Journal of King Saud University – Science (2016) 28, 17–20

King Saud University
Journal of King Saud University – Science
www.ksu.edu.sa
www.sciencedirect.com

ORIGINAL ARTICLE

Lévy stable distribution and space-fractional Fokker–Planck type equation

Jun-Sheng Duan [a,*], Temuer Chaolu [b], Zhong Wang [c], Shou-Zhong Fu [c]

[a] School of Sciences, Shanghai Institute of Technology, Shanghai 201418, PR China
[b] College of Sciences and Arts, Shanghai Maritime University, Shanghai 200135, PR China
[c] Department of Mathematics, Zhaoqing University, Zhaoqing, Guang Dong 526061, PR China

Received 16 February 2015; accepted 30 March 2015
Available online 8 April 2015

KEYWORDS
Fokker–Planck equation;
Laplace operator;
Fourier transform;
Fox H function

Abstract The space-fractional Fokker–Planck type equation $\frac{\partial p}{\partial t}+\gamma\frac{\partial p}{\partial x}=-D(-\Delta)^{\alpha/2}p$ $(0<\alpha\leqslant 2)$ subject to the initial condition $p(x,0)=\delta(x)$ is solved in terms of Fox H functions. The solution as $\gamma=0$ expresses the Lévy stable distribution with the index α. From the properties of Fox H functions, the series representation and asymptotic behavior for the solution are also obtained. Lévy stable distribution as $0<\alpha<2$ describes anomalous superdiffusion and its diffusion velocity is characterized by $x_d\propto(Dt)^{1/\alpha}$.

1. Introduction

In recent decades, the theory and applications of the fractional calculus have developed rapidly. The applied fields include viscoelasticity, anomalous diffusion, heat transfer, signal processing, dynamics and control, and so on (Podlubny, 1999; Kilbas et al., 2006). Different definitions and methods have also been proposed (Yang et al., 2013; Yang and Baleanu, 2013; Yang, 2012; Li et al., 2011a,b). Scientists and engineers have found the description of some phenomena is more accurate when the fractional derivative is used. In particular, anomalous diffusion can be characterized by fractional differential equations.

Let random variable $X(t)$ denote the location of diffusing particle with $X(0)=0$ and $p(x,t)$ be the probability density function for $X(t)$. The time-fractional Fokker–Planck type equations are derived and solved in Hilfer (1995) and Rangarajan and Ding (2000); Space-fractional Fokker–Planck type equation is obtained in Compte (1996) and Yanovsky et al. (2000) using statistical methods. One-dimensional case with convective term without asymmetric term reads (Yanovsky et al., 2000)

$$\frac{\partial p}{\partial t}+\gamma\frac{\partial p}{\partial x}=-D(-\Delta)^{\alpha/2}p, \tag{1}$$

with the initial condition

$$p(x,0)=\delta(x). \tag{2}$$

Here α,γ,D are real constants $(0<\alpha\leq 2, D>0)$, $\delta(x)$ is the Dirac delta function and fractional Laplace operator is defined by

* Corresponding author.
E-mail address: duanjs@sit.edu.cn (J.-S. Duan).
Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.03.007
1018-3647

$$(-\Delta)^{\alpha/2}f(x) = \mathcal{F}^{-1}[|k|^{\alpha}\mathcal{F}[f(x)]], \tag{3}$$

where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote, respectively, the Fourier transform and its inverse:

$$\mathcal{F}[f] = \hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}dx, \tag{4}$$

$$\mathcal{F}^{-1}[\hat{f}] = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(k)e^{-ikx}dk. \tag{5}$$

Space-fractional Fokker–Planck type equations similar to the Eq. (1) are also considered in Chaves (1998) and Chechkin et al. (2002), but therein the exact analytic solutions for the problem (1) and (2) are given only for the cases of $\alpha = 1$ and $\alpha = 2$. In the general case of $0 < \alpha \le 2$, the solution for the problem (1) and (2) is studied in the sequel. We obtain the exact analytic solution in terms of Fox H functions (Mathai and Saxena, 1978; Srivastava et al., 1982), and its series representation and asymptotic behavior are investigated.

