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The minimal components of the Mayr–Meyer ideals

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Grete Hermann proved in [8] that for any ideal I in an n-dimensional polynomial ring over the field of rational numbers, if I is generated by polynomials $f_1, \dots, f_k$ of degree at most d, then it is possible to write $f = \sum r_i f_i$ such that each r_i has degree at most $\deg f + (kd)^{(2^n)}$. Mayr and Meyer in [11] found (generators of) a family of ideals for which a doubly exponential bound in n is indeed achieved. Bayer and Stillman [1] showed that for these Mayr–Meyer ideals any minimal generating set of syzygies has elements of doubly exponential degree in n. Koh [9] modified the original ideals to obtain homogeneous quadric ideals with doubly exponential syzygies and ideal membership equations.

Bayer, Huneke, and Stillman asked whether the doubly exponential behavior is due to the number of associated prime ideals, or to the nature of one of them? By comparing to Kollár's effective Nullstellensatz [10], the suspicion is that the exponential behavior is due to some deeply embedded component. This paper examines the minimal components and minimal prime ideals of the Mayr–Meyer ideals. In particular, in Section 2 it is proved that the intersection of the minimal components of the Mayr–Meyer ideals does not satisfy the doubly exponential property, so that the doubly exponential behavior of the Mayr–Meyer ideals must be due to the embedded prime ideals.

The structure of the embedded prime ideals of the Mayr–Meyer ideals is examined in [14].

There exist algorithms for computing primary decompositions of ideals (see Gianni et al. [5], Eisenbud et al. [3], or Shimoyama and Yokoyama [12]), and they have been partially implemented on the symbolic computer algebra programs Singular [7] and

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Macaulay2 [6]. However, the Mayr–Meyer ideals have variable degree and a variable number of variables over an arbitrary field, and there are no algorithms to deal with this generality. Thus any primary decomposition of the Mayr–Meyer ideals has to be accomplished with traditional proof methods. Small cases of the primary decomposition analysis were partially verified on Macaulay2 and Singular, and the emphasis here is on "partially": the computers quickly run out of memory.

The Mayr–Meyer ideals are binomial, so by the results of Eisenbud and Sturmfels in [4] all the associated prime ideals themselves are also binomial ideals. It turns out that many minimal prime ideals are even monomial, which simplifies the calculations.

The Mayr–Meyer ideals depend on two parameters, n and d, where the number of variables in the ring is $\mathrm{O}(n)$ and the degree of the given generators of the ideal is $\mathrm{O}(d)$. Both n and d are positive integers.

Here is the definition of the Mayr–Meyer ideals: let $n, d \geqslant 1$ be integers and k a field of arbitrary characteristic. Let $s, f, s_{r+1}, f_{r+1}, b_{r1}, b_{r2}, b_{r3}, b_{r4}, c_{r1}, c_{r2}, c_{r3}, c_{r4}$ be variables over k, with $r = 0, 1, \ldots, n-1$. The notation here closely follows that of [9]. Set

$$S = k[s = s_0, f = f_0, s_{r+1}, f_{r+1}, b_{ri}, c_{ri} \mid r = 0, \ldots, n-1;\ i = 1, \ldots, 4].$$

Thus S is a polynomial ring of dimension $10n + 2$. The following generators define the Mayr–Meyer ideal $J_l(n, d)$ (subscript l for "long," there will be a "shortened" version later on): first the four level 0 generators:

$$H_{0i} = c_{0i}\left(s - f b_{0i}^d\right), \quad i = 1, 2, 3, 4;$$

then the first six level r generators, $r = 1, \ldots, n$:

$$\begin{aligned}
H_{r1} &= s_r - s_{r-1}c_{r-1,1}, \\
H_{r2} &= f_r - s_{r-1}c_{r-1,4}, \\
H_{r3} &= f_{r-1}c_{r-1,1} - s_{r-1}c_{r-1,2}, \\
H_{r4} &= f_{r-1}c_{r-1,4} - s_{r-1}c_{r-1,3}, \\
H_{r5} &= s_{r-1}(c_{r-1,3} - c_{r-1,2}), \\
H_{r6} &= f_{r-1}(c_{r-1,2}b_{r-1,1} - c_{r-1,3}b_{r-1,4}),
\end{aligned}$$

the last four level r generators, $r = 1, \ldots, n-1$:

$$H_{r,6+i} = f_{r-1}c_{r-1,2}c_{ri}(b_{r-1,2} - b_{ri}b_{r-1,3}), \quad i = 1, \ldots, 4,$$

and the last level n generator:

$$H_{n7} = f_{n-1}c_{n-1,2}(b_{n-1,2} - b_{n-1,3}).$$

The maximum degree of a given generator of $J_l(n, d)$ is $\max\{d+2, 4, 5\delta_{n\geqslant 2}\}$, where $\delta_{n\geqslant 2}$ is the (extended) Kronecker delta function: it is 1 if $n \geqslant 2$ and is 0 otherwise. The degree 1 element $s_n - f_n$ of S is in $J_l(n,d)$, and when written as an S-linear combination of the given generators, the S-coefficient of H_{04} has degree which is doubly exponential in n (see any of [1,2,9,11]).

The main result of this paper is the computation of the minimal prime ideals and the minimal primary components of these Mayr–Meyer ideals. Another result is the computation of the intersection of all the minimal components, which also shows that the doubly exponential behavior of the $J_l(n,d)$ is due to the embedded prime ideals.

The following summarizes the elementary facts used in the paper:

Facts.

0.1. For any ideals I, I' and I'' with $I \subseteq I''$, $(I + I') \cap I'' = I + I' \cap I''$.
0.2. For any ideal I and element x, $(x) \cap I = x(I : x)$.
0.3. Let $x_1, \ldots, x_n$ be variables over a ring R. Let $S = R[x_1, \ldots, x_n]$. For any $f_1 \in R$, $f_2 \in R[x_1], \ldots, f_n \in R[x_1, \ldots, x_{n-1}]$, let L be the ideal $(x_1 - f_1, \ldots, x_n - f_n)S$ in S. Then an ideal I in R is primary (respectively, prime) if and only if $IS + L$ is primary (respectively, prime) in S. Furthermore, $\bigcap_i q_i = I$ is a primary decomposition of I if and only if $\bigcap_i (q_i S + L)$ is a primary decomposition of $IS + L$.
0.4. Let x be an element of a ring R and I an ideal. Suppose that there is an integer k such that for all m, $I : x^m \subseteq I : x^k$. Then $I = (I : x^k) \cap (I + (x^k))$. Thus to find a (possibly redundant) primary decomposition of I it suffices to find primary decompositions of (possibly larger) $I : x^k$ and of $I + (x^k)$.

We immediately apply this: in order to find a primary decomposition of the Mayr–Meyer ideals $J_l(n,d)$, by the structure of the H_{r1}, H_{r2} and by Fact 0.3, it suffices to find a primary decomposition of the ideals $J(n,d)$ obtained from $J_l(n,d)$ by rewriting the variables s_r, f_r in terms of other variables, and then omitting the generators H_{r1}, H_{r2}, $r \geqslant 1$. An ideal q is a component (respectively associated prime) of $J(n,d)$ if and only if $(q + (H_{r1}, H_{r2} | r))S$ is a component (respectively associated prime) of $J_l(n,d)$. Thus to simplify the notation, throughout we will be searching for the primary components and associated prime ideals of the "shortened" Mayr–Meyer ideals $J(n,d)$ in a smaller polynomial ring R obtained as above. When we list the new generators explicitly, the case $n = 1$ is rather special. In fact, the primary decomposition in the case $n = 1$ is very different from the case $n \geqslant 2$, and is given in [13]. In this paper it is always assumed that $n \geqslant 2$.

