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Fractal curvature measures and Minkowski content for self-conformal subsets of the real line

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Abstract

We show that the fractal curvature measures of invariant sets of one-dimensional conformal iterated function systems satisfying the open set condition exist if the associated geometric potential function is nonlattice. Moreover, we prove that if the maps of the conformal iterated function system are all analytic, then the fractal curvature measures do not exist in the lattice case. Further, in the nonlattice situation we obtain that the Minkowski content exists and prove that the fractal curvature measures are constant multiples of the δ-conformal measure, where δ denotes the Minkowski dimension of the invariant set. For the first fractal curvature measure, this constant factor coincides with the Minkowski content of the invariant set. In the lattice situation we give sufficient conditions for the Minkowski content of the invariant set to exist, disproving a conjecture of Lapidus and standing in contrast with the fact that the Minkowski content of a self-similar lattice fractal never exists. However, every self-similar set satisfying the open set condition exhibits a Minkowski measurable $\mathcal{C}^{1+\alpha}$ diffeomorphic image. Both in the lattice and in the nonlattice situation, average versions of the fractal curvature measures are shown to always exist.

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1. A brief introduction

Notions of curvature constitute an important tool for describing the geometric structure of sets and have been introduced and intensively studied for broad classes of sets. Originally, the idea of characterising sets in terms of their curvature stems from the study of smooth manifolds as well as from the theory of convex bodies with sufficiently smooth boundaries. In his fundamental paper *Curvature measures* [8], Federer localises, extends and unifies the previously existing notions of curvature to sets of positive reach. This is where he introduces curvature measures, which can be viewed as a measure theoretical substitute for the notion of curvature for sets without a differentiable structure. Federer's curvature measures were studied and generalised in various ways. An extension to finite unions of convex bodies is given in [10,28] and to finite unions of sets with positive reach in [33]. In [32], Winter extends the curvature measures to fractal sets in $\mathbb{R}^d$, which typically cannot be expressed as finite unions of sets with positive reach. These measures are referred to as fractal curvature measures and are defined as weak limits of rescaled versions of the curvature measures introduced by Federer, Groemer and Schneider. Winter also examines conditions for their existence in the self-similar case. However, fractal sets arising in geometry (for instance as limit sets of Fuchsian groups) or in number theory (for instance as sets defined by Diophantine inequalities) are typically not self-similar but rather self-conformal. In order to have a notion of curvature at hand also for this important class of fractal sets, in this paper, we study nonempty compact subsets of $\mathbb{R}$ which occur as the invariant sets of finite conformal iterated function systems satisfying the open set condition and call them self-conformal sets (see Definition 2.6). We remark that the zeroth and the first fractal curvature measures respectively correspond to the limiting behaviours of rescaled versions of the gap counting number and the lengths of the parallel neighbourhoods. Although we consider subsets of the real line, we follow the systematic description by Winter and call these measures fractal curvature measures. We obtain conditions under which the fractal curvature measures exist and others under which these measures do not exist. Further, we establish links to the Minkowski content.

The Minkowski content was proposed in [21] as a measure of "lacunarity" for a fractal set. Indeed, the value of the Minkowski content allows one to compare the lacunarity of sets of the same Minkowski dimension. Besides the geometric interpretation, results on the existence of the Minkowski content play an important role in the context of the Weyl–Berry conjecture concerning the asymptotic distribution of the eigenvalues of the Laplacian on sets with fractal boundaries. We refer the reader to [5, Section 4] for an overview and references concerning these studies. An additional motivation for studying the Minkowski content of fractal sets arises from noncommutative geometry. In Connes' seminal book [3], the notion of a noncommutative fractal geometry is developed. There, it is shown that the natural analogue of the volume of a compact smooth Riemannian spin^c manifold for a fractal set in $\mathbb{R}$ is that of the Minkowski content. This idea is also reflected in the works [7,11,27].

The paper is organised as follows. In Section 2 we provide the central definitions and state the main results. They include results on the existence of the fractal curvature measures for general self-conformal sets as well as a complete answer to the question on the existence of the fractal curvature measures for $\mathcal{C}^{1+\alpha}$ diffeomorphic images of self-similar sets. The precise definitions and background information as well as the relevant properties and auxiliary results are presented in Section 3. In Section 4, the proofs of our main theorems for self-conformal sets (Theorems 2.11 and 2.12) are provided. Finally, in Section 5, we conclude the paper by proving our results for the special cases of self-similar sets and $\mathcal{C}^{1+\alpha}$ diffeomorphic images of self-similar sets.

2. The main results and central definitions

We start this section by briefly introducing the fractal curvature measures, the Minkowski content, and self-conformal sets in order to subsequently present our main results. For further, more detailed background concerning these quantities, we refer the reader to Section 3.

The introduction of the fractal curvature measures relies on the definition of scaling exponents, for which we require the following notation. Let λ^0 denote the counting measure on $\mathbb{R}$ and let λ^1 be the one-dimensional Lebesgue measure on $\mathbb{R}$. For $\varepsilon > 0$ we define

$$Y_\varepsilon := \left\{ t \in \mathbb{R} \mid \inf_{y \in Y} |t - y| \leq \varepsilon \right\}$$

to be the *ε-parallel neighbourhood* of $Y \subset \mathbb{R}$ and let ∂Y denote the boundary of Y.

Definition 2.1. For a nonempty compact set $Y \subset \mathbb{R}$ the zeroth and first *curvature scaling exponents* of Y are respectively defined to be

$$s_0(Y) := \inf\{t \in \mathbb{R} \mid \varepsilon^t \lambda^0(\partial Y_\varepsilon) \to 0 \text{ as } \varepsilon \to 0\} \quad \text{and}$$
$$s_1(Y) := \inf\{t \in \mathbb{R} \mid \varepsilon^t \lambda^1(Y_\varepsilon) \to 0 \text{ as } \varepsilon \to 0\}.$$

Definition 2.2. Let $Y \subset \mathbb{R}$ denote a nonempty compact set. Provided that the weak limit

$$C_0^f(Y, \cdot) := \operatorname*{w-lim}_{\varepsilon \to 0} \varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap \cdot)/2$$

of the finite Borel measures $\varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap \cdot)/2$ exists, we call it the *zeroth fractal curvature measure* of Y. Likewise, the weak limit

$$C_1^f(Y, \cdot) := \operatorname*{w-lim}_{\varepsilon \to 0} \varepsilon^{s_1(Y)} \lambda^1(Y_\varepsilon \cap \cdot)$$

is called the *first fractal curvature measure* of Y, if it exists. Moreover, for a Borel set $B \subseteq \mathbb{R}$ we set

$$\overline{C}_0^f(Y, B) := \limsup_{\varepsilon \to 0} \varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap B)/2,$$
$$\underline{C}_0^f(Y, B) := \liminf_{\varepsilon \to 0} \varepsilon^{s_0(Y)} \lambda^0(\partial Y_\varepsilon \cap B)/2 \quad \text{and}$$
$$\overline{C}_1^f(Y, B) := \limsup_{\varepsilon \to 0} \varepsilon^{s_1(Y)} \lambda^1(Y_\varepsilon \cap B),$$
$$\underline{C}_1^f(Y, B) := \liminf_{\varepsilon \to 0} \varepsilon^{s_1(Y)} \lambda^1(Y_\varepsilon \cap B).$$

Provided it exists, we write $C_k^f(Y) := C_k^f(Y, \mathbb{R})$ for $k \in \{0, 1\}$.

The central question arising in this context is that of identifying those sets $Y \subset \mathbb{R}$ for which the fractal curvature measures exist. In [32] it has been shown that the fractal curvature measures exist for self-similar sets with positive Lebesgue measure as well as for self-similar sets which are nonlattice (see Definition 3.4) and satisfy the open set condition (see Definition 2.5). In the lattice case, Winter [32] shows that average versions of the fractal curvature measures exist, which are defined as follows.

Definition 2.3. Let $Y \subset \mathbb{R}$ denote a nonempty compact set. Provided that the weak limit exists, we let

$$\widetilde{C}_0^f(Y,\cdot) := \operatorname*{w-lim}_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{s_0(Y)-1} \lambda^0(\partial Y_\varepsilon \cap \cdot)\, \mathrm{d}\varepsilon/2$$

denote the *zeroth average fractal curvature measure* of Y and let the weak limit

$$\widetilde{C}_1^f(Y,\cdot) := \operatorname*{w-lim}_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{s_1(Y)-1} \lambda^1(Y_\varepsilon \cap \cdot)\, \mathrm{d}\varepsilon$$

likewise denote the *first average fractal curvature measure* of Y.

Note that the definition of the first curvature scaling exponent resembles the definition of the Minkowski dimension, which coincides with the box counting dimension (see [6, Claim 3.1]), and is defined as follows.

Definition 2.4. For a nonempty compact set $Y \subset \mathbb{R}$ the *upper* and *lower Minkowski dimensions* are respectively defined to be

$$\overline{\dim}_M(Y) := 1 - \liminf_{\varepsilon \searrow 0} \frac{\ln \lambda^1(Y_\varepsilon)}{\ln \varepsilon} \quad \text{and} \quad \underline{\dim}_M(Y) := 1 - \limsup_{\varepsilon \searrow 0} \frac{\ln \lambda^1(Y_\varepsilon)}{\ln \varepsilon}.$$

In the case where the upper and lower Minkowski dimensions coincide, we call the common value the *Minkowski dimension* of Y and denote it by $\dim_M(Y)$.

Our interest lies in determining for which self-conformal sets the fractal curvature measures exist. In order to introduce self-conformal sets, we first give the following definition. The notation that we introduce in the following definitions will be fixed throughout the paper.

Definition 2.5. Let $X \subset \mathbb{R}$ be a nonempty compact interval. We call $Q := \{Q_i : X \to X \mid i \in \{1,\ldots,N\}\}$ a *conformal iterated function system (cIFS)* acting on X provided:

(i) $N \geq 2$,
(ii) $Q_1,\ldots,Q_N$ are differentiable contractions with α-Hölder continuous derivatives $Q_1',\ldots,Q_N'$, where $\alpha \in (0,1]$ and $|Q_1'|,\ldots,|Q_N'|$ are bounded away from both 0 and 1 and
(iii) the *open set condition (OSC)* is satisfied with $O := \operatorname{int}X$ as feasible set, that is $\bigcup_{i=1}^N Q_i(O) \subseteq O$ and $Q_i(O) \cap Q_j(O) = \varnothing$ for $i,j \in \{1,\ldots,N\}$, $i \neq j$, where $\operatorname{int}X$ denotes the interior of X.

A famous theorem of Hutchinson (see for instance [6, Theorem 9.1]) implies that for a cIFS Q as defined above, there exists a unique nonempty compact set $F \subseteq X$ which is *invariant* under Q, that is $F = \bigcup_{i=1}^N Q_i F =: QF$.

Definition 2.6. Let Q denote a cIFS. We call the unique nonempty compact invariant set of Q the *self-conformal* set associated with Q. A cIFS which solely consists of similarities will be called an *sIFS*. The unique nonempty compact invariant set of an sIFS R is called the *self-similar set* associated with R.

Remark 2.7. (i) One easily verifies that our definition of a cIFS coincides with the definition of a finite conformal iterated function system in $\mathbb{R}$ given in [22].

(ii) By Definition 2.6, an sIFS necessarily satisfies the OSC with int$X$ as the feasible set. Regarding the definition of the OSC for sIFS in the literature, note that the feasible set is commonly not required to be connected. For further discussions concerning the nonconnected case, see Remark 2.23(i).

For self-conformal sets it is well-known that the Minkowski dimension exists (see Theorem 3.6). As we will see, a self-conformal set is either a nonempty compact interval or has one-dimensional Lebesgue measure 0 (Proposition 3.1). In order to determine the curvature scaling exponents we have to distinguish between these two cases.

Proposition 2.8. *Let F denote a self-conformal set. If $\lambda^1(F) = 0$, then $s_0(F) = \dim_M(F)$ and $s_1(F) = \dim_M(F) - 1$. If F is a nonempty compact interval, then $s_0(F) = s_1(F) = 0$.*

Let us first consider the latter situation of the above proposition. As an immediate consequence of Proposition 2.8 we obtain the following complete description.

Corollary 2.9. *If $Y \subset \mathbb{R}$ is a nonempty compact interval, then both the zeroth and first fractal curvature measures exist and satisfy*

$$C_0^f(Y, \cdot) = \lambda^0(\partial Y \cap \cdot)/2 \quad \textit{and} \quad C_1^f(Y, \cdot) = \lambda^1(Y \cap \cdot).$$

Let us now focus on self-conformal sets with one-dimensional Lebesgue measure 0. Here, as the total mass of the first (average) fractal curvature measure, the (average) Minkowski content appears. This is defined as follows.

Definition 2.10. Let $Y \subset \mathbb{R}$ denote a set whose Minkowski dimension exists. The *upper Minkowski content* $\overline{\mathcal{M}}(Y)$ and the *lower Minkowski content* $\underline{\mathcal{M}}(Y)$ of Y are respectively defined to be

$$\overline{\mathcal{M}}(Y) := \limsup_{\varepsilon\to 0} \varepsilon^{\dim_M(Y)-1}\lambda^1(Y_\varepsilon) \quad \text{and} \quad \underline{\mathcal{M}}(Y) := \liminf_{\varepsilon\to 0} \varepsilon^{\dim_M(Y)-1}\lambda^1(Y_\varepsilon).$$

If the upper and lower Minkowski contents coincide, we denote the common value by $\mathcal{M}(Y)$ and call it the *Minkowski content* of Y. In the case where the Minkowski content exists, is positive and finite we call Y *Minkowski measurable*. The *average Minkowski content* of Y is defined to be the following limit provided it exists:

$$\widetilde{\mathcal{M}}(Y) := \lim_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\dim_M(Y)-2}\lambda^1(Y_\varepsilon)\,\mathrm{d}\varepsilon.$$

Next, we introduce important notation, which will be explained in more detail in Section 3. Recall that we fix the notation from Definitions 2.5 and 2.6.

For $\Sigma := \{1, \ldots, N\}$ let (Σ^∞, σ) denote the full shift space on N symbols and let $\pi\colon \Sigma^\infty \to F$ be the natural code map. It turns out that existence of the fractal curvature measures of F is guaranteed if the *geometric potential function* $\xi\colon \Sigma^\infty \to \mathbb{R}$ given by $\xi(\omega) := -\ln|Q'_{\omega_1}(\pi(\sigma\omega))|$ for $\omega := \omega_1\omega_2\cdots \in \Sigma^\infty$ is nonlattice (see Definition 3.3). In this case we call Q (resp. F) nonlattice; otherwise Q (resp. F) is called lattice (see Definition 3.4).

By applying Q to the convex hull of F one obtains a family of L gap intervals $G^1, \ldots, G^L$, which we call the *primary gaps* of F, where we have that $1 \le L \le N-1$ since $\lambda^1(F) = 0$. Given an $n \in \mathbb{N}$ and an $\omega := \omega_1\cdots\omega_n \in \Sigma^n$, we let $G^1_\omega, \ldots, G^L_\omega$ respectively denote the images of the primary gaps under the map $Q_\omega := Q_{\omega_1} \circ \cdots \circ Q_{\omega_n}$ and call these sets the *main gaps* of

$Q_\omega F$. Further, letting δ denote the Minkowski dimension of F, we call the unique probability measure ν supported on F, which satisfies

$$\nu(Q_i X \cap Q_j X) = 0 \quad \text{for } i \neq j \in \Sigma \quad \text{and} \quad \nu(Q_i B) = \int_B |Q_i'|^\delta \, \mathrm{d}\nu \tag{2.1}$$

for all $i \in \{1, \ldots, N\}$ and for all Borel sets $B \subseteq X$, the δ-*conformal measure* associated with Q. The statement on the uniqueness and existence is shown in [22] and goes back to the work of [4,24,30]. Finally, let $H(\mu_{-\delta\xi})$ denote the measure theoretical entropy of the shift map with respect to the unique shift-invariant Gibbs measure $\mu_{-\delta\xi}$ for the potential function $-\delta\xi$ (see Eq. (3.3)).

Theorem 2.11 (*Self-Conformal Sets — Fractal Curvature Measures*). *Let Q denote a cIFS and let F denote the self-conformal set associated with Q. Assume that $\lambda^1(F) = 0$, let $\delta := \dim_M(F)$ denote the Minkowski dimension of F and let ξ denote the geometric potential function associated with Q. Then the following hold.*

(i) *The average fractal curvature measures always exist and are both constant multiples of the δ-conformal measure ν associated with F, that is*

$$\widetilde{C}_0^f(F, \cdot) = \frac{2^{-\delta} c}{H(\mu_{-\delta\xi})} \cdot \nu(\cdot) \quad \text{and} \quad \widetilde{C}_1^f(F, \cdot) = \frac{2^{1-\delta} c}{(1-\delta) H(\mu_{-\delta\xi})} \cdot \nu(\cdot),$$

where the constant $c > 0$ is given by the well-defined limit

$$c := \lim_{n \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^n} |G_\omega^i|^\delta. \tag{2.2}$$

(ii) *If ξ is nonlattice, then both the zeroth and first fractal curvature measures exist and satisfy $C_k^f(F, \cdot) = \widetilde{C}_k^f(F, \cdot)$ for $k \in \{0, 1\}$.*

(iii) *If ξ is lattice, then there exists a constant $\overline{c} \in \mathbb{R}$ such that $\overline{C}_k^f(F, B) \leq \overline{c}$ for every Borel set $B \subseteq \mathbb{R}$ and $k \in \{0, 1\}$. Moreover, $\underline{C}_k^f(F, \mathbb{R})$ is positive for $k \in \{0, 1\}$. If additionally the system Q consists of analytic maps, then neither the zeroth nor the first fractal curvature measure exists.*

Note that (ii) and (iii) in particular show that the scaling exponents of F can alternatively be characterised by $s_0(F) = \sup\{t \in \mathbb{R} \mid \varepsilon^t \lambda^0(\partial F_\varepsilon) \to \infty \text{ as } \varepsilon \to 0\}$ and $s_1(F) = \sup\{t \in \mathbb{R} \mid \varepsilon^t \lambda^1(F_\varepsilon) \to \infty \text{ as } \varepsilon \to 0\}$ respectively. The boundedness in (iii) will moreover be used for showing (i).