2. Solution to the problem

Taking the Fourier transform for the problem (1) and (2) with respect to x we get

$$\hat{p}(k,t) = e^{i\gamma tk}e^{-Dt|k|^{\alpha}}. \tag{6}$$

Let

$$U(x,t) = \mathcal{F}^{-1}[e^{-Dt|k|^{\alpha}}]. \tag{7}$$

Then the inverse Fourier transform of (6) is

$$p(x,t) = U(x-\gamma t, t). \tag{8}$$

Lévy stable distribution $\rho(x)$ with index γ $(0 < \gamma \le 2)$ is defined through the Fourier transform as (Feller, 1971; Fogedby et al., 1992; Zanette, 1997)

$$\mathcal{F}[\rho(x)] = \hat{\rho}(k) = \exp(-|k/k_0|^{\gamma}), \quad (k_0 = \text{const.}). \tag{9}$$

So $U(x,t)$ is Lévy stable distribution with the index α. It follows from (8) that $p(x,t)$ describes further the convection of particles by the constant velocity γ in contrast with $U(x,t)$. We focus our attention on the discussion for $U(x,t)$ in the following.

In order to obtain the inverse in (7) we rewrite it as

$$U(x,t) = \frac{1}{\pi}\mathcal{F}_c[e^{-Dt|k|^{\alpha}}, |k| \to |x|], \tag{10}$$

where $\mathcal{F}_c$ denotes the Fourier cosine transform

$$\mathcal{F}_c[g(u), u \to v] = \int_0^{\infty} g(u)\cos uv du. \tag{11}$$

Using the Fox function representation (Mathai and Saxena, 1978; Srivastava et al., 1982; Duan, 2005)

$$e^{-Dt|k|^{\alpha}} = \frac{1}{\alpha}H^{1,0}_{0,1}\left((Dt)^{1/\alpha}|k|\big|_{(0,1/\alpha)}\right) \tag{12}$$

and the Fourier cosine transform of Fox functions (Glöckle and Nonnenmacher, 1993)

$$\mathcal{F}_c[H^{m,n}_{p,q}(z), z\to v] = \frac{\pi}{v}H^{n+1,m}_{q+1,p+2}\left(v\left|\begin{matrix}(1-b_j,\beta_j),(1,1/2)\\(1,1),(1-a_j,\alpha_j),(1,1/2)\end{matrix}\right.\right), \quad \mu \le 1, \tag{13}$$

$$\mathcal{F}_c[H^{m,n}_{p,q}(z), z\to v] = \frac{\pi}{v}H^{m,n+1}_{p+2,q+1}\left(\frac{1}{v}\left|\begin{matrix}(0,1),(a_j,\alpha_j),(0,1/2)\\(b_j,\beta_j),(0,1/2)\end{matrix}\right.\right), \quad \mu \geqslant 1, \tag{14}$$

where $\mu = \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j$, we obtain from Eq. (10)

$$U(x,t) = \frac{1}{\alpha|x|}H^{1,1}_{2,2}\left(\frac{(Dt)^{1/\alpha}}{|x|}\left|\begin{matrix}(0,1),(0,1/2)\\(0,1/\alpha),(0,1/2)\end{matrix}\right.\right), \quad 0<\alpha\le 1,$$

$$U(x,t) = \frac{1}{\alpha|x|}H^{1,1}_{2,2}\left(\frac{|x|}{(Dt)^{1/\alpha}}\left|\begin{matrix}(1,1/\alpha),(1,1/2)\\(1,1),(1,1/2)\end{matrix}\right.\right), \quad 1\leqslant\alpha\le 2. \tag{15}$$

As $\alpha = 1$, the first expression in Eq. (15) permits $|Dt/x| < 1$, while the second permits $|Dt/x| > 1$. Making use of the series expression of Fox functions and the formula

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z},$$

we get the series representations

$$U(x,t) = \frac{1}{\pi|x|}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n!}\sin\frac{\pi\alpha n}{2}\Gamma(1+\alpha n)\left(\frac{Dt}{|x|^{\alpha}}\right)^n, 0<\alpha\le 1, \tag{16}$$

$$U(x,t) = \frac{1}{\pi\alpha(Dt)^{1/\alpha}}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\Gamma\left(\frac{1+2n}{\alpha}\right)\left(\frac{|x|}{(Dt)^{1/\alpha}}\right)^{2n}, 1\leqslant\alpha\leq 2. \tag{17}$$

3. Discussions and conclusions

As $\alpha = 1$, the Cauchy distribution is obtained from Eqs. (16) and (17)

$$U(x,t)|_{\alpha=1} = \frac{Dt}{\pi(x^2+(Dt)^2)}. \tag{18}$$

As $\alpha = 2$, with the help of the identity

$$\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}(2n)!}{4^n n!}$$

and the expression in Eq. (17), the Gauss distribution is obtained

$$U(x,t)|_{\alpha=2} = \frac{1}{2\sqrt{\pi Dt}}\exp\left(-\frac{x^2}{4Dt}\right). \tag{19}$$