Thus explicitly, we will be working with the following "shortened" Mayr–Meyer ideals: for any fixed integers $n \geqslant 2, d$, set $R = k[s, f, b_{ri}, c_{ri} \mid r = 0, \ldots, n-1;\ i = 1, \ldots, 4]$, a polynomial ring in $8n + 2$ variables, and set $J(n,d)$ to be the ideal in R generated by the following polynomials h_{ri}: first the four level 0 generators:

$$h_{0i} = c_{0i}\bigl(s - f b_{0i}^d\bigr), \quad i = 1, 2, 3, 4;$$

then the eight level 1 generators:

$$h_{13} = f c_{01} - s c_{02},$$
$$h_{14} = f c_{04} - s c_{03},$$

$$h_{15} = s(c_{03} - c_{02}),$$
$$h_{16} = f(c_{02}b_{01} - c_{03}b_{04}),$$
$$h_{1,6+i} = fc_{02}c_{1i}(b_{02} - b_{1i}b_{03}), \quad i = 1, \ldots, 4,$$

the first four level r generators, $r = 1, \ldots, n$:

$$\begin{aligned} h_{r3} &= sc_{01}c_{11} \cdots c_{r-3,1}(c_{r-2,4}c_{r-1,1} - c_{r-2,1}c_{r-1,2}), \\ h_{r4} &= sc_{01}c_{11} \cdots c_{r-3,1}(c_{r-2,4}c_{r-1,4} - c_{r-2,1}c_{r-1,3}), \\ h_{r5} &= sc_{01}c_{11} \cdots c_{r-2,1}(c_{r-1,3} - c_{r-1,2}), \\ h_{r6} &= sc_{01}c_{11} \cdots c_{r-3,1}c_{r-2,4}(c_{r-1,2}b_{r-1,1} - c_{r-1,3}b_{r-1,4}), \end{aligned}$$

the last four level r generators, $r = 1, \ldots, n-1$:

$$h_{r,6+i} = sc_{01}c_{11} \cdots c_{r-3,1}c_{r-2,4}c_{r-1,2}c_{ri}(b_{r-1,2} - b_{ri}b_{r-1,3}), \quad i = 1, \ldots, 4,$$

and the last level n generator:

$$h_{n7} = sc_{01}c_{11} \cdots c_{n-3,1}c_{n-2,4}c_{n-1,2}(b_{n-1,2} - b_{n-1,3}).$$

For simpler notation, $J(n,d)$ will often be abbreviated to J.

Observe that the maximum degree of the given generators of $J(n,d)$ is $\max\{n+2, 5, d+2\}$. The image $sc_{01}c_{11} \cdots c_{n-2,1}(c_{n-1,1} - c_{n-1,4})$ of $s_n - f_n$ by construction lies in $J(n,d)$ and has degree $n+1$. When this element is written as an R-linear combination of the h_{ri}, the coefficient of h_{04} is doubly exponential in n. Note that the contrast between a number doubly exponential in n and the degree $n+1$ of the input polynomial arising from this instance of the ideal membership problem for $J(n,d)$ is not as striking as the contrast between a number doubly exponential in n and the degree 1 of the input polynomial arising from the ideal membership example $s_n - f_n$ for $J_l(n,d)$.

Thus while $J(n,d)$ is a useful simplification of $J_l(n,d)$ as far as the primary decomposition and associated prime ideals are concerned, its doubly exponential nature is partially concealed.

This paper consists of two sections. Section 1 is about all the minimal prime ideals, their components, and their heights. For simplicity we assume that the underlying field k is algebraically closed. Then the number of minimal prime ideals over $J(n,d)$ is $n(d')^2 + 20$ (Proposition 1.6), where d' is the largest divisor of d which is relatively prime to the characteristic of the field. Most of the minimal components are simply the prime ideals (Proposition 1.7). Section 2 shows that the doubly exponential behavior of the Mayr–Meyer ideals is due to the existence of embedded prime ideals.

The computation of embedded prime ideals is tackled in [14]. [14] also constructs a new family of ideals with the doubly exponential ideal membership problem. Recursion can be applied to this new family in the construction of the embedded prime ideals, see [15].

1. Minimal prime ideals and their components

The minimal prime ideals over $J(n,d)$ and their components are easy to compute. Let d' denote the largest divisor of d which is relatively prime to the characteristic of the field. Then there are $n(d')^2 + 20$ minimal prime ideals, many of which are their own primary components of $J(n,d)$.

The minimal prime ideals are analyzed in two groups: those on which s and f are non-zerodivisors, and the rest of them. The first group consists of $n(d')^2 + 1$ prime ideals.

The minimal prime ideals not containing sf are denoted $P_{r\_}$, where r varies from 0 to n, and the other part _ of the subscript depends on r. For the rest of the minimal prime ideals the front part of the subscript varies from -1 to -4.

Lemma 1.1. *Let P be an ideal of R containing J such that s and f are non-zero-divisors modulo P (in particular $sf \notin P$). Let $r \in \{0,\dots,n-1\}$. Suppose that for all $j < r$ and all $i = 1,2,3,4$, c_{ji} is not a zero-divisor modulo P. Then*

(1) *For all $j \in \{0,\dots,r\}$,*

$$c_{j3} - c_{j2},\, c_{j4} - c_{j1},\, c_{01} - c_{02}b_{01}^d \in P,$$

and if $j > 0$,

$$c_{j2} - c_{j1} \in P.$$

(2) *If $r > 0$, $c_{ri} \in P$ for some $i \in \{1,2,3,4\}$ if and only if $c_{ri} \in P$ for all $i \in \{1,2,3,4\}$.*

(3) *For all $j \in \{0,\dots,r-1\}$,*

$$b_{j4} - b_{j1} \in P.$$

Also, for all $j \in \{0,\dots,r-2\}$,

$$b_{j2} - b_{j+1,i}b_{j3} \in P, \quad i = 1,2,3,4.$$

(4) *Assume that $r > 0$. Then for all $i,j \in \{1,2,3,4\}$,*

$$s - fb_{01}^d,\ b_{0i}^d - b_{0j}^d \in P.$$

(5) *Assume that $r > 1$ and that P is a prime ideal such that no b_{0i} lies in P. Then whenever $1 \leqslant i < j \leqslant 4$, there exists a (d')th root of unity $\alpha_{ij} \in k$ such that $b_{0i} - \alpha_{ij}b_{0j} \in P$ and*

$$\alpha_{14} = 1, \qquad \alpha_{24} = \alpha_{12}^{-1}, \qquad \alpha_{34} = \alpha_{13}^{-1}.$$

Proof. By the assumption that sf is a non-zerodivisor modulo J, if $j = 0$, $h_{15} = s(c_{03} - c_{02})$ being in P implies that $c_{03} - c_{02}$ is in P. Also, $h_{14} - h_{13}$ equals $f(c_{04} - c_{01})$, so that $c_{04} - c_{01} \in P$. Note that $h_{01} + b_{01}^d h_{13} = s(c_{01} - c_{02}b_{01}^d)$, so that $c_{01} - c_{02}b_{01}^d \in P$. This proves (1) for $j = 0$.

