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On the Stability of Processes Defined by Stochastic Difference-Differential Equations*

H. J. KUSHNER

Center for Dynamical Systems, Brown University, Providence, Rhode Island 02912

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1. INTRODUCTION

In this paper we extend previous work (Kushner [*1*], [*2*], [*3*]) on the stability of strong Markov processes with values in a finite-dimensional space, to processes defined by difference-differential Itô equations of type (1.1). The extension is analogous to the extension of the Liapunov stability theorems to theorems on the stability of the solutions of ordinary difference-differential equations as, for example, presented in Hale [*4*].

Let C be the space of continuous functions on the real interval $[-r, 0]$, $r > 0$, and let $x(t)$ be a vector-valued stochastic process. Define the process x_t, with values in C, by $x_t(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. Let $|x(t)|^2 = \sum x_i^2(t)$ and $\| x_t \| = \sup_\theta \| x(t + \theta)\|$, $\theta \in [-r, 0]$. Suppose $x(t)$ satisfies the vector stochastic difference-differential equation

$$\begin{aligned} dx(t) &= f(x_t)\, dt + g(x_t)\, dz(t) \\ x(t) &= x(0) + \int_0^t f(x_s)\, ds + \int_0^t g(x_s)\, dz(s), \end{aligned} \tag{1.1}$$

where x_0, f and g satisfy Properties (A1)–(A3) or (A1), (A2) and (A4) of Section 2, and $z(s)$ is a vector-valued normalized Wiener process with independent components. Equations of type (1.1) have been studied by Itô and Nisio [*5*] and Fleming and Nisio [*6*]. Their result concerning existence is stated in Lemma 2.1.

We are concerned with criteria, of the *stochastic Liapunov function* type, which assure that the solution paths of (1.1) have certain "stability"

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properties; e.g., for some set R, we may want to prove that $x(t) \to R$ w.p.l.†, or (with initial condition $x_0 = x$) obtain an estimate of $P_x\{\sup_{\infty > t \geqslant 0} | x(t)| \geqslant \epsilon\}$, or prove that $P_x\{\sup_{\infty > t \geqslant 0} | x(t)| \geqslant \epsilon > 0\} \to 0$, as $\| x \| \to 0$, or estimate $P_x\{\sup_{\infty > t \geqslant 0} V(x_t) \geqslant \epsilon\}$ for a suitable real-valued function V. Some definitions concerning stochastic stability are given in [*1*]–[*3*]. Here, in lieu of stating definitions, we merely concern ourselves with the properties the definitions imply and establish criteria for properties of the type just mentioned. Results concerning first-passage times and moment estimates as well as applications to control are also available, although our attention here is confined to 'asymptotic' results. In addition to the intrinsic interest in the problem attacked an important motivation for the work is to provide a foundation for the stabilization and control of processes, defined by stochastic *differential* (Itô) equations, with controls depending on delayed arguments. Such delays are often an unavoidable part of the control problem. Also for an example of a deterministic system which cannot be stabilized by a control depending on the state, but which can be stabilized by a control depending on delayed values of the state, see Krasovskii [*7*].

In Section 2 we derive some useful estimates concerning the probabilistic behavior of the solution of (1.1). These are used subsequently to establish stochastic continuity, the strong Markov character of the x_t process, and some needed characterizations of the weak infinitesimal operator of the x_t process. Sections 3 and 4 establish the strong Markov nature of x_t and corresponding stopped processes, respectively. Section 5 gives some results on the weak infinitesimal operator. In Section 6 these results are used to prove some stability theorems, and examples appear in Section 7. The stability results depend on stochastic continuity, a formula of Dynkin ([*8*], Theorems 5, 6 and Corollary to 6) and supermartingale theorems. Unfortunately, in order to make the first property explicit and to apply the latter results, much of the analysis in Section 2–5 is needed. As in the deterministic case (Hale [*4*]), the natural process to deal with seems to be x_t [rather than $x(t)$], since then much of the theory of Markov processes can be applied.

2. Properties of the Solution of Equations (1.1)

Let f_i and g_{ij} be the components of the vector and matrix-valued functions f and g, respectively, and define the vector and matrix norms as $| f |^2 = \sum_i f_i^2$, $| g |^2 = \sum_{i,j} g_{ij}^2$, respectively. Throughout, K and K_i are positive real numbers whose values may change from theorem to theorem.

(A1) $f_i(\cdot)$ and $g_{ij}(\cdot)$ are continuous real-valued functions on C.

† w-p-1 denotes with probability one.

(A2) In the interval $[-r, 0]$, $x(t)$ is continuous w.p.l. and independent of $z(s) - z(0)$, $s \geqslant 0$, and $E|x(t)|^4 < \infty$.

(A3) There is a constant $M < \infty$ and a bounded measure μ on $[-r, 0]$ so that for φ and $\psi \in C$,

$$|f(\varphi) - f(\psi)| + |g(\varphi) - g(\psi)| \leqslant \int_{-r}^{0} |\varphi(\theta) - \psi(\theta)|\, d\mu(\theta) \qquad (2.1)$$
$$|f(0)| + |g(0)| \leqslant M.$$

Note that (A3) implies (A3′):

(A3′) There is a constant $M < \infty$ and a bounded measure (also denoted by μ) on $[-r, 0]$ so that $|f(0)| + |g(0)| \leqslant M$ and

$$|f(\varphi) - f(\psi)|^2 + |g(\varphi) - g(\psi)|^2 \leqslant \int_{-r}^{0} |\varphi(\theta) - \psi(\theta)|^2\, d\mu(\theta). \qquad (2.2)$$

Eventually (A3) [or (A3′)] will be replaced by the local condition (A4) [or stronger condition (A4′)].

(A4)[(A4′)] For each positive real number ρ there is a bounded measure μ_ρ on $[-r, 0]$ so that for $\|\psi\| \leqslant \rho$ and $\|\varphi\| \leqslant \rho$, (2.1)[(2.2)] is valid with μ_ρ replacing μ. Also, $|f(0)| + |g(0)| \leqslant M < \infty$.

LEMMA 2.1. (See Itô and Nisio [5], Section 5, or Fleming and Nisio [6], for proof.) *Suppose* (A1) *to* (A3). *Then there is a continuous solution to* (1.1) w.p.l. *with* $E|x(t)|^4 \leqslant \gamma e^{\gamma t}$ *for some* $\gamma < \infty$. $x(s)$ *is independent of the collection* $z(t) - z(s)$, *for all* $t \geqslant s \geqslant 0$.

LEMMA 2.2. *Assume* (A1) *to* (A3). *For initial condition* $x = x_0$, *the stochastic integral*

$$w_i(t) = \int_0^t \sum_j g_{ij}(x_s)\, dz_j(s) \qquad (2.3a)$$

is a martingale, and for $\alpha > 1$

$$E \max_{T \geqslant t \geqslant 0} |w_i(t)|^\alpha \leqslant \left(\frac{\alpha}{\alpha - 1}\right)^\alpha E\,|w_i(T)|^\alpha$$
$$E \max_{T \geqslant t \geqslant 0} |w'(t)\, w(t)| \leqslant 4Ew'(T)\, w(T) = 4 \int_0^T E\,|g(x_t)|^2\, dt. \qquad (2.3b)$$

Proof. By Lemma 2.1 and (A3), the integral on the right side of (2.3b) exists and is finite. Then, since $x(t)$ and x_t are nonanticipative, the $w_i(t)$ are continuous martingales (Doob [9], Chapter IX, Theorem 5.2) and (2.3b) is the continuous parameter version of Doob [9], Chapter VII, Theorem 3.4.

