a) \geq \mu(a)$. So μ becomes a fuzzy right ideal of S. So μ is a fuzzy ideal of S. Hence, S is fuzzy left duo.

(2) ⇒ (1) Let S be fuzzy left duo. Let A be any left ideal of S. Then, the characteristic function χ_A of A is a fuzzy left ideal of S. Then, by our assumption χ_A is a fuzzy ideal of S. So A is an ideal of S. Hence, S is left duo. Similarly we can prove the other cases.

Theorem 3.7 *Let S be a regular po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *Every bi-ideal of S is a right ideal (resp. left ideal, two-sided ideal) of S.*

(2) *Every fuzzy bi-ideal of S is a fuzzy right ideal (resp. fuzzy left ideal, fuzzy ideal) of S.*

Proof (1) $\Rightarrow$ (2) Let (1) hold, μ be a fuzzy bi-ideal of S, $a, b \in S$ and $\gamma \in \Gamma$. Then, $(a\Gamma S\Gamma a]$ is a bi-ideal of S. So, by hypothesis $(a\Gamma S\Gamma a]$ is a right ideal of S. Since S is regular, $a \in (a\Gamma S\Gamma a]$. So, by definition of right ideal we have $a\gamma b \in (a\Gamma S\Gamma a]\Gamma S \subseteq (a\Gamma S\Gamma a]$. Then, there exist $s \in S$, $\alpha, \beta \in \Gamma$ such that $a\gamma b \leq a\alpha s\beta a$. So $\mu(a\gamma b) \geq \mu(a\alpha s\beta a) \geq \min\{\mu(a), \mu(a)\} = \mu(a)$ (cf. Definition 3.8). Hence, μ is a fuzzy right ideal of S.

(2) $\Rightarrow$ (1) Let (2) hold and A be any bi-ideal of S. Then, it follows from Theorem 3.1 that the characteristic function χ_A of A denotes a fuzzy bi-ideal of S. Hence, by hypothesis it is a fuzzy right ideal of S. Then, since A is non-empty, A is a right ideal of S. Similarly, we can prove all other cases.

Now, we obtain below theorems on characterizations on regular, intra-regular po-Γ-semigroup in terms of fuzzy ideals and fuzzy bi-ideals.

Theorem 3.8 *Let S be a po-Γ-semigroup. Then, the following are equivalent:*

(1) *S is regular.*

(2) *$\mu \cap \lambda = \mu \circ \lambda \circ \mu$ for every fuzzy bi-ideal μ and every fuzzy both-sided ideal λ of S.*

Proof (1) $\Rightarrow$ (2) Let S be regular, μ be a fuzzy bi-ideal and λ be a fuzzy both-sided ideal of S. Since μ is a fuzzy bi-ideal, from Theorem 3.3 we can say $\mu \circ \chi \circ \mu \subseteq \mu$ (where χ is the characteristic function of S). So $\mu \circ \lambda \circ \mu \subseteq \mu \circ \chi \circ \mu \subseteq \mu$. Again $\mu \circ \lambda \circ \mu \subseteq \chi \circ \lambda \circ \chi \subseteq \lambda$ (since λ is a fuzzy both-sided ideal). Hence, $\mu \circ \lambda \circ \mu \subseteq \mu \cap \lambda$. Now, let $a \in S$. Since S is regular, $\exists\, x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$ and so $a \leq a\alpha x\beta a\alpha x\beta a$. Then,

$$\begin{aligned}(\mu \circ \lambda \circ \mu)(a) &= \sup_{a \leq u\gamma v}\{\min\{\mu(u), (\lambda \circ \mu)(v)\}\}\\ &\geq \min\{\mu(a), (\lambda \circ \mu)(x\beta a\alpha x\beta a)\}\\ &\geq \min\{\mu(a), \min\{\lambda(x\beta a\alpha x), \mu(a)\}\}\\ &\geq \min\{\mu(a), \lambda(x\beta a\alpha x), \mu(a)\}\\ &\geq \min\{\mu(a), \lambda(a)\} \text{ (since } \lambda \text{ is a fuzzy both-sided ideal).}\end{aligned}$$

So $\mu \cap \lambda \subseteq \mu \circ \lambda \circ \mu$. Hence, $\mu \cap \lambda = \mu \circ \lambda \circ \mu$.