Now assume that $j > 0$. If $j \leqslant r < n$, $h_{j+1,5} = sc_{01}c_{11}\cdots c_{j-1,1}(c_{j3} - c_{j2})$ being in P implies that $c_{j3} - c_{j2}$ is in P. Furthermore,

$$h_{j+1,4} - h_{j+1,3} + h_{j+1,5} = sc_{01}c_{11}\cdots c_{j-2,1}c_{j-1,4}(c_{j4} - c_{j1}) \in P,$$

so that $c_{j4} - c_{j1}$ is in P. Then $h_{j+1,3}$ equals $sc_{01}c_{11}\cdots c_{j-1,1}(c_{j1} - c_{j2})$ modulo $(c_{j-1,4} - c_{j-1,1})$, so that $c_{j1} - c_{j2}$ lies in P. This proves (1).

With (1) established, (2) is an easy consequence.

To prove (3), observe that modulo $(c_{03} - c_{02}) \subseteq P$, h_{16} equals $fc_{02}(b_{01} - b_{04})$. Hence if $r > 0$, $b_{01} - b_{04}$ is in P. If $0 \leqslant j < r$,

$$h_{j+1,6} \equiv sc_{01}c_{11}\cdots c_{j-2,1}c_{j-1,4}c_{j2}(b_{j1} - b_{j4}) \quad \text{modulo } (c_{j3} - c_{j2}),$$

hence $b_{j1} - b_{j4}$ is in P. Furthermore, for all $i = 1, \ldots, 4$,

$$\begin{aligned} h_{1,6+i} &\equiv fc_{02}c_{14}(b_{02} - b_{1i}b_{03}) \in P, \\ h_{j,6+i} &\equiv sc_{01}c_{11}\cdots c_{j-3,1}c_{j-2,4}c_{j-1,2}c_{ji}(b_{j-1,2} - b_{ji}b_{j-1,3}) \in P \quad \text{for } j > 1, \end{aligned}$$

so that $b_{j-1,2} - b_{ji}b_{j-1,3}$ is in P for all $j = 1, \ldots, r-1$ and all $i = 1, \ldots, 4$. This proves (3).

If $r > 0$, $h_{0i} = c_{0i}(s - fb_{0i}^d) \in P$ implies that $s - fb_{0i}^d \in P$. Hence whenever $1 \leqslant i < j \leqslant 4$, $f(b_{0i}^d - b_{0j}^d)$ is in P so that $b_{0i}^d - b_{0j}^d$ is in P. This proves (4), and then (5) follows easily. □

For notational purposes define the following ideals in R:

$$\begin{aligned} E &= \bigl(s - fb_{01}^d\bigr) + \bigl(b_{01} - b_{04}, b_{02}^d - b_{03}^d, b_{01}^d - b_{02}^d\bigr), \\ F &= \bigl(b_{02} - b_{11}b_{03}, b_{14} - b_{11}, b_{13} - b_{11}, b_{12} - b_{11}, b_{12}^d - 1\bigr), \\ C_r &= (c_{r1}, c_{r2}, c_{r3}, c_{r4}), \quad r = 0, \ldots, n-1, \\ C_n &= (0), \\ D_0 &= \bigl(c_{04} - c_{01}, c_{03} - c_{02}, c_{01} - c_{02}b_{01}^d\bigr), \\ D_r &= (c_{r4} - c_{r1}, c_{r3} - c_{r2}, c_{r2} - c_{r1}), \quad r = 1, \ldots, n-1, \\ D_n &= (0), \\ B_0 &= B_1 = (0), \\ B_r &= (1 - b_{2i}, 1 - b_{3i}, \ldots, 1 - b_{ri} \mid i = 1, \ldots, 4), \quad r = 2, \ldots, n-1. \end{aligned}$$

With the previous lemma and this notation then:

Proposition 1.2. *Let P be a minimal prime ideal containing J and not containing sf.*

(1) *If P contains one of the c_{0i}, then P equals the height four prime ideal*

$$P_0 = (c_{01}, c_{02}, c_{03}, c_{04}) = C_0.$$

(2) *If P contains no c_{ji}, set $r = n$, otherwise set r to be the smallest integer such that P contains some c_{ri}. If $r = 1$, P contains*

$$p_1 = C_1 + E + D_0,$$

and if $r > 1$, P contains

$$p_r = C_r + E + F + B_{r-1} + D_0 + D_1 + \cdots + D_{r-1}.$$

(3) *For all $r = 1, \ldots, n$, $J \subseteq p_r$.*

Proof. Suppose that P contains c_{01} or c_{04}. By Lemma 1.1, P contains both c_{01} and c_{04}. Similarly, if P contains c_{02} or c_{03}, then P contains P_0. As P contains $fc_{01} - sc_{02}$, $fc_{04} - sc_{03}$ and does not contain s, then P contains all the c_{0i}, and thus P_0. As P_0 contains J, this verifies (1).

If $r \geqslant 1$, p_r obviously contains J, thus verifying (3). By Lemma 1.1, $C_r + E + D_0 + D_1 + \cdots + D_{r-1}$ lies in P. Thus it remains to prove that $F + B_{r-1} \subseteq P$ when $r > 1$. As $b_{j2} - b_{j+1,i}b_{j3} \in P$ for all $j = 0, \ldots, r-2$, $i = 1, \ldots, 4$, it follows that $(b_{j+1,i} - b_{j+1,i'})b_{j3}$ is in P for any $i, i' \in \{1, 2, 3, 4\}$. If $b_{j3} \in P$, by an application of Lemma 1.1(3), $b_{j-1,2} \in P$, whence $b_{j-2,2} \in P, \ldots, b_{02}$ is in P. But then $c_{02}s = c_{02}(s - fb_{02}^d) + c_{02}fb_{02}^d \in J$, contradicting the assumptions. Thus necessarily $b_{j+1,i} - b_{j+1,i'}$ is in P for all $j = 0, \ldots, r-2$, or that $b_{j-1,i} - b_{j-1,i'}$ is in P for all $j = 2, \ldots, r$. Once this is established, then $h_{j,6+i}$ equals $sc_{01}c_{11} \cdots c_{j-3,1}c_{j-2,4}c_{j-1,2}c_{ji}b_{j-1,3}(1 - b_{ji})$ modulo P so that $1 - b_{ji}$ is in P for all $i = 1, \ldots, 4$ and all $j = 2, \ldots, r-1$. A similar argument shows that $b_{11}^d - 1$ is in P.

The remaining case $r = n$ has essentially the same proof. □

From this one can read off the minimal prime ideals and components:

Proposition 1.3. *Let d' be the largest divisor of d which is relatively prime to the characteristic of the field. Write $d = d'e$ for some integer e. All the minimal prime ideals over J which do not contain sf are*

$$\begin{aligned}
&P_0,\\
&P_{1\alpha\beta} = p_1 + (b_{01} - \alpha b_{02}, b_{02} - \beta b_{03}),\\
&P_{r\alpha\beta} = p_r + (b_{01} - \alpha b_{02}, b_{02} - \beta b_{03}, \beta - b_{1i} \mid i = 1, \ldots, 4),
\end{aligned}$$

where α and β vary over the (d')th roots of unity. The heights of these ideals are as follows: $\mathrm{ht}(P_0) = 4$, *for* $r \in \{1, \ldots, n-1\}$, $\mathrm{ht}(P_{r\alpha\beta}) = 7r + 4$, *and* $\mathrm{ht}(P_{n\alpha\beta}) = 7n$.