THEOREM 2.1. *Assume* (A1) *and* (A3). *Let* $x(t)$ *and* $y(t)$ *be solutions to* (1.1) *corresponding to initial condition* $x_0 = x$ *and* $y_0 = y$, *resp., where* x *and* y *satisfy* (A2). *Then*

$$E \max_{T \geqslant t \geqslant 0} |x(t) - y(t)|^2 \leqslant K\left\{E|x(0) - y(0)|^2 + \int_{-r}^{0} E|x(\theta) - y(\theta)|^2 \, d\mu(\theta)\right\}, \tag{2.4}$$

where K *depends only on* T, *and the* μ *and* M *of* (A3), *and is bounded for bounded* T. *The solution of* (1.1) *is unique in the sense that if* $x = x_0$ *satisfies* (A2), *then any two solutions with bounded second moments must coincide* w.p.1.

Remark. The right side of (2.4) depends only on the initial data.

Proof. (2.4) implies the uniqueness. From

$$x(t) - y(t) = x(0) - y(0) + \int_0^t (f(x_s) - f(y_s)) \, ds + \int_0^t (g(x_s) - g(y_s)) \, dz(s),$$

(2.3), and the bound $\max_{t \leqslant T} |\int_0^t k(s)\, ds|^2 \leqslant T \int_0^T k^2(s)\, ds$, we obtain

$$E \max_{T \geqslant t \geqslant 0} |x(t) - y(t)|^2 \leqslant K_1 E|x(0) - y(0)|^2 + K_1 T E \int_0^T |f(x_s) - f(y_s)|^2 \, ds$$
$$+ K_1 E \int_0^T |g(x_s) - g(y_s)|^2 \, ds.$$

Now (A3) gives

$$E \max_{T \geqslant t \geqslant 0} |x(t) - y(t)|^2 \leqslant K_2 E|x(0) - y(0)|^2$$
$$+ K_2 \int_0^T ds \int_{-r}^{0} E|x(s+\theta) - y(s+\theta)|^2 \, d\mu(\theta). \tag{2.5}$$

By separating out the contribution of the initial condition $x - y$, (2.5) can be written as

$$E \max_{T \geqslant t \geqslant 0} |x(t) - y(t)|^2$$
$$\leqslant \Delta_I + K_2 \int_0^T ds \int_{m(-r,-s)}^{0} E|x(s+\theta) - y(s+\theta)|^2 \, d\mu(\theta), \tag{2.6}$$

where $m(-r, -s) = \max(-r, -s) = -\min(r, s)$ [both r and s are non-negative] and

$$\Delta_I = K_2 E|x(0) - y(0)|^2 + K_2 \int_0^r ds \int_{-r}^{m(-r,-s)} \Delta_{s+\theta} \, d\mu(\theta),$$
$$\Delta_s = E|x(s) - y(s)|^2. \tag{2.7}$$

To evaluate the right side of (2.6), we first evaluate Δ_t which by (2.6) satisfies, for $t \leqslant T$,

$$\Delta_t \leqslant \Delta_I + K_2 \int_0^t ds \int_{m(-r,-s)}^0 \Delta_{s+\theta}\, d\mu(\theta). \tag{2.8}$$

Define $U =$ variation of μ and $B = \max_{T \geqslant t \geqslant 0} \Delta_t$ (which is finite, by Lemma 2.1), and

$$Q_n(t) \equiv \Delta_I \left(1 + UK_2 t + \cdots + \frac{U^n K_2{}^n t^n}{n!}\right) + \frac{U^n K_2{}^n t^n B}{n!}.$$

By (2.8), $\Delta_t \leqslant Q_1(t)$. By induction, it is easy to show that $\Delta_t \leqslant Q_p(t)$. Thus, since $B < \infty$,

$$\Delta_t \leqslant \Delta_I\, e^{UK_2 t}. \tag{2.9}$$

After substituting (2.9) into (2.6), it is easy to see that (2.4) holds for some finite K independent of x and y. Q.E.D.

THEOREM 2.2. *Assume* (A1) *to* (A3). *Then*

$$E \max_{T \geqslant t \geqslant 0} | x(t) - x(0)|^2 \leqslant KTE \left\{1 + \int_{-r}^0 (| x(\theta)|^2 + | x(\theta) - x(0)|^2)\, d\mu(\theta)\right\}, \tag{2.10}$$

where K depends only on T and μ and M, and is bounded for bounded T. Also, with $x_0 \in C$ fixed,

$$| Ex(h) - x(0) - hf(x_0)| = o(h), \tag{2.11}$$

$$| E(x(h) - x(0))(x(h) - x(0))' - hg(x_0)\, g'(x_0)| = o(h). \tag{2.12}$$

Proof. By (A3′),

$$\begin{aligned} &|f(x_s)|^2 + | g(x_s)|^2 \\ &\quad\leqslant 2\,|f(x_s) - f(x_0)|^2 + 2\,| g(x_s) - g(x_0)|^2 + 2\,|f(x_0)|^2 + 2\,| g(x_0)|^2 \\ &\quad\leqslant K_1 \left[1 + \int_{-r}^0 | x(s+\theta) - x(\theta)|^2\, d\mu(\theta) + \int_{-r}^0 | x(\theta)|^2\, d\mu(\theta)\right] \\ &\quad\leqslant K_2 \left[1 + \int_{-r}^0 | x(s+\theta) - x(0)|^2\, d\mu(\theta) \right. \\ &\qquad\qquad \left. + \int_{-r}^0 (| x(\theta) - x(0)|^2 + | x(\theta)|^2)\, d\mu(\theta)\right]. \end{aligned}$$

Thus, from

$$x(t) - x(0) = \int_0^t f(x_s)\, ds + \int_0^t g(x_s)\, dz(s)$$

and Lemma 2.2, we get

$$E \max_{T \geqslant t \geqslant 0} | x(t) - x(0)|^2 \leqslant K_3 E \max_{T \geqslant t \geqslant 0} \left| \int_0^t f(x_s)\, ds \right|^2 + K_3 \int_0^T E\, | g(x_s)|^2\, ds$$

$$\leqslant K_3 T \int_0^T E\, | f(x_s)|^2\, ds + K_3 \int_0^T E\, | g(x_s)|^2\, ds$$

$$\leqslant K_4 \Big[T + T^2 + \int_0^T ds$$

$$\times \int_{-r}^0 \{E\, | x(s+\theta) - x(0)|^2 + | x(\theta) - x(0)|^2 + | x(\theta)|^2\}\, d\mu(\theta) \Big].$$

Separating out the contribution of the initial condition gives

$$E \max_{T \geqslant t \geqslant 0} | x(t) - x(0)|^2$$

$$\leqslant K_4 \Big[T + T^2 + T \int_{-r}^0 E(| x(\theta)|^2 + | x(\theta) - x(0)|^2)\, d\mu(\theta)$$

$$+ \int_0^T ds \int_{-r}^{m(-r,-s)} E\, | x(s+\theta) - x(0)|^2\, d\mu(\theta)$$

$$+ \int_0^T ds \int_{m(-r,-s)}^0 E\, | x(s+\theta) - x(0)|^2\, d\mu(\theta) \Big]$$

$$\leqslant K_6 T \delta_I + K_6 \int_0^T ds \int_{m(-r,-s)}^0 \delta_{s+\theta}\, d\mu(\theta), \tag{2.13}$$

where $\delta_s = E| x(s) - x(0)|^2$ and

$$\delta_I = 1 + \int_{-r}^0 E(| x(\theta)|^2 + | x(\theta) - x(0)|^2)\, d\mu(\theta).$$

Now, proceeding as in Theorem 2.1, we have ($t \leqslant T$)

$$\delta_t \leqslant \delta_I K_6 t + K_6 \int_0^t ds \int_{m(-r,-s)}^0 \delta_{s+\theta}\, d\mu(\theta)$$

and

$$\delta_t \leqslant \delta_I K_6 t e^{K_6 U t}. \tag{2.14}$$

Substituting (2.14) into (2.13) yields (2.10).