Using the definition of the Minkowski content and Proposition 2.8, we see that the existence of the fractal curvature measures immediately implies the existence of the Minkowski content. However, it is important to remark that nonexistence of the fractal curvature measures does not generally imply nonexistence of the Minkowski content. We will return to this when investigating the case of $\mathcal{C}^{1+\alpha}$ diffeomorphic images of self-similar sets in Corollary 2.18.

The next theorem deals with general self-conformal sets and in particular states that for a self-conformal set which is lattice, the Minkowski content may or may not exist. To formulate the theorem, let Σ^∞ be equipped with the topology of pointwise convergence and let $\mathcal{C}(\Sigma^\infty)$ denote the space of continuous real valued functions on Σ^∞. For an α-Hölder continuous function $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ (see Section 3.3) we let ν_f denote the unique eigenmeasure corresponding to the eigenvalue 1 of the dual of the Perron–Frobenius operator for the potential function f (see Section 3.3).

Theorem 2.12 (*Self-Conformal Sets — Minkowski Content*)**.** *Under the conditions of Theorem 2.11 and letting c denote the constant given in Eq. (2.2), the following hold.*

(i) *The average Minkowski content exists and is equal to*

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}c}{(1-\delta)H(\mu_{-\delta\xi})}.$$

(ii) *If ξ is nonlattice, then the Minkowski content $\mathcal{M}(F)$ of F exists and coincides with $\widetilde{\mathcal{M}}(F)$.*

(iii) *If ξ is lattice, then we have that*

$$0 < \underline{\mathcal{M}}(F) \le \overline{\mathcal{M}}(F) < \infty.$$

What is more, equality in the above equation can be attained. More precisely let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ denote functions satisfying $\xi - \zeta = \psi - \psi \circ \sigma$, where the range of ζ is contained in a discrete subgroup of $\mathbb{R}$ and $a \in \mathbb{R}$ is maximal such that $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. If, for every $t \in [0, a)$, we have

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\delta a n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, na+t)) = \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{\delta a}-1} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\delta a n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, (n+1)a)), \tag{2.3}$$

then it follows that $0 < \underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F) < \infty$.

Remark 2.13. (i) The sums occurring in Eq. (2.3) are finite, since $\psi \in \mathcal{C}(\Sigma^\infty)$.

(ii) Eq. (2.3) in fact not only implies the existence of the Minkowski content but also that $0 < \underline{C}_0^f(F, \mathbb{R}) = \overline{C}_0^f(F, \mathbb{R}) < \infty$ (see the proof of Theorem 2.12(iii)).

(iii) A comment on how existence of the Minkowski content can be shown for self-conformal sets arising from nonlattice systems consisting of conformal $\mathcal{C}^2$ contractions is given in [5, around Proposition 5].

An example for a lattice self-conformal set F which satisfies Eq. (2.3) and thus is Minkowski measurable is given in Example 2.20. However, in the special case where F is a self-similar set, Eq. (2.3) cannot be satisfied. In this case it even turns out that F is Minkowski measurable if and only if F is nonlattice. The special case of self-similar sets has also been considered in the literature and we give an exposition of those results in Remark 2.15. With the following theorem, we provide a proof alternative to the existing ones, which uses the methods of proof that we developed for showing Theorem 2.11.

Theorem 2.14 (*Self-Similar Sets — Fractal Curvature Measures*)**.** *Let $R := \{R_1, \ldots, R_N\}$ denote an sIFS and let E denote the self-similar set associated with R. Assume that $\lambda^1(E) = 0$ and let $r_1, \ldots, r_N$ denote the respective similarity ratios of $R_1, \ldots, R_N$. Let δ denote the Minkowski dimension of E and let ν be the δ-conformal measure associated with E. Then, additionally to the statements of Theorem 2.11, the following hold.*

(i) *The formulae for the average fractal curvature measures simplify to*

$$\widetilde{C}_0^f(E, \cdot) = \frac{2^{-\delta} \sum_{i=1}^{L} |G^i|^\delta}{-\delta \sum_{i\in\Sigma} \ln(r_i) r_i^\delta} \cdot \nu(\cdot) \quad \textit{and} \quad \widetilde{C}_1^f(E, \cdot) = \frac{2^{1-\delta} \sum_{i=1}^{L} |G^i|^\delta}{(\delta-1)\delta \sum_{i\in\Sigma} \ln(r_i) r_i^\delta} \cdot \nu(\cdot).$$

(ii) *If R is lattice, the following holds. For $k \in \{0, 1\}$ and for every Borel set $B \subseteq \mathbb{R}$ for which $E \cap B$ is a nonempty finite union of sets of the form $R_\omega E$, where $\omega \in \Sigma^*$, and for which $E_\varepsilon \cap B = (E \cap B)_\varepsilon$ for all sufficiently small $\varepsilon > 0$ we have that*

$$0 < \underline{C}_k^f(E, B) < \overline{C}_k^f(E, B) < \infty. \tag{2.4}$$

This in particular shows that the fractal curvature measures do not exist.

Remark 2.15. (i) For self-similar systems the measure ν coincides with the δ-dimensional Hausdorff measure normalised on E, that is with $\mathcal{H}^\delta(\cdot \cap E)/\mathcal{H}^\delta(E)$.

(ii) For self-similar sets the existence of the average fractal curvature measures and of the fractal curvature measures in the nonlattice case has also been shown in [32, Theorem 2.5.1]. The formulae for the coefficients of the measures obtained in [32] are given by an integration over a certain "overlap function". The formulae from Theorem 2.14 can be deduced from the one in [32] by writing the overlap function $\widetilde{R}_k(\varepsilon)$ for $k \in \{0, 1\}$ in the following way: $\widetilde{R}_k(\varepsilon) = C_k(E_\varepsilon) - \sum_{i=1}^N C_k(R_i E_\varepsilon) + \sum_{i=1}^N \mathbb{1}_{(r_i,1]}(\varepsilon) C_k(R_i E_\varepsilon)$. We thank Steffen Winter for pointing this out to us.

(iii) An important case in Theorem 2.14 is the case $B = \mathbb{R}$. For $k = 1$ this case yields statements on the Minkowski content (see Corollary 2.16). Existence of the Minkowski content for self-similar subsets of $\mathbb{R}$ had already been investigated well before the fractal curvature measures. Relevant works are [5,18,19], where Minkowski measurability of fractal strings is investigated. In [5,19] the theory of fractal strings is applied to retrieve results for self-similar sets arising from nonlattice IFS, where the strong separation condition is assumed in [5] and the OSC (with possibly disconnected feasible set) is assumed in [19]. In [19] moreover, nonexistence of the Minkowski content in the lattice case is proven under the OSC. Note that the formulae in the above mentioned literature coincide with the prefactor of the measures in Theorem 2.14. That $C_0^f(E, \mathbb{R})$ does not exist either follows from [19] with a result from [25] (see also Theorem 3.13).

Nonexistence of the Minkowski content and of $C_0^f(E, \mathbb{R})$ clearly implies nonexistence of the fractal curvature measures. In Theorem 2.14(ii) we chose the more involved exposition with arbitrary sets B to indicate the analogy with Theorem 2.17(iii). These two theorems together provide a distinguishing characteristic of self-similar sets and $\mathcal{C}^{1+\alpha}$ diffeomorphic images of self-similar sets.

For the Minkowski content, Theorem 2.14 immediately implies the following corollary which we present without a proof. As stated in the above remark, the results and formulae of the next corollary coincide with the ones obtained in [19, Chapter 8.4] and in the case (ii) with [5, Proposition 4], where the strong separation condition is assumed.

Corollary 2.16 (*Self-Similar Sets — Minkowski Content*)**.** *Under the conditions of Theorem 2.14 the following hold.*

(i) *The average Minkowski content of E exists and is given by*

$$\widetilde{\mathcal{M}}(E) = \frac{2^{1-\delta} \sum_{i=1}^{L} |G^i|^\delta}{(\delta - 1)\delta \sum_{i \in \Sigma} \ln(r_i) r_i^\delta}.$$

(ii) *If R is nonlattice, the Minkowski content $\mathcal{M}(E)$ of E exists and is equal to $\widetilde{\mathcal{M}}(E)$.*

(iii) *If R is lattice, then*

$$0 < \underline{\mathcal{M}}(E) < \overline{\mathcal{M}}(E) < \infty.$$

Another special case of self-conformal sets is provided by the $\mathcal{C}^{1+\alpha}$ diffeomorphic images of self-similar sets. Here, $\mathcal{C}^{1+\alpha}$ is the class of real valued functions which are differentiable with α-Hölder continuous derivative, where $\alpha \in (0, 1]$. For these sets, Theorem 2.11(i) and (ii) yield interesting relationships between the (average) fractal curvature measures of the self-similar set and those of its $\mathcal{C}^{1+\alpha}$ diffeomorphic image which are stated in the following theorem.

Theorem 2.17 ($\mathcal{C}^{1+\alpha}$ *Images — Fractal Curvature Measures*)**.** *Let R denote an sIFS acting on $X \subset \mathbb{R}$, let E denote the self-similar set associated with R, and let δ denote its Minkowski dimension. Further, let $\mathcal{U} \supset X$ be a connected open neighbourhood of X in $\mathbb{R}$ and let $g{:}\,\mathcal{U} \to \mathbb{R}$ be a $\mathcal{C}^{1+\alpha}(\mathcal{U})$ map, for which $|g'|$ is bounded away from 0, where $\alpha \in (0, 1]$. Assume that $\lambda^1(E) = 0$ and set $F := g(E)$.*

(i) *The average fractal curvature measures of both E and F exist. Moreover, they are absolutely continuous and for $k \in \{0, 1\}$ their Radon–Nikodym derivatives are given by*

$$\frac{\mathrm{d}\widetilde{C}_k^f(F, \cdot)}{\mathrm{d}\widetilde{C}_k^f(E, \cdot) \circ g^{-1}} = |g' \circ g^{-1}|^\delta.$$

(ii) *If R is nonlattice, then the fractal curvature measures of both E and F exist and are absolutely continuous with Radon–Nikodym derivatives*

$$\frac{\mathrm{d}C_k^f(F, \cdot)}{\mathrm{d}C_k^f(E, \cdot) \circ g^{-1}} = |g' \circ g^{-1}|^\delta,$$

for $k \in \{0, 1\}$.

(iii) *If R is lattice, then there always exists a Borel set B which satisfies $(B \cap F)_\varepsilon = B \cap F_\varepsilon$ for all sufficiently small $\varepsilon > 0$ such that $\underline{C}_k(F, B) < \overline{C}_k(F, B)$ for $k \in \{0, 1\}$. Thus, neither the zeroth nor the first fractal curvature measure of F exists.*

In contrast to the self-similar setting, the Minkowski content of a $\mathcal{C}^{1+\alpha}$ diffeomorphic image of a lattice self-similar set may or may not exist. In fact, for every lattice fractal self-similar set E there exist diffeomorphisms $g \in \mathcal{C}^{1+\alpha}$ such that $g(E)$ is Minkowski measurable. The explicit form of such diffeomorphisms is given in (iii) of the next corollary. Items (i) and (ii) of the next corollary are immediate consequences of Theorem 2.17.

Corollary 2.18 ($\mathcal{C}^{1+\alpha}$ *Images — Minkowski Content*)**.** *Suppose we are in the situation of Theorem 2.17 and let ν denote the δ-conformal measure associated with E.*

(i) *The average Minkowski contents of both E and F exist and satisfy*

$$\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(E) \cdot \int |g'|^\delta \,\mathrm{d}\nu.$$

(ii) *If R is nonlattice, then the Minkowski contents of both E and F exist and satisfy*

$$\mathcal{M}(F) = \mathcal{M}(E) \cdot \int |g'|^\delta \,\mathrm{d}\nu.$$

(iii) *If R is lattice, then the Minkowski content of E does not exist, whereas the Minkowski content of F might or might not exist. More precisely, assume that $E \subseteq [0,1]$ and that the geometric potential function ζ associated with R is lattice. Let $a > 0$ be maximal such that the range of ζ is contained in $a\mathbb{Z}$. Define $\widetilde{g}: \mathbb{R} \to \mathbb{R}$, $\widetilde{g}(x) := \nu((-\infty, x])$ to be the distribution function of ν. For $n \in \mathbb{N}$ define the function $g_n: [-1, \infty) \to \mathbb{R}$ by*

$$g_n(x) := \int_{-1}^{x} \left(\widetilde{g}(t)(\mathrm{e}^{\delta a n} - 1) + 1\right)^{-1/\delta} \mathrm{d}t$$

and set $F^n := g_n(E)$. Then for every $n \in \mathbb{N}$ we have $\underline{\mathcal{M}}(F^n) = \overline{\mathcal{M}}(F^n)$.

Remark 2.19. The sets F^n constructed in Corollary 2.18(iii) are actually not only Minkowski measurable but also satisfy $\underline{C}_0^f(F^n) = \overline{C}_0^f(F^n)$ (see the proof of Corollary 2.18(iii)).

The results stated in Theorem 2.17 and Corollary 2.18 have recently been obtained also for higher dimensions in [9]. There, $\mathcal{C}^{1+\alpha}$ diffeomorphic images of self-similar sets satisfying the strong separation condition are considered.

The above results enable us to construct examples of lattice self-conformal sets for which the Minkowski content exists.

Example 2.20. Let $E \subseteq [0,1]$ be the middle third Cantor set and let ν denote the $\ln 2/\ln 3$-conformal measure associated with E. Let $\widetilde{g}: \mathbb{R} \to \mathbb{R}$ denote the Devil's Staircase Function defined by $\widetilde{g}(r) := \nu((-\infty, r])$, define the function $g: [-1, \infty) \to \mathbb{R}$ by

$$g(x) := \int_{-1}^{x} (\widetilde{g}(t) + 1)^{-\ln 3/\ln 2} \mathrm{d}t$$

and set $F := g(E)$. Then we have $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$, although $\underline{\mathcal{M}}(E) < \overline{\mathcal{M}}(E)$. This is a consequence of Corollaries 2.16 and 2.18.

Remark 2.21. Example 2.20 together with Corollary 2.3 of [18] shows that there exist fractal strings whose set of boundary points coincides with an invariant set of a lattice cIFS with Lebesgue measure 0, for which the asymptotic second term of the eigenvalue counting function of the Laplacian is monotonic. In Conjecture 4 of [17] it was conjectured that for 'approximately' self-similar sets, monotonic behaviour of the asymptotic second term occurs if and only if the system is nonlattice. Conformal maps locally behave like similarities and thus Example 2.20 disproves the conjecture for self-conformal sets. We thank Michel Lapidus for drawing our attention to this connection. In [12] we will show that Conjecture 4 of [17] is true for limit sets of Fuchsian groups of Schottky type.

Lattice cIFS which arise via $\mathcal{C}^{1+\alpha}$ conjugation of IFS consisting of similarities play an important role in the general theory of lattice cIFS. Namely, if a lattice cIFS is analytic, then it is automatically conjugate to a lattice system consisting of similarities.

Theorem 2.22. *Let Q be a lattice cIFS acting on $X \subset \mathbb{R}$ and consisting of analytic maps. Let F denote the associated self-conformal set. Then there exists a self-similar set $E \subset \mathbb{R}$ and a map g which is analytic on an open neighbourhood of E such that $F = g(E)$.*

The above result is of interest, since it allows one to transfer the results of Theorem 2.17 and Corollary 2.18 over to general self-conformal sets.

Remark 2.23. (i) In the above results we assume that the OSC is satisfied with int$X$ as the open set. Assuming this condition allows for a simple definition of those gaps, which have the property that all other gaps arise as their images under maps of the form Q_ω. However, this assumption excludes cases like the following. Take $X := [0, 1]$ and define $Q_1, Q_2, Q_3 : X \to X$ by $Q_1(x) := x/3$, $Q_2(x) := x/3 + 2/3$ and $Q_3(x) := x/9 + 1/9$. The system satisfies the OSC with $(0, 1/3) \cup (2/3, 1)$ as feasible open set but not with int$X$ as feasible open set. Such systems can be investigated in the more general context of graph directed systems. This will be done in the forthcoming paper [12] by the authors.

(ii) The results concerning the nonlattice and average case have recently been generalised to higher dimensional ambient spaces. In the higher dimensional setting existence of the localised Minkowski content (which corresponds to the first fractal curvature measure) could be shown in the nonlattice case and existence of the average local Minkowski content could also be shown. These results can be found in [15].

3. Preliminaries

3.1. Fractal curvature measures

The works of Groemer and Schneider [10,28] play a vital role in the introduction of Winter's fractal curvature measures. In what follows, we focus on the construction in the one-dimensional setting. For a set $Y \subset \mathbb{R}$ which is a finite union of compact convex sets, there exist two curvature measures, namely the zeroth and the first curvature measures of Y. Originally, these measures were defined through a localised Steiner formula (see [8,28]), but an equivalent and simpler characterisation is the following. The first curvature measure of Y equals $\lambda^1(Y \cap \cdot)$ and under the additional assumption that Y is the closure of its interior, the zeroth curvature measure is equal to $\lambda^0(\partial Y \cap \cdot)/2$.