Furthermore, with notation as in the previous proposition, for all $r \geqslant 1$,

$$\bigcap_{\alpha,\beta} P_{r\alpha\beta} = \sqrt{p_r} = p_r + \left(b_{02}^{d'} - b_{03}^{d'}, b_{01}^{d'} - b_{02}^{d'}, b_{12}^{d'} - 1\right).$$

Proof. The case of P_0 is trivial. It is easy to see that for $r > 0$, the listed prime ideals $P_{r\alpha\beta}$ are minimal over p_r and that the intersection of the $(d')^2$ $P_{r\alpha\beta}$ equals p_r. It is trivial to calculate the heights, and it is straightforward to prove the last statement. □

This completes the list of all the minimal prime ideals over $J(n,d)$ which do not contain s and f. Their primary components follow easily:

Proposition 1.4. *Adopt the notation of Proposition* 1.3*. The* P_0*-primary component of* J *is* P_0*. Whenever* $r \geqslant 1$*, and* α *and* β *are* (d')*th roots of unity, the* $P_{r\alpha\beta}$*-primary component* $p_{r\alpha\beta}$ *of* J *is*

$$\begin{aligned} p_{1\alpha\beta} &= p_1 + \left(b_{01}^e - \alpha^e b_{02}^e, b_{02}^e - \beta^e b_{03}^e\right), \\ p_{r\alpha\beta} &= p_r + \left(b_{01}^e - \alpha^e b_{02}^e, b_{02}^e - \beta^e b_{03}^e, \beta^e - b_{1i}^e \mid i = 1, \ldots, 4\right). \end{aligned}$$

Furthermore, for all $r \geqslant 1$, $\bigcap_{\alpha,\beta} p_{r\alpha\beta} = p_r$.

The next group of minimal prime ideals all contain s:

Proposition 1.5. *Let* P *be a prime ideal minimal over* J*. If* P *contains* s*, then* P *is one of the following* 19 *prime ideals*:

$$\begin{aligned} P_{-1} &= (s, f), \\ P_{-2} &= (s, c_{01}, c_{02}, c_{04}, b_{03}, b_{04}), \\ P_{-3} &= (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04}), \\ P_{-4\Lambda} &= (c_{1i} \mid i \notin \Lambda) + (b_{1i} \mid i \in \Lambda) + (s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}), \end{aligned}$$

as Λ *varies over the subsets of* $\{1,2,3,4\}$*. The heights of these prime ideals are* 2, 6, 6 *and* 10*, respectively.*

Proof. Note that

$$\begin{aligned} J + (s) = {} & \left(c_{0i} f b_{0i}^d, f c_{02} c_{1i}(b_{02} - b_{1i} b_{03}) \mid i = 1,2,3,4\right) \\ & + \left(s, f c_{01}, f c_{04}, f(c_{02}b_{01} - c_{03}b_{04})\right). \end{aligned}$$

If P contains f, it certainly equals P_{-1}. Now assume that P does not contain f. Then P is minimal over

$$\left(c_{0i} b_{0i}^d, c_{02} c_{1i}(b_{02} - b_{1i} b_{03}) \mid i = 1,2,3,4\right) + (s, c_{01}, c_{04}, c_{02}b_{01} - c_{03}b_{04}).$$

If $c_{02} \in P$, then P is minimal over

$$\left(c_{03}b_{03}^{d}, s, c_{01}, c_{02}, c_{04}, c_{03}b_{04}\right),$$

so it is either $(s, c_{01}, c_{02}, c_{03}, c_{04})$ or $(s, c_{01}, c_{02}, c_{04}, b_{03}, b_{04}) = P_{-2}$. However, the first option is not minimal over J as it strictly contains P_0 from Proposition 1.2.

Now assume that P does not contain fc_{02}. Then P is minimal over

$$\left(b_{02}, c_{03}b_{03}^{d}\right) + (c_{1i}b_{1i}b_{03} \mid i = 1, 2, 3, 4) + (s, c_{01}, c_{04}, c_{02}b_{01} - c_{03}b_{04}).$$

If P contains b_{03}, then $P = (s, c_{01}, c_{04}, b_{02}, b_{03}, c_{02}b_{01} - c_{03}b_{04})$, which is P_{-3}.

Finally, assume that P does not contain $fc_{02}b_{03}$. Then P is minimal over

$$(b_{02}, c_{03}) + (c_{1i}b_{1i} \mid i = 1, 2, 3, 4) + (s, c_{01}, c_{04}, b_{01}),$$

whence P is one of the $P_{-4\Lambda}$. $\square$

It turns out that there are no other minimal prime ideals over $J(n, d)$:

Proposition 1.6. *The prime ideals from the previous three propositions are the only prime ideals minimal over J. Thus there are $1 + n(d')^2 + 3 + 2^4 = n(d')^2 + 20$ minimal prime ideals.*

Proof. Proposition 1.3 determined all the minimal prime ideals over J not containing sf, and Proposition 1.5 determined all those minimal prime ideals which contain s. It remains to find all the prime ideals containing f and J but not s. As $J + (f)$ contains $(c_{0i}s \mid i = 1, 2, 3, 4)$, a minimal prime ideal containing $J + (f)$ but not s contains, and even equals $(f, c_{01}, c_{02}, c_{03}, c_{04})$. However, this prime ideal properly contains P_0, and hence is not minimal over J. The proposition follows as there are no containment relations among the given prime ideals. $\square$

The $n(d')^2 + 20$ minimal primary components can be easily computed:

Proposition 1.7. *For all possible subscripts $\circ$, let $p_\circ$ be the $P_\circ$-primary component of J. Then*

$$\begin{aligned} p_{-2} &= \left(s, c_{01}, c_{02}, c_{04}, b_{03}^{d}, b_{04}\right), \\ p_{-4\Lambda} &= (c_{1i} \mid i \notin \Lambda) + \left(b_{1i}^{d}, b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} \mid i, j \in \Lambda\right) \\ &\quad + \left(s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^{d}\right), \end{aligned}$$

and $p_{-1} = P_{-1}$, $p_{-4\Lambda} = P_{-4\Lambda}$.

Proof. By Proposition 1.4, it remains to calculate p_{-1}, p_{-2}, p_{-3} and $p_{-4\Lambda}$. As $c_{03} - c_{02}$ is not an element of P_{-1}, P_{-2}, P_{-3} and $P_{-4\Lambda}$, and since $h_{15} = s(c_{03} - c_{02})$ is in J, it follows that $s \in p_{-1}, p_{-2}, p_{-3}$ and $p_{-4\Lambda}$. Then $c_{01} f b_{01}^d \in p_{-1}$, so that $f \in p_{-1}$, and so $p_{-1} = P_{-1}$.

As $h_{13} = fc_{01} - sc_{02}$, $h_{14} = fc_{04} - sc_{03}$ are in J, then $fc_{01}, fc_{04} \in p_{-2}, p_{-3}$ and $p_{-4\Lambda}$, whence $c_{01}, c_{04} \in p_{-2}, p_{-3}, p_{-4\Lambda}$. For all $i = 1, \dots, 4$, as $h_{1,6+i} = fc_{02}c_{1i}(b_{02} - b_{1i}b_{03}) \in J$, it follows that $b_{02} - b_{1i}b_{03} \in p_{-3}$. Thus as $b_{11} - b_{12} \notin P_{-3}$, it follows that b_{03} and hence also b_{02} are in P_{-3}. Now it is clear that p_{-3} is the P_{-3}-primary component of J.

Further, for $i = 2, 3$, $c_{0i} f b_{0i}^d \in p_j$ implies that $b_{02}^d \in p_{-4\Lambda}$, $b_{03}^d \in p_{-2}$, $c_{02} \in p_{-2}$, and $c_{03} \in p_{-4\Lambda}$. As $h_{16} = f(c_{02}b_{01} - c_{03}b_{04})$ is in J, then $fc_{03}b_{04}$ is in p_{-2} so that b_{04} is in p_{-2}. Also $fc_{02}b_{01}$ is in $p_{-4\Lambda}$ so that b_{01} is in $p_{-4\Lambda}$. Thus the P_{-2}-primary component contains p_{-2}. But p_{-2} contains J and p_{-2} is clearly primary, so p_{-2} is the P_{-2}-primary component of J.