To prove (2.11), *fix* $x_0 \in C$. Then (2.10), the continuity of $x(t)$ for $t \in [-r, 0]$, and (A3′) imply

$$E \max_{h \geqslant t \geqslant 0} \| x_h - x_0 \|^2 \to 0$$

and

$$\max_{h\geqslant t\geqslant 0} E\,|f(x_s)-f(x_0)|^2 \to 0 \qquad \text{as} \quad h\to 0.$$

This result, with the evaluation

$$| Ex(h)-x(0)-hf(x_0)|^2 = \left| E\int_0^h (f(x_s)-f(x_0))\,ds\right|^2$$

$$\leqslant h\int_0^h E\,|f(x_s)-f(x_0)|^2\,ds \leqslant h^2 \max_{h\geqslant t\geqslant 0} E\,|f(x_s)-f(x_0)|^2,$$

proves (2.11). Eq. (2.12) is proved in a similar way.

3. Markov Properties of the Process x_t

Let $\mathscr{C}$ be the collection of open sets in C (with topology determined by the norm $\| x \| = \sup| x(\theta)|$, $\theta \in [-r, 0]$), and $\mathfrak{B}$ the Borel field over $\mathscr{C}$. The triple $\{C, \mathscr{C}, \mathfrak{B}\}$ is a topological state space (Dynkin [*8*], Appendix). Let x, the initial condition for (1.1), satisfy (A2). We suppose that all probability measure spaces introduced in the sequel are *complete* with respect to whatever measures are imposed on them. Let Ω denote the probability sample space, and ω the generic element of Ω. Define $\tilde{M}_t^x$ and $\tilde{N}_t^x$ as the least σ-fields on Ω over which $x(s)$, $-r \leqslant s \leqslant t$ and $x(s)$, $t-r\leqslant s \leqslant t$, are measurable, resp., for fixed $x_0 = x \in C$. Let P_x be the probability measure on

$$\tilde{M}^x = \bigcup_{t\geqslant 0} \tilde{M}_t^x.$$

Consider the collection of ω sets S defined by, for some $y\in C$, some $\epsilon > 0$, and any $0\leqslant s\leqslant t$,

$$S = \{\omega : \| x_s - y \| < \epsilon\} = \{\omega : \sup_{-r\leqslant\theta\leqslant 0} | x(s+\theta) - y(\theta)| < \epsilon\}.$$

Such S are in $\tilde{M}_t^x$ and, in fact, for any $\Gamma\in\mathfrak{B}$, the set $\{\omega : x_s\in\Gamma\}$, $s\leqslant t$, is contained in the least sub σ-field of $\tilde{M}_t^x$ containing all such S (for all $\epsilon>0$, $y\in C$). Denote this sub σ-field by M_t^x. Since $x(t)$, $t\geqslant -r$, is continuous w.p.l., so is x_t, $t\geqslant 0$, (in the topology induced by the norm $\| x\|$). Thus we have

Lemma 3.1. *Suppose* (A1) *to* (A3) *and fix* $x_0 = x\in C$. *Each* x_s, $0\leqslant s\leqslant t$, *is a random variable on* $\{\Omega, M_t^x, P_x\}$ *to* $\{C, \mathscr{C}, \mathfrak{B}\}$, *and* x_t *is continuous* w.p.l.;

x_t is measurable on $\{\Omega, N_t^x, P_x\}$ where $N_t^x = M_t^x \cap \tilde{N}_t^x$. For any function q whose expectation exists we have w.p.l.

$$E\{q(x_{t+s}) \mid M_t^x\} = E\{q(x_{t+s}) \mid N_t^x\}, \qquad s \geqslant 0. \tag{3.1}$$

THEOREM 3.1. *Assume* (A1) *to* (A3) *and let $x_0 = x \in C$. Then x_t is a continuous strong Markov process on the topological state space $\{C, \mathscr{C}, \mathfrak{B}\}$ with killing time $\xi(\omega) = \infty$* w.p.l.

Proof. The last statement merely says that the solution paths are defined for all $t < \infty$ w.p.l. To prove the Markov property we check the conditions of Dynkin [8], 77–80. For each fixed initial condition $x \in C$, the process $x(t)$ is defined by Lemma 2.1, and x_t by Lemma 3.1.

To prove the Markov property, we have only to show that

(i) the function p defined by $p(t, x, \Gamma) = P_x\{x_t \in \Gamma\}$, for arbitrary $\Gamma \in \mathfrak{B}$, is $\mathfrak{B}$ measurable, and

(ii) $P_x\{x_{t+h} \in \Gamma \mid M_t^x\} = p(h, x_t, \Gamma)$ w.p.l.

(i) is true since by Theorem 2.1, $p(t, x, \Gamma)$ is measurable on C. The "Markov" property (ii) is also true by Theorem 2.1 and Lemma 3.1, since the paths $x(s)$, $s \geqslant t$, (or x_s, $s \geqslant t$) of (1.1) are uniquely determined by the initial condition x_t w.p.l., and x_t is independent of $z(s) - z(t)$, $s \geqslant t$.

To prove that x_t is a strong Markov process, it suffices to prove (Dynkin [8], Theorem 3.10) that if $\alpha(x)$ is bounded and continuous on C, then $E_x\alpha(x_t) = \beta(x)$ is continuous in x. (E_x is the expectation operator corresponding to P_x.) Let x_t, y_t^n correspond to fixed initial conditions x, y^n. Then $\| x_t - y_t^n \| \to 0$ w.p.l. $t \geqslant 0$, as $\| x - y^n \| \to 0$ (Theorem 2.1). Then, the ω function α_n defined by $| \alpha(x_t) - \alpha(y_t^n)| \equiv \alpha_n(\omega)$ goes to zero as $n \to \infty$. Since $\alpha_n(\omega)$ is bounded, we have $E\alpha_n \to 0$ which implies that $\beta(x)$ is continuous in x. Q.E.D.

4. STOPPED PROCESSES [AND (A4) REPLACING (A3)]

Let R be some bounded open set in C and $\tau = \inf\{t : x_t \notin Q\}$. If $x_t \in Q$, all $0 \leqslant t < \infty$, set $\tau = \infty$. τ is a Markov time (Dynkin [8], Theorem 10.2); i.e., $\{\omega : \tau \leqslant t\} \in M_t^x$. Define the *stopped process* $\tilde{x}_t$

$$\tilde{x}_t = x_t, \qquad t \leqslant \tau$$

$$\tilde{x}_t = x_\tau, \qquad t > \tau.$$

$\tilde{x}_t$ is also a strong Markov process [under (A1)–(A3)] with infinite escape time, hence the paths of $\tilde{x}_t$ do not depend (w.p.l.) on the values of f and g [of (1.1)] outside of R.