If $Y \subset \mathbb{R}$ is not a finite union of compact convex sets, but an arbitrary compact set, we still have that the ε-parallel neighbourhood Y_ε of Y is a finite union of convex compact sets, for each $\varepsilon > 0$. Moreover, Y_ε is the closure of its interior, for each $\varepsilon > 0$. Thus, the zeroth and first curvature measures are defined on Y_ε and respectively coincide with the measures $\lambda^0(\partial Y_\varepsilon \cap \cdot)/2$ and $\lambda^1(Y_\varepsilon \cap \cdot)$. The fractal curvature measures now arise on taking the weak limit as $\varepsilon \to 0$. However, before taking the limit, we observe that for a fractal set $F \subset \mathbb{R}$ one typically obtains that the number of boundary points of F_ε tends to infinity as $\varepsilon \to 0$, whereas the volume of F_ε tends to zero as $\varepsilon \to 0$. In order to obtain nontrivial measures, we need to introduce the curvature scaling exponents $s_0(F)$ and $s_1(F)$ as in Definition 2.1. By taking the weak limits of the rescaled curvature measures $\varepsilon^{s_0(F)} \cdot \lambda^0(\partial F_\varepsilon \cap \cdot)/2$ and $\varepsilon^{s_1(F)} \cdot \lambda^1(F_\varepsilon \cap \cdot)$ as $\varepsilon \to 0$, we obtain the fractal curvature measures $C_0^f(F, \cdot)$ and $C_1^f(F, \cdot)$ (Definition 2.2), whenever the weak limits exist. The average fractal curvature measures are gained by taking the weak limit over the average rescaled curvature measures if these limits exist (Definition 2.3).

Besides extending the notions of curvature, the fractal curvature measures also provide a set of geometric characteristics of a fractal set which can be used to distinguish fractal sets of the same Minkowski dimension. More precisely, considering two fractal sets $F_1, F_2 \subseteq [0, 1]$ with $\{0, 1\} \subseteq F_1, F_2$ which are of the same Minkowski dimension, the first fractal curvature measure compares the local rate of decay of the lengths of the ε-parallel neighbourhood of F_1 and F_2. In this way it can be interpreted as the "local fractal length". Since, by the inclusion exclusion principle, the above mentioned rate of decay correlates with the length of the overlap of sets of the form $(Q_\omega F_i)_\varepsilon$, where $\omega \in \Sigma^n$ for $n \in \mathbb{N}$ and $i \in \{0, 1\}$, the value of the first fractal curvature measure describes the distribution of the gaps. That is, the more equally spread the gaps are over the fractal, the greater its fractal curvature measure is. Analogously, the value of the zeroth fractal curvature measure can be interpreted as the "local fractal number of boundary points" or "local fractal Euler number". For further details on the geometric interpretation in higher dimensions, we refer the reader to [14,20,32].

3.2. Self-conformal sets and the shift space

Recall the definition of a cIFS and a self-conformal set from Definitions 2.5 and 2.6.

Proposition 3.1. *Let Q be a cIFS and let F be the self-conformal set associated with Q. Then F is either a nonempty compact interval or has one-dimensional Lebesgue measure 0.*

Proof. Set $Q := \{Q_1, \ldots, Q_N\}$, define $X =: [a, b]$ to be the set which Q acts on and set $Q_i(X) =: [a_i, b_i]$, where $a_i, b_i \in \mathbb{R}$ for $i \in \{1, \ldots, N\}$ and $a, b \in \mathbb{R}$. Assume without loss of generality that $Q_1, \ldots, Q_N$ are ordered such that $a_1 < a_2 < \cdots < a_N$. If $[a_i, b_i] \cap [a_{i+1}, b_{i+1}] \neq \varnothing$ for all $i \in \{1, \ldots, N-1\}$, then clearly $F = [a, b]$. Now assume that there exists an $i \in \{1, \ldots, N-1\}$ such that $[a_i, b_i] \cap [a_{i+1}, b_{i+1}] = \varnothing$. Then [22, Proposition 4.4] gives that F has Lebesgue measure 0. $\square$

It turns out to be useful to view a self-conformal set on a symbolic level. For the following, we fix a cIFS $Q := \{Q_1, \ldots, Q_N\}$ and let F denote the self-conformal set associated with Q. We introduce the *full shift space on N symbols* (Σ^∞, σ) as follows.

Let $\Sigma := \{1, \ldots, N\}$ denote the *alphabet* and let Σ^n denote the set of words of length $n \in \mathbb{N}$ over Σ. Further, let $\Sigma^* := \bigcup_{n\in\mathbb{N}_0} \Sigma^n$ be the set of all finite words over Σ including the empty word $\varnothing$. We call the set Σ^∞ of infinite words over Σ the *code space*. The *shift map* is defined to be the map $\sigma \colon \Sigma^* \cup \Sigma^\infty \to \Sigma^* \cup \Sigma^\infty$ given by $\sigma(\omega) := \varnothing$ for $\omega \in \{\varnothing\} \cup \Sigma^1$, $\sigma(\omega_1 \cdots \omega_n) := \omega_2 \cdots \omega_n \in \Sigma^{n-1}$ for $\omega_1 \cdots \omega_n \in \Sigma^n$, where $n \geq 2$ and $\sigma(\omega_1\omega_2 \cdots) := \omega_2\omega_3 \cdots \in \Sigma^\infty$ for $\omega_1\omega_2 \cdots \in \Sigma^\infty$. For a finite word $\omega \in \Sigma^*$ we let $n(\omega)$ denote its length.

Note that Σ^∞ gives a coding of the self-conformal set F as can be seen as follows. For $\omega = \omega_1 \cdots \omega_n \in \Sigma^*$ we set $Q_\omega := Q_{\omega_1} \circ \cdots \circ Q_{\omega_n}$ and define $Q_\varnothing := \mathrm{id}|_X$ to be the identity map on X. For $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$ and $n \in \mathbb{N}$ we denote the initial word of length n of ω by $\omega|_n := \omega_1\omega_2 \cdots \omega_n$. For each $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$ the intersection $\bigcap_{n\in\mathbb{N}} Q_{\omega|_n}(X)$ contains exactly one point $x_\omega \in F$ and gives rise to a surjection $\pi \colon \Sigma^\infty \to F,\ \omega \mapsto x_\omega$ which we call the *code map*.

One of the key properties of a cIFS is the bounded distortion property. Our results require the following refinement of this property, which we could not find in this precise form in the literature. Therefore, we give a short proof of this statement.

Lemma 3.2 (*Bounded Distortion*)**.** *There exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ with $\varrho_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varrho_n = 1$ such that for all $\omega, u \in \Sigma^*$ and $x, y \in Q_\omega X$ we have*

$$\varrho_{n(\omega)}^{-1} \leq \frac{|Q_u'(x)|}{|Q_u'(y)|} \leq \varrho_{n(\omega)}.$$

Proof. Fix $\omega \in \Sigma^n$ and let $x, y \in Q_\omega X$ and $u = u_1 \cdots u_{n(u)} \in \Sigma^*$ be arbitrarily chosen. Then

$$\frac{|Q_u'(x)|}{|Q_u'(y)|} \leq \exp\left(\sum_{k=1}^{n(u)} \underbrace{\left|\ln|Q_{u_k}'(Q_{\sigma^k u}(x))| - \ln|Q_{u_k}'(Q_{\sigma^k u}(y))|\right|}_{=:A_k}\right).$$

Since Q_i' is α-Hölder continuous and bounded away from 0, it follows that $\ln|Q_i'|$ is α-Hölder continuous for each $i \in \{1, \ldots, N\}$. Let c_i be the corresponding Hölder constant and set $c := \max_{i\in\{1,\ldots,N\}} c_i$. Further, let $r < 1$ be a common upper bound for the contraction ratios of the maps $Q_1, \ldots, Q_N$. Then we have

$$A_k \leq c|Q_{\sigma^k u}(x) - Q_{\sigma^k u}(y)|^\alpha \leq c \cdot \left(r^{n(u)-k}|x-y|\right)^\alpha$$

and thus

$$\sum_{k=1}^{n(u)} A_k \leq \frac{c}{1-r^\alpha}|x-y|^\alpha \leq \frac{c}{1-r^\alpha} \max_{\omega\in\Sigma^n} \sup_{x,y\in Q_\omega X} |x-y|^\alpha =: \widetilde{\varrho}_n.$$

Since $\widetilde{\varrho}_n$ converges to 0 as $n \to \infty$, $\varrho_n := \exp(\widetilde{\varrho}_n)$ converges to 1 as $n \to \infty$. The estimate for the lower bound can be obtained by just interchanging the roles of x and y. □

3.3. Perron–Frobenius theory and the geometric potential function

In order to provide the necessary background for defining the constants in our main statements and also to set up the tools needed in the proofs, we now recall some facts from the Perron–Frobenius theory. For $f \in \mathcal{C}(\Sigma^\infty)$, $\alpha \in (0,1)$ and $n \in \mathbb{N} \cup \{0\}$ define

$$\begin{aligned}
&\mathrm{var}_n(f) := \sup\{|f(\omega) - f(u)| \mid \omega, u \in \Sigma^\infty \text{ and } \omega_i = u_i \text{ for all } i \in \{1,\ldots,n\}\},\\
&|f|_\alpha := \sup_{n\geq 0} \frac{\mathrm{var}_n(f)}{\alpha^n} \quad \text{and}\\
&\mathcal{F}_\alpha(\Sigma^\infty) := \{f \in \mathcal{C}(\Sigma^\infty) \mid |f|_\alpha < \infty\}.
\end{aligned}$$

Elements of $\mathcal{F}_\alpha(\Sigma^\infty)$ are called *α-Hölder continuous* functions on Σ^∞. For $f \in \mathcal{C}(\Sigma^\infty)$, define the *Perron–Frobenius operator* $\mathcal{L}_f : \mathcal{C}(\Sigma^\infty) \to \mathcal{C}(\Sigma^\infty)$ by

$$\mathcal{L}_f \psi(x) := \sum_{y:\sigma y = x} \mathrm{e}^{f(y)} \psi(y) \tag{3.1}$$

for $x \in \Sigma^\infty$ and let $\mathcal{L}_f^*$ be the dual of $\mathcal{L}_f$ acting on the set of Borel probability measures on Σ^∞. By [31, Theorem 2.16 and Corollary 2.17] and [2, Theorem 1.7], for each real valued Hölder continuous $f \in \mathcal{F}_\alpha(\Sigma^\infty)$, for some $\alpha \in (0,1)$, there exists a unique Borel probability measure ν_f on Σ^∞ such that $\mathcal{L}_f^* \nu_f = \gamma_f \nu_f$ for some $\gamma_f > 0$. Moreover, γ_f is uniquely determined by this equation and satisfies $\gamma_f = \exp(P(f))$. Here $P : \mathcal{C}(\Sigma^\infty) \to \mathbb{R}$ denotes the *topological pressure function* which for $\psi \in \mathcal{C}(\Sigma^\infty)$ is defined by

$$P(\psi) := \lim_{n\to\infty} n^{-1} \ln \sum_{\omega\in\Sigma^n} \exp \sup_{u\in[\omega]} \sum_{k=0}^{n-1} \psi \circ \sigma^k(u)$$

(see [2, Lemma 1.20]), where $[\omega] := \{u \in \Sigma^\infty \mid u_i = \omega_i \text{ for } 1 \leq i \leq n(\omega)\}$ is the *ω-cylinder set*.

Further, there exists a unique strictly positive eigenfunction h_f of $\mathcal{L}_f$ satisfying $\mathcal{L}_f h_f = \gamma_f h_f$. We take h_f to be normalised such that $\int h_f \, \mathrm{d}\nu_f = 1$. By μ_f we denote the σ-invariant probability measure defined by $\frac{\mathrm{d}\mu_f}{\mathrm{d}\nu_f} = h_f$. This is the unique σ-invariant Gibbs measure for the potential function f. Under some normalisation assumptions we have convergence of the iterates of the Perron–Frobenius operator to the projection onto its eigenfunction h_f. To be more precise we have for all $\psi \in \mathcal{C}(\Sigma^\infty)$,

$$\lim_{m\to\infty} \left\| \gamma_f^{-m} \mathcal{L}_f^m \psi - \int \psi \, \mathrm{d}\nu_f \cdot h_f \right\| = 0, \tag{3.2}$$

where $\|\cdot\|$ denotes the supremum-norm on $\mathcal{C}(\Sigma^\infty)$. The results on the Perron–Frobenius operator quoted above originate mainly from the work of Ruelle [26].

A central object of our investigations is the geometric potential function associated with the cIFS Q and its property of being lattice or nonlattice, which we now define.

Definition 3.3. Two functions $f_1, f_2 \in \mathcal{C}(\Sigma^\infty)$ are called *cohomologous* if there exists a $\psi \in \mathcal{C}(\Sigma^\infty)$ such that $f_1 - f_2 = \psi - \psi \circ \sigma$. A function $f \in \mathcal{C}(\Sigma^\infty)$ is said to be a *lattice* function if f is cohomologous to a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Otherwise, we say that f is a *nonlattice* function.

The notion of being lattice or not carries over to Q and its self-conformal set F via considering the geometric potential function associated with Q:

Definition 3.4. Fix a cIFS $Q := \{Q_1, \dots, Q_N\}$. Denote by F the self-conformal set associated with Q and let Σ^∞ be the associated code space. Define the *geometric potential function* to be the map $\xi : \Sigma^\infty \to \mathbb{R}$ given by $\xi(\omega) := -\ln |Q'_{\omega_1}(\pi\sigma\omega)|$ for $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$. If ξ is nonlattice, then we call Q (and also F) *nonlattice*. On the other hand, if ξ is a lattice function, then we call Q (and also F) *lattice*.

Remark 3.5. The geometric potential function ξ associated with a cIFS $Q := \{Q_1, \dots, Q_N\}$ satisfies $\xi \in \mathcal{F}_{\widetilde{\alpha}}(\Sigma^\infty)$ for some $\widetilde{\alpha} \in (0, 1)$. To see this, we let $r < 1$ be a common upper bound for the contraction ratios of $Q_1, \dots, Q_N$. Because of the α-Hölder continuity of $Q'_1, \dots, Q'_N$ and the fact that $Q'_1, \dots, Q'_N$ are bounded away from zero we obtain that there exists a constant $c \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ we have $\mathrm{var}_n(\xi) \leq c r^{\alpha(n-1)}$. Thus, $\xi \in \mathcal{F}_{\widetilde{\alpha}}(\Sigma^\infty)$, where $\widetilde{\alpha} := r^\alpha \in (0, 1)$.

For the geometric potential function $\xi \in \mathcal{C}(\Sigma^\infty)$ it can be shown that the *measure theoretical entropy* $H(\mu_{-\delta\xi})$ of the shift map σ with respect to $\mu_{-\delta\xi}$ is given by

$$H(\mu_{-\delta\xi}) = \delta \int \xi \, \mathrm{d}\mu_{-\delta\xi}, \tag{3.3}$$

where δ denotes the Minkowski dimension of F. This observation follows for example from the variational principle, [2, Theorem 1.22] and the following result of [1] which will also be needed in the proof of Theorem 2.11.

Theorem 3.6. *The Minkowski and the Hausdorff dimensions of F are both equal to the unique real number $t > 0$ such that $P(-t\xi) = 0$, where P denotes the topological pressure function.*

3.4. Renewal theory and geometric measure theory

In the proof of Theorem 2.11 we are going to make use of a renewal theory argument for counting measures in symbolic dynamics. For this we first fix the following notation.

For a map $f\colon \Sigma^\infty \to \mathbb{R}$ and $n \in \mathbb{N}$ define the *nth ergodic sum* to be

$$S_n f := \sum_{k=0}^{n-1} f \circ \sigma^k \quad \text{and} \quad S_0 f := 0.$$

Moreover, we call a function $f_1\colon (0,\infty) \to \mathbb{R}$ *asymptotic* to a function $f_2\colon (0,\infty) \to \mathbb{R}$ as $\varepsilon \to 0$, in symbols $f_1(\varepsilon) \sim f_2(\varepsilon)$ as $\varepsilon \to 0$, if $\lim_{\varepsilon \to 0} f_1(\varepsilon)/f_2(\varepsilon) = 1$. Similarly, we say that f_1 is *asymptotic* to f_2 as $t \to \infty$, in symbols $f_1(t) \sim f_2(t)$ as $t \to \infty$, if $\lim_{t\to\infty} f_1(t)/f_2(t) = 1$.

The following proposition is a well-known fact which is stated in for example [16, Proposition 2.1].

Proposition 3.7. *Let $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ for some $\alpha \in (0,1)$ be such that for some $n \geq 1$ the function $S_n f$ is strictly positive on Σ^∞. Then there exists a unique $s > 0$ such that*

$$\gamma_{-sf} = 1. \tag{3.4}$$

The following two theorems play a crucial role in the proof of Theorem 2.11. The first of the two theorems is [16, Theorem 1]. The second one is a refinement and generalisation of [16, Theorem 3] and hence we will give a proof.

Proposition 3.8 (*Lalley*)**.** *Assume that f lies in $\mathcal{F}_\alpha(\Sigma^\infty)$ for some $\alpha \in (0,1)$, is nonlattice and is such that for some $n \geq 1$ the function $S_n f$ is strictly positive. Let $g \in \mathcal{F}_\alpha(\Sigma^\infty)$ be nonnegative but not identically zero and let $s > 0$ be implicitly given by Eq. (3.4). Then we have that*

$$\sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \leq t\}} \sim \frac{\int g \, \mathrm{d}\nu_{-sf}}{s \int f \, \mathrm{d}\mu_{-sf}} h_{-sf}(x) \mathrm{e}^{st}$$

as $t \to \infty$ uniformly for $x \in \Sigma^\infty$.

For $b \in \mathbb{R}$, we denote by $\lceil b \rceil$ the smallest integer which is greater than or equal to b, by $\lfloor b \rfloor$ the greatest integer which is less than or equal to b, and by $\{b\}$ the fractional part of b, that is $\{b\} := b - \lfloor b \rfloor$.

Theorem 3.9. *Assume that f lies in $\mathcal{F}_\alpha(\Sigma^\infty)$ for some $\alpha \in (0,1)$ and that for some $n \geq 1$ the function $S_n f$ is strictly positive. Further assume that f is lattice and let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ denote functions which satisfy*

$$f - \zeta = \psi - \psi \circ \sigma,$$

where ζ is a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be maximal such that $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. Further, let $g \in \mathcal{F}_\alpha(\Sigma^\infty)$ be nonnegative but not identically zero and $s > 0$ be implicitly given by Eq. (3.4). Then we have that

$$\sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \leq t\}} \sim \frac{a h_{-s\zeta}(x) \int g(y) \mathrm{e}^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} - \frac{t}{a} \right\rceil} \mathrm{d}\nu_{-s\zeta}(y)}{\left(1 - \mathrm{e}^{-sa}\right) \int \zeta \, \mathrm{d}\mu_{-s\zeta}} \tag{3.5}$$

as $t \to \infty$ uniformly for $x \in \Sigma^\infty$.