Lastly, as J contains $h_{1,6+i}$, $i = 1, \dots, 4$, each $p_{-4\Lambda}$ contains each $c_{1i}(b_{02} - b_{1i}b_{03})$. If $i \notin \Lambda$, then $b_{02} - b_{1i}b_{03}$ is not in $P_{-4\Lambda}$, so that $c_{1i} \in p_{-4\Lambda}$. If instead $i \in \Lambda$, then $c_{1i} \notin P_{-4\Lambda}$, so that $b_{02} - b_{1i}b_{03}$ is in $p_{-4\Lambda}$. Hence $b_{02}^d - b_{1i}^d b_{03}^d$ is in $p_{-4\Lambda}$, so that as $b_{02}^d \in p_{-4\Lambda}$, so is $b_{1i}^d b_{03}^d$. Hence b_{1i}^d is in $p_{-4\Lambda}$. Furthermore, for $i, j \in \Lambda$, $b_{03}(b_{1j} - b_{1i}) = (b_{02} - b_{1i}b_{03}) - (b_{02} - b_{1j}b_{03})$ is in $p_{-4\Lambda}$, so that $b_{1j} - b_{1i}$ is in $p_{-4\Lambda}$. Thus

$$p_{-4\Lambda} \supseteq (c_{1i} \mid i \notin \Lambda) + \bigl(b_{1i}^d, b_{02} - b_{1i}b_{03}, b_{1i} - b_{1j} \mid i, j \in \Lambda\bigr) + \bigl(s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d\bigr),$$

but the latter ideal is primary and contains J, so equality holds. □

The structure of p_{-2} shows that:

Proposition 1.8. *For $n, d \geqslant 2$, $J(n,d)$ is not a radical ideal.*

Table 1 contains all the minimal prime ideals over $J(n,d)$. There, $d = d'e$ with d' the greatest divisor of d relatively prime to the characteristic of the field, and α and β are varying over the (d')th roots of unity.

Table 1

Minimal prime ideal	Height	Component of $J(n,d)$
$P_0 = (c_{01}, c_{02}, c_{03}, c_{04})$	4	$p_0 = P_0$
$P_{1\alpha\beta} = p_1 + (b_{01} - \alpha b_{02}, b_{02} - \beta b_{03})$	11	$P_{1\alpha\beta} = p_1 + (b_{01}^e - \alpha^e b_{02}^e, b_{02}^e - \beta^e b_{03}^e)$
$P_{r\alpha\beta} = p_r + (b_{01} - \alpha b_{02}, b_{02} - \beta b_{03}, \beta - b_{1i})$	$7r + 4\delta_{r<n}$	$p_{r\alpha\beta} = p_r + (b_{01}^e - \alpha^e b_{02}^e, b_{02}^e - \beta^e b_{03}^e, b_{12}^e - \beta^e)$
$P_{-1} = (s, f)$	2	$p_{-1} = P_{-1}$
$P_{-2} = (s, c_{01}, c_{02}, c_{04}, b_{03}, b_{04})$	6	$p_{-2} = (s, c_{01}, c_{02}, c_{04}, b_{03}^d, b_{04})$
$P_{-3} = (s, c_{01}, c_{04}, b_{02}, b_{03}) + (c_{02}b_{01} - c_{03}b_{04})$	6	$p_{-3} = P_{-3}$
$P_{-4\Lambda} = (s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}) + (c_{1i}, b_{1j} \mid i \notin \Lambda, j \in \Lambda)$	10	$p_{-4\Lambda} = (s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d) + (c_{1i} \mid i \notin \Lambda) + (b_{1j}^d, b_{02} - b_{1j}b_{03}, b_{1j} - b_{1j'} \mid j, j' \in \Lambda)$

2. Doubly exponential behavior is due to embedded prime ideals

The aim of this section is to compute the intersection of all the minimal components.

Not surprisingly, the doubly exponential behavior is not due to the minimal prime ideals or their components. We prove this below. Explicitly, it is straightforward to see that the element $fc_{02}b_{02}^d c_{11}c_{21}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4})$ lies in every minimal component of J. (This will be verified below explicitly in the computation of the intersection of all the minimal components.) But any ideal containing J and the element above does not exhibit the doubly exponential behavior. Namely, set $c'=c_{11}c_{21}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4})$. Then

$$\begin{aligned}
&sc_{01}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4})\\
&\quad = c_{01}\big(s-fb_{01}^d\big)c' + (fc_{01}-sc_{02})b_{01}^d c' + c_{02}\big(s-fb_{02}^d\big)b_{01}^d c' + b_{01}^d fc_{02}b_{02}^d c',
\end{aligned}$$

and the degrees of the coefficients c', $b_{01}^d c'$, $b_{01}^d c'$ and b_{01}^d of the elements h_{01}, h_{13}, h_{02} and $fc_{02}b_{02}^d c'$, respectively, are not doubly exponential in n. This proves:

Proposition 2.1. *The doubly exponential ideal membership problem of the Mayr–Meyer ideals $J(n,d)$ and $J_l(n,d)$ for the element $sc_{01}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4})$ is not due to the minimal components, but to some embedded prime ideal.*

We compute below the intersection of all the minimal components of $J(n,d)$, and show that the element $fc_{02}b_{02}^d c_{11}c_{21}\cdots c_{n-2,1}(c_{n-1,1}-c_{n-1,4})$ is even in a natural minimal generating set of this intersection.

First define the ideal

$$\begin{aligned}
p_{-4} = {} & \big(s, c_{01}, c_{03}, c_{04}, b_{01}, b_{02}^d\big)\\
& + \big(c_{1i}(b_{02}-b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i}-b_{1j}) \mid i,j=1,\dots,4\big).
\end{aligned}$$

Note that $p_{-4}:c_{11}=p_{-4}:c_{11}^2$, so that by Fact 0.4, $p_{-4}=(p_{-4}:c_{11})\cap(p_{-4}+(c_{11}))$. Similarly,

$$\begin{aligned}
p_{-4} &= (p_{-4}:c_{11}c_{12})\cap\big((p_{-4}:c_{11})+(c_{12})\big)\cap\big(\big(p_{-4}+(c_{11})\big):c_{12}\big)\cap\big(p_{-4}+(c_{11},c_{12})\big)\\
&= \cdots\\
&= \bigcap_{\Lambda}\big(\big(\big((p_{-4}*_1 c_{11})*_2 c_{12}\big)*_3 c_{13}\ \big)*_4 c_{14}\big),
\end{aligned}$$

where $*_i$ vary over the operations colon and addition. But the resulting component ideals are just the various $p_{-4\Lambda}$, so that

$$p_{-4}=\bigcap_{\Lambda} p_{-4\Lambda}.$$

Next we compute the intersection of p_{-4} and p_{-2} (using Fact 0.1):

$$
\begin{aligned}
p_{-2} \cap p_{-4} &= \bigl((s, c_{01}, c_{04}) + \bigl(c_{02}, b_{03}^d, b_{04}\bigr)\bigr) \cap p_{-4} \\
&= (s, c_{01}, c_{04}) + \bigl(c_{02}, b_{03}^d, b_{04}\bigr) \\
&\qquad \cdot \bigl(c_{03}, b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr),
\end{aligned}
$$

so that

$$
\begin{aligned}
p_{-2} \cap p_{-3} \cap p_{-4} &= (s, c_{01}, c_{04}, c_{02}b_{01} - c_{03}b_{04}) \\
&\quad + \bigl(b_{03}^d\bigr)\bigl(c_{03}, b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr) \\
&\quad + (c_{02}, b_{04}) \cdot \bigl(b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03})\bigr) \\
&\quad + (c_{02}, b_{04}) \cdot \bigl(c_{03}, b_{01}, c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr) \cdot (b_{02}, b_{03}) \\
&= (s, c_{01}, c_{04}, c_{02}b_{01} - c_{03}b_{04}) \\
&\quad + \bigl(b_{03}^d\bigr)\bigl(c_{03}, b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr) \\
&\quad + (c_{02}, b_{04})\bigl(b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, \\
&\qquad\qquad c_{1i}b_{1i}^d b_{02}, c_{1i}b_{1i}^d b_{03}\bigr).
\end{aligned}
$$