From Eqs. (16) and (17) we have the following asymptotic expressions

$$U(x,t) \sim \sin\frac{\pi\alpha}{2}\Gamma(1+\alpha)\frac{Dt}{\pi|x|^{\alpha+1}}, \quad \frac{Dt}{|x|^{\alpha}} \to 0 \tag{20}$$

for $0 < \alpha \le 1$, and

$$U(x,t) \sim \Gamma\left(\frac{1}{\alpha}\right)\frac{1}{\pi\alpha(Dt)^{1/\alpha}}, \quad \frac{|x|}{(Dt)^{1/\alpha}} \to 0 \tag{21}$$

for $1 \leqslant \alpha \le 2$. Using the asymptotic expansion of Fox functions we get

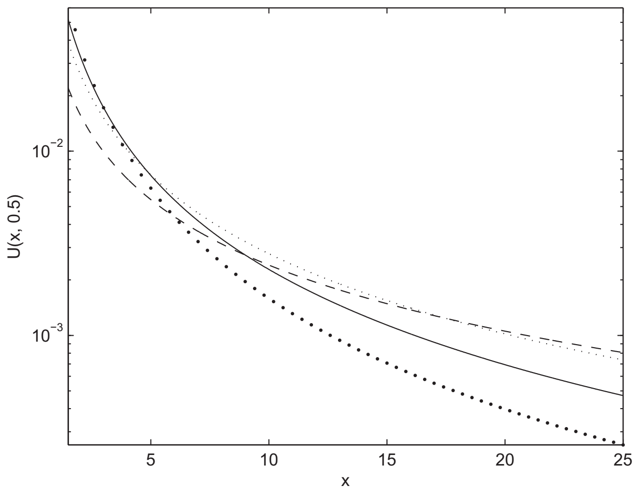

Figure 1 Curves of $U(x, 0.5)$ versus x on the interval $1.5 \leqslant x \leqslant 25$ for $D = 1$ and $\alpha = 1$ (thick dot line), 0.75 (solid line), 0.5 (dot line) and 0.25 (dash line).

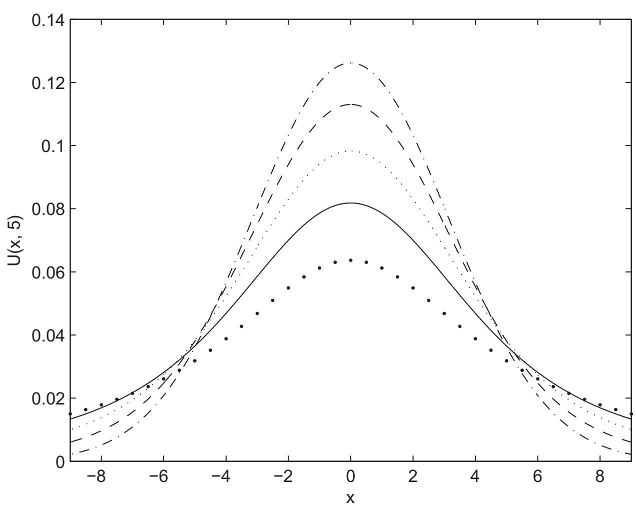

Figure 2 Curves of $U(x, 5)$ versus x on the interval $-9 \leqslant x \leqslant 9$ for $D = 1$ and $\alpha = 1$ (thick dot line), 1.25 (solid line), 1.5 (dot line), 1.75 (dash line) and 2 (dot–dash line).

$$U(x,t) \sim \Gamma\left(\frac{1}{\alpha}\right)\frac{1}{\pi\alpha(Dt)^{1/\alpha}}, \quad \frac{Dt}{|x|^{\alpha}} \to \infty \tag{22}$$

for $0 < \alpha \le 1$, and

$$U(x,t) \sim \sin\frac{\pi\alpha}{2}\Gamma(1+\alpha)\frac{Dt}{\pi|x|^{\alpha+1}}, \quad \frac{|x|}{(Dt)^{1/\alpha}} \to \infty \tag{23}$$

for $1 \leqslant \alpha < 2$.

In Figs. 1 and 2, we plot the curves of $U(x,t)$ versus x for fixed values of t and different values of α. Since for fixed t, the series in (16) converges faster for large $|x|$ while the second converges faster for small $|x|$, the different ranges of x are chosen in Figs. 1 and 2.