Remark 3.10. Proposition 3.8 and Theorem 3.9 are also valid in the more general situation of (Σ^∞, σ) being a subshift of finite type. See also [16, Theorem 3] where the exact asymptotic is not provided.

Proof of Theorem 3.9. For the proof we first assume that $a = 1$, which implies that ζ is integer valued and not cohomologous to any function taking its values in a proper subgroup of $\mathbb{Z}$. We first follow the lines of the proofs of [16, Theorems 2 and 3] and then refine the last steps of the proof of [16, Theorem 3] to obtain the exact asymptotics.

In [16], Lalley introduces the following functions for $t \in \mathbb{R}$ and $x \in \Sigma^\infty$:

$$N_f(t,x) := \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = x} g(y) \mathbb{1}_{\{S_n f(y) \le t\}},$$
$$N^*(t,x) := N_f(t - \psi(x), x)$$

and for $\beta \in [0,1)$ and $z \in \mathbb{C}$ the Fourier–Laplace transform

$$\hat{N}^*_\beta(z,x) := \sum_{n=-\infty}^{\infty} \mathrm{e}^{nz} N^*(n+\beta, x).$$

It is easy to verify that $N_f(t,x)$ satisfies a renewal equation (see [16, Eq. (2.2)])

$$N_f(t,x) = \sum_{y:\sigma y = x} N_f(t - f(y), y) + g(x) \mathbb{1}_{\{t \ge 0\}}$$

from which one can deduce that $\hat{N}^*_\beta$ satisfies the following equation:

$$\hat{N}^*_\beta(z,x) = (I - \mathcal{L}_{z\zeta})^{-1} g(x) \frac{\mathrm{e}^{z\lceil \psi(x) - \beta \rceil}}{1 - \mathrm{e}^z}, \tag{3.6}$$

where I denotes the identity operator. We remark that Eq. (3.6) differs slightly from the respective equation in [16], in that Lalley obtains $z\lfloor \psi(x) + 1 - \beta \rfloor$ as the argument of the exponential, whereas our calculations result in $z\lceil \psi(x) - \beta \rceil$ being the right expression instead.

By arguments in the proof of [16, Theorem 2], the function $z \mapsto (I - \mathcal{L}_{z\zeta})^{-1} g(x)$ is meromorphic in $\{z \in \mathbb{C} \mid 0 \le \mathrm{Im}(z) \le \pi,\ \mathrm{Re}(z) < -s + \varepsilon\}$ for some $\varepsilon > 0$ and the only singularity in this region is a simple pole at $z = -s$ with residue

$$-\frac{h_{-s\zeta}(x) \int g \, \mathrm{d}\nu_{-s\zeta}}{\int \zeta \, \mathrm{d}\mu_{-s\zeta}}.$$

Since $z \mapsto \mathrm{e}^{z\lceil \psi(x) - \beta \rceil}$ and $z \mapsto (1 - \mathrm{e}^z)^{-1}$ are holomorphic in $\{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$, we deduce from this that $z \mapsto \hat{N}^*_\beta(z,x)$ is meromorphic in $\{z \in \mathbb{C} \mid 0 \le \mathrm{Im}(z) \le \pi,\ \mathrm{Re}(z) < -s + \varepsilon\}$ for some $\varepsilon > 0$ and that the only singularity in this region is a simple pole at $z = -s$ with residue

$$-\frac{h_{-s\zeta}(x) \int g(y) \mathrm{e}^{-s\lceil \psi(y) - \beta \rceil} \, \mathrm{d}\nu_{-s\zeta}(y)}{(1 - \mathrm{e}^{-s}) \int \zeta \, \mathrm{d}\mu_{-s\zeta}} =: C(\beta, x).$$

Now, again following the lines of the proof of [16, Theorem 2], it follows that

$$N^*(n+\beta,x) \sim -C(\beta,x)\mathrm{e}^{sn}$$

as $n \to \infty$ uniformly for $x \in \Sigma^\infty$. Thus for $t \in (0,\infty)$,

$$N_f(t,x) = N_f\left(\underbrace{\lfloor \psi(x)+t \rfloor}_{=:n} + \underbrace{\{\psi(x)+t\}}_{=:\beta} - \psi(x), x\right) = N^*(n+\beta,x)$$

$$\sim -C(\beta,x)\mathrm{e}^{sn} = \frac{h_{-s\zeta}(x)\int g(y)\mathrm{e}^{-s\lceil \psi(y)-\psi(x)-t\rceil}\,\mathrm{d}\nu_{-s\zeta}(y)}{(1-\mathrm{e}^{-s})\int \zeta\,\mathrm{d}\mu_{-s\zeta}} \tag{3.7}$$

as $n \to \infty$ uniformly for $x \in \Sigma^\infty$. This proves the case $a = 1$.

The case where $a \neq 1$ is not covered in [16]. If $a > 0$ is arbitrary, then we consider the function $a^{-1}f = a^{-1}\zeta + a^{-1}\psi - (a^{-1}\psi)\circ\sigma$. Since by Proposition 3.7, $s > 0$ satisfying Eq. (3.4) is the unique positive real number such that $\gamma_{-sf} = 1$, $\widetilde{s} := sa$ is the unique positive real number satisfying $\gamma_{-\widetilde{s}a^{-1}f} = 1$. Therefore, Eq. (3.7) implies

$$\sum_{n=0}^{\infty}\sum_{y:\sigma^n y = x} g(y)\mathbb{1}_{\{S_n f(y)\le t\}} = \sum_{n=0}^{\infty}\sum_{y:\sigma^n y = x} g(y)\mathbb{1}_{\{S_n a^{-1} f(y)\le t a^{-1}\}}$$

$$\sim \frac{h_{-s\zeta}(x)\int g(y)\mathrm{e}^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} - \frac{t}{a}\right\rceil}\,\mathrm{d}\nu_{-s\zeta}(y)}{(1-\mathrm{e}^{-sa})\int a^{-1}\zeta\,\mathrm{d}\mu_{-s\zeta}}$$

as $t \to \infty$ uniformly for $x \in \Sigma^\infty$. □

In view of the existence of the average fractal curvature measures, the following corollary is essential.

Corollary 3.11. *Under the assumptions of Theorem 3.9,*

$$\lim_{T\to\infty} T^{-1}\int_0^T \mathrm{e}^{-st}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y = x} g(y)\mathbb{1}_{\{S_n f(y)\le t\}}\,\mathrm{d}t$$

exists and is equal to

$$\frac{h_{-sf}(x)\int g\,\mathrm{d}\nu_{-sf}}{s\int f\,\mathrm{d}\mu_{-sf}}.$$

Proof. First, observe that for two functions $f_1, f_2\colon (0,\infty) \to \mathbb{R}$ which satisfy $f_1(t) \sim f_2(t)$ as $t \to \infty$, the existence of $A_1 := \lim_{T\to\infty} T^{-1}\int_0^T f_1(t)\,\mathrm{d}t$ implies the existence of $A_2 := \lim_{T\to\infty} T^{-1}\int_0^T f_2(t)\,\mathrm{d}t$ and $A_1 = A_2$. In view of Theorem 3.9, we hence consider the function $\eta\colon [0,\infty) \to \mathbb{R}$ given by

$$\eta(t) := \mathrm{e}^{-st}\int g(y)\mathrm{e}^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} - \frac{t}{a}\right\rceil}\,\mathrm{d}\nu_{-s\zeta}(y).$$

Since $\eta(t+a)=\eta(t)$ for all $t\in(0,\infty)$, η is periodic with period a. As η is moreover locally integrable, this implies

$$\begin{aligned}\lim_{T\to\infty} T^{-1}\int_0^T \eta(t)\,\mathrm{d}t &= \lim_{T\to\infty} T^{-1}\left(\sum_{k=0}^{\lfloor a^{-1}T\rfloor -1}\int_{T-a(k+1)}^{T-ak}\eta(t)\,\mathrm{d}t+\int_0^{T-a\lfloor a^{-1}T\rfloor}\eta(t)\,\mathrm{d}t\right)\\ &= \lim_{T\to\infty} T^{-1}\left\lfloor a^{-1}T\right\rfloor\int_0^a\eta(t)\,\mathrm{d}t = a^{-1}\int_0^a \eta(t)\,\mathrm{d}t.\end{aligned}$$

Applying Fubini's theorem yields

$$\int_0^a\eta(t)\,\mathrm{d}t=\int_{\Sigma^\infty}\int_0^a \mathrm{e}^{-st}g(y)\mathrm{e}^{-sa\left\lceil\frac{\psi(y)-\psi(x)}{a}-\frac{t}{a}\right\rceil}\,\mathrm{d}t\,\mathrm{d}\nu_{-s\zeta}(y).$$

Define $D(y):=a\{a^{-1}(\psi(y)-\psi(x))\}$. This is the unique real number in the interval $[0,a)$ such that $a^{-1}(\psi(y)-\psi(x)-D(y))\in\mathbb{Z}$. Since $a^{-1}t\in[0,1)$ for $t\in[0,a)$, we hence have

$$\begin{aligned}&\int_0^a\eta(t)\,\mathrm{d}t\\ &=\int_{\Sigma^\infty}\left(\int_0^{D(y)}\mathrm{e}^{-st}g(y)\mathrm{e}^{-sa\left\lceil\frac{\psi(y)-\psi(x)}{a}\right\rceil}\mathrm{d}t+\int_{D(y)}^a \mathrm{e}^{-st}g(y)\mathrm{e}^{-sa\left\lfloor\frac{\psi(y)-\psi(x)}{a}\right\rfloor}\mathrm{d}t\right)\mathrm{d}\nu_{-s\zeta}(y)\\ &=\int_{\Sigma^\infty}\frac{g(y)}{s}\left(\mathrm{e}^{-sa\left\lceil\frac{\psi(y)-\psi(x)}{a}\right\rceil}\left(1-\mathrm{e}^{-sD(y)}\right)+\mathrm{e}^{-sa\left\lfloor\frac{\psi(y)-\psi(x)}{a}\right\rfloor}\left(\mathrm{e}^{-sD(y)}-\mathrm{e}^{-sa}\right)\right)\mathrm{d}\nu_{-s\zeta}(y)\\ &=\frac{1-\mathrm{e}^{-sa}}{s}\mathrm{e}^{s\psi(x)}\int_{\Sigma^\infty}g(y)\mathrm{e}^{-s\psi(y)}\,\mathrm{d}\nu_{-s\zeta}(y),\end{aligned}$$

where the last equality can be obtained by distinguishing the cases $D(y)\neq 0$ and $D(y)=0$, that is $a^{-1}(\psi(y)-\psi(x))\in\mathbb{Z}$. As by Theorem 3.9

$$\mathrm{e}^{-st}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x}g(y)\mathbb{1}_{\{S_nf(y)\le t\}}\sim\frac{ah_{-s\zeta}(x)}{\left(1-\mathrm{e}^{-sa}\right)\int\zeta\,\mathrm{d}\mu_{-s\zeta}}\eta(t)$$

as $t\to\infty$ uniformly for $x\in\Sigma^\infty$, the initial remark of this proof now implies that

$$\begin{aligned}&\lim_{T\to\infty}T^{-1}\int_0^T\mathrm{e}^{-st}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x}g(y)\mathbb{1}_{\{S_nf(y)\le t\}}\,\mathrm{d}t\\ &=\frac{\mathrm{e}^{s\psi(x)}h_{-s\zeta}(x)}{s\int\zeta\,\mathrm{d}\mu_{-s\zeta}}\int g(y)\mathrm{e}^{-s\psi(y)}\,\mathrm{d}\nu_{-s\zeta}(y).\end{aligned}$$

Finally, one easily verifies that $\mathrm{e}^{s\psi}h_{-s\zeta}=h_{-sf}$, $\mathrm{e}^{-s\psi}\,\mathrm{d}\nu_{-s\zeta}=\mathrm{d}\nu_{-sf}$ and $\int\zeta\,\mathrm{d}\mu_{-s\zeta}=\int f\,\mathrm{d}\mu_{-sf}$, which completes the proof. □

In order to prove Theorem 2.12(iii), the following lemma, which is closely related to Theorem 3.9, is needed.

Lemma 3.12. *Assume the conditions of Theorem 3.9 and fix a nonempty Borel set $B \subseteq \mathbb{R}$. For $x \in \Sigma^\infty$ define the function $\eta_B\colon (0,\infty) \to \mathbb{R}$ by*

$$\eta_B(t) := \mathrm{e}^{-st} \int \mathbb{1}_{\psi^{-1}B}(y)\mathrm{e}^{-sa\left\lceil \frac{\psi(y)-\psi(x)}{a} - \frac{t}{a}\right\rceil} \mathrm{d}\nu_{-s\zeta}(y).$$

Then $\lim_{t\to\infty} \eta_B(t)$ exists if and only if for every $t \in [0,a)$ we have

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-san}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, na+t)\big) = \frac{\mathrm{e}^{st}-1}{\mathrm{e}^{sa}-1} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-san}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, (n+1)a)\big).$$

Proof. First, note that the above sums are finite. η_B is a periodic function with period a, meaning that $\eta_B(t+a) = \eta_B(t)$ for all $t \in (0,\infty)$. Therefore, $\lim_{t\to\infty} \eta_B(t)$ exists if and only if η_B is a constant function. For $t \in [\psi(x), \psi(x)+a)$ we have

$$\begin{aligned}
&\eta_B(t-\psi(x)) \\
&= \mathrm{e}^{s\psi(x)-st} \int_{\Sigma^\infty} \mathbb{1}_{\psi^{-1}B}(y)\mathrm{e}^{-sa\left\lceil \frac{\psi(y)-t}{a}\right\rceil} \mathrm{d}\nu_{-s\zeta}(y) \\
&= \mathrm{e}^{s\psi(x)-st} \sum_{n\in\mathbb{Z}} \int_{na}^{(n+1)a} \mathbb{1}_B(y)\mathrm{e}^{-sa\left\lceil \frac{y-t}{a}\right\rceil} \mathrm{d}\nu_{-s\zeta} \circ \psi^{-1}(y) \\
&= \mathrm{e}^{s\psi(x)-st+sa\lfloor \frac{t}{a}\rfloor} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-san} \Big(\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, na+a\{a^{-1}t\}]\big) \\
&\quad + \mathrm{e}^{-sa}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap (na+a\{a^{-1}t\}, (n+1)a)\big)\Big) \\
&= \mathrm{e}^{s\psi(x)-sa\{\frac{t}{a}\}} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-san} \Big((1-\mathrm{e}^{-sa})\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, na+a\{a^{-1}t\}]\big) \\
&\quad + \mathrm{e}^{-sa}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, (n+1)a)\big)\Big).
\end{aligned}$$

Thus, $\lim_{t\to\infty} \eta_B(t)$ exists if and only if there is a $\widetilde{c} \in \mathbb{R}$ such that for every $t \in [0,a)$,

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-san}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, na+t]\big) = (1-\mathrm{e}^{-sa})^{-1} \left(\widetilde{c}\mathrm{e}^{st-s\psi(x)} - \mathrm{e}^{-sa} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-san}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, (n+1)a)\big)\right).$$

Taking the limit as t tends to a we hence obtain

$$\widetilde{c} = \mathrm{e}^{s\psi(x)-sa} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-san}\nu_{-s\zeta} \circ \psi^{-1}\big(B \cap [na, (n+1)a)\big)$$

which proves the statement. □

Another important tool in the proofs of our results is a relationship between the zeroth and the first (average) fractal curvature measures. In order to show that the existence of the zeroth fractal curvature measure implies the existence of the first fractal curvature measure we use [25, Corollary 3.2] which is a higher dimensional and more general version of the following theorem.

Theorem 3.13 (*Rataj, Winter*). *Let $Y \subset \mathbb{R}$ be a nonempty and compact set such that $\lambda^1(Y) = 0$. Assume that the Minkowski dimension δ of Y exists. Then*

$$\liminf_{\varepsilon\to 0} \frac{\varepsilon^{\delta}\lambda^0(\partial Y_\varepsilon)}{1-\delta} \le \liminf_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon) \le \limsup_{\varepsilon\to 0} \varepsilon^{\delta-1}\lambda^1(Y_\varepsilon) \le \limsup_{\varepsilon\to 0} \frac{\varepsilon^{\delta}\lambda^0(\partial Y_\varepsilon)}{1-\delta}.$$

The proof is based on an interesting relationship between the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^1(F_\varepsilon)$ which exists Lebesgue almost everywhere and the quantity $\lambda^0(\partial F_\varepsilon)$ which was established in [29] for arbitrary bounded subsets of $\mathbb{R}^d$ and builds on the work of [13]. As this relationship is also of use to us, we state it in the form of [25, Corollary 2.5].

Proposition 3.14 (*Stachó*). *Let $Y \subset \mathbb{R}$ be compact. Then the function $\varepsilon \mapsto \lambda^1(Y_\varepsilon)$ is differentiable for all but a countable number of $\varepsilon > 0$ with derivative*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^1(Y_\varepsilon) = \lambda^0(\partial Y_\varepsilon).$$

For the results on the average fractal curvature measures we use [25, Lemma 4.6(ii)] which is a higher dimensional version of the next proposition.

Proposition 3.15 (*Rataj, Winter*). *Let $Y \subset \mathbb{R}$ be nonempty and compact and such that its Minkowski dimension δ exists and is strictly less than 1. If $\overline{\mathcal{M}}(Y) < \infty$, then*

$$\limsup_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2}\lambda^1(Y_\varepsilon)\,\mathrm{d}\varepsilon = (1-\delta)^{-1} \limsup_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1}\lambda^0(Y_\varepsilon)\,\mathrm{d}\varepsilon,$$
$$\liminf_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-2}\lambda^1(Y_\varepsilon)\,\mathrm{d}\varepsilon = (1-\delta)^{-1} \liminf_{T\searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1}\lambda^0(Y_\varepsilon)\,\mathrm{d}\varepsilon.$$

4. Proofs of Theorems 2.11 and 2.12

In this section we provide the proofs of Theorems 2.11 and 2.12. Since (i) to (iii) of Theorem 2.11 require different methods of proof, we are going to split this section into three subsections, each of which deals with one of these parts. But before subdividing the section, we make the following observations which are needed in the proofs of Theorem 2.11(i) and (ii), and for Theorem 2.12.