Thus the intersection of the minimal components of $J(n, d)$ which contain s equals:

$$
\begin{aligned}
&p_{-1} \cap p_{-2} \cap p_{-3} \cap p_{-4} \\
&\quad = \bigl(s, fc_{01}, fc_{04}, f(c_{02}b_{01} - c_{03}b_{04})\bigr) \\
&\qquad + f\bigl(b_{03}^d\bigr)\bigl(c_{03}, b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr) \\
&\qquad + f(c_{02}, b_{04})\bigl(b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{02}, \\
&\qquad\qquad c_{1i}b_{1i}^d b_{03}\bigr) \\
&\quad = J + (s) + fb_{03}^d\bigl(b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr) \\
&\qquad + fc_{02}\bigl(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}\bigr) \\
&\qquad + fb_{04}\bigl(b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{02}, c_{1i}b_{1i}^d b_{03}\bigr).
\end{aligned}
$$

We can simplify this intersection in terms of the generators of J if we first intersect the intersection with the minimal component p_0:

$$
\begin{aligned}
p_0 \cap \cdots \cap p_{-4} &= J + sC_0 \\
&\quad + fb_{03}^d C_0\bigl(b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\bigr) \\
&\quad + fc_{02}\bigl(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}\bigr) \\
&\quad + fb_{04}C_0\bigl(b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{02}, c_{1i}b_{1i}^d b_{03}\bigr).
\end{aligned}
$$

As, $J + fC_0 = J + f(c_{02}, c_{03})$ and $J + sC_0 = J + sD_0 + (fc_{02}b_{02}^d)$, it follows that

$$
\begin{aligned}
&p_0 \cap \cdots \cap p_{-4} \\
&\quad = J + sC_0 + f b_{03}^d(c_{02}, c_{03})\big(b_{01}, b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), c_{1i}b_{1i}^d, c_{1i}c_{1j}(b_{1i} - b_{1j})\big) \\
&\qquad + f c_{02}\big(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}\big) \\
&\qquad + f b_{04}(c_{02}, c_{03})\big(b_{02}^d, c_{1i}(b_{02} - b_{1i}b_{03}), b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{02}, c_{1i}b_{1i}^d b_{03}\big) \\
&\quad = J + sC_0 + f c_{02}\big(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}\big) \\
&\quad = J + sD_0 + f c_{02}\big(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}, b_{02}^d\big).
\end{aligned}
$$

Next we compute the intersection of all the minimal components of $J(n,d)$ which do not contain s and are different from p_0:

Lemma 2.2. *For* $2 \leqslant r \leqslant n$,

$$
p_1 \cap p_2 \cap \cdots \cap p_r = E + D_0 + C_1 F + \sum_{i=0}^{r-1} C_1 C_2 \cdots C_i (D_{i+1} + B_i) + C_1 C_2 \cdots C_r.
$$

Proof. When $r = 2$,

$$
\begin{aligned}
p_1 \cap p_2 &= (C_1 + E + D_0) \cap (C_2 + E + F + D_0 + D_1 + B_1) \\
&= E + D_0 + C_1 \cap (C_2 + E + F + D_0 + D_1 + B_1) \\
&= E + D_0 + D_1 + C_1 \cap (C_2 + E + F + D_0 + B_1) \\
&= E + D_0 + D_1 + C_1 \cdot (C_2 + E + F + D_0 + B_1) \\
&= E + D_0 + D_1 + C_1 F + C_1 \cdot (C_2 + B_1),
\end{aligned}
$$

which starts the induction. Then by induction assumption for $r \geqslant 2$ and $r \leqslant n-1$,

$$
\begin{aligned}
p_1 \cap \cdots \cap p_{r+1} &= \left(E + D_0 + C_1 F + \sum_{i=0}^{r-1} C_1 C_2 \cdots C_i (D_{i+1} + B_i) + C_1 \cdots C_r\right) \cap p_{r+1} \\
&= E + D_0 + C_1 F + \sum_{i=0}^{r-1} C_1 C_2 \cdots C_i (D_{i+1} + B_i) + (C_1 \cdots C_r) \cap p_{r+1},
\end{aligned}
$$

and by multihomogeneity, the last intersection equals

$$
C_1 \cdots C_r (C_{r+1} + E + F + B_r) + \sum_{j=1}^{r} \left(D_j \prod_{k \neq j}^{r} C_k \right).
$$

Combining the last two displays proves the lemma. □

Thus the intersection of all the minimal components of $J(n,d)$ equals:

$$\begin{aligned}\bigcap_{r=-4}^{n} p_r &= \bigl(J + sD_0 + fc_{02}\bigl(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}, b_{02}^d\bigr)\bigr) \cap \bigcap_{r=1}^{n} p_r \\ &= J + sD_0 + \bigl(fc_{02}\bigl(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}, b_{02}^d\bigr)\bigr) \cap \bigcap_{r=1}^{n} p_r \\ &= J + sD_0 + fc_{02}\left(\bigl(c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}, b_{02}^d\bigr) \cap \bigcap_{r=1}^{n} p_r\right).\end{aligned}$$

Let $A = (c_{03}b_{02}, c_{03}b_{03}, b_{01}b_{02}, b_{01}b_{03}, c_{1i}b_{1i}^d b_{03}, b_{02}^d) \cap \bigcap_{r=1}^n p_r$. Thus the intersection of all the minimal components of $J(n,d)$ equals $J + sD_0 + fc_{02}A$. Finding the generators of A takes up most of the rest of this section. We will use the decomposition

$$A = \bigl(c_{03}, b_{01}, c_{1i}b_{1i}^d, b_{02}^d\bigr) \cap \bigl(c_{03}, b_{01}, b_{03}, b_{02}^d\bigr) \cap (b_{02}, b_{03}) \cap \bigcap_{r=1}^{n} p_r,$$

and start computing A via the indicated partial intersections, again using Fact 0.1:

$$\begin{aligned}(b_{02}, b_{03}) \cap \bigcap_{r=1}^{n} p_r &= \bigl(b_{02}^d - b_{03}^d, c_{11}(b_{02} - b_{11}b_{03})\bigr) + (b_{02}, b_{03}) \cdot L' \\ &= \bigl(b_{02}^d - b_{03}^d, c_{11}(b_{02} - b_{11}b_{03})\bigr) + b_{03} \cdot L' + b_{02}L'',\end{aligned}$$

where

$$\begin{aligned}L' &= L'' + c_{11}\bigl(b_{1i} - b_{1j}, b_{12}^d - 1\bigr) + \sum_{i=0}^{n-1} c_{11} \cdots c_{i1}(D_{i+1} + B_i), \\ L'' &= \bigl(s - fb_{01}^d, b_{01} - b_{04}, b_{01}^d - b_{03}^d\bigr) + D_0 + D_1.\end{aligned}$$