Now, suppose that (A3) is replaced by (A4). The solution to (1.1) is defined as follows: Let $R_n = \{x : \| x \| < n\}$. Define functions f^n, g^n equal to f, g in

R_n and satisfying (A1) and (A3) for $\mu = \mu_n$. Define $x^n(t)$ (or x_t^n) as the solution to (1.1) corresponding to f^n, g^n. Let $\tau_n = \inf\{t : x_t^n \notin R_n\} = \inf\{t : |x^n(t)| \geqslant n\}$. If $x_0 \in R_n$, then $\tau_n > 0$ w.p.1. and x_t^n is a strong Markov process for each n; hence, w.p.1., $x_t^n = x_t^m$ for $m > n$ and $t \leqslant \tau_n$. Let $\xi = \lim \tau_n$. The solution to (1.1), under (A4), is defined as the process x_t which equals x_t^n up to τ_n for all n. If $\xi < \infty$ with a probability δ, the escape (or killing time) is finite w.p.δ. x_t (with the appropriate probability space) is a strong Markov process with killing time ξ.

For most of sequel we will be concerned only with the paths x_t only up to a time $\tau = \inf\{t : x_t \notin Q\}$ for some bounded open set Q, and only the properties of f, g in Q will be important. Since in application (A4) occurs frequently, we suppose that (A4) holds [in lieu of (A3)] and use the above interpretation of the solution of (1.1).

5. The Domain of the Weak Infinitesimal Operator

A real-valued function F on C is said to be in the domain of $\tilde{A}$, the weak infinitesimal operator, if the limits

$$\lim_{t\to 0} \frac{E_x F(x_t) - F(x)}{t} = q(x)$$

$$\lim_{t\to 0} E_x q(x_t) = q(x)$$

exist pointwise in C and the sequence is uniformly bounded in x. Then we write $q(x) = \tilde{A}F(x)$. Write $\tilde{A}_R$ for the weak infinitesimal operator of $\tilde{x}_t = x_t$ stopped at $\tau = \inf\{t : x_t \notin R\}$ for an open set R.

Lemma 5.1. *Let* (A1), (A2) *and* (A4) *hold for* (1.1). *Let $\hat{A}$ be the weak infinitesimal operator of a process $(\hat{x}(t))$ satisfying* (1.1) *with $\hat{f}$, $\hat{g}$ replacing f, g and satisfing* (A1)–(A3). *Let $\hat{f} = f$, $\hat{g} = g$ in the bounded open set R. $\hat{f}$ and $\hat{g}$ can be arranged outside of R so that $\| \hat{x}_t \| \leqslant K < \infty$. Let F be continuous and bounded on bounded sets. Then if $F \in \mathscr{D}(\hat{A})$ and $\hat{A}F = q$ is bounded on bounded sets, the restriction of F to R is in $\mathscr{D}(\tilde{A}_R)$ and on R, $\hat{A}F = \tilde{A}_R F$.*

Proof. That $\hat{f}$ and $\hat{g}$ can be arranged so that $\| \hat{x}_t \| < K$ is clear, since we can always find $\hat{f}$, $\hat{g}$ satisfying the other conditions and which are identically zero outside of some bounded open set containing the closure of R.

Next, let $x \in R$ and suppose $F \in \mathscr{D}(\hat{A})$ and $\hat{A}F = q$. Define $\tau = \inf\{t: x_t \notin R\}$. Then[1]

$$| E_x q(\hat{x}_t) - E_x q(\tilde{x}_t)| \leqslant | E_x[\chi_{\tau<t}(q(\hat{x}_t) - q(\tilde{x}_t))]| \to 0$$

[1] $\chi_{\tau<t}$ is the characteristic function of the set $[\omega: \tau < t]$, and $\tilde{x}_t$ is the x_t process stopped at τ.

or

$$E_x q(\tilde{x}_t) \to q(x),$$

as $t \to 0$, since $q(\tilde{x}_t)$ and $q(\hat{x}_t)$ are uniformly bounded and $\chi_{\tau<t} \to 0$ w.p.l. as $t \to 0$ by (2.10). To complete the proof we need only verify that

$$[E_x F(\tilde{x}_t) - F(x)]/t \to q(x).$$

But, since $[E_x F(\hat{x}_t) - F(x)]/t \to q(x)$, it suffices to verify that[2]

$$\begin{aligned} 0 &= \lim_t \frac{E_x F(\hat{x}_t) - F(x)}{t} - \lim_t \frac{E_x F(\hat{x}_t) - F(x)}{t} \\ &= \lim_t \frac{E_x \chi_{\tau<t}[F(\hat{x}_t) - F(\tilde{x}_{t\cap\tau})]}{t}, \end{aligned}$$

and that the sequence is uniformly bounded in x, as $t \to 0$.

For $1 < \gamma < 2$, the evaluation (6.4) and Chebyshev's inequality imply that $E_x(\chi_{\tau>t}/t)^\gamma \to 0$ as $t \to 0$ uniformly in R. Also, $F(\hat{x}_t) - F(\tilde{x}_{t\cap\tau})$ is uniformly bounded. Then, Holders inequality implies that the last expression is zero.

Q.E.D.

We have not been able to completely characterize the domain of the weak infinitesimal operator of either the x_t or $\tilde{x}_t$ process. For example, $F(x) = x(-a)$, $r > a > 0$, is not necessarily in $\mathscr{D}(\tilde{A})$, since $x(t)$ is not necessarily differentiable. Basically we are able to study functions $F(x)$ whose dependence on $x(\theta)$, for $-r \leqslant \theta < 0$, is in the form of an integral. The dependence of $F(x)$ on $x(0)$ can be more arbitrary. Fortunately the stochastic analogs of the available and useful deterministic Liapunov functions have this property. Theorems 5.1 and 5.2 give some results on the weak infinitesimal operator of $\tilde{x}_t$, where R is some open bounded set, $\tau = \inf\{t : x_t \notin R\}$ and (1.1) is interpreted in the sense of Sections 3 and 4, and (A4) is used. [(A4) is assumed since it appears in applications]. The proofs are only sketched since they involve only routine calculations.

THEOREM 5.1. *Assume* (A1), (A2) *and* (A4) *and* $x_0 = x \in C$. *Let* $F(x) \equiv G(x(0))$ *have continuous second derivatives with respect to* $x(0)$. *Then* $F(x) \in \mathscr{D}(\tilde{A}_R)$ *and*[3]

$$\tilde{A}_R F(x) \equiv LG(x(0)) = q(x) = G_u'(x(0)) f(x) + \tfrac{1}{2} \sum_{i,j} G_{u_i u_j}(x(0))\, \sigma_{ij}(x) \quad (5.1)$$

[2] $t \cap \tau = \min(t, \tau)$.