(2) $\Rightarrow$ (1) Let (2) hold. So $\mu \cap \chi = \mu \circ \chi \circ \mu$ (where χ is the characteristic function of S), which means $\mu = \mu \circ \chi \circ \mu$ for every fuzzy bi-ideal μ of S. Let $a \in S$. Consider $B(a) = (a\Gamma S\Gamma a] \cup (a\Gamma a] \cup (\{a\}]$ which is the bi-ideal generated by a in S. Then, $\chi_{B(a)}$, characteristic function of $B(a)$, is a fuzzy bi-ideal of S (cf. Theorem 3.1). Therefore, $\chi_{B(a)} = \chi_{B(a)} \circ \chi \circ \chi_{B(a)}$. Since $a \in B(a)$, $(\chi_{B(a)} \circ \chi \circ \chi_{B(a)})(a) = \chi_{B(a)}(a) = 1$. So $\exists\, x, y \in S$ and $\alpha \in \Gamma$ with $a \leq x\alpha y$ such that $\chi_{B(a)}(x) = 1$ and $(\chi \circ \chi_{B(a)})(y) = 1$. So $\exists\, p, q \in S$ and $\beta \in \Gamma$ with $y \leq p\beta q$ such that $\chi(p) = 1$ and $\chi_{B(a)}(q) = 1$. Hence, we can assert that $\exists$ $x, q \in B(a)$, $p \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq x\alpha p\beta q$, i.e., $a \in B(a)\Gamma S\Gamma B(a)$ which indicates the regularity of S.

Theorem 3.9 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is regular.*

(2) *$\mu \cap \lambda \subseteq \mu \circ \lambda$ for every fuzzy bi-ideal μ and every fuzzy left ideal λ of S.*

Proof (1) $\Rightarrow$ (2) Let S be regular, μ be a fuzzy bi-ideal and λ be a fuzzy left ideal of S. Now, let $a \in S$. Since S is regular, $\exists\ x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. Then,

$$\begin{aligned}(\mu \circ \lambda)(a) &= \sup_{a \leq u\gamma v} \{\min\{\mu(u), \lambda(v)\}\} \\ &\geq \min\{\mu(a), \lambda(x\beta a)\} \\ &\geq \min\{\mu(a), \lambda(a)\} \text{ (since } \lambda \text{ is a fuzzy left ideal).}\end{aligned}$$

Hence, $\mu \cap \lambda \subseteq \mu \circ \lambda$.

(2) $\Rightarrow$ (1) Let us assume (2) hold, μ be a fuzzy right ideal and λ be a fuzzy left ideal of S. Since every fuzzy right ideal is a fuzzy bi-ideal of S, so μ is a fuzzy bi-ideal of S. Then, $\mu \cap \lambda \subseteq \mu \circ \lambda$. Again $\mu \circ \lambda \subseteq \mu \cap \lambda$. So $\mu \circ \lambda = \mu \cap \lambda$. Hence, (cf. Theorem 3.17 [23]) S is regular.

Theorem 3.10 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is regular.*

(2) *$\mu \cap \lambda \subseteq \lambda \circ \mu$ for every fuzzy bi-ideal μ and every fuzzy right ideal λ of S.*

Proof The proof is very similar to that of Theorem 3.9.

Theorem 3.11 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is regular.*

(2) *$\nu \cap \mu \cap \lambda \subseteq \nu \circ \mu \circ \lambda$ for every fuzzy right ideal ν, fuzzy bi-ideal μ and every fuzzy left ideal λ of S.*

Proof (1) $\Rightarrow$ (2) Let S be regular, ν be a fuzzy right ideal, μ be a fuzzy left ideal and λ be a fuzzy both-sided ideal of S. Now, let $a \in S$. Since S is regular, $\exists x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$. Then,

$$\begin{aligned}(\nu \circ \mu \circ \lambda)(a) &= \sup_{a \leq u\gamma v} \{\min\{\nu(u), (\mu \circ \lambda)(v)\}\} \\ &\geq \min\{\nu(a\alpha x), (\mu \circ \lambda)(a)\} \\ &= \min\{\nu(a\alpha x), \sup_{a \leq p\eta q} \{\min\{\mu(p), \lambda(q)\}\}\} \\ &\geq \min\{\nu(a), \min\{\mu(a), \lambda(x\beta a)\}\} \text{ (since } \nu \text{ is a fuzzy right ideal)} \\ &\geq \min\{\nu(a), \mu(a), \lambda(a)\} \text{ (since } \lambda \text{ is a fuzzy left ideal).}\end{aligned}$$

Hence, $\nu \cap \mu \cap \lambda \subseteq \nu \circ \mu \circ \lambda$.

(2) $\Rightarrow$ (1) Let (2) hold, ν be a fuzzy right ideal and λ be a fuzzy left ideal of S. Now, χ (the characteristic function of S) is a fuzzy bi-ideal of S. Then, $\nu \cap \lambda = \nu \cap \chi \cap \lambda \subseteq \nu \circ \chi \circ \lambda \subseteq \nu \circ \lambda$. Again $\nu \circ \lambda \subseteq \nu \cap \lambda$. So $\nu \circ \lambda = \nu \cap \lambda$. Hence, (cf. Theorem 3.17 [23]) S is regular.