Note that L' is generated by all the generators of $\bigcap_{r\geqslant 1} p_r$ other than $b_{02}^d - b_3^d$. Then the intersection of the last three components of A is

$$\begin{aligned}&\bigl(c_{03}, b_{01}, b_{03}, b_{02}^d\bigr) \cap (b_{02}, b_{03}) \cap \bigcap_{r=1}^{n} p_r \\ &\quad= \bigl(b_{02}^d - b_{03}^d\bigr) + b_{03} \cdot L' + \bigl(c_{03}, b_{01}, b_{03}, b_{02}^d\bigr) \cap \bigl(\bigl(c_{11}(b_{02} - b_{11}b_{03})\bigr) + b_{02}L''\bigr) \\ &\quad= \bigl(b_{02}^d - b_{03}^d, b_{02}\bigl(b_{01}^d - b_{03}^d\bigr)\bigr) + b_{03} \cdot L' \\ &\qquad+ \bigl(c_{03}, b_{01}, b_{03}, b_{02}^d\bigr) \cap \bigl(\bigl(c_{11}(b_{02} - b_{11}b_{03})\bigr) \\ &\qquad\qquad+ b_{02}\bigl(\bigl(s - fb_{01}^d, b_{01} - b_{04}\bigr) + D_0 + D_1\bigr)\bigr) \\ &\quad= \bigl(b_{02}^d - b_{03}^d, b_{02}\bigl(b_{01}^d - b_{03}^d\bigr)\bigr) + b_{03} \cdot L'\end{aligned}$$

$$
\begin{aligned}
&+ (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&+ \big(b_{03}, b_{02}^d\big) \cap \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
= {}& \big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big)\big) + b_{03} \cdot L' \\
&+ (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&+ \big(b_{03}, b_{02}^d\big) \cap \big(c_{11}(b_{02} - b_{11}b_{03}), b_{02}\big) \\
&\cap \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + \big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big) \\
= {}& \big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big)\big) + b_{03} \cdot L' \\
&+ (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&+ \big(b_{03}c_{11}b_{11}, b_{02}^d, b_{02}b_{03}\big) \cap \big(\big(c_{11}(b_{02} - b_{11}b_{03}), s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big) \\
= {}& \big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big)\big) + b_{03} \cdot L' \\
&+ (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&+ \big(b_{03}c_{11}b_{11}, b_{02}^d, b_{02}b_{03}\big) \cdot \big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big) \\
&+ \big(b_{03}c_{11}b_{11}, b_{02}^d, b_{02}b_{03}\big) \cap \big(c_{11}(b_{02} - b_{11}b_{03})\big) \\
= {}& \big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big)\big) + b_{03} \cdot L' \\
&+ (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&+ c_{11}(b_{02} - b_{11}b_{03})\big(b_{03}, b_{02}^{d-1}\big) \\
= {}& \big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\big) + b_{03}\bigcap_{r=1}^n p_r \\
&+ (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big).
\end{aligned}
$$

Hence A, the intersection of all of its components, equals

$$
\begin{aligned}
A &= \big(c_{03}, b_{01}, c_{1i}b_{1i}^d, b_{02}^d\big) \cap \big(c_{03}, b_{01}, b_{03}, b_{02}^d\big) \cap (b_{02}, b_{03}) \cap \bigcap_{r=1}^n p_r \\
&= (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&\quad + \big(c_{03}, b_{01}, c_{1i}b_{1i}^d, b_{02}^d\big) \\
&\qquad \cap \left(\big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\big) + b_{03}\bigcap_{r=1}^n p_r\right) \\
&= (c_{03}, b_{01}) \cdot \big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big)
\end{aligned}
$$

$$+ c_{03}\left(\left(b_{02}^d - b_{03}^d, b_{02}\left(b_{01}^d - b_{03}^d\right), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\right) + b_{03}\bigcap_{r=1}^n p_r\right)$$
$$+ \left(b_{01}, c_{1i}b_{1i}^d, b_{02}^d\right)$$
$$\cap \left(\left(b_{02}^d - b_{03}^d, b_{02}\left(b_{01}^d - b_{03}^d\right), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\right) + b_{03}\bigcap_{r=1}^n p_r\right).$$

Let A' be the last intersection above. The second intersectand ideal of A' decomposes as

$$\begin{aligned}&\left(b_{02}^d - b_{03}^d, b_{02}\left(b_{01}^d - b_{03}^d\right), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\right) + b_{03}\bigcap_{r=1}^n p_r\\ &\quad = \bigcap_{r=1}^n p_r \cap \left(\left(b_{02}^d, b_{03}^d, b_{02}\left(b_{01}^d - b_{03}^d\right), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\right) + b_{03}\bigcap_{r=1}^n p_r\right),\end{aligned}$$

so that

$$\begin{aligned}A' &= \bigcap_{r=1}^n p_r \cap \left(b_{01}, c_{1i}b_{1i}^d, b_{02}^d\right) \cap \left(\left(b_{02}^d, b_{03}^d, b_{02}b_{01}^d, b_{02}^{d-1}c_{11}b_{11}b_{03}\right) + b_{03}\bigcap_{r=1}^n p_r\right)\\
&= \bigcap_{r=1}^n p_r \cap \left(\left(b_{02}^d, b_{02}b_{01}^d\right)\right.\\
&\qquad \left. + \left(b_{01}, c_{1i}b_{1i}^d, b_{02}^d\right) \cap \left(\left(b_{03}^d, b_{02}^{d-1}c_{11}b_{11}b_{03}\right) + b_{03}\bigcap_{r=1}^n p_r\right)\right)\\
&= \bigcap_{r=1}^n p_r \cap \left(\left(b_{02}^d, b_{02}b_{01}^d\right)\right.\\
&\qquad \left. + b_{03}\left(\left(b_{01}, c_{1i}b_{1i}^d, b_{02}^d\right) \cap \left(\left(b_{03}^{d-1}, b_{02}^{d-1}c_{11}b_{11}\right) + \bigcap_{r=1}^n p_r\right)\right)\right)\\
&= \bigcap_{r=1}^n p_r \cap \left(\left(b_{02}^d, b_{02}b_{01}^d, b_{03}\left(c_{1i}b_{1i}^d - c_{11}b_{11}^d\right), b_{03}b_{01}^d\right)\right.\\
&\qquad \left. + b_{03}\left(\left(b_{01}, c_{11}b_{11}^d\right) \cap \left(\left(b_{03}^{d-1}, b_{02}^{d-1}c_{11}b_{11}\right) + \bigcap_{r=1}^n p_r\right)\right)\right)\\
&= \bigcap_{r=1}^n p_r \cap \left(\left(b_{02}^d, b_{02}b_{01}^d, b_{03}\left(c_{1i}b_{1i}^d - c_{11}b_{11}^d\right), b_{03}b_{01}^d\right)\right.\\
&\qquad \left. + b_{03}\left(\left(b_{01}, c_{11}b_{11}^d\right) \cap \left(\left(b_{03}^{d-1}, b_{02}^{d-1}c_{11}b_{11}\right) + L'''\right)\right)\right),\end{aligned}$$

where L''' is generated by all the given generators of $\bigcap_{r\geqslant 1} p_r$ other than $b_{01}^d - b_{03}^d$:

$$L''' = \big(s - f b_{01}^d, b_{01} - b_{04}, b_{02}^d - b_{03}^d\big) + D_0 + C_1 F + \sum_{i=0}^{n-1} C_1 \cdots C_i (D_{i+1} + B_i).$$