Without loss of generality we assume that $\{0,1\} \subset F \subseteq [0,1]$ as otherwise the result follows by rescaling. We start by giving the proof for the zeroth fractal curvature measure. For that we fix an $\varepsilon > 0$ and consider the expression $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])/2$ for some $b \in \mathbb{R}$. Since λ^0 is the counting measure, $\lambda^0(\partial F_\varepsilon \cap (-\infty, b])$ gives the number of endpoints of the connected components of F_ε in $(-\infty, b]$. This number can be obtained by looking at how many complementary intervals of lengths greater than 2ε exist in $(-\infty, b]$:

$$\lambda^0\big(\partial F_\varepsilon \cap (-\infty, b]\big)/2 = \underbrace{\sum_{i=1}^{L} \#\{\omega \in \Sigma^* \mid G_\omega^i \subseteq (-\infty, b],\ |G_\omega^i| > 2\varepsilon\}}_{=:\Xi(\varepsilon)} + c_1/2, \tag{4.1}$$

where $c_1 \in \{1, 2, 3\}$ depends on the value of b. Next, we need to find appropriate bounds for $\Xi(\varepsilon)$. For this, we choose an $m \in \mathbb{N} \cup \{0\}$ such that for all $\omega \in \Sigma^m$ all main gaps $G^1_\omega, \ldots, G^L_\omega$ of the sets $Q_\omega(F)$ are greater than 2ε and set

$$\Xi^i_\omega(\varepsilon) := \#\{u \in \Sigma^* \mid G^i_{u\omega} \subseteq (-\infty, b\,],\ |G^i_{u\omega}| > 2\varepsilon\}$$

for each $\omega \in \Sigma^m$ and $i \in \{1, \ldots, L\}$. We have the following connection:

$$\sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \Xi^i_\omega(\varepsilon) \le \Xi(\varepsilon) \le \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \Xi^i_\omega(\varepsilon) + \sum_{j=1}^{m} L \cdot N^{j-1}. \tag{4.2}$$

For the following, we fix $b \in \mathbb{R} \setminus F$. Then $F \cap (-\infty, b\,]$ can be expressed as a finite union of sets of the form $Q_\kappa F$, where $\kappa \in \Sigma^*$. To be more precise, let $l \in \mathbb{N}$ be minimal such that there exist $\kappa_1, \ldots, \kappa_l \in \Sigma^*$ satisfying:

(i) $F \cap (-\infty, b\,] = \bigcup_{j=1}^{l} Q_{\kappa_j} F$ and

(ii) $Q_{\kappa_i} F \cap Q_{\kappa_j} F$ contains at most one point for all $i \neq j$, where $i, j \in \{1, \ldots, l\}$.

Then for $\kappa := \bigcup_{j=1}^{l} [\kappa_j]$ the function $\mathbb{1}_\kappa$ is Hölder continuous. Making use of the existence of the bounded distortion constant $\varrho_{n(\omega)}$ of Q on $Q_\omega X$ (see Lemma 3.2), we can give estimates for $\Xi^i_\omega(\varepsilon)$, namely for an arbitrary $x \in \Sigma^\infty$ we have

$$\begin{aligned} \Xi^i_\omega(\varepsilon) &\le \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_\kappa(u\omega x) \mathbb{1}_{\{|Q'_u(Q_\omega \pi x)| \cdot \varrho_{n(\omega)} \cdot |G^i_\omega| > 2\varepsilon\}} + \overline{c}_2(x, \kappa) \\ &\le \underbrace{\sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_\kappa(u\omega x) \mathbb{1}_{\{|Q'_u(Q_\omega \pi x)| \cdot \varrho_{n(\omega)} \cdot |G^i_\omega| \ge 2\varepsilon\}}}_{=:\overline{A}^i_\omega(x,\varepsilon,\kappa)} + \overline{c}_2(x, \kappa), \end{aligned} \tag{4.3}$$

where we need to insert the constant $\overline{c}_2(x, \kappa)$ for the following reason: $G^i_{u\omega} \subseteq (-\infty, b\,]$ does not necessarily imply $u\omega x \in \kappa$ for an arbitrary $x \in \Sigma^\infty$. However, if $n(u) \ge \max_{j=1,\ldots,l} n(\kappa_j)$, either $[u\omega] \subseteq \kappa$ or $[u\omega] \cap \kappa = \varnothing$. Hence, there are only finitely many $u \in \Sigma^*$ such that $G^i_{u\omega} \subseteq (-\infty, b\,]$ does not imply $u\omega x \in \kappa$ for all $x \in \Sigma^\infty$. Letting $\overline{c}_2(x, \kappa) \in \mathbb{R}$ denote this finite number shows the validity of Eq. (4.3) for all $\varepsilon > 0$. Likewise, there exists a constant $\underline{c}_2(x, \kappa) \in \mathbb{R}$ such that for all $\varepsilon > 0$ and $\beta > 1$

$$\begin{aligned} \Xi^i_\omega(\varepsilon) &\ge \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_\kappa(u\omega x) \cdot \mathbb{1}_{\{|Q'_u(Q_\omega \pi x)| \cdot \varrho^{-1}_{n(\omega)} \cdot |G^i_\omega| > 2\varepsilon\}} - \underline{c}_2(x, \kappa) \\ &\ge \underbrace{\sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_\kappa(u\omega x) \cdot \mathbb{1}_{\{|Q'_u(Q_\omega \pi x)| \cdot \varrho^{-1}_{n(\omega)} \cdot |G^i_\omega| \ge 2\varepsilon\beta\}}}_{=:\underline{A}^i_\omega(x,\varepsilon\beta,\kappa)} - \underline{c}_2(x, \kappa). \end{aligned} \tag{4.4}$$

Combining Eqs. (4.1)–(4.4) we obtain that for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$,

$$\overline{C}_0^f(F, (-\infty, b]) \le \limsup_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \overline{A}_\omega^i(x, \varepsilon, \kappa) \quad \text{and} \tag{4.5}$$

$$\underline{C}_0^f(F, (-\infty, b]) \ge \liminf_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \underline{A}_\omega^i(x, \varepsilon\beta, \kappa) \tag{4.6}$$

for any $\beta > 1$. In order to prove Theorems 2.11 and 2.12 we want to apply Proposition 3.8 and Theorem 3.9 to get asymptotics for both the expressions $\overline{A}_\omega^i(x, \varepsilon, \kappa)$ and $\underline{A}_\omega^i(x, \varepsilon\beta, \kappa)$. For this, note that

$$\begin{aligned}
&\sum_{u \in \Sigma^n} \mathbb{1}_\kappa(u\omega x) \cdot \mathbb{1}_{\{|Q_u'(Q_\omega \pi x)| \cdot \varrho_{n(\omega)}^{\pm 1} \cdot |G_\omega^i| \ge 2\varepsilon\}} \\
&\quad = \sum_{y:\sigma^n y = \omega x} \mathbb{1}_\kappa(y) \cdot \mathbb{1}_{\left\{\sum_{k=1}^{n} -\ln |Q_{y_k}'(\pi \sigma^k y)| \le -\ln \frac{2\varepsilon}{|G_\omega^i| \varrho_{n(\omega)}^{\pm 1}}\right\}} \\
&\quad = \sum_{y:\sigma^n y = \omega x} \mathbb{1}_\kappa(y) \cdot \mathbb{1}_{\left\{S_n \xi(y) \le -\ln \frac{2\varepsilon}{|G_\omega^i| \varrho_{n(\omega)}^{\pm 1}}\right\}}.
\end{aligned} \tag{4.7}$$

The hypotheses and Remark 3.5 imply that the geometric potential function ξ is Hölder continuous and strictly positive. The unique $s > 0$ for which $\gamma_{-s\xi} = 1$ is precisely the Minkowski dimension δ of F, which results from combining the fact that $\gamma_{-s\xi} = \exp(P(-s\xi))$ for each $s > 0$ and Theorem 3.6.

Before we distinguish between the lattice and nonlattice cases and give the proof of Theorem 2.11, we prove the following lemma, which is needed in the proofs of all three parts of Theorem 2.11.

Lemma 4.1. *For an arbitrary $x \in \Sigma^\infty$ and $\Upsilon \in \mathbb{R}$ we have that:*

(i) $\Upsilon \le \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} h_{-\delta\xi}(\omega x) \big(|G_\omega^i| \varrho_m\big)^\delta$ *for all $m \in \mathbb{N}$ implies*

$$\Upsilon \le \liminf_{m \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta.$$

(ii) $\Upsilon \ge \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} h_{-\delta\xi}(\omega x) \big(|G_\omega^i| \varrho_m^{-1}\big)^\delta$ *for all $m \in \mathbb{N}$ implies*

$$\Upsilon \ge \limsup_{m \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta.$$

Proof. We are first going to approximate the eigenfunction $h_{-\delta\xi}$ of the Perron–Frobenius operator $\mathcal{L}_{-\delta\xi}$. For that we claim that $\mathcal{L}_{-\delta\xi}^n 1(x) = \sum_{u \in \Sigma^n} |Q_u'(\pi x)|^\delta$ for each $x \in \Sigma^\infty$ and $n \in \mathbb{N}$, where 1 is the constant 1-function. This can be easily seen by induction. Since $\mathcal{L}_{-\delta\xi}^n 1$ converges uniformly to the eigenfunction $h_{-\delta\xi}$ on taking $n \to \infty$ (see Eq. (3.2)), we have that

$$\forall t > 0\, \exists M \in \mathbb{N}\colon \forall n \ge M,\ \forall x \in \Sigma^\infty\colon \left| \sum_{u \in \Sigma^n} |Q_u'(\pi x)|^\delta - h_{-\delta\xi}(x) \right| < t.$$

Furthermore, through Lemma 3.2 we know that

$$\forall t' > 0\,\exists M' \in \mathbb{N}\colon \forall m \geq M'\colon |\varrho_m - 1| < t'.$$

Thus, for all $n \geq M$ and $m \geq M'$,

$$\begin{aligned}
\Upsilon &\leq \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} h_{-\delta\xi}(\omega x)\left(|G_\omega^i|\varrho_m\right)^\delta \\
&\leq \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \left(\sum_{u\in\Sigma^n} |Q_u'(Q_\omega \pi x)|^\delta + t\right)\left(|G_\omega^i|\varrho_m\right)^\delta \\
&\leq \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} |Q_u(G_\omega^i)|^\delta \varrho_m^{2\delta} + t\sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \left(|G_\omega^i|\varrho_m\right)^\delta \\
&\leq \left(1+t'\right)^{2\delta} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} |G_{u\omega}^i|^\delta + t(1+t')^\delta \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta =: A_{m,n}
\end{aligned}$$

Hence, for all $t, t' > 0$,

$$\begin{aligned}
\Upsilon &\leq \liminf_{m\to\infty}\liminf_{n\to\infty} A_{m,n} \\
&\leq \left(1+t'\right)^{2\delta} \liminf_{m\to\infty}\liminf_{n\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} |G_{u\omega}^i|^\delta \\
&\quad + t(1+t')^\delta \limsup_{m\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta.
\end{aligned}$$

Because we have $\sum_{i=1}^{L}\sum_{\omega\in\Sigma^m}|G_\omega^i|^\delta \leq \sum_{i=1}^{L}\sum_{\omega\in\Sigma^m}\|Q_\omega'\|^\delta =: a_m$, where $\|\cdot\|$ denotes the supremum-norm on $\mathcal{C}(X)$, and the sequence $(a_m)_{m\in\mathbb{N}}$ is bounded by [23, Lemma 4.2.12], letting t and t' tend to zero then gives the assertion.

The same arguments can be used to show that $\limsup_{m\to\infty}\sum_{i=1}^{L}\sum_{\omega\in\Sigma^m}|G_\omega^i|^\delta$ is a lower bound in the second case. □

4.1. The nonlattice case

Proof of Theorem 2.11(ii). In this proof we fix the notation from the beginning of Section 4.

If $\mathbb{1}_\kappa$ is identically zero, we immediately obtain $C_0^f(F, (-\infty, b]) = 0 = \nu(F \cap (-\infty, b])$. Therefore, in the following, we assume that $\mathbb{1}_\kappa$ is not identically zero. Since $\mathbb{1}_\kappa$ is Hölder continuous, on combining Eqs. (4.3), (4.4) and (4.7), we see that Proposition 3.8 can be applied to $\overline{A}_\omega^i(x, \varepsilon, \kappa)$ and $\underline{A}_\omega^i(x, \varepsilon\beta, \kappa)$ giving the following asymptotics:

$$\overline{A}_\omega^i(x, \varepsilon, \kappa) \sim \frac{\nu_{-\delta\xi}(\kappa)}{\delta\int \xi\, \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x)\cdot(2\varepsilon)^{-\delta}\left(|G_\omega^i|\varrho_{n(\omega)}\right)^\delta \quad \text{and} \tag{4.8}$$

$$\underline{A}_\omega^i(x, \varepsilon\beta, \kappa) \sim \frac{\nu_{-\delta\xi}(\kappa)}{\delta\int \xi\, \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x)\cdot(2\varepsilon\beta)^{-\delta}\left(|G_\omega^i|\varrho_{n(\omega)}^{-1}\right)^\delta \tag{4.9}$$

as $\varepsilon \to 0$ uniformly for $x \in \Sigma^\infty$. We first put our focus on finding an upper bound for $\overline{C}_0^f(F, (-\infty, b])$. As in the statement of this theorem, set $H(\mu_{-\delta\xi}) := \delta\int\xi\,\mathrm{d}\mu_{-\delta\xi}$. Combining

Eqs. (4.5) and (4.8), we obtain for $x \in \Sigma^\infty$ and all $m \in \mathbb{N}$,

$$\overline{C}_0^f(F,(-\infty,b]) \le \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} h_{-\delta\xi}(\omega x)\left(|G_\omega^i|\varrho_m\right)^\delta \nu_{-\delta\xi}(\kappa).$$

Now an application of Lemma 4.1 implies

$$\overline{C}_0^f(F,(-\infty,b]) \le \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \liminf_{m\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta \nu_{-\delta\xi}(\kappa). \tag{4.10}$$

Analogously, one can conclude that for all $\beta > 1$ we have

$$\underline{C}_0^f(F,(-\infty,b]) \ge \frac{(2\beta)^{-\delta}}{H(\mu_{-\delta\xi})} \limsup_{m\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta \nu_{-\delta\xi}(\kappa).$$

Thus,

$$\underline{C}_0^f(F,(-\infty,b]) \ge \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \limsup_{m\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta \nu_{-\delta\xi}(\kappa). \tag{4.11}$$

Combining the inequalities (4.10) and (4.11) yields that all the limits occurring therein exist and are equal. Moreover, the δ-conformal measure introduced in Eq. (2.1) and $\nu_{-\delta\xi}$ satisfy the relation $\nu_{-\delta\xi}(\mathbb{1}_\kappa) = \nu((-\infty,b])$. Therefore,

$$C_0^f(F,(-\infty,b]) = \frac{2^{-\delta}}{H(\mu_{-\delta\xi})} \lim_{n\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^n} |G_\omega^i|^\delta \cdot \nu(F\cap(-\infty,b])$$

holds for every $b \in \mathbb{R}\setminus F$. As $\mathbb{R}\setminus F$ is dense in $\mathbb{R}$, the assertion concerning the zeroth fractal curvature measure follows. The result on the first fractal curvature measure now follows on applying Theorem 3.13, as for every $b \in \mathbb{R}\setminus F$ we have that $F_\varepsilon \cap (-\infty,b] = (F\cap(-\infty,b])_\varepsilon$ for sufficiently small $\varepsilon > 0$. □

Proof of Theorem 2.12(ii). This part is an immediate consequence of Theorem 2.11(ii). □

4.2. The lattice case

This subsection addresses part (iii) of Theorem 2.11 and of Theorem 2.12.

Proof of Theorem 2.11(iii). The statement concerning the nonexistence of the fractal curvature measures under the condition that Q consists of analytic maps follows from Theorem 2.17(iii) together with Theorem 2.22. Both of these theorems will be proven in Section 5. Thus, we now concentrate on the first part of Theorem 2.11(iii), namely the boundedness of $\overline{C}_k^f(F,\mathbb{R})$ and positivity of $\underline{C}_k^f(F,\mathbb{R})$. Since ξ is a lattice function, there exist $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ such that

$$\xi - \zeta = \psi - \psi\circ\sigma$$

and such that ζ is a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be the maximal real number such that $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. Recall from the beginning of Section 4 that the hypotheses and Remark 3.5 imply that ξ is Hölder continuous and strictly positive and that the unique $s > 0$ for which $\gamma_{-s\xi} = 1$ is the Minkowski dimension δ of F.