[3] G_u is the gradient with respect to the vector argument and the subscript $u_i u_j$ denotes a second partial derivative. Recall that R is a bounded open set.

where

$$\sigma_{ij} = \sum_k g_{ik} g_{jk} .$$

Proof. To compute $\tilde{A}_R F$ it suffices to assume, by Lemma 5.1, that (A1)–(A3) hold and $\| x_t \| \leqslant K$ for some sufficiently large but finite K, and to compute $\hat{A}F$ for the modified process (denoted also by x_t). Define $\delta x(0) = x(s) - x(0)$. Then

$$\begin{aligned} \frac{1}{s}[E_x G(x(t)) - G(x(0))] &= \frac{1}{s} G_u'(x(0)) E_x \, \delta x(0) \\ &+ \frac{1}{2s} \sum_{i,j} G_{u_i u_j}(x(0)) E_x \, \delta x_i(0)) \, \delta x_j(0) \qquad (5.2) \\ &+ \frac{1}{2s} \sum_{i,j} E_x [G_{u_i u_j}(x(0) + \alpha(\omega) \, \delta x(0)) - G_{u_i u_j}(x(0))] \, \delta x_i(0) \, \delta x_j(0), \end{aligned}$$

where $0 \leqslant \alpha(\omega) \leqslant 1$ and $\delta x_j(0)$ is the jth component of $\delta x(0)$. By (2.11) and (2.12), the limits (as $s \to 0$) of the first two terms on the right side of (5.2) exist (uniformly in x) and are the first two terms on the right side of (5.1). Now

$$[G_{u_i u_j}(x(0) + \alpha(\omega) \, \delta x(0)) - G_{u_i u_j}(x(0))]$$

is bounded and tends to zero w.p.l. as $s \to 0$. Then, applying Schwarz's inequality and the estimate (6.4) to the 3rd term in (5.2) yields that the term tends to zero (uniformly in x) as $s \to 0$.

Since we have assumed that $\| x_t \| \leqslant K < \infty$, and (A1), the f_i and σ_{ij} may be assumed to be bounded and continuous. Since, in addition, G_u and $G_{u_i u_j}$ are bounded on bounded sets and $\| x_s - x \| \to 0$ w.p.l., we have $E_x q(x_t) \to q(x)$ as $t \to 0$. Thus, by Lemma 5.1, $F(x) \in \mathscr{D}(\tilde{A}_R)$.

THEOREM 5.2. *Assume the conditions of Theorem* 5.1. *except that*

$$F(x) = \int_{-r}^{0} h(\theta) \, H(x(\theta), x(0)) \, d\theta. \qquad (5.3)$$

Let h be defined and have a continuous derivative on some open set containing $[-r, 0]$. *Let $H(\alpha, \beta)$, $H_{\beta_i}(\alpha, \beta)$ and $H_{\beta_i \beta_j}(\alpha, \beta)$ be continuous in α and β. Then* $F(x) \in \mathscr{D}(\tilde{A}_R)$ *and*

$$\begin{aligned} \tilde{A}_R F(x) = q(x) &= h(0) \, H(x(0), x(0)) - h(-r) \, H(x(-r), x(0)) \\ &- \int_{-r}^{0} h_\theta(\theta) \, H(x(\theta), x(0)) \, d\theta + \int_{-r}^{0} h(\theta) \, LH(x(\theta), x(0)) \, d\theta, \qquad (5.4) \end{aligned}$$

where the operator L is defined by (5.1) *and acts on H as a function of* $x(0)$ *only.*

Proof. As in the proof Theorem 5.1, we appeal to Lemma 5.1 and suppose that $\| x_t \| \leqslant K < \infty$ and (A1)–(A3) hold. Then, for small s,

$$
\begin{aligned}
\frac{1}{s}[E_x F(x_s) - F(x)] &= \frac{E_x}{s} \int_{-r}^{0} h(\theta)[H(x(s+\theta), x(s)) - H(x(\theta), x(0))]\, d\theta \\
&= \frac{E_x}{s} \int_{s-r}^{s} h(\theta - s)\, H(x(\theta), x(s))\, d\theta - \frac{E_x}{s} \int_{-r}^{0} h(\theta)\, H(x(\theta), x(0))\, d\theta \\
&= \int_{-r}^{0} \frac{E_x}{s} [h(\theta - s)\, H(x(\theta), x(s)) - h(\theta)\, H(x(\theta), x(0))]\, d\theta \\
&\quad + \frac{1}{s} \int_{0}^{s} E_x h(\theta - s)\, H(x(\theta), x(s))\, d\theta - \frac{1}{s} \int_{-r}^{-r+s} E_x h(\theta - s)\, H(x(\theta), x(s)) ds.
\end{aligned}
\tag{5.5}
$$

The last two terms tend, uniformly in x, to the first two terms of (5.4), resp. (In fact the last integral is not random for $s \leqslant r$). This is easily seen by virtue of the boundedness of H (for $\| x \| \leqslant K < \infty$), the continuity of h and H and (2.10)

By a straightforward calculation similar to that in the proof of Theorem 5.1, it is easy to show that the first terms of (5.5) tends (uniformly in x) to the last two terms of (5.4).

That $E_x q(x_t) \to q(x)$ also follows easily from (2.10), $\| x_t \| \leqslant K < \infty$, and the assumed boundedness and continuity of properties of h, h_θ, H and LH.

Theorem 5.3 and its corollary are useful extensions of Theorems 5.1 and 5.2. Their proof are also straightforward computations and will not be given. Loosely speaking, for Theorem 5.3, (see statement of Theorem)

$$
\begin{aligned}
\tilde{A}_R G = \lim_{s \to 0} G_F(F(x))\, E_x \left[\frac{F(x_s) - F(x)}{s}\right] \\
+ \lim_{s \to 0} \frac{G_{FF}(F(x))}{2} E_x \frac{[F(x_s) - F(x)]^2}{s}.
\end{aligned}
$$

The first and second terms correspond to the first and second terms of (5.6), resp. The second term reduces to merely

$$
\lim_{s \to 0} \frac{E_x}{s} \left[\int_{-r}^{0} h(\theta)[H(x(\theta), x(s)) - H(x(\theta), x(0))]\, d\theta\right]^2 \cdot G_{FF}(F(x)).
$$

THEOREM 5.3. *Let G be a twice continuously-differentiable real-valued function of a real argument. Assume the conditions of Theorem 5.2. Then $F_1(x) \equiv G(F(x)) \in \mathscr{D}(\tilde{A}_R)$ and*

$$
\begin{aligned}
\tilde{A}_R F_1(x) &= G_F(F(x))\, \tilde{A}_R F(x) + \tfrac{1}{2} G_{FF}(F(x)) \cdot B \\
B &= \int_{-r}^{0} \int_{-r}^{0} h(\theta)\, h(\rho) \sum_{i,j} H_{\beta_i}(x(\theta), x(0))\, H_{\beta_i}(x(\rho), x(0))\, \sigma_{ij}(x)\, d\theta\, d\rho.
\end{aligned}
\tag{5.6}
$$

where the derivatives H_{β_i} *are with respect to the* ith *component of the second vector argument of* $H(\alpha, \beta)$.

COROLLARY. *Let* $F^a(\beta)$ *and* $F^b(\alpha, \beta)$, *resp., satisfy the conditions on the respective* F's *of Theorems* 5.1 *and* 5.2. *Then, if* G *is twice continuously-differentiable,* $F_1(x) = G(F^a(x) + F^b(x)) \in \mathscr{D}(\tilde{A}_R)$ *and*

$$\tilde{A}_R F_1(x) = G_F(F^a(x) + F^b(x))(\tilde{A}_R F^a(x) + \tilde{A}_R F^b(x)) + \tfrac{1}{2} G_{FF}(F^a(x) + F^b(x)) \cdot B$$

$$B = \sum_{i,j} \sigma_{ij}(x)[F^a_{\beta_i}(x(0)) + C_i(x)][F^b_{\beta_j}(x(0)) + C_j(x)]$$

$$C_i(x) = \int_{-r}^{0} h(\theta)\, H_{\beta_i}(x(\theta), x(0))\, d\theta.$$

The differentiations $F^a_{\beta_i}$ *and* H_{β_i} *are with respect to the* ith *component of* $x(0)$ (the second argument of the second function).