Definition 3.16 *A po-Γ-semigroup S is called intra-regular if for each $a \in S$, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq x\alpha a\beta a\gamma y$.*

Theorem 3.12 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is regular and intra-regular.*

(2) *$\mu \cap \lambda \subseteq \mu \circ \lambda$ for every fuzzy bi-ideals μ and λ of S.*

Proof (1) $\Rightarrow$ (2) Let (1) hold, μ and λ be any two fuzzy bi-ideals of S. Now, let $a \in S$. Since S is regular, $\exists x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$ and so $a \leq a\alpha x\beta a\alpha x\beta a$. Again since S is intra-regular, $\exists\, y, z \in S$ and $\alpha_1, \beta_1, \gamma_1 \in \Gamma$ such that $a \leq y\alpha_1 a\beta_1 a\gamma_1 z$. Then, $a \leq a\alpha x\beta y\alpha_1 a\beta_1 a\gamma_1 z\alpha x\beta a$. So,

$$\begin{aligned}(\mu \circ \lambda)(a) &= \sup_{a \leq u\gamma v} \{\min\{\mu(u), \lambda(v)\}\} \\ &\geq \min\{\mu(a\alpha x\beta y\alpha_1 a), \lambda(a\gamma_1 z\alpha x\beta a)\} \\ &\geq \min\{\min\{\mu(a), \mu(a)\}, \min\{\lambda(a), \lambda(a)\}\} \\ &\quad \text{(since } \mu \text{ and } \lambda \text{ are fuzzy bi-ideals)} \\ &= \min\{\mu(a), \lambda(a)\}.\end{aligned}$$

Hence, $\mu \cap \lambda \subseteq \mu \circ \lambda$.

(2) $\Rightarrow$ (1) Let (2) hold, μ be a fuzzy right ideal and λ be a fuzzy left ideal of S. Then, μ and λ are fuzzy bi-ideals of S. So $\mu \cap \lambda \subseteq \mu \circ \lambda$. Again $\mu \circ \lambda \subseteq \mu \cap \lambda$. So $\mu \circ \lambda = \mu \cap \lambda$. Hence, S is regular (cf. Theorem 3.17 [23]). Again let $a \in S$ and $B(a)$ be the bi-ideal in S generated by a. So $B(a) = (a\Gamma S\Gamma a] \cup (a\Gamma a] \cup (\{a\}]$. Now, $\chi_{B(a)}$, the characteristic function of $B(a)$, is a fuzzy bi-ideal of S (cf. Theorem 3.1). Hence, by hypothesis $\chi_{B(a)} \cap \chi_{B(a)} \subseteq \chi_{B(a)} \circ \chi_{B(a)}$. Since $a \in B(a)$, $\chi_{B(a)}(a) = 1$ whence $(\chi_{B(a)} \circ \chi_{B(a)})(a) = 1$. So $\exists\, x, y \in S$, $\alpha \in \Gamma$ with $a \leq x\alpha y$ such that $\chi_{B(a)}(x) = 1 = \chi_{B(a)}(y)$, i.e., $x, y \in B(a)$. Then, we will get $p, q \in S$ and $\beta, \gamma, \eta \in \Gamma$ such that $a \leq p\beta a\gamma a\eta q$. Hence, S is intra-regular.

Theorem 3.13 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is regular and intra-regular.*

(2) *$\mu \cap \lambda \subseteq (\mu \circ \lambda) \cap (\lambda \circ \mu)$ for every fuzzy bi-ideals μ and λ of S.*

Proof (1) $\Rightarrow$ (2) Let (1) hold and μ, λ be any two fuzzy bi-ideals of S. Then, we have $\mu \cap \lambda \subseteq \mu \circ \lambda$. Similarly, $\mu \cap \lambda = \lambda \cap \mu \subseteq \lambda \circ \mu$. Hence, $\mu \cap \lambda \subseteq (\mu \circ \lambda) \cap (\lambda \circ \mu)$.

(2) $\Rightarrow$ (1) Let (2) hold and μ, λ be two fuzzy bi-ideals of S. So $\mu \cap \lambda \subseteq (\mu \circ \lambda) \cap (\lambda \circ \mu) \subseteq \mu \circ \lambda$. Hence, by Theorem 3.28 we can conclude that S is regular and intra-regular.