Inserting Eqs. (15)–(17), (20)–(23) into Eq. (8), we can obtain the Fox function representations, series representations and asymptotic behavior for the probability density $p(x,t)$.

As $\alpha = 2$, the Gauss distribution (19) describes the standard diffusion process. The mean square displacement is given by $\langle x^2\rangle = 2Dt$. But as $0 < \alpha < 2$, from the asymptotic representation the mean square displacement $\langle x^2\rangle$ is infinite. The Lévy stable distribution $U(x,t)$ describes anomalous diffusion (Shlesinger et al., 1993). In order to determine the diffusion velocity we adopt the following method.

Taking a fixed number d $(0 < d < 1)$, we investigate x_d such that

$$2\int_0^{x_d} U(x,t)dx = d \tag{24}$$

for all t. Calculating the integration with the help of (15) and (18) leads to

$$\frac{x_d}{(Dt)^{1/\alpha}} = \text{const.} \tag{25}$$

Therefore we have

$$x_d \propto (Dt)^{1/\alpha}. \tag{26}$$

This gives the characteristic of anomalous superdiffusion. The smaller the value of α is, the faster the diffusion spreads. This is shown in Figs. 1 and 2.

Acknowledgements

This work was supported by the Natural Science Foundation of Shanghai (No. 14ZR1440800), the Innovation Program of the Shanghai Municipal Education Commission (No. 14ZZ161) and the National Natural Science Foundation of China (Nos. 11171295; 11201308).

Appendix: Fox H function

The Fox H function (Fox function or H function) is defined by the contour integral (Mathai and Saxena, 1978; Srivastava et al., 1982; Glöckle and Nonnenmacher, 1993)

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left(z\,\middle|\,\begin{matrix}(a_1,\alpha_1),\ldots,(a_p,\alpha_p)\\(b_1,\beta_1),\ldots,(b_q,\beta_q)\end{matrix}\right) := \frac{1}{2\pi i}\int_L h(s)z^s ds, \tag{A1}$$

where $h(s)$ is given by

$$h(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j s)}. \tag{A2}$$

Here m,n,p,q are integers satisfying $0 \leqslant n \leqslant p$, $1 \leqslant m \leqslant q$ and empty products are interpreted as unity. The parameters a_j $(j = 1,\ldots,p)$ and b_j $(j = 1,\ldots,q)$ are complex numbers and α_j $(j = 1,\ldots,p)$ and β_j $(j = 1,\ldots,q)$ are positive numbers satisfying $P_a \cap P_b = \varnothing$, where

$$P_a = \{s = (b_j + k)/\beta_j,\ j = 1,2,\ldots,m;\ k = 0,1,2,\ldots\}$$

and

$$P_b = \{s = (a_j - 1 - k)/\alpha_j,\ j = 1,2,\ldots,n;\ k = 0,1,2,\ldots\}.$$

The integration contour L runs from $s = c - i\infty$ to $s = c + i\infty$ such that P_a lies to the right of L and P_b to the left of L. Fox H function is an analytic function of z which makes sense (i) for every $z \neq 0$ if $\mu > 0$ and (ii) for $0 < |z| < \beta^{-1}$ if $\mu = 0$, where

$$\mu = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j$$

and

$$\beta = \prod_{j=1}^{p} \alpha_j^{\alpha_j} \prod_{j=1}^{q} \beta_j^{-\beta_j}$$

and it can be expanded into series

$$H_{p,q}^{m,n}(z) = -\sum_{s \in P_a} \mathrm{Res}(h(s)z^s). \quad \text{(A3)}$$

Due to the factor z^s in (A1), the H function is in general multiple-valued, but is one-valued on the Riemann surface of $\log z$.

As $\mu > 0$ and $n \neq 0$ the asymptotic expansion

$$H_{p,q}^{m,n}(z) \sim \sum_{s \in P_b} \mathrm{Res}(h(s)z^s), \quad |z| \to \infty \quad \text{(A4)}$$

holds uniformly on every closed subsector of $|\arg z| \leqslant \pi\lambda/2$, where λ is defined by

$$\lambda = \sum_{j=1}^{m} \beta_j + \sum_{j=1}^{n} \alpha_j - \sum_{j=m+1}^{q} \beta_j - \sum_{j=n+1}^{p} \alpha_j.$$