6. STABILITY THEOREMS

Various definitions concerning stochastic stability appear in [1] and [3]. In lieu of definitions, we merely concern ourselves with the properties which the definitions codify and which appear in the theorems. Theorem 6.1 is a generalization of Lemma 1 and Theorems 1, 2 of Kushner [3], Chapter 2, where the state space is supposed to be Euclidean.

THEOREM 6.1. *Let* x_t *be a right-continuous strong Markov process on a topological state space* $\{C, \mathscr{C}, \mathfrak{B}\}$ *with weak infinitesimal operator* $\tilde{A}$. *Let the norm* $\|\ \|$ *generate* $\mathscr{C}$. *Let the nonnegative continuous real-valued function* $V(x)$ *in* $\mathscr{D}(\tilde{A})$. *Let* $Q = \{x : V(x) < q\}$ *and let* $\tau = \inf\{t : x_t \notin Q\}$. *Set* $\tau = \infty$ *if* $x_t \in Q$ *for all* $t < \infty$. *Let* $\tilde{A}V(x) = -k(x) \leqslant 0$ *in* Q. *Then, for* $x = x_0 \in Q$,

(B1) $V(x_{t\cap\tau}) \equiv w_t$ *is a nonnegative supermartingale,*

(B2) $P_x\{\sup_{\infty > t \geqslant 0} V(x_t) \geqslant q\} \leqslant V(x)/q$,

(B3) $V(x_{t\cap\tau}) \to v \geqslant 0$ w.p.l.

If, in addition,

(i) *k is uniformly continuous on the nonempty open set*

$$R_\delta = \{x : k(x) < \delta\} \cap Q,$$

for some $\delta > 0$, *and*

(ii) *for all sufficiently large but finite Markov times t, and all sufficiently small* ϵ,

$$P_x\{\max_{t+h \geqslant s \geqslant t} \| x_s - x_t \| \geqslant \epsilon \quad \text{and} \quad x_r \in Q, \quad \text{all} \quad r \leqslant t\} \to 0$$

as $h \to 0$, *uniformly in t for sufficiently large t, and any* $x \in Q$. *Then*

(B4) $k(x_t) \to 0$ w.p.l. (*relative to* $\Omega_Q = \{\omega : \sup_{\infty > t \geqslant 0} V(x_t(\omega)) < q\}$).

Proof. Fix the initial conditions $x = x_0 \in Q$. Since $V(x) \in \mathscr{D}(\tilde{A})$ and $\tau \cap t$ is a finite-valued Markov time, Dynkins formula [8], Theorem 5.1 and corollary) gives

$$E_x V(x_{t\cap\tau}) - V(x) = -E_x \int_0^{t\cap\tau} k(x_s)\, ds \leqslant 0. \tag{6.1}$$

(6.1) together with fact that $V(x) \in \mathscr{D}(\tilde{A})$ yields that $V(x_{t\cap\tau}) \equiv w_t$ is a nonnegative supermartingale (see proof of Theorem 12.6 in Dynkin, [8]). Then (B2) and (B3) follow immediately as properties of nonnegative supermartingales.

Let $0 < \delta < \hat{\delta}$ and $R_\delta \equiv \{x : k(x) < \delta\} \cap Q$. Let $I_x(\delta, \omega, s)$ be the indicator of the (s, ω) set where $k \geqslant \delta$ (for $x_0 = x$) and let $\int_{t\cap\tau}^{\tau} I_x(\delta, \omega, s)\, ds = T_x(\delta, t)$. Then, by the facts that the left side of (6.1) is bounded below by $-V(x)$, and that $V(x) \geqslant 0$, we have $E_x T_x(\delta, 0) \leqslant V(x)/\delta$. $T_x(\delta, t)$ is the *total* time that x_t spends in $Q - R_\delta$ before either $t = +\infty$ (if $\tau = \infty$) or the first exit time from Q (if $\tau < \infty$). Furthermore $T_x(\delta, t) < \infty$ w.p.l. and $T_x(\delta, t) \to 0$ w.p.l. as $t \to \infty$.

Now, $\min\{\| x - y \| : x \in R_{\delta/2}, \ y \in Q - R_\delta\} = \epsilon$, where $\epsilon > 0$ by (i). Define $\Omega_Q = \{\omega : x_t \in Q, \text{ all } t < \infty\}$. $P(\Omega_Q\} \geqslant 1 - V(x)/q$ by (B2). For each fixed positive h and γ, there is a $t_x(h, \gamma)$ so that $t > t_x(h, \gamma)$ implies $T_x(\delta, t) \leqslant T_x(\delta/2, t) < h$ with probability $\geqslant 1 - \gamma$. Let $t_x(h, \gamma)$ be sufficiently large and h sufficiently small so that the probability on the left side of (ii) is less than γ. Suppose that there is a finite Markov time $t > t_x(h, \gamma)$ for which $x_t \in Q - R_\delta$. The probability of the event $\{x_t \in Q - R_\delta, x_{t+\alpha} \in Q - R_{\delta/2}\}$ for some $h \geqslant \alpha \geqslant 0$ is no greater than γ (relative to Ω_Q). Thus, since $T_x(\delta_1, t) > h$ with probability $\geqslant 1 - \gamma$, we conclude that the probability of never leaving R_δ in $[t, \infty)$ goes to 1 (relative to Ω_Q) as $t \to \infty$. Since δ is arbitrary, we conclude that $k(x_t) \to 0$ w.p.l. (relative to Ω_Q). Q.E.D.

An apparent difficulty with the sets $\{x : V(x) < q\}$ defined in Theorem 6.1 is that they are not bounded for typical cases (see Section 7, Examples), and hence the characterization of the weak infinitesimal operator is much harder than the work in Section 5. This is also the situation in the deterministic case (as in Hale [4]). However, in our examples (as well as in the determinstic cases studied (Hale [4])), it turns out that if

$$x_0 = x \in \{x : V(x) < q\} \subset C,$$

then there is a constant K *independent of* x_0 so that $\| \tilde{x}_t \|$ (for $t \geqslant r$) and $| \tilde{x}(t)|$ (for $t \geqslant 0$) are no greater than K. In other words, up until the first exit time from $\{x : V(x) < q\}$, it will turn out that $| x(t)| \leqslant K < \infty$. (See Section 7) Since any initial $x_0 \in C$ is bounded, there is no loss in generality in supposing that there is a bounded open set B whose radius is K_1, $\infty > K_1 \geqslant K$, so that if $x_0 \in Q \equiv \{x : V(x) < q\} \cap B$, then $\| x_t \| \leqslant K_1$ until $\tau = \inf\{t : x_t \notin Q\}$. B can always be made large enough to include any desired initial condition which satisfies $x \in \{x : V(x) < q\}$. The resulting boundedness, besides not appearing to be a serious restriction, enables us to use the results of Section 5.