Theorem 3.14 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is regular and intra-regular.*

(2) *$\mu \cap \lambda \subseteq \mu \circ \lambda \circ \mu$ for every fuzzy bi-ideals μ and λ of S.*

Proof (1) $\Rightarrow$ (2) Let (1) hold and μ, λ be any two fuzzy bi-ideals of S. Now, let $a \in S$. Since S is regular, $\exists\ x \in S$ and $\alpha, \beta \in \Gamma$ such that $a \leq a\alpha x\beta a$ and so $a \leq a\alpha x\beta a\alpha x\beta a\alpha x\beta a$. Again since S is intra-regular, $\exists\ y, z \in S$ and $\alpha_1, \beta_1, \gamma_1 \in \Gamma$ such that $a \leq y\alpha_1 a\beta_1 a\gamma_1 z$. Then, $a \leq (a\alpha x\beta y\alpha_1 a)\beta_1(a\gamma_1 z\alpha x\beta y\alpha_1 a)\beta_1(a\gamma_1 z\alpha x\beta a)$. Then,

$$\begin{aligned}(\mu \circ \lambda \circ \mu)(a) &= \sup_{a \leq u\gamma v}\{\min\{\mu(u), (\lambda \circ \mu)(v)\}\} \\ &\geq \min\{\mu(a\alpha x\beta y\alpha_1 a), (\lambda \circ \mu)((a\gamma_1 z\alpha x\beta y\alpha_1 a)\beta_1(a\gamma_1 z\alpha x\beta a))\} \\ &\geq \min\{\min\{\mu(a), \mu(a)\}, \min\{\lambda(a\gamma_1 z\alpha x\beta y\alpha_1 a), \mu(a\gamma_1 z\alpha x\beta a)\}\} \\ &\geq \min\{\mu(a), \lambda(a), \mu(a)\} \text{ (since } \mu \text{ and } \lambda \text{ are fuzzy bi-ideals)} \\ &= (\mu \cap \lambda)(a).\end{aligned}$$

Hence, $\mu \cap \lambda \subseteq \mu \circ \lambda \circ \mu$.

(2) $\Rightarrow$ (1) Let (2) hold and $a \in S$. Consider $B(a) = (a\Gamma S\Gamma a] \cup (a\Gamma a] \cup (\{a\}]$ which is the bi-ideal generated by a in S. Then, $\chi_{B(a)}$, the characteristic function of $B(a)$, is a fuzzy bi-ideal of S (cf. Theorem 3.1). Then, by the assumption we have $\chi_{B(a)} \cap \chi_{B(a)} \subseteq \chi_{B(a)} \circ \chi_{B(a)} \circ \chi_{B(a)}$. Since $a \in B(a)$, $\chi_{B(a)} = 1$ and then $(\chi_{B(a)} \circ \chi_{B(a)} \circ \chi_{B(a)})(a) = 1$. So $\exists\ x, y \in S$ and $\alpha \in \Gamma$ with $a \leq x\alpha y$ such that $\chi_{B(a)}(x) = 1$ and $(\chi_{B(a)} \circ \chi_{B(a)})(y) = 1$. So $\exists\ p, q \in S$ and $\beta \in \Gamma$ with $y \leq p\beta q$ such that $\chi_{B(a)}(p) = 1$ and $\chi_{B(a)}(q) = 1$. Hence, we can assert that $\exists\ x, p, q \in B(a)$ and $\alpha, \beta \in \Gamma$ such that $a \leq x\alpha p\beta q$, i.e., $a \in B(a)\Gamma B(a)\Gamma B(a)$ which indicates the regularity and intra-regularity of S.

Definition 3.17 *A po-Γ-semigroup S is called left zero (right zero) if $x \leq x\gamma y$ (resp. $y \leq x\gamma y$) $\forall x, y \in S$, $\forall \gamma \in \Gamma$.*

Proposition 3.1 *For a left zero (right zero) po-Γ-semigroup S, every fuzzy left (resp. fuzzy right) ideal is a constant function.*

Proof Let S be a left zero po-Γ-semigroup and μ be a fuzzy left ideal of S. Let $x, y \in S$. Then, $x \leq x\gamma y$ and $y \leq y\gamma x$ $\forall \gamma \in \Gamma$ as S is left zero. Then, $\mu(x) \geq \mu(x\gamma y) \geq \mu(y)$ and $\mu(y) \geq \mu(y\gamma x) \geq \mu(x)$. So $\mu(x) = \mu(y)$ $\forall x, y \in S$. Hence, every fuzzy left ideal is a constant function. Similarly we can prove the other case also.

Theorem 3.15 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is intra-regular.*

(2) *If μ is any fuzzy ideal of S, then corresponding to each $a \in S$, there exists $\beta \in \Gamma$ such that $\mu(a) = \mu(a\beta a)$.*

Proof (1) $\Rightarrow$ (2) Let S be intra-regular, μ be a fuzzy ideal of S and $a \in S$. Then, there exist $x, y \in S$ and $\alpha, \beta, \gamma \in \Gamma$ such that $a \leq x\alpha a\beta a\gamma y$. Again by definition of fuzzy ideal we see that $\mu(a) \geq \mu(x\alpha a\beta a\gamma y) \geq \mu(x\alpha a\beta a) \geq \mu(a\beta a) \geq \mu(a)$. So $\mu(a) = \mu(a\beta a)$.