With this,

$$\begin{aligned}
A' &= \bigcap_{r=1}^{n} p_r \cap \Big(\big(b_{02}^d, b_{02}b_{01}^d, b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}b_{01}^d\big) \\
&\qquad + b_{03}b_{01}\big(\big(b_{03}^{d-1}, b_{02}^{d-1}c_{11}b_{11}\big) + L'''\big) \\
&\qquad + b_{03}\big(\big(c_{11}b_{11}^d\big) \cap \big(\big(b_{03}^{d-1}, b_{02}^{d-1}c_{11}b_{11}\big) + L'''\big)\big)\Big) \\
&= \bigcap_{r=1}^{n} p_r \cap \Big(\big(b_{02}^d, b_{02}b_{01}^d, b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}b_{01}^d\big) \\
&\qquad + b_{03}b_{01}\big(\big(b_{03}^{d-1}, b_{02}^{d-1}c_{11}b_{11}\big) + L'''\big) \\
&\qquad + b_{03}c_{11}b_{11}^d\big(\big(b_{03}^{d-1}, b_{02}^{d-1}\big) + L''' : c_{11}\big)\Big) \\
&= \big(b_{02}\big(b_{01}^d - b_{02}^d\big), b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}\big(b_{01}^d - b_{02}^d\big), b_{01}\big(b_{03}^d - b_{02}^d\big)\big) \\
&\quad + \big(b_{01}b_{02}^{d-1}c_{11}(b_{02} - b_{03}b_{11})\big) + b_{03}b_{01}L''' \\
&\quad + \big(c_{11}b_{11}^d\big(b_{03}^d - b_{02}^d\big), b_{02}^{d-1}b_{11}^{d-1}c_{11}(b_{02} - b_{03}b_{11})\big) \\
&\quad + b_{03}c_{11}b_{11}^d\big(L''' : c_{11}\big) + \bigcap_{r=1}^{n} p_r \cap \big(b_{02}^d\big) \\
&= \big(b_{02}\big(b_{01}^d - b_{02}^d\big), b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}\big(b_{01}^d - b_{02}^d\big), b_{01}\big(b_{03}^d - b_{02}^d\big)\big) \\
&\quad + \big(b_{01}b_{02}^{d-1}c_{11}(b_{02} - b_{03}b_{11})\big) + b_{03}b_{01}L''' \\
&\quad + \big(c_{11}b_{11}^d\big(b_{03}^d - b_{02}^d\big), b_{02}^{d-1}b_{11}^{d-1}c_{11}(b_{02} - b_{03}b_{11})\big) \\
&\quad + b_{03}c_{11}b_{11}^d\big(L''' : c_{11}\big) + b_{02}^d \cdot \bigcap_{r=1}^{n} p_r.
\end{aligned}$$

Hence A equals

$$\begin{aligned}
A &= (c_{03}, b_{01}) \cdot \Big(\big(c_{11}(b_{02} - b_{11}b_{03})\big) + b_{02}\big(\big(s - f b_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\Big) \\
&\quad + c_{03}\left(\big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big), b_{02}^{d-1}c_{11}(b_{02} - b_{11}b_{03})\big) + b_{03}\bigcap_{r=1}^{n} p_r\right) \\
&\quad + \big(b_{02}\big(b_{01}^d - b_{02}^d\big), b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}\big(b_{01}^d - b_{02}^d\big), b_{01}\big(b_{03}^d - b_{02}^d\big)\big)
\end{aligned}$$

$$
\begin{aligned}
&+ \big(b_{01}b_{02}^{d-1}c_{11}(b_{02}-b_{03}b_{11})\big) + b_{03}b_{01}L''' \\
&+ \big(c_{11}b_{11}^d\big(b_{03}^d - b_{02}^d\big), b_{02}^{d-1}b_{11}^{d-1}c_{11}(b_{02}-b_{03}b_{11})\big) \\
&+ b_{03}c_{11}b_{11}^d\big(L''' : c_{11}\big) + b_{02}^d \cdot \bigcap_{r=1}^{n} p_r.
\end{aligned}
$$

Thus finally,

$$
\begin{aligned}
\bigcap_{r=-4}^{n} p_r &= J + sD_0 + fc_{02}A \\
&= J + sD_0 + fc_{02}(c_{03}, b_{01}) \cdot \big(b_{02}\big(\big(s - fb_{01}^d, b_{01} - b_{04}\big) + D_0 + D_1\big)\big) \\
&\quad + fc_{02}c_{03}\left(\big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big)\big) + b_{03}\bigcap_{r=1}^{n} p_r\right) \\
&\quad + fc_{02}\big(b_{02}\big(b_{01}^d - b_{02}^d\big), b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}\big(b_{01}^d - b_{02}^d\big), \\
&\qquad b_{01}\big(b_{03}^d - b_{02}^d\big), c_{11}b_{11}^d\big(b_{03}^d - b_{02}^d\big)\big) \\
&\quad + fc_{02}b_{03}b_{01}L''' + fc_{02}b_{03}c_{11}b_{11}^d\big(L''' : c_{11}\big) + fc_{02}b_{02}^d \cdot \bigcap_{r=1}^{n} p_r,
\end{aligned}
$$

or, in nicer form:

Theorem 2.3. *The intersection of all the minimal components of $J(n, d)$ equals*

$$
\begin{aligned}
\bigcap_{r=-4}^{n} p_r &= J + sD_0 + fc_{02}b_{02}(c_{03}, b_{01})\big(s - fb_{01}^d, b_{01} - b_{04}\big) \\
&\quad + fc_{02}b_{02}(c_{03}, b_{01})(D_0 + D_1) + fc_{02}c_{03}\big(b_{02}^d - b_{03}^d, b_{02}\big(b_{01}^d - b_{03}^d\big)\big) \\
&\quad + fc_{02}c_{03}b_{03} \cdot \left(E + D_0 + C_1F + \sum_{i=0}^{n-1} c_{11}\cdots c_{i1}(D_{i+1} + B_i)\right) \\
&\quad + fc_{02}\big(b_{02}\big(b_{01}^d - b_{02}^d\big), b_{03}\big(c_{1i}b_{1i}^d - c_{11}b_{11}^d\big), b_{03}\big(b_{01}^d - b_{02}^d\big)\big) \\
&\quad + fc_{02}\big(b_{01}\big(b_{03}^d - b_{02}^d\big), c_{11}b_{11}^d\big(b_{03}^d - b_{02}^d\big)\big) \\
&\quad + fc_{02}b_{03}b_{01}\left(E''' + D_0 + C_1F + \sum_{i=0}^{n-1} c_{11}\cdots c_{i1}(D_{i+1} + B_i)\right) \\
&\quad + fc_{02}b_{03}b_{11}^d c_{11}\left(E''' + D_0 + F + \sum_{i=0}^{n-1} c_{21}\cdots c_{i1}(D_{i+1} + B_i)\right)
\end{aligned}
$$

$$+ f c_{02} b_{02}^d \cdot \left(E + D_0 + C_1 F + \sum_{i=0}^{n-1} c_{11} \cdots c_{i1} (D_{i+1} + B_i) \right),$$

where

$$E''' = \left(s - f b_{01}^d, b_{01} - b_{04}, b_{03}^d - b_{02}^d\right).$$

(With Macaulay2 I verified this theorem and intermediate computations in the proof above for the case $n = 3$, $d = 2$.)

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