Fix the notation from the beginning of Section 4. Since $\mathbb{1}_\kappa$ is Hölder continuous and since we can assume that $\mathbb{1}_\kappa$ is not identically zero, on combining Eqs. (4.3), (4.4) and (4.7), we see that an application of Theorem 3.9 to $\overline{A}_\omega^i(x, \varepsilon, \kappa)$ and $\underline{A}_\omega^i(x, \varepsilon\beta, \kappa)$ for $\beta > 1$ gives the following asymptotics:

$$\overline{A}_\omega^i(x, \varepsilon, \kappa) \sim W_\omega(x) \int_\kappa \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|G_\omega^i|\varrho_{n(\omega)}} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) \quad \text{and} \tag{4.12}$$

$$\underline{A}_\omega^i(x, \varepsilon\beta, \kappa) \sim W_\omega(x) \int_\kappa \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon\beta\varrho_{n(\omega)}}{|G_\omega^i|} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) \tag{4.13}$$

as $\varepsilon \to 0$ uniformly for $x \in \Sigma^\infty$, where

$$W_\omega(x) := \frac{a h_{-\delta\zeta}(\omega x)}{(1 - \mathrm{e}^{-\delta a}) \int \zeta \, \mathrm{d}\mu_{-\delta\zeta}}. \tag{4.14}$$

For the boundedness we first remark that $\overline{C}_0^f(F, \cdot)$ is monotonically increasing as a set function in the second component. Therefore, in order to find an upper bound for $\overline{C}_0^f(F, \cdot)$ it suffices to consider $\overline{C}_0^f(F, \mathbb{R})$. For all $m \in \mathbb{N}$ we have

$$\begin{aligned}
\overline{C}_0^f(F, \mathbb{R}) &\overset{(4.5)}{\leq} \limsup_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^L \sum_{\omega \in \Sigma^m} \overline{A}_\omega^i(x, \varepsilon, \Sigma^\infty) \\
&\overset{(4.12)}{=} \limsup_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^L \sum_{\omega \in \Sigma^m} W_\omega(x) \int \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|G_\omega^i|\varrho_m} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y) \\
&\leq \limsup_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^L \sum_{\omega \in \Sigma^m} W_\omega(x) \int \mathrm{e}^{-\delta a \left(\frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|G_\omega^i|\varrho_m} \right)} \mathrm{d}\nu_{-\delta\zeta}(y) \\
&= \sum_{i=1}^L \sum_{\omega \in \Sigma^m} \frac{a \mathrm{e}^{\delta\psi(\omega x)} h_{-\delta\zeta}(\omega x)}{(1 - \mathrm{e}^{-\delta a}) \int \zeta \, \mathrm{d}\mu_{-\delta\zeta}} \left(\frac{|G_\omega^i|\varrho_m}{2} \right)^\delta \int \mathrm{e}^{-\delta\psi(y)} \, \mathrm{d}\nu_{-\delta\zeta}(y).
\end{aligned}$$

Note that $h_{-\delta\xi} = \mathrm{e}^{\delta\psi} h_{-\delta\zeta}$ and $\mathrm{d}\nu_{-\delta\xi} = \mathrm{e}^{-\delta\psi} \mathrm{d}\nu_{-\delta\zeta}$. Hence, by Lemma 4.1,

$$\overline{C}_0^f(F, \mathbb{R}) \leq \liminf_{m \to \infty} \sum_{i=1}^L \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta \frac{a 2^{-\delta}}{(1 - \mathrm{e}^{-\delta a}) \int \zeta \, \mathrm{d}\mu_{-\delta\zeta}} =: c_0.$$

$c_0 \in (0, \infty)$ because $\sum_{i=1}^L \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta \leq \sum_{i=1}^L \sum_{\omega \in \Sigma^m} \|Q'_\omega\|^\delta =: a_m$, where $\|\cdot\|$ denotes the supremum-norm on $\mathcal{C}(X)$ and the sequence $(a_m)_{m \in \mathbb{N}}$ is bounded by [23, Lemma 4.2.12].

That $\underline{C}_0^f(F, \mathbb{R})$ is positive can be seen by the following, which holds for all $\beta > 1$:

$$\begin{aligned}\underline{C}_0^f(F, \mathbb{R}) &\geq \liminf_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \underline{A}_\omega^i(x, \varepsilon\beta, \Sigma^\infty) \\ &\overset{(4.13)}{\geq} \liminf_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} W_\omega(x) \int \mathrm{e}^{-\delta a \left(\frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon\beta\varrho_m}{|G_\omega^i|} + 1 \right)} \,\mathrm{d}\nu_{-\delta\zeta}(y) \\ &= \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \frac{a h_{-\delta\zeta}(\omega x)}{(1-\mathrm{e}^{-\delta a}) \int \zeta \,\mathrm{d}\mu_{-\delta\zeta}} \mathrm{e}^{\delta\psi(\omega x) - \delta a} \left(\frac{|G_\omega^i|}{2\beta\varrho_m} \right)^\delta \\ &\quad \times \int \mathrm{e}^{-\delta\psi(y)} \,\mathrm{d}\nu_{-\delta\zeta}(y).\end{aligned}$$

By using $h_{-\delta\xi} = \mathrm{e}^{\delta\psi} h_{-\delta\zeta}$ and $\mathrm{d}\nu_{-\delta\xi} = \mathrm{e}^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta}$ and Lemma 4.1, we hence obtain

$$\underline{C}_0^f(F, \mathbb{R}) \geq \limsup_{m \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta \frac{a(2\beta)^{-\delta}\mathrm{e}^{-\delta a}}{(1-\mathrm{e}^{-\delta a}) \int \zeta \,\mathrm{d}\mu_{-\delta\zeta}} > 0$$

for all $\beta > 1$.

The results on $\underline{C}_1^f(F, B)$ and $\overline{C}_1^f(F, B)$ are now a straightforward application of Theorem 3.13. □

Proof of Theorem 2.12(iii). We fix the notation and use the observations from the beginning of Section 4. Recall from Eqs. (4.5) and (4.6) that for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$ we have that

$$\overline{C}_0^f(F, \mathbb{R}) \leq \limsup_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \overline{A}_\omega^i(x, \varepsilon, \Sigma^\infty) \quad \text{and}$$

$$\underline{C}_0^f(F, \mathbb{R}) \geq \liminf_{\varepsilon \to 0} \varepsilon^\delta \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \underline{A}_\omega^i(x, \varepsilon\beta, \Sigma^\infty)$$

for any $\beta > 1$. Theorem 3.9 implies that

$$\overline{A}_\omega^i(x, \varepsilon, \Sigma^\infty) \sim W_w(x) \int \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon}{|G_\omega^i| \varrho_{n(\omega)}} \right\rceil} \,\mathrm{d}\nu_{-\delta\zeta}(y)$$

$$\underline{A}_\omega^i(x, \varepsilon\beta, \Sigma^\infty) \sim W_w(x) \int \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a} \ln \frac{2\varepsilon\beta\varrho_{n(\omega)}}{|G_\omega^i|} \right\rceil} \,\mathrm{d}\nu_{-\delta\zeta}(y),$$

where

$$W_\omega(x) := \frac{a h_{-\delta\zeta}(\omega x)}{(1-\mathrm{e}^{-\delta a}) \int \zeta \,\mathrm{d}\mu_{-\delta\zeta}}.$$

Now, we use that the hypotheses of Theorem 2.12(iii) and Lemma 3.12 together imply that

$$\overline{A} := \lim_{\varepsilon\to 0} \varepsilon^{\delta} \int \mathrm{e}^{-\delta a\left[\frac{\psi(y)-\psi(\omega x)}{a}+\frac{1}{a}\ln\frac{2\varepsilon}{|G^i_\omega|\varrho_m}\right]}\,\mathrm{d}\nu_{-\delta\zeta}(y)\cdot\left(\frac{2}{|G^i_\omega|\varrho_m}\right)^{\delta} \quad \text{and}$$

$$\underline{A} := \lim_{\varepsilon\to 0} \varepsilon^{\delta} \int \mathrm{e}^{-\delta a\left[\frac{\psi(y)-\psi(\omega x)}{a}+\frac{1}{a}\ln\frac{2\varepsilon\varrho_m}{|G^i_\omega|}\right]}\,\mathrm{d}\nu_{-\delta\zeta}(y)\cdot\left(\frac{2\varrho_m}{|G^i_\omega|}\right)^{\delta}$$

exist for every $\omega \in \Sigma^m$ and $i \in \{1,\dots,L\}$, are independent of ω and i, and are equal, that is $\overline{A} = \underline{A} =: A$. Combining the above equations we conclude that

$$\overline{C}^f_0(F,\mathbb{R}) \le \sum_{i=1}^{L}\sum_{\omega\in\Sigma^m} W_\omega(x)\left(\frac{|G^i_\omega|\varrho_m}{2}\right)^{\delta}\cdot A \quad \text{and}$$

$$\underline{C}^f_0(F,\mathbb{R}) \ge \sum_{i=1}^{L}\sum_{\omega\in\Sigma^m} W_\omega(x)\left(\frac{|G^i_\omega|}{2\beta\varrho_m}\right)^{\delta}\cdot A$$

for all $\beta > 1$. Thus,

$$\underline{C}^f_0(F,\mathbb{R}) \ge \sum_{i=1}^{L}\sum_{\omega\in\Sigma^m} W_\omega(x)\left(\frac{|G^i_\omega|}{2\varrho_m}\right)^{\delta}\cdot A.$$

Applying Lemma 4.1 we obtain that $0 < \overline{C}^f_0(F,\mathbb{R}) = \underline{C}^f_0(F,\mathbb{R}) < \infty$. An application of Theorem 3.13 then completes the proof. □

4.3. Average fractal curvature measures

Proof of Theorem 2.11(i). If ξ is nonlattice, Theorem 2.11(i) immediately follows from Theorem 2.11(ii), and the fact that $f(\varepsilon) \sim c$ as $\varepsilon \to 0$ for some constant $c \in \mathbb{R}$ implies $\lim_{T\searrow 0}|\ln T|^{-1}\int_T^1 \varepsilon^{-1} f(\varepsilon)\,\mathrm{d}\varepsilon = c$ for every locally integrable function $f\colon (0,\infty)\to\mathbb{R}$.

Thus for the rest of the proof we assume that ξ is lattice and fix the notation from the beginning of Section 4. In particular, recall that $b \in \mathbb{R}\setminus F$. We begin with showing the result on the zeroth average fractal curvature measure.

Observe that $\lim_{T\searrow 0}|\ln T|^{-1}\int_T^1 c\varepsilon^{\delta-1}\,\mathrm{d}\varepsilon = \lim_{T\to\infty}|T|^{-1}\int_0^T c\mathrm{e}^{-\delta t}\,\mathrm{d}t = 0$ for every constant $c \in \mathbb{R}$. For a fixed $m \in \mathbb{N}$ define $M := \min\{|G^i_\omega| \mid i \in \{1,\dots,L\}, \omega \in \Sigma^m\}/2$. From Eqs. (4.1)–(4.3) we deduce the following for $x \in \Sigma^\infty$:

$$\begin{aligned}\overline{D} &:= \limsup_{T\searrow 0} |2\ln T|^{-1}\int_T^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon \cap (-\infty,b])\,\mathrm{d}\varepsilon \\ &\le \limsup_{T\searrow 0} |\ln T|^{-1}\left(\int_T^M \varepsilon^{\delta-1}\sum_{i=1}^{L}\sum_{\omega\in\Sigma^m}\overline{A}^i_\omega(x,\varepsilon,\kappa)\,\mathrm{d}\varepsilon \right.\\ &\quad \left. +\frac{1}{2}\int_M^1 \varepsilon^{\delta-1}\lambda^0(\partial F_\varepsilon\cap(-\infty,b])\,\mathrm{d}\varepsilon\right).\end{aligned}$$

Local integrability of the integrands implies that we have the following equation for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$:

$$\begin{aligned}
\overline{D} \leq\; & \limsup_{T \searrow 0} |\ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \overline{A}_\omega^i(x, \varepsilon, \kappa)\, \mathrm{d}\varepsilon \\
=\; & \limsup_{T \to \infty} T^{-1} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \int_0^T \mathrm{e}^{-\delta t} \overline{A}_\omega^i(x, \mathrm{e}^{-t}, \kappa)\, \mathrm{d}t \\
\overset{(4.7)}{=}\; & \limsup_{T \to \infty} T^{-1} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \int_0^T \mathrm{e}^{-\delta t} \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = \omega x} \mathbb{1}_\kappa(y) \cdot \mathbb{1}_{\left\{S_n \xi(y) \leq t - \ln \frac{2}{|G_\omega^i| \varrho_m}\right\}}\, \mathrm{d}t \\
\leq\; & \limsup_{T \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \left(\frac{|G_\omega^i| \varrho_m}{2} \right)^\delta \frac{T - \ln \frac{2}{|G_\omega^i| \varrho_m}}{T} \\
& \times \left(T - \ln \frac{2}{|G_\omega^i| \varrho_m} \right)^{-1} \int_0^{T - \ln \frac{2}{|G_\omega^i| \varrho_m}} \mathrm{e}^{-\delta t} \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = \omega x} \mathbb{1}_\kappa(y) \cdot \mathbb{1}_{\{S_n \xi(y) \leq t\}}\, \mathrm{d}t \\
=\; & \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \left(\frac{|G_\omega^i| \varrho_m}{2} \right)^\delta \frac{h_{-\delta\xi}(\omega x) \nu_{-\delta\xi}(\kappa)}{\delta \int \xi \,\mathrm{d}\mu_{-\delta\xi}}.
\end{aligned} \tag{4.15}$$

The last equality is an application of Corollary 3.11. Because Eq. (4.15) holds for all $m \in \mathbb{N}$, applying Lemma 4.1 yields

$$\begin{aligned}
& \limsup_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b])\, \mathrm{d}\varepsilon \\
& \leq \frac{2^{-\delta} \nu_{-\delta\xi}(\kappa)}{\delta \int \xi \,\mathrm{d}\mu_{-\delta\xi}} \liminf_{m \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta.
\end{aligned} \tag{4.16}$$

Analogous estimates give

$$\begin{aligned}
& \liminf_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b])\, \mathrm{d}\varepsilon \\
& \geq \frac{(2\beta)^{-\delta} \nu_{-\delta\xi}(\kappa)}{\delta \int \xi \,\mathrm{d}\mu_{-\delta\xi}} \limsup_{m \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta
\end{aligned}$$

for all $\beta > 1$ and hence

$$\begin{aligned}
& \liminf_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b])\, \mathrm{d}\varepsilon \\
& \geq \frac{2^{-\delta} \nu_{-\delta\xi}(\kappa)}{\delta \int \xi \,\mathrm{d}\mu_{-\delta\xi}} \limsup_{m \to \infty} \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} |G_\omega^i|^\delta.
\end{aligned} \tag{4.17}$$

Eqs. (4.16) and (4.17) together imply that for every $b \in \mathbb{R} \setminus F$,

$$\lim_{T \searrow 0} |2 \ln T|^{-1} \int_T^1 \varepsilon^{\delta-1} \lambda^0(\partial F_\varepsilon \cap (-\infty, b])\, \mathrm{d}\varepsilon = \frac{2^{-\delta} c}{H(\mu_{-\delta\xi})} \nu(F \cap (-\infty, b]),$$

where the constant $c := \lim_{m\to\infty} \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta$ is well-defined. Since $\mathbb{R} \setminus F$ is dense in $\mathbb{R}$, the statement on the zeroth average fractal curvature measure in Theorem 2.11(i) follows.

For the statement on the first average fractal curvature measure, we use Theorem 2.11(iii) which says that $\overline{C}_0^f(F, (-\infty, b]) < \infty$ for every $b \in \mathbb{R} \setminus F$. Applying Theorem 3.13 hence yields that $\overline{\mathcal{M}}(F \cap (-\infty, b]) < \infty$ for every $b \in \mathbb{R} \setminus F$. Since for every $b \in \mathbb{R} \setminus F$ we have that $(F \cap (-\infty, b])_\varepsilon = F_\varepsilon \cap (-\infty, b]$ for sufficiently small $\varepsilon > 0$, we can apply Proposition 3.15 to $F \cap (-\infty, b]$ and obtain the desired statement. □

Proof of Theorem 2.12(i). This is an immediate consequence of Theorem 2.11(i). □

5. Proofs concerning the special cases

In order to show nonexistence of the fractal curvature measures in Theorem 2.14(ii) and Theorem 2.17(iii), we make use of the following lemma.