(2) $\Rightarrow$ (1) Let (2) hold and $a \in S$. Then, the characteristic function $\chi_{(S\Gamma a\Gamma a\Gamma S]}$ of the both sided ideal $(S\Gamma a\Gamma a\Gamma S]$ of S is a fuzzy ideal of S. So, there exists $\beta \in \Gamma$ such that $\chi_{(S\Gamma a\Gamma a\Gamma S]}(a) = \chi_{(S\Gamma a\Gamma a\Gamma S]}(a\beta a)$. Now, for this $a\beta a$ there exists $\alpha \in \Gamma$ such that $\chi_{(S\Gamma a\Gamma a\Gamma S]}(a\beta a) = \chi_{(S\Gamma a\Gamma a\Gamma S]}((a\beta a)\alpha(a\beta a)) = \chi_{(S\Gamma a\Gamma a\Gamma S]}(a\beta(a\alpha a)\beta a) = 1$ (since $a\beta(a\alpha a)\beta a \in (S\Gamma a\Gamma a\Gamma S]$). So, $\chi_{(S\Gamma a\Gamma a\Gamma S]}(a) = 1$. Hence, $a \in (S\Gamma a\Gamma a\Gamma S]$. This proves that S is intra-regular and so (1) follows.

Remark 3.1 Let S be a regular duo po-Γ-semigroup. Then, the following conditions are equivalent:

(1) S is intra-regular.

(2) If μ is any fuzzy bi-ideal of S, then corresponding to each $a \in S$, there exists $\beta \in \Gamma$ such that $\mu(a) = \mu(a\beta a)$.

Theorem 3.16 *Let S be a po-Γ-semigroup. Then, the following conditions are equivalent:*

(1) *S is left regular (right regular).*

(2) *If μ is any fuzzy left ideal (resp. fuzzy right ideal) of S, then corresponding to each $a \in S$, there exists $\gamma \in \Gamma$ such that $\mu(a) = \mu(a\gamma a)$.*

Proof (1) $\Rightarrow$ (2) Let S be left regular, μ be a fuzzy left ideal of S and $a \in S$. Then, there exist $x \in S$ and $\alpha, \gamma \in \Gamma$ such that $a \leq x\alpha a\gamma a$. Since μ is a fuzzy left ideal, we have $\mu(a) \geq \mu(x\alpha a\gamma a) \geq \mu(a\gamma a) \geq \mu(a)$. So $\mu(a) = \mu(a\gamma a)$.

(2) $\Rightarrow$ (1) Let (2) hold and $a \in S$. Then, the characteristic function $\chi_{(S\Gamma a\Gamma a]}$ of the left ideal $(S\Gamma a\Gamma a]$ of S is a fuzzy left ideal of S. So there exists $\gamma \in \Gamma$ such that $\chi_{(S\Gamma a\Gamma a]}(a) = \chi_{(S\Gamma a\Gamma a]}(a\gamma a)$. Now, for this $a\gamma a$ there exists $\alpha \in \Gamma$ such that $\chi_{(S\Gamma a\Gamma a]}(a\gamma a) = \chi_{(S\Gamma a\Gamma a]}((a\gamma a)\alpha(a\gamma a)) = \chi_{(S\Gamma a\Gamma a]}((a\gamma a)\alpha a\gamma a) = 1$ (since $(a\gamma a)\alpha a\gamma a \in (S\Gamma a\Gamma a]$). So, $\chi_{(S\Gamma a\Gamma a]}(a) = 1$. Hence, $a \in (S\Gamma a\Gamma a]$. This proves that S is left regular. Similarly we can prove the other case.

Remark 3.2 Let S be a regular right duo (left duo) po-Γ-semigroup. Then, the following conditions are equivalent:

(1) S is left regular (resp. right regular).

(2) If μ is any fuzzy bi-ideal of S, then corresponding to each $a \in S$, there exists $\gamma \in \Gamma$ such that $\mu(a) = \mu(a\gamma a)$.

Theorem 3.17 *Suppose that S is both regular and intra-regular po-Γ-semigroup. Then,*

(1) $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$,

(2) $(\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) \supseteq \mu_1 \cap \mu_2$, *where* μ_1, μ_2 *are fuzzy bi-ideals of* S.

Proof Let $a \in S$. Then, there exist $x, y, z \in S$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5 \in \Gamma$ such that

$$\begin{aligned} a \leq a\gamma_1 x\gamma_2 a &\leq a\gamma_1 x\gamma_2 a\gamma_1 x\gamma_2 a \\ &\leq a\gamma_1 x\gamma_2 (y\gamma_3 a\gamma_4 a\gamma_5 z)\gamma_1 x\gamma_2 a \\ &\leq (a\gamma_1 x\gamma_2 y\gamma_3 a)\gamma_4 (a\gamma_5 z\gamma_1 x\gamma_2 a). \end{aligned}$$

Again by definition of fuzzy bi-ideal, we see that $\mu_1(a\gamma_1 x\gamma_2 y\gamma_3 a) \geq \mu_1(a)$ and $\mu_2(a\gamma_5 z\gamma_1 x\,\gamma_2 a) \geq \mu_2(a)$. Then,

$$\begin{aligned} (\mu_1 \circ \mu_2)(a) &\geq \min\{\mu_1(a\gamma_1 x\gamma_2 y\gamma_3 a), \mu_2(a\gamma_5 z\gamma_1 x\gamma_2 a)\} \\ &\geq \min\{\mu_1(a), \mu_2(a)\} = (\mu_1 \cap \mu_2)(a). \end{aligned}$$

Hence, $\mu_1 \circ \mu_2 \supseteq \mu_1 \cap \mu_2$. Similarly, we can show that $\mu_2 \circ \mu_1 \supseteq \mu_1 \cap \mu_2$. Therefore, $(\mu_1 \circ \mu_2) \cap (\mu_2 \circ \mu_1) \supseteq \mu_1 \cap \mu_2$.

Definition 3.18 *A po-*Γ*-semigroup* S *is said to be left (right) simple if* S *has no proper left (resp. right) ideals.*

Definition 3.19 *A po-*Γ*-semigroup* S *is said to be simple if* S *has no proper ideals.*

Definition 3.20 *A po-*Γ*-semigroup* S *is said to be fuzzy left (fuzzy right) simple if every fuzzy left (resp. fuzzy right) ideal of* S *is a constant function.*

Definition 3.21 *A po-*Γ*-semigroup* S *is said to be fuzzy simple if every fuzzy ideal of* S *is a constant function.*

Theorem 3.18 *Let* S *be a po-*Γ*-semigroup. Then, the following conditions are equivalent:*

(1) S *is left simple (resp. right simple, simple).*

(2) S *is fuzzy left simple (resp. fuzzy right simple, fuzzy simple).*

Proof (1) $\Rightarrow$ (2) Assume that S is left simple, μ be any fuzzy left ideal of S and $a, b \in S$. Then, $(S\Gamma a]$, $(S\Gamma b]$ are left ideals of S which means $(S\Gamma a] = S = (S\Gamma b]$. So there exist $x, y \in S$ and $\alpha, \beta \in \Gamma$ such that $b \leq x\alpha a$ and $a \leq y\beta b$ and so we obtain $\mu(a) \geq \mu(y\beta b) \geq \mu(b) \geq \mu(x\alpha a) \geq \mu(a)$ (cf. Definition 3.7). Thus, $\mu(a) = \mu(b)$. Hence, μ is a constant function. So, S is fuzzy left simple.

(2) $\Rightarrow$ (1) Suppose that S is fuzzy left simple and A be any left ideal of S. Thus, the characteristic function χ_A of A is a fuzzy left ideal of S and hence a constant function. Since A is non-empty, the constant value is 1. So every element of S is in A and so S is left simple. Similarly, we can prove the other cases.

Theorem 3.19 *Let* S *be a left (right) simple po-*Γ*-semigroup. Then, every fuzzy bi-ideal of* S *is a fuzzy right ideal (resp. fuzzy left ideal) of* S.

Proof Suppose that μ is any fuzzy bi-ideal of S and $a, b \in S$. Then, $(S\Gamma a]$ is a left ideal of S which means $(S\Gamma a] = S$. So there exist $x \in S$, $\gamma \in \Gamma$ such that $b \leq x\gamma a$ and so for any $\alpha \in \Gamma$, $a\alpha b \leq a\alpha x\gamma a$. Then, by definition of fuzzy bi-ideal we see that $\mu(a\alpha b) \geq \mu(a\alpha x\gamma a) \geq \min\{\mu(a), \mu(a)\} = \mu(a)$. Hence, μ is a fuzzy right ideal of S. Similarly, we can prove the other case also.

Now, to conclude this section we deduce from the previous two theorems the following characterization of simple po-Γ-semigroup in terms of fuzzy bi-ideals.