THEOREM 6.2. *Assume* (A1), (A2) *and* (A4). *Let* $V(x)$ *be a continuous nonnegative real valued function on* C. *Suppose that*

(iii) *there is a bounded open set* B *such that*

$$x_0 = x \in Q \equiv \{x : V(x) < q\} \cap B$$

and $\sup_{t > s \geqslant 0} V(x_s) < q$ *imply that* $x_s \in Q$, *all* $0 < s < t$. *Let* $V(x) \in \mathscr{D}(\tilde{A}_Q)$ *and* $\tilde{A}_Q V(x) = -k(x) \leqslant 0$ *in* Q, *and* $x \in Q$. *Then* (B1)–(B3) *hold, and* $P(\Omega_Q) \geqslant 1 - V(x)/q$. *If* k *is uniformly continuous on* $\{R_\delta = x : k(x) < \delta\}$ *for some* $\delta > 0$, *then* $k(x_t) \to 0$ w.p.1. (*relative to* Ω_Q).

Remark. For $V(x) \in \mathscr{D}(\tilde{A}_Q)$, it suffices, by the hypothesis and Lemma 5.1, that $V(x) \in \mathscr{D}(\hat{A})$ where $\hat{A}$ is the weak infinitesimal operator of any modification of (1.1) with $\hat{f} = f$, $\hat{g} = g$ in Q and which has uniformly bounded paths (where the bound is at least the outer radius of B).

Proof. Condition (iii) and Theorem 6.1 imply (B1)–(B3). To complete the proof we have only to show that (ii) of Theorem 6.1 is true. According to Lemma 5.1, it suffices to show this under assumptions (A1)–(A3) and with the paths $\| x_t \| \leqslant K_2$ for some finite K_2. Condition (ii) is equivalent to

$$P_x\{\max_{h \geqslant s \geqslant 0} \max_{-r \leqslant \theta \leqslant 0} | x(t + s + \theta) - x(t + \theta)| \geqslant \epsilon, x_u \in Q, u \leqslant t\} \to 0 \quad (6.2)$$

as $h \to 0$, uniformly in t for large t, and any $\epsilon > 0$. (6.2) is majorized by

$$\begin{aligned} &\sup_{x \in Q} P_x\{\max_{h \geqslant s \geqslant 0} \max_{-r \leqslant \theta \leqslant 0} | x(r + s + \theta) - x(r + \theta)| \geqslant \epsilon\} \\ &\quad = \sup_{x \in Q} P_x\{\max_{h \geqslant s \geqslant 0} | x(r + \theta + s) - x(r + \theta)| \geqslant \epsilon \quad \text{for some} \quad \theta \in [-r, 0]\} \\ &\quad \leqslant \sum_{n=0}^{r/h} \sup_{x \in Q} P_x \left\{\max_{h \geqslant s \geqslant 0} | x(nh + s) - x(nh)| \geqslant \frac{\epsilon}{2}\right\}. \end{aligned} \quad (6.3)$$

To complete the proof, we need the evaluation

$$E \max_{h\geqslant s\geqslant 0} | x(t+s) - x(t)|^4$$
$$\leqslant K_3 \left(E \int_t^{t+h} |f(x_s)|\, ds\right)^4 + K_3 \left(\int_t^{t+h} E|\sigma(x_s)|^2\, ds\right)^2 \leqslant K_4 h^2, \quad (6.4)$$

where K_4 is independent of h, t and x, for $x \in Q$. In (6.4) we used the assumption (Lemma 5.1) that the paths $\| x_t \|$ are bounded (hence $|f|$ and $|\sigma|$ are bounded), the first line of (2.3) and $Ew_i^4(T) = 3(Ew_i^2(T))^2$ [see (2.3)].

By (6.4) and Chebyshev's inequality

$$P_x \left\{ \max_{h\geqslant s\geqslant 0} | x(nh+s) - x(nh)| \geqslant \frac{\epsilon}{2} \right\} \leqslant \frac{E \max_{h\geqslant s\geqslant 0} | x(nh+s) - x(nh)|}{(\epsilon/2)^4}$$
$$\leqslant \frac{16K_4h^2}{\epsilon^4} = \frac{K_5h^2}{\epsilon^4}$$

for $x \in Q$. Then each entry of the right-hand sum of (6.3) is bounded by K_5h^2/ϵ^4 and hence the sum is bounded by $(r+h)\, K_5 h/\epsilon^4$, which completes the proof.

7. Examples

Example 1. Let $x(t)$ be scalar and

$$dx(t) \doteq -ax(t)\, dt - bx(t-\tau)\, dt + \sigma x(t-\rho)\, dz(t).$$

$$V(x) = x^2(0)/2 + \alpha \int_{-\tau}^0 x^2(\theta)\, d\theta + \beta \int_{-\rho}^0 x^2(\theta)\, d\theta, \qquad \alpha \geqslant 0, \quad \beta \geqslant 0. \quad (7.1)$$

Fix $q < \infty$, and $x_0 = x \in C$. Let $\| x \| = K_2$. Note that if $V(x_s) < q$ for all $s < t$, then $x^2(s) < 2q$ for all $0 \leqslant s < t$, and $\| x_s \| \leqslant \max([2q]^{1/2}, K_2)$ for all $s < t$. Then, any bounded open set B, containing the origin and with radius at least $\max([2q]^{1/2}, K_2)$, satisfies the condition on the set B of Theorem 6.2. Let $Q = \{x : V(x) < q\} \cap B$. Then $V(x) \in \mathscr{D}(\tilde{A}_Q)$ by Theorems 5.1 and 5.2, and

$$\tilde{A}_Q V(x) = x^2(0)(-a + \alpha + \beta) - bx(0)\, x(-\tau)$$
$$- \alpha x^2(-\tau) - \beta x^2(-\rho) + \frac{\sigma^2}{2} x^2(-\rho). \quad (7.2)$$

Suppose that there is an $\alpha > 0$ and $\beta > 0$ so that the quadratic form (7.2) [in $x(0)$, $x(-\tau)$, $x(-\rho)$] is negative definite. Then, by Theorem 6.2

$$P_x\{ \sup_{\infty > t \geqslant 0} V(x_t) \geqslant q\} \leqslant V(x)/q. \quad (7.3)$$

Since q is arbitrary, we also have, w.p.l.

$$V(x_t) \to v$$
$$k(x_t) \to 0$$
$$x_t \to \{x : x(t) = x(t-\rho) = x(t-\tau) = 0\}.$$

where $v(\omega)$ is some random variable. Hence $x_t \to 0$ w.p.l.

For small noise the estimate (7.3) can be improved. Let $\beta = \rho = 0$ for ease of computation. Let $F(x) = e^{\lambda V(x)}$, where $\lambda > 0$. $F(x) \in \mathscr{D}(\tilde{A}_Q)$ (for any sufficiently large B) and, by the corollary to Theorem 5.3,

$$\begin{aligned}\tilde{A}_Q F(x) &= \lambda F(x)\, \tilde{A}_Q V(x) + \frac{\lambda^2}{2} F(x) \cdot x^2(0)\, \sigma^2 \\ &= \lambda F(x) \left\{ x^2(0) \left(-a + \frac{\sigma^2}{2} + \alpha + \frac{\lambda\sigma^2}{2} \right) - \alpha x^2(-\tau) - b x(0)\, x(-\tau) \right\}.\end{aligned}$$

If

$$\alpha \left(a - \frac{\sigma^2}{2} - \frac{\lambda\sigma^2}{2} - \alpha \right) \geqslant b^2/4, \tag{7.4}$$

then $F(x)$ is a Liapunov function, and

$$P_x\{ \sup_{\infty > t \geqslant 0} V(x_t) \geqslant q \} = P_x\{ \sup_{\infty > t \geqslant 0} e^{\lambda V(x_t)} \geqslant e^{\lambda q} \} \leqslant e^{\lambda (V(x) - q)}. \tag{7.5}$$

Clearly, as λ increases, within constraint (7.4), the estimate (7.5) improves.