Lemma 5.1. *Let F denote a self-conformal set associated with the cIFS $Q := \{Q_1, \dots, Q_N\}$. Let $\delta := \dim_M(F)$ denote the Minkowski dimension of F and let $B \subseteq \mathbb{R}$ denote a Borel set for which $F_\varepsilon \cap B = (F \cap B)_\varepsilon$ for all sufficiently small $\varepsilon > 0$. Assume that there exists a positive, bounded, periodic and Borel-measurable function $f: \mathbb{R}^+ \to \mathbb{R}^+$ which has the following properties.*

(i) *f is not equal to an almost everywhere constant function.*
(ii) *There exist sequences $(a_m)_{m\in\mathbb{N}}$ and $(c_m)_{m\in\mathbb{N}}$ where $a_m, c_m > 0$ for all $m \in \mathbb{N}$ and $a_m \to 1$ as $m \to \infty$ such that the following property is satisfied. For all $t > 0$ and $m \in \mathbb{N}$ there exists an $M \in \mathbb{N}$ such that for all $T \geq M$,*

$$\begin{aligned}&(1-t)a_m^{-\delta} f(T - \ln a_m) - c_m \mathrm{e}^{-\delta T}\\ &\quad\leq \mathrm{e}^{-\delta T} \lambda^0(\partial F_{\mathrm{e}^{-T}} \cap B) \leq (1+t)a_m^{\delta} f(T + \ln a_m) + c_m \mathrm{e}^{-\delta T}.\end{aligned} \tag{5.1}$$

Then for $k \in \{0, 1\}$ we have that

$$\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B).$$

Proof. We first cover the case $k = 0$. Since f is positive and not equal to an almost everywhere constant function, there exist $\widetilde{T}_1, \widetilde{T}_2 > 0$ such that $D := f(\widetilde{T}_2)/f(\widetilde{T}_1) > 1$. Choose $m \in \mathbb{N}$ such that $a_m^{2\delta} < \sqrt{D}$ and choose $t > 0$ such that $(1+t)/(1-t) < \sqrt{D}$. Then $\widetilde{D} := (1-t)a_m^{-\delta} f(\widetilde{T}_2) - (1+t)a_m^{\delta} f(\widetilde{T}_1) > 0$. By (ii) we can find an $M \in \mathbb{N}$ for these t and m such that Eq. (5.1) is satisfied for all $T \geq M$. Because of the periodicity of f we can find $T_1, T_2 \geq M$ such that $f(\widetilde{T}_1) = f(T_1 + \ln a_m)$ and $f(\widetilde{T}_2) = f(T_2 - \ln a_m)$. Moreover, we can assume that T_1, T_2 are so large that $c_m \mathrm{e}^{-\delta T_1} + c_m \mathrm{e}^{-\delta T_2} \leq \widetilde{D}/2$. Then

$$\begin{aligned}\mathrm{e}^{-\delta T_1} \lambda^0(\partial F_{\mathrm{e}^{-T_1}} \cap B) &\leq (1+t)a_m^{\delta} f(T_1 + \ln a_m) + c_m \mathrm{e}^{-\delta T_1}\\ &\leq (1-t)a_m^{-\delta} f(T_2 - \ln a_m) - \widetilde{D}/2 - c_m \mathrm{e}^{-\delta T_2}\\ &< \mathrm{e}^{-\delta T_2} \lambda^0(\partial F_{\mathrm{e}^{-T_2}} \cap B).\end{aligned}$$

Because of the periodicity of f this proves the case $k = 0$. For $k = 1$, observe that the function $g: \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g(T) := \int_0^\infty f(s+T)\mathrm{e}^{(\delta-1)s}\,\mathrm{d}s$$

is periodic. Also, g is not a constant function, since if it was, then we would have $0 = g(0) - g(T)$ for all $T \geq 0$, and this would imply $\int_T^\infty f(s)\mathrm{e}^{(\delta-1)s}\,\mathrm{d}s = \mathrm{e}^{(\delta-1)T}\int_0^\infty f(s)\mathrm{e}^{(\delta-1)s}\,\mathrm{d}s$ for all $T \geq 0$; differentiating with respect to T would then imply that f itself is constant almost everywhere, which is a contradiction. Using that $F_\varepsilon \cap B = (F \cap B)_\varepsilon$ for sufficiently small $\varepsilon > 0$ and Stachó's Theorem (Proposition 3.14), we obtain for sufficiently large $T \geq 0$,

$$\begin{aligned}\mathrm{e}^{-T(\delta-1)}\lambda^1(F_{\mathrm{e}^{-T}} \cap B) &= \mathrm{e}^{-T(\delta-1)}\int_T^\infty \lambda^0(\partial F_{\mathrm{e}^{-s}} \cap B)\mathrm{e}^{-s}\,\mathrm{d}s \\ &\leq \mathrm{e}^{-T(\delta-1)}(1+t)a_m^\delta \int_T^\infty f(s+\ln a_m)\mathrm{e}^{s(\delta-1)}\,\mathrm{d}s + c_m\mathrm{e}^{-T\delta} \\ &= (1+t)a_m^\delta g(T+\ln a_m) + c_m\mathrm{e}^{-\delta T}.\end{aligned}$$

Analogously, we obtain

$$\mathrm{e}^{-T(\delta-1)}\lambda^1(F_{\mathrm{e}^{-T}} \cap B) \geq (1-t)a_m^{-\delta} g(T-\ln a_m) - c_m\mathrm{e}^{-\delta T}.$$

Therefore, the same arguments as were used in the proof of the case $k = 0$ imply that

$$\liminf_{\varepsilon\to 0}\varepsilon^{\delta-1}\lambda^1(F_\varepsilon \cap B) < \limsup_{\varepsilon\to 0}\varepsilon^{\delta-1}\lambda^1(F_\varepsilon \cap B). \quad \square$$

5.1. Self-similar sets; proof of Theorem 2.14

Self-similar sets satisfying the open set condition form a special class of self-conformal sets, namely those which are generated by an iterated function system R consisting of similarities $R_1, \ldots, R_N$. We let $r_1, \ldots, r_N$ denote the respective similarity ratios of $R_1, \ldots, R_N$ and set $r_\omega := r_{\omega_1} \cdots r_{\omega_n}$ for a finite word $\omega = \omega_1 \cdots \omega_n \in \Sigma^n$. Further, E denotes the self-similar set associated with the sIFS R and ξ the associated geometric potential function. When considering self-similar sets, some of the formulae simplify significantly:

(A) The geometric potential function is constant on the 1-cylinders, meaning that $\xi(\omega) = -\ln r_{\omega_1}$ for every $\omega \in [\omega_1]$.
(B) The unique σ-invariant Gibbs measure $\mu_{-\delta\xi}$ for the potential function $-\delta\xi$ coincides with the δ-dimensional normalised Hausdorff measure on E. Thus, $\mu_{-\delta\xi}([i]) = r_i^\delta$, where $[i]$ will denote the cylinder of $i \in \Sigma$. Therefore we have that $H(\mu_{-\delta\xi}) = -\delta\sum_{i\in\Sigma}\ln(r_i)r_i^\delta$.
(C) The lengths of the main gaps G_ω^i of $R_\omega E$ are just multiples of the lengths of the primary gaps G^i of E, that is $|G_\omega^i| = r_\omega|G^i|$ for each $i \in \{1, \ldots, L\}$ and $\omega \in \Sigma^*$.
(D) By the Moran–Hutchinson formula (see for instance [6, Theorem 9.3]) we have that $\sum_{\omega\in\Sigma^n} r_\omega^\delta = 1$ for each $n \in \mathbb{N}$.

Proof of Theorem 2.14. Combining (A)–(D) with Theorem 2.11, we obtain Theorem 2.14(i).

As $E \cap B$ has a representation as a nonempty finite union of sets of the form $R_\omega E$ with $\omega \in \Sigma^* \setminus \{\varnothing\}$, there is a set $\kappa \subseteq \Sigma^\infty$ which is a finite union of cylinder sets and which satisfies $\pi\kappa = E \cap B$. For this κ, $\mathbb{1}_\kappa$ is Hölder continuous. Furthermore, the range of the geometric potential function of a lattice self-similar set itself is contained in a discrete subgroup of $\mathbb{R}$. Thus, ψ is a constant function and $\zeta = \xi$. Moreover, $\varrho_m = 1$ for all $m \in \mathbb{N}$ and one easily verifies that $h_{-\delta\xi} \equiv 1$. For these reasons the methods at the beginning of Section 4 simplify in the following way.

Let $T \geq 0$ be sufficiently large that $E_{\mathrm{e}^{-T}} \cap B = (E \cap B)_{\mathrm{e}^{-T}}$ and let $x \in \Sigma^\infty$ be arbitrary. Then there exists a constant $c \geq 0$, which depends on the number of sets $R_\omega E$ whose union is $E \cap B$, such that

$$\begin{aligned}
\lambda^0\big(\partial E_{\mathrm{e}^{-T}} \cap B\big)/2 &\overset{(4.1)}{=} \sum_{i=1}^{L} \#\{\omega \in \Sigma^* \mid G_\omega^i \subseteq B,\ |G_\omega^i| > 2\mathrm{e}^{-T}\} + c_1/2 \\
&= \sum_{i=1}^{L}\sum_{n=0}^{\infty}\sum_{\omega\in\Sigma^n} \mathbb{1}_\kappa(\pi\omega x)\mathbb{1}_{\{|R'_\omega(\pi x)|\cdot|G^i|>2\mathrm{e}^{-T}\}} + c_1/2 \\
&\leq \sum_{i=1}^{L}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \mathbb{1}_\kappa(y)\mathbb{1}_{\left\{S_n\xi(y)\leq -\ln\frac{2\mathrm{e}^{-T}}{|G^i|}\right\}} + c_1/2 \\
&\sim \sum_{i=1}^{L} \frac{a\nu_{-\delta\xi}(\kappa)}{(1-\mathrm{e}^{-\delta a})\int \xi\,\mathrm{d}\mu_{-\delta\xi}} \cdot \mathrm{e}^{-\delta a\left\lceil \ln\frac{2\mathrm{e}^{-T}}{|G^i|}\right\rceil} + c_1/2 \qquad (5.2)
\end{aligned}$$

as $T \to \infty$, where the last asymptotic is obtained by applying Theorem 3.9. Likewise,

$$\begin{aligned}
\lambda^0\big(\partial E_{\mathrm{e}^{-T}} \cap B\big)/2 &\geq \sum_{i=1}^{L}\sum_{n=0}^{\infty}\sum_{y:\sigma^n y=x} \mathbb{1}_\kappa(y)\mathbb{1}_{\left\{S_n\xi(y)\leq -\ln\frac{2\mathrm{e}^{-T}\beta}{|G^i|}\right\}} + c_1/2 \\
&\sim \sum_{i=1}^{L} \frac{a\nu_{-\delta\xi}(\kappa)}{(1-\mathrm{e}^{-\delta a})\int \xi\,\mathrm{d}\mu_{-\delta\xi}} \cdot \mathrm{e}^{-\delta a\left\lceil \ln\frac{2\mathrm{e}^{-T}\beta}{|G^i|}\right\rceil} + c_1/2 \qquad (5.3)
\end{aligned}$$

as $T \to \infty$.

We introduce the function $f: \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$f(T) := \mathrm{e}^{-\delta T}\frac{a\nu(B)}{(1-\mathrm{e}^{-\delta a})H(\mu_{-\delta\xi})}\sum_{i=1}^{L}\mathrm{e}^{-\delta a\left\lceil \frac{1}{a}\ln\frac{2\mathrm{e}^{-T}}{|G^i|}\right\rceil},$$

where ν denotes the δ-conformal measure associated with R. By the asymptotics given in Eqs. (5.2) and (5.3), we know that for all $t > 0$ there exists an $M \in \mathbb{N}$ such that for all $T \geq M$ we have

$$(1-t)\beta^{-\delta}f(T-\ln\beta) \leq \mathrm{e}^{-\delta T}\lambda^0(\partial E_{\mathrm{e}^{-T}} \cap B)/2 \leq (1+t)f(T) + c\mathrm{e}^{-\delta T}$$

for all $\beta > 1$ and thus

$$(1-t)f(T) \leq \mathrm{e}^{-\delta T}\lambda^0(\partial E_{\mathrm{e}^{-T}} \cap B)/2 \leq (1+t)f(T) + c\mathrm{e}^{-\delta T}.$$

Clearly, f is a periodic function with period a. Moreover, f is piecewise continuous with a finite number of discontinuities in an interval of length a. Additionally, on every interval where f is continuous, f is strictly decreasing. Therefore f is not equal to an almost everywhere constant function. Thus, all conditions of Lemma 5.1 are satisfied, which finishes the proof. □

5.2. $\mathcal{C}^{1+\alpha}$ *images of self-similar sets; proofs of Theorems 2.17 and 2.22 and Corollary 2.18*

In this subsection we consider the case where F is an image of a self-similar set $E \subseteq X$ under a diffeomorphism $g \in \mathcal{C}^{1+\alpha}(\mathcal{U})$, where $\alpha \in (0, 1]$ and $\mathcal{U}$ is a convex neighbourhood of the compact connected set X. We assume that $|g'|$ is bounded away from 0 on its domain of definition. Thus, g is bi-Lipschitz and therefore the Minkowski dimension of F coincides with the Minkowski dimension of E (see for instance [6, Corollary 2.4]). We denote the common value by δ.

The similarities $R_1, \ldots, R_N$ generating E and the mappings $Q_1, \ldots, Q_N$ generating F are connected through the equations $Q_i = g \circ R_i \circ g^{-1}$ for each $i \in \Sigma$. We denote by $\widetilde{\pi}$ and π respectively the code maps from Σ^∞ to E and F. If we further let $\mathcal{H}_E^\delta$ denote the normalised δ-dimensional Hausdorff measure on E, that is $\mathcal{H}_E^\delta(\cdot) := \mathcal{H}^\delta(\cdot \cap E)/\mathcal{H}^\delta(E)$, and let $r_1, \ldots, r_N$ denote the respective similarity ratios of $R_1, \ldots, R_N$, we have the following list of observations.

(A′) Q_i is differentiable for every $i \in \Sigma$ with derivative

$$Q_i'(y) = \frac{g'(R_i \circ g^{-1}(y))}{g'(g^{-1}(y))} \cdot r_i,$$

where $y \in Y$ and Y is the nonempty compact interval which each Q_i is defined on.

(B′) The geometric potential function ξ associated with F is given by $\xi(\omega) = -\ln |g'(g^{-1}(\pi\omega))| + \ln |g'(g^{-1}(\pi\sigma\omega))| - \ln r_{\omega_1}$, where $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$. The geometric potential function ζ associated with E is given by $\zeta(\omega) = -\ln r_{\omega_1}$. Thus ξ is nonlattice if and only if ζ is nonlattice.

(C′) The unique σ-invariant Gibbs measure for the potential function $-\delta\xi$ is $\mu_{-\delta\xi} = \mathcal{H}_E^\delta \circ g^{-1} \circ \pi$; the one associated with $-\delta\zeta$ is $\mu_{-\delta\zeta} = \mathcal{H}_E^\delta \circ \pi$.

(D′) From (B′) and (C′) we obtain for the measure theoretical entropies

$$H(\mu_{-\delta\xi}) = \delta \int \xi \, \mathrm{d}\mu_{-\delta\xi} = -\delta \sum_{i \in \Sigma} \ln r_i \cdot r_i^\delta = \delta \int \zeta \, \mathrm{d}\mu_{-\delta\zeta} = H(\mu_{-\delta\zeta}).$$

Further, let $\widetilde{G}^1, \ldots, \widetilde{G}^L$ denote the primary gaps of E and $\widetilde{G}_\omega^1, \ldots, \widetilde{G}_\omega^L$ the main gaps of $R_\omega E$ for each $\omega \in \Sigma^*$ and recall that $G^1, \ldots, G^L$ and $G_\omega^1, \ldots, G_\omega^L$ respectively denote the primary gaps of F and the main gaps of $Q_\omega F$. Then:

(E′) $G_\omega^i = g(\widetilde{G}_\omega^i)$ for $i \in \{1, \ldots, L\}$ and $\omega \in \Sigma^*$. Since furthermore $|\widetilde{G}_\omega^i| = r_\omega |\widetilde{G}^i|$, we have

$$\lim_{n\to\infty} \sum_{i=1}^L \sum_{\omega \in \Sigma^n} |G_\omega^i|^\delta = \lim_{n\to\infty} \sum_{i=1}^L \sum_{\omega \in \Sigma^n} \left(r_\omega |\widetilde{G}^i| \cdot |g'(x_\omega)| \right)^\delta = \sum_{i=1}^L |\widetilde{G}^i|^\delta \int_E |g'|^\delta \, \mathrm{d}\mathcal{H}_E^\delta,$$

where $x_\omega \in [\omega]$ for each $\omega \in \Sigma^*$. Note that the above line can be rigorously proven by using the Bounded Distortion Lemma (Lemma 3.2).

(F′) The δ-conformal measure ν associated with F and the δ-conformal measure $\widetilde{\nu}$ associated with E are absolutely continuous with Radon–Nikodym derivative

$$\frac{\mathrm{d}\nu}{\mathrm{d}\widetilde{\nu} \circ g^{-1}} = |g' \circ g^{-1}|^\delta \left(\int_E |g'|^\delta \, \mathrm{d}\mathcal{H}_E^\delta \right)^{-1}.$$

(G′) From the fact that $R_1, \ldots, R_N$ are contractions and g' is Hölder continuous and bounded away from 0, one can deduce that there exists an iterate of $Q := \{Q_1, \ldots, Q_N\}$ which solely consists of contractions. As this iterate also generates F, it follows that F is a self-conformal set.

Proof of Theorem 2.17. Using items (A′)–(G′), an application of Theorem 2.11(i) and (ii) to F and of Theorem 2.14 to E proves Theorem 2.17(i) and (ii). The proof of Theorem 2.17(iii) is more involved and will be presented now.

We want to apply Lemma 5.1 in order to show that there exists a Borel set $B \subseteq \mathbb{R}$ for which $\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B)$ for $k \in \{0, 1\}$ from which we then deduce that the fractal curvature measures do not exist. For applying Lemma 5.1 we first introduce a family Δ of nonempty Borel subsets of Σ^∞, where Σ^∞ denotes the code space associated with R. For every $\kappa \in \Delta$ we then construct a pair $(B(\kappa), f_\kappa)$ which consists of a nonempty Borel set $B(\kappa) \subseteq \mathbb{R}$ satisfying $F_\varepsilon \cap B(\kappa) = (F \cap B(\kappa))_\varepsilon$ for all sufficiently small $\varepsilon > 0$ and a positive bounded periodic Borel-measurable function $f_\kappa : \mathbb{R}^+ \to \mathbb{R}^+$ such that Lemma 5.1(ii) is satisfied for $B = B(\kappa)$ and $f = f_\kappa$. Then, we show that there always exists a $\kappa \in \Delta$ for which f_κ is not equal to an almost everywhere constant function, verifying Lemma 5.1(i).

Fix the sIFS $R =: \{R_1, \dots, R_N\}$ and let $r_1, \dots, r_N$ denote their respective similarity ratios, that is $r_i := R_i'(x)$ for any $x \in X$. Note that g is a bijective function by definition. For $i \in \{1, \dots, N\}$ define $Q_i := g \circ R_i \circ g^{-1}$ and set $Q := \{Q_1, \dots, Q_N\}$. From the fact that $R_1, \dots, R_N$ are contractions and g' is Hölder continuous and bounded away from zero, one can deduce that there exists an iterate $\widetilde{Q}$ of Q which solely consists of contractions. Without loss of generality we assume that $Q_1, \dots, Q_N$ are contractions themselves. Then Q is a cIFS with open set $\operatorname{int}(gX)$ and bounded distortion constants $\varrho_m = 1 + \max_{\omega \in \Sigma^m} c|R_\omega X|^\alpha / k_g$, where $k_g > 0$ is such that $|g'| \geq k_g$ on $\mathcal{U}$ and c is a constant depending on the Hölder constant of g. Clearly, $\varrho_m \to 1$ as $m \to \infty$. Moreover, $F := g(E)$ is its associated self-conformal set, since $\bigcup_{i=1}^N Q_i F = \bigcup_{i=1}^N g R_i g^{-1} g(E) = \bigcup_{i=1}^N g R_i(K) = F$.

Let us begin by introducing the family Δ. First, fix an $n \in \mathbb{N} \cup \{0\}$, let $\langle Y \rangle$ denote the convex hull of a compact set $Y \subset \mathbb{R}$ and define

$$\Delta_n := \left\{ \bigcup_{i=1}^{l} [\kappa^{(i)}] \mid \kappa^{(i)} \in \Sigma^n,\ l \in \{1, \dots, N^n\},\ \bigcup_{i=1}^{l} \langle Q_{\kappa^{(i)}} F \rangle \text{ is an interval}, \right.$$
$$\left. \bigcup_{i=1}^{l} Q_{\kappa^{(i)}} F \cap Q_\omega F = \varnothing \text{ for every } \omega \in \Sigma^n \setminus \{\kappa^{(1)}, \dots, \kappa^{(l)}\} \right\}.$$

(Note that if the strong separation condition was satisfied, then $\Delta_n = \{[\omega] \mid \omega \in \Sigma^n\}$.) We remark that the condition $\lambda^1(F) = 0$ implies that $\kappa \subsetneq \Sigma^\infty$ for every $\kappa \in \Delta_n$, whenever $n \in \mathbb{N}$. Further, note that $\Delta_n \neq \varnothing$ for all $n \in \mathbb{N}$ because of the OSC and set $\Delta := \bigcup_{n \in \mathbb{N} \cup \{0\}} \Delta_n$. Now, fix an $n \in \mathbb{N} \cup \{0\}$ and a $\kappa = \bigcup_{i=1}^l [\kappa^{(i)}] \in \Delta_n$ and choose $\theta > 0$ such that $\bigcup_{i=1}^l \langle Q_{\kappa^{(i)}} F \rangle_{3\theta} \cap Q_\omega F = \varnothing$ for every $\omega \in \Sigma^n \setminus \{\kappa^{(1)}, \dots, \kappa^{(l)}\}$. Then $B(\kappa) := \bigcup_{i=1}^l \langle Q_{\kappa^{(i)}} F \rangle_\theta$ is a nonempty Borel subset of $\mathbb{R}$ satisfying $F_\varepsilon \cap B(\kappa) = (F \cap B(\kappa))_\varepsilon$ for all $\varepsilon < \theta$. Denote by $G^1, \dots, G^L$ the primary gaps of F and by $G^1_\omega, \dots, G^L_\omega$ the associated main gaps for $\omega \in \Sigma^*$.

For constructing the function f_κ fix an $m \in \mathbb{N}$ and choose $M \in \mathbb{N}$ such that $\mathrm{e}^{-M} < \theta$ and that for every $\omega \in \Sigma^m$ all main gaps $G^1_\omega, \dots, G^L_\omega$ which lie in $B(\kappa)$ are of length greater than $2\mathrm{e}^{-M}$. Then for all $T \geq M$ we have

$$\begin{aligned}
\lambda^0\left(\partial F_{\mathrm{e}^{-T}} \cap B(\kappa)\right)/2 &= \sum_{i=1}^{L} \#\{\omega \in \Sigma^* \mid G^i_\omega \subseteq B(\kappa),\ |G^i_\omega| > 2\mathrm{e}^{-T}\} + 1 \\
&\leq \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \Xi^i_\omega(\mathrm{e}^{-T}) + \underbrace{\sum_{j=1}^{m-n} L \cdot N^{j-1} + 1}_{=: c_m},
\end{aligned}$$

where we agree that $\sum_{j=1}^{m-n-1} L \cdot N^{j-1} = 0$ if $m - n < 1$ and where

$$\Xi^i_\omega(\mathrm{e}^{-T}) := \#\{u \in \Sigma^* \mid G^i_{u\omega} \subseteq B(\kappa),\ |G^i_{u\omega}| \geq 2\mathrm{e}^{-T}\}.$$

Likewise, for all $\beta > 1$ we have that

$$\lambda^0 \left(\partial F_{\mathrm{e}^{-T}} \cap B(\kappa)\right)/2 \geq \sum_{i=1}^{L} \sum_{\omega \in \Sigma^m} \Xi_\omega^i(\mathrm{e}^{-T}\beta).$$

We let ξ and ζ respectively denote the geometric potential functions associated with Q and R. Note that Σ^∞ is also the code space associated with Q and let π and $\widetilde{\pi}$ respectively denote the code maps from Σ^∞ to F and E. They satisfy $\pi = g\widetilde{\pi}$. For $x \in \Sigma^\infty$ we have the following relation:

$$\begin{aligned}\xi(x) &= -\ln|Q'_{x_1}(\pi\sigma x)| \\ &= -\ln|g'(R_{x_1}g^{-1}\pi\sigma x)| - \ln|R'_{x_1}(g^{-1}\pi\sigma x)| + \ln|g'(g^{-1}\pi\sigma x)| \\ &= -\ln|g'(\widetilde{\pi}x)| + \zeta(x) + \ln|g'(\widetilde{\pi}\sigma x)|.\end{aligned}$$

Therefore, $\psi: \Sigma^\infty \to \mathbb{R}$ given by $\psi(x) := -\ln|g'(\widetilde{\pi}x)|$ defines a continuous function which satisfies

$$\xi - \zeta = \psi - \psi \circ \sigma.$$

Let c_g be the Hölder constant of g' and let $k_g > 0$ be such that $|g'| \geq k_g$ on $\mathcal{U}$. Also g satisfies a bounded distortion property, since we have for all $x, y \in \langle R_\omega E\rangle$ where $\omega \in \Sigma^n$ and $n \in \mathbb{N}$ that

$$\left|\frac{g'(x)}{g'(y)}\right| \leq \left|\frac{g'(x) - g'(y)}{g'(y)}\right| + 1 \leq \frac{c_g|x-y|^\alpha}{k_g} + 1 \leq \max_{\omega\in\Sigma^n} \frac{c_g|\langle R_\omega E\rangle|^\alpha}{k_g} + 1 =: p_n. \quad (5.4)$$

Clearly, $p_n \to 1$ as $n \to \infty$. Denote by $\widetilde{G}^i$ the primary gaps of E and by $\widetilde{G}^i_\omega$ the main gaps of $R_\omega E$, where $i \in \{1, \ldots, N\}$ and $\omega \in \Sigma^*$. Thus, for an arbitrary $x \in \Sigma^\infty$, $i \in \{1, \ldots, L\}$, $\omega \in \Sigma^m$ and $u \in \Sigma^n$ we have that

$$\begin{aligned}|G^i_{u\omega}| &= |g\widetilde{G}^i_{u\omega}| \leq |g'(R_{u\omega}\widetilde{\pi}x)|p_m|R_{u\omega}\widetilde{G}^i| \\ &= \exp(\ln p_m - \psi(u\omega x) - S_n\zeta(u\omega x) + \ln|R_\omega\widetilde{G}^i|) \\ &= \exp(\ln p_m - \psi(\omega x) - S_n\xi(u\omega x) + \ln|R_\omega\widetilde{G}^i|)\end{aligned}$$

Therefore, for $x \in \Sigma^\infty$, $m \geq M$ and $\omega \in \Sigma^m$,

$$\begin{aligned}&\Xi^i_\omega(\mathrm{e}^{-T}) \\ &\quad \leq \#\{u \in \Sigma^* \mid G^i_{u\omega} \subseteq B(\kappa), \ln(2/p_m) + \psi(\omega x) - \ln|R_\omega\widetilde{G}^i| \leq T - S_n\xi(u\omega x)\}.\end{aligned}$$

By construction, κ is a finite union of cylinder sets and thus $\mathbb{1}_\kappa \in \mathcal{F}_\alpha(\Sigma^\infty)$. Recalling that $a > 0$ denotes the maximal real number for which $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$, an application of Theorem 3.9 hence yields

$$\begin{aligned}&\lambda^0(\partial F_{\mathrm{e}^{-T}} \cap B(\kappa))/2 - c_m \\ &\quad \leq \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{y:\sigma^n y = \omega x} \mathbb{1}_\kappa(y) \cdot \mathbb{1}_{\{S_n\xi(y) \leq T - \ln(2/p_m) - \psi(\omega x) + \ln|R_\omega\widetilde{G}^i|\}} \\ &\quad \sim \sum_{i=1}^{L} \sum_{\omega\in\Sigma^m} \frac{a h_{-\delta\zeta}(\omega x) \int_\kappa \mathrm{e}^{-\delta a\left\lceil \frac{\psi(y)-\psi(\omega x)}{a} + \frac{1}{a}(\psi(\omega x) - \ln(r_\omega) + \ln 2 - \ln(p_m|\widetilde{G}^i|) - T)\right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y)}{\left(1 - \mathrm{e}^{-\delta a}\right)\int \zeta \, \mathrm{d}\mu_{-\delta\zeta}}. \quad (5.5)\end{aligned}$$

Define $W := a\left(1-\mathrm{e}^{-\delta a}\right)^{-1}\left(\int \zeta\,\mathrm{d}\mu_{-\delta\zeta}\right)^{-1}$ and note that $h_{-\delta\zeta}\equiv 1$. Using that $\ln r_\omega \in a\mathbb{Z}$ for every $\omega\in\Sigma^*$ and that $\sum_{\omega\in\Sigma^m}(r_\omega)^\delta = 1$ for every $m\in\mathbb{N}$, Eq. (5.5) simplifies to

$$\sum_{i=1}^{L}\sum_{\omega\in\Sigma^m} W r_\omega^\delta \int_\kappa \mathrm{e}^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2\mathrm{e}^{-T}}{p_m|\widetilde{G}^i|}\right\rceil}\mathrm{d}\nu_{-\delta\zeta}(y)$$
$$=\sum_{i=1}^{L} W \int_\kappa \mathrm{e}^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2\mathrm{e}^{-T}}{p_m|\widetilde{G}^i|}\right\rceil}\mathrm{d}\nu_{-\delta\zeta}(y).$$

Hence, for all $t>0$ there exists an $M'\geq M$ such that for all $T\geq M'$ we have

$$\mathrm{e}^{-\delta T}\lambda^0(\partial F\mathrm{e}^{-T}\cap B(\kappa))/2$$
$$\leq (1+t)\mathrm{e}^{-\delta T}\sum_{i=1}^{L} W \int_\kappa \mathrm{e}^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2\mathrm{e}^{-T}}{p_m|\widetilde{G}^i|}\right\rceil}\mathrm{d}\nu_{-\delta\zeta}(y)+c_m\mathrm{e}^{-\delta T}.$$

Defining the function $f_\kappa:\mathbb{R}^+\to\mathbb{R}^+$ by

$$f_\kappa(T) := \mathrm{e}^{-\delta T}\sum_{i=1}^{L} W \int_\kappa \mathrm{e}^{-\delta a\left\lceil \frac{\psi(y)}{a}+\frac{1}{a}\ln\frac{2\mathrm{e}^{-T}}{|\widetilde{G}^i|}\right\rceil}\mathrm{d}\nu_{-\delta\zeta}(y)$$

we thus have

$$\mathrm{e}^{-\delta T}\lambda^0(\partial F_{\mathrm{e}^{-T}}\cap B(\kappa))/2 \leq (1+t)p_m^\delta f_\kappa(T+\ln p_m)+c_m\mathrm{e}^{-\delta T}.$$

Likewise, for all $\beta>1$ we have

$$\mathrm{e}^{-\delta T}\lambda^0(\partial F_{\mathrm{e}^{-T}}\cap B(\kappa))/2 \geq (1-t)p_m^{-\delta} f_\kappa(T-\ln(p_m\beta)),$$

which implies that

$$\mathrm{e}^{-\delta T}\lambda^0(\partial F_{\mathrm{e}^{-T}}\cap B(\kappa))/2 \geq (1-t)p_m^{-\delta} f_\kappa(T-\ln p_m).$$

Clearly, f_κ is periodic with period a. Thus, Lemma 5.1(ii) is satisfied for $B=B(\kappa)$ and $f=f_\kappa$.

In order to apply Lemma 5.1 it remains to prove the validity of Lemma 5.1(i), that is that there exists a $\kappa\in\Delta$ for which f_κ is not equal to an almost everywhere constant function. Set $\underline{\beta} := \min\{\{a^{-1}\ln|\widetilde{G}^i|\} \mid i=1,\ldots,L\}$ and $\overline{\beta} := \max\{\{a^{-1}\ln|\widetilde{G}^i|\} \mid i=1,\ldots,L\}$. We first assume that $\underline{\beta}>0$ and consider the following four cases.

Case 1: $\underline{D} := \{y\in\Sigma^\infty \mid \{a^{-1}\psi(y)\}<\underline{\beta}\}\neq\varnothing$.

Since $\psi\in\mathcal{C}(\Sigma^\infty)$ and thus $\underline{D}$ is open, there exists a $\kappa\in\Delta$ such that $\kappa\subseteq\underline{D}$. For $n\in\mathbb{N}$ and $r\in(0,1-\overline{\beta})$ define $T_n(r) := a(n+r)+\ln 2$. Then

$$f_\kappa(T_n(r)) = \mathrm{e}^{-\delta ar}\cdot 2^{-\delta}\sum_{i=1}^{L} W\int_\kappa \mathrm{e}^{-\delta a\left\lceil\frac{\psi(y)}{a}\right\rceil+\delta a}\mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that f_κ is strictly decreasing on $(an+\ln 2, a(n+1-\overline{\beta})+\ln 2)$ for every $n\in\mathbb{N}$. Therefore, f_κ is not equal to an almost everywhere constant function.

Case 2: $\overline{D} := \{y\in\Sigma^\infty \mid \{a^{-1}\psi(y)\}>\overline{\beta}\}\neq\varnothing$.

Like in Case 1, there exists a $\kappa\in\Delta$ such that $\kappa\subseteq\overline{D}$. For $n\in\mathbb{N}$ and $r\in(0,\underline{\beta})$ set $T_n(r) := a(n-r)+\ln 2$. Then

$$f_\kappa(T_n(r)) = \mathrm{e}^{\delta ar}\cdot 2^{-\delta}\sum_{i=1}^{L} W\int_\kappa \mathrm{e}^{-\delta a\left\lceil\frac{\psi(y)}{a}\right\rceil}\mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that f_κ is strictly decreasing on $(a(n-\underline{\beta})+\ln 2, an+\ln 2)$ for every $n \in \mathbb{N}$. Therefore, f_κ is not equal to an almost everywhere constant function.

For the remaining cases we let $q^* \in \mathbb{N} \cup \{0\}$ be maximal such that $\underline{\beta} + q^*(1-\overline{\beta}) \le \overline{\beta}$.

Case 3: There exists a $q \in \{0, \ldots, q^*\}$ such that

$$D_q := \{y \in \Sigma^\infty \mid \underline{\beta} + q(1-\overline{\beta}) < \{a^{-1}\psi(y)\} < \underline{\beta} + (q+1)(1-\overline{\beta})\} \neq \varnothing.$$

As in the above cases, there exists a $\kappa \in \Delta$ such that $\kappa \subseteq D_q$. For $n \in \mathbb{N}$ and $r \in (0, \underline{\beta})$ set $T_n^q(r) := a(n - \overline{\beta} + \underline{\beta} + q(1-\overline{\beta}) - r) + \ln 2$. Then

$$f_\kappa(T_n^q(r)) = \mathrm{e}^{\delta ar} \cdot 2^{-\delta} \mathrm{e}^{\delta a(\overline{\beta} - \underline{\beta} - q(1-\overline{\beta}))} \sum_{i=1}^{L} W \int_\kappa \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)}{a} \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that f_κ is strictly decreasing on $(a(n - \overline{\beta} + q(1-\overline{\beta})) + \ln 2, a(n - \overline{\beta} + \underline{\beta} + q(1-\overline{\beta})) + \ln 2)$. Therefore, f_κ is not equal to an almost everywhere constant function.

If neither of the cases 1–3 obtains, then the following case obtains.

Case 4: $\{y \in \Sigma^\infty \mid \{a^{-1}\psi(y)\} \subseteq \{\underline{\beta} + q(1-\overline{\beta}) \mid q \in \{0, \ldots, q^*\}\}\} = \Sigma^\infty$.

Define $q_i := \min(\{\underline{\beta} + q(1-\overline{\beta}) - \{a^{-1} \ln |\widetilde{G}^i_{\omega'}|\} > 0 \mid q \in \{0, \ldots, q^*\}, \omega' \in \Sigma^{\widetilde{M}}\} \cup \{1\})$ and $p := \min\{q_1, \ldots, q_N, 1 - \overline{\beta} + \underline{\beta}\}$. For $n \in \mathbb{N}$ and $r \in (0, p/2)$ define $T_n(r) := a(n+r) + \ln 2$. Then

$$f_\varnothing(T_n(r)) = \mathrm{e}^{-\delta ar} \cdot 2^{-\delta} \sum_{i=1}^{L} W \int_{\Sigma^\infty} \mathrm{e}^{-\delta a \left\lceil \frac{\psi(y)}{a} - \frac{1}{a} \ln |\widetilde{G}^i_{\omega'}| \right\rceil} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that $f_\varnothing$ is strictly decreasing on $(an + \ln 2, a(n + p/2) + \ln 2)$. Therefore, $f_\varnothing$ is not equal to an almost everywhere constant function.

If $\underline{\beta} = 0$, then the same methods can be applied after shifting the origin by $(1-\overline{\beta})/2$ to the left.

Thus, we can apply Lemma 5.1 in all four cases and obtain that there always exists a Borel set $B(\kappa)$ such that $\underline{C}_k^f(F, B(\kappa)) < \overline{C}_k^f(F, B(\kappa))$ for $k \in \{0, 1\}$.

In order to deduce that the fractal curvature measures do not exist, construct a function $\eta : \mathbb{R} \to [0, 1]$ which is continuous, equal to 1 on $B(\kappa)$ and equal to 0 on $\mathbb{R} \setminus B(\kappa)_\theta$. Then we have $\liminf_{\varepsilon \to 0} \int \eta \varepsilon^\delta \, \mathrm{d}\lambda^0(\partial F_\varepsilon \cap \cdot)/2 = \underline{C}_0^f(F, B(\kappa)) < \overline{C}_0^f(F, B(\kappa)) = \limsup_{\varepsilon \to 0} \int \eta \varepsilon^\delta \, \mathrm{