Theorem 3.20 *A po Γ-semigroup S is left and right simple if and only if S does not contain proper fuzzy bi-ideals.*

4. F-Regularity of Fuzzy Subsemigroups

Definition 4.1 *Let S be a po-Γ-semigroup. Let $a \in S$ and $t \in (0, 1]$. We now define a fuzzy subset a_t of S as follows:*

$$(a_t)(x) = \begin{cases} t, & \text{if } x \leq a \\ 0, & \text{otherwise} \end{cases}$$

for all $x \in S$. We call a_t a fuzzy point or fuzzy singleton of S.

Definition 4.2 *Let a_t and b_r be two fuzzy points of a po-Γ-semigroup S. Then,*

$$(a_t \circ b_r)(x) = \begin{cases} \sup\limits_{x \leq y\gamma z} \{\min\{a_t(y), b_r(z)\}\}, & \text{if } \exists y, z \in S, \gamma \in \Gamma \text{ with } x \leq y\gamma z, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 4.1 *Let a_t and b_r be two fuzzy points of a po-Γ-semigroup S. Then,*

$$a_t \circ b_r = \bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r}.$$

Proof Suppose that $x \in S$. If there do not exist any $y, z \in S$ and $\gamma \in \Gamma$ such that $x \leq y\gamma z$, then $(a_t \circ b_r)(x) = 0 = (\bigcup\limits_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r})(x)$.

Again, for $y, z \in S$ and $\gamma \in \Gamma$ if $x \leq y\gamma z$ implies either $y \nleq a$ or $z \nleq b$, then also $(a_t \circ b_r)(x) = 0 = (\bigcup\limits_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r})(x)$.

Now, if $x \leq a\gamma' b$ for some $\gamma' \in \Gamma$, then

$$(a_t \circ b_r)(x) = a_t(a) \wedge b_r(b) = t \wedge r = (a\gamma' b)_{t \wedge r}(x) = (\bigcup_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r})(x).$$

Hence, $a_t \circ b_r = \bigcup\limits_{\gamma \in \Gamma} (a\gamma b)_{t \wedge r}$.

Definition 4.3 *Let μ be a fuzzy subsemigroup of S and $x_t \in \mu$. If there exists $y_r \in \mu$ with $x_t \subseteq x_t \circ y_r \circ x_t$, then x_t is called a fuzzy regular point or an F-regular point in μ.*

Proposition 3.1 *Let μ be a fuzzy subsemigroup of S and $x_t \in \mu$. Then, x_t is an F-regular point in μ if and only if there exists $y_r \in \mu$ with $x_t \subseteq x_t \circ y_r \circ x_t$.*

Proof If for any $x_t \in \mu$, $x_t \subseteq x_t \circ y_t \circ x_t$, for some $y_t \in \mu$, then we are done.

Conversely, let μ be a fuzzy subsemigroup of S and x_t be an F-regular point of μ, i.e., $\mu(x) \geq t$ and $\exists y_r \in \mu$ such that $x_t \subseteq x_t \circ y_r \circ x_t$. Now, for any $z \in S$,

$$(x_t \circ y_r \circ x_t)(z) = ((\bigcup_{\gamma\in\Gamma}(x\gamma y)_{t\wedge r}) \circ x_t)(z) = \sup_{z \leq p\alpha q}\{\min\{\sup_{\gamma}\{(x\gamma y)_{t\wedge r}(p)\}, x_t(q)\}\}.$$

If $z \leq p\alpha q$ such that either $p \nleq x\gamma y$ for any $\gamma \in \Gamma$ or $q \nleq x$, then

$$(x_t \circ y_r \circ x_t)(z) = ((\bigcup_{\gamma\in\Gamma}(x\gamma y)_{t\wedge r}) \circ x_t)(z) = 0 = \bigcup_{\gamma_1,\gamma_2\in\Gamma}(x\gamma_1 y\gamma_2 x)_{t\wedge r}.$$

If $\exists\, \gamma_1, \gamma_2 \in \Gamma$ such that $z \leq x\gamma_1 y\gamma_2 x$, then

$$\begin{aligned}(x_t \circ y_r \circ x_t)(z) &= ((\bigcup_{\gamma\in\Gamma}(x\gamma y)_{t\wedge r}) \circ x_t)(z)\\ &= \min\{\sup_{\gamma}\{(x\gamma y)_{t\wedge r}(x\gamma_1 y)\}, x_t(x)\}\\ &= \min\{t \wedge r, t\}\\ &= t \wedge r\\ &= (\bigcup_{\gamma_1,\gamma_2\in\Gamma}(x\gamma_1 y\gamma_2 x)_{t\wedge r})(z).\end{aligned}$$

Hence, $x_t \subseteq \bigcup_{\gamma_1,\gamma_2\in\Gamma}(x\gamma_1 y\gamma_2 x)_{t\wedge r}$. So $\exists\, \gamma_1, \gamma_2 \in \Gamma$ such that $x_t \subseteq (x\gamma_1 y\gamma_2 x)_{t\wedge r}$. Now, $(x\gamma_1 y\gamma_2 x)_{t\wedge r} \subseteq (x\gamma_1 y\gamma_2 x)_t$. So $x_t \subseteq (x\gamma_1 y\gamma_2 x)_t$ for some $\gamma_1, \gamma_2 \in \Gamma$. Therefore, $x_t \subseteq \bigcup_{\gamma_1,\gamma_2\in\Gamma}(x\gamma_1 y\gamma_2 x)_t$. Hence, $x_t \subseteq x_t \circ y_t \circ x_t$.

Now, suppose that if possible $y_t \notin \mu$, i.e., $r \leq \mu(y) < t$. Then, from the fact that $x_t \subseteq (x\gamma_1 y\gamma_2 x)_{t\wedge r}$ for some $\gamma_1, \gamma_2 \in \Gamma$, we can conclude that $x_t \subseteq (x\gamma_1 y\gamma_2 x)_r$, i.e., $x_t(x) = t \leq (x\gamma_1 y\gamma_2 x)_r(x) \implies x \leq x\gamma_1 y\gamma_2 x$ and $t \leq r$ which contradicts our assumption.

Definition 4.4 *Let μ be a fuzzy subsemigroup of S. Then, μ is said to be an F-regular fuzzy subsemigroup of S if for each $x \in S, \exists\, x' \in R_x = \{y \in S \mid x \leq x\alpha y\beta x$ for some $\alpha, \beta \in \Gamma\}$, with $\mu(x') \geq \mu(x)$ provided $\mu(x) \neq 0$. As $y_r \in \mu$, so $\mu(y) \geq r$.*

To conclude the paper we obtain the following characterization of F-regular fuzzy subsemigroup in terms of fuzzy points.

Theorem 4.2 *Let μ be an F-regular fuzzy subsemigroup of S. Then, the following conditions are equivalent:*

(1) *μ is F-regular fuzzy subsemigroup of S.*

(2) *For all $x_t \in \mu$, x_t is an F-regular point in μ.*

(3) *For each $x_t \in \mu$, $\exists\, y_t \in \mu$ such that $x_t \subseteq x_t \circ y_t \circ x_t$.*

Proof (2) $\Leftrightarrow$ (3) It is clear.

(3) $\Rightarrow$ (1) Let (3) hold and $x \in S$. Let $\mu(x) = t$ where $0 < t \leq 1$. Then, $x_t \in \mu$. So $\exists\, y_t \in \mu$ such that $x_t \subseteq x_t \circ y_t \circ x_t$. Now, $x_t \subseteq \bigcup_{\gamma_1,\gamma_2\in\Gamma}(x\gamma_1 y\gamma_2 x)_t$. So $\exists\, \gamma_1, \gamma_2 \in \Gamma$ such

that $x_t \subseteq (x\gamma_1 y\gamma_2 x)_t$. Then, $t = x_t(x) \leq (x\gamma_1 y\gamma_2 x)_t(x) \Rightarrow (x\gamma_1 y\gamma_2 x)_t(x) = t \Rightarrow x \leq x\gamma_1 y\gamma_2 x \Rightarrow y \in R_x$ and $\mu(y) \geq t = \mu(x)$.

(1) $\Rightarrow$ (3) Let (1) hold and $x_t \in \mu$. Then, from the definition of fuzzy point in po-Γ-semigroup S, $x \in Supp\ \mu$. Now, by the assumption we can say that $\exists\ y \in R_x = \{z \in S \mid x \leq x\alpha z\beta x$ for some $\alpha, \beta \in \Gamma\}$, with $\mu(y) \geq \mu(x)$. So $\exists\ \alpha, \beta \in \Gamma$ such that $x \leq x\alpha y\beta x$. Now, let $z \in S$ such that $z \leq x$. Then, $z \leq x\alpha y\beta x$. So $x_t(z) = t = (x\alpha y\beta x)_t(z)$ which means $x_t \subseteq (x\alpha y\beta x)_t$. Therefore, clearly $x_t \subseteq \bigcup_{\gamma_1,\gamma_2\in\Gamma} (x\gamma_1 y\gamma_2 x)_t$. Hence, $x_t \subseteq x_t \circ y_t \circ x_t$.

5. Conclusion

We make a new connection between semigroups and fuzzy sets. We investigate some properties of fuzzy ideals, fuzzy bi-ideals and fuzzy $(1, 2)$-ideals. Also, we give a pointwise characterization of fuzzy regular subsemigroups in a po-Γ-semigroup. The object of the paper can be considered as a generalization of fuzzy semigroups.

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