EXAMPLE 2. Let

$$dx_1(t) = x_2(t)\, dt$$
$$dx_2(t) = \left\{ -h(x_1(t)) + \int_{-r}^{0} f(\theta)\, g(x_1(t+\theta) - x_1(t))\, d\theta \right\} dt + \sigma(x_t)\, dz(t).$$

Suppose that $w \neq 0$ implies that $h(w)\, w > 0$ and $g(w)\, w > 0$ and let $h(0) = g(0) = \sigma(0) = 0$. Let $f(\theta)$, $g(w)$ and $h(w)$ have continuous derivatives and suppose that (A1), (A2) and (A4) hold. Define

$$V(x) = x_2^2(0)/2 + H(x_1(0)) + \int_{-r}^{0} f(\theta)\, G(x_1(\theta) - x_1(0))\, d\theta \tag{7.6}$$

where

$$H(w) \equiv \int_0^w h(\lambda)\, d\lambda \to \infty \qquad \text{as} \quad |w| \to \infty$$

and

$$G(w) = \int_0^w g(\lambda)\, d\lambda.$$

Fix $q < \infty$ and $x = x_0 \in C$. Let $\| x \| = K_2$. Note that if $V(x_s) < q$ for all $s < t$, then $x_2^2(s) < 2q$ and $H(x_1(s)) < q$ for all $0 \leqslant s < t$ and hence, for $0 \leqslant s < t$,

$$\| x_s \| \leqslant \max\{K_2, (2q + \max\{| x_1 |^2 : H(x_1) = q\})^{1/2}\} = K_1 .$$

Any bounded open set B, with radius at least K_1 and which contains the origin, satisfies the conditions on the B of Theorem 6.2. Then $V(x) \in \mathscr{D}(\tilde{A}_Q)$ and Theorems 5.1 and 5.2 yield

$$\begin{aligned}
\tilde{A}_Q V(x) &= x_2(0) \left\{ -h(x_1(0)) + \int_{-r}^{0} f(\theta)\, g(x_1(\theta) - x_1(0))\, d\theta \right\} \\
&\quad + \sigma^2(x)/2 + h(x_1(0))\, x_2(0) - f(-r)\, G(x_1(-r) - x_1(0)) \\
&\quad + \int_{-r}^{0} f_\theta(\theta)\, G(x_1(\theta) - x_1(0))\, d\theta - \int_{-r}^{0} f(\theta)\, g(x_1(\theta) - x_1(0))\, x_2(0)\, d\theta \\
&= \sigma^2(x)/2 + \int_{-r}^{0} f_\theta(\theta)\, G(x_1(\theta) - x_1(0))\, d\theta - f(-r)\, G(x_1(-r) - x_1(0)).
\end{aligned} \tag{7.7}$$

To complete the analysis, in analogy to the method of Hale [4], suppose that $f(\theta) > 0$, $f_\theta(\theta) \leqslant 0$ and $f_\theta(\rho) < 0$ for some $\rho \in [-r, 0]$, and that for some $\gamma > 0$,

$$\sigma^2(x)/2 - f(-r)\, G(x_1(-r) - x_1(0)) \leqslant -\gamma f(-r)\, G(x_1(-r) - x_1(0)).$$

Note that, by continuity, $f_\theta(\theta) < 0$ for $\rho - \beta < \theta < \rho + \alpha$, for some $\alpha > 0$, $\beta > 0$. Then

$$\tilde{A}_Q V(x) \leqslant \int_{-r}^{0} f_\theta(\theta)\, G(x_1(\theta) - x_1(0))\, d\theta - \gamma f(-r)\, G(x_1(-r) - x_1(0)) \leqslant 0.$$

$$P_x\{ \sup_{\infty > t \geqslant 0} V(x_t) \geqslant q\} \leqslant V(x)/q. \tag{7.8}$$

Since q is arbitrary, Theorem 6.2 implies that $k(x_t) \to 0$ w.p.1., and that $V(x_t)$ converges w.p.1. Eq. (7.8) will be useful in the sequel for it says that the paths x_t are uniformly bounded with a probability as close to one as desired. Note that $G(x_1(t - r) - x_1(t)) \to 0$ w.p.1. implies that

$$x_1(t - r) - x_1(t) \to 0 \text{ w.p.1.}$$

We now show that $x(t) \to 0$ w.p.1. Since $k(x_t) \to 0$ w.p.1.,

$$\int_{-\rho-\beta+\epsilon}^{-\rho+\alpha-\epsilon} G(x_1(t + \theta) - x_1(t))\, d\theta \to 0$$

w.p.l., for $0 < \epsilon < \min(\alpha, \beta)$. Thus, using the positive definiteness of G and the boundedness of the paths,

$$\begin{aligned} &\int_{-T}^{0} G(x_1(t+\theta) - x_1(t))\, d\theta \to 0 \\ &\int_{-T}^{0} | x_1(t+\theta) - x_1(t)|\, d\theta \to 0 \end{aligned} \tag{7.9}$$

w.p.l., as $t \to 0$, for any finite T. Also, using (7.9) and the fact that $V(x_t) \to v(\omega) \geqslant 0$ w.p.l., we have w.p.l.

$$x_2^2(t)/2 + H(x_1(t)) \to v(\omega) \geqslant 0. \tag{7.10}$$

Now integrating the defining equations between $t - s$ and s gives

$$\begin{aligned} x_2(t) - x_2(t-s) = &- \int_{t-s}^{t} h(x_1(u))\, du \\ &+ \int_{t-s}^{t} du \int_{-r}^{0} f(\theta)\, g(x_1(u+\theta) - x_1(u))\, d\theta + \int_{t-s}^{t} \sigma(x_u)\, dz(u) \end{aligned} \tag{7.11}$$

$$x_1(t) - x_1(t-s) = \int_{t-s}^{t} x_2(u)\, du. \tag{7.12}$$

Using (7.9) and the boundedness of the paths, the second term on the right of (7.11) goes to zero w.p.l. as $t \to \infty$ for any $s > 0$. Also, (7.9) and (7.12), together with the stochastic continuity of $x_2(u)$ (Theorem 2.2), imply that $x_1(t) - x_1(t-s) \to 0$ w.p.l. for any finite s. Then, using this fact and stochastic continuity, (7.10) implies that $x_2(t) - x_2(t-s) \to 0$ w.p.l. The latter fact implies, via (7.12), that $x_2(u) \to 0$ w.p.l. as $t \to \infty$. Finally, (7.11) gives

$$- \int_{t-s}^{t} h(x_1(u))\, du + \int_{t-s}^{t} \sigma(x_u)\, dz(u) \to 0 \tag{7.13}$$

w.p.l. Eq. (7.13), together with the fact that $x_1(u)$ is asymptotically constant over time intervals of fixed length (i.e., $x_1(t) - x_1(t-s) \to 0$ w.p.l. for all $s > 0$ as $t \to \infty$) implies that $h(x_1(u)) \to 0$ w.p.l., and hence that $x(t) \to 0$ w.p.l.

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