We list some properties of the H function:

$$H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right) = H_{q,p}^{n,m}\left(\frac{1}{z} \left| \begin{matrix} (1-b_1, \beta_1), \ldots, (1-b_q, \beta_q) \\ (1-a_1, \alpha_1), \ldots, (1-a_p, \alpha_p) \end{matrix} \right.\right), \quad \text{(A5)}$$

$$H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}), (a_1, \alpha_1) \end{matrix} \right.\right) = H_{p-1,q-1}^{m,n-1}\left(z \left| \begin{matrix} (a_2, \alpha_2), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_{q-1}, \beta_{q-1}) \end{matrix} \right.\right), \quad n > 0, \; q > m, \quad \text{(A6)}$$

$$\frac{1}{k} H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right) = H_{p,q}^{m,n}\left(z^k \left| \begin{matrix} (a_1, k\alpha_1), \ldots, (a_p, k\alpha_p) \\ (b_1, k\beta_1), \ldots, (b_q, k\beta_q) \end{matrix} \right.\right), k > 0, \quad \text{(A7)}$$

$$z^\sigma H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{matrix} \right.\right) = H_{p,q}^{m,n}\left(z \left| \begin{matrix} (a_1 + \sigma\alpha_1, \alpha_1), \ldots, (a_p + \sigma\alpha_p, \alpha_p) \\ (b_1 + \sigma\beta_1, \beta_1), \ldots, (b_q + \sigma\beta_q, \beta_q) \end{matrix} \right.\right). \quad \text{(A8)}$$

References

Chaves, A.S., 1998. A fractional diffusion equation to describe Lévy flights. Phys. Lett. A 239 (1–2), 13–16.

Chechkin, A.V., Gonchar, V.Y., Szydowski, M., 2002. Fractional kinetics for relaxation and superdiffusion in a magnetic field. Phys. Plasmas 9 (1), 78–88.

Compte, A., 1996. Stochastic foundations of fractional dynamics. Phys. Rev. E 53 (4), 4191–4193.

Duan, J.S., 2005. Time- and space-fractional partial differential equations. J. Math. Phys. 46, 13504–13511.

Feller, W., 1971. An Introduction to Probability Theory and Its Applications. Wiley, New York.

Fogedby, H.C., Bohr, T., Jensen, H.J., 1992. Fluctuations in a Lévy fight gas. J. Stat. Phys. 66, 583–594.

Glöckle, W.G., Nonnenmacher, T.F., 1993. Fox function representation of Non-Debye relaxation processes. J. Stat. Phys. 71 (3), 741–757.

Hilfer, R., 1995. Exact solutions for a class of fractal time random walks. Fractals 3 (1), 211–216.

Kilbas, A.A., Srivastava, H.M., Trujillo, J.J., 2006. Theory and Applications of Fractional Differential Equations. Elsevier, Amsterdam.

Li, M., Lim, S.C., Chen, S., 2011a. Exact solution of impulse response to a class of fractional oscillators and its stability. Math. Probl. Eng. 2011 (Article ID 657839) 9 p.

Li, M., Cattani, C., Chen, S.Y., 2011b. Viewing sea level by a one dimensional random function with long memory. Math. Probl. Eng. 2011 (Article ID 654284) 13 p.

Mathai, A.M., Saxena, R.K., 1978. The H-function with Applications in Statistics and Other Disciplines. Wiley Eastern Limited, New Delhi.

Podlubny, I., 1999. Fractional Differential Equations. Academic, San Diego.

Rangarajan, G., Ding, M., 2000. First passage time distribution for anomalous diffusion. Phys. Lett. A 273, 322–330.

Shlesinger, M.F., Zaslavsky, G.M., Klafter, J., 1993. Strange kinetics. Nature 363, 31–37.

Srivastava, H.M., Gupta, K.C., Goyal, S.P., 1982. The H-functions of One and Two Variables with Applications. South Asian Publishers, New Delhi/Madras.

Yang, X.J., 2012. Advanced Local Fractional Calculus and Its Applications. World Science Publisher, New York.

Yang, X.J., Baleanu, D., 2013. Fractal heat conduction problem solved by local fractional variation iteration method. Therm. Sci. 17 (2), 625–628.

Yang, X.J., Srivastava, H.M., He, J.H., Baleanu, D., 2013. Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives. Phys. Lett. A 377 (28–30), 1696–1700.

Yanovsky, V.V., Chechkin, A.V., Schertzer, D., Tur, A.V., 2000. Lévy anomalous diffusion and fractional Fokker–Planck equation. Physica A 282, 13–34.

Zanette, D.H., 1997. Wave fronts in bistable reactions with anomalous Lévy-flight diffusion. Phys. Rev. E 55 (1), 1181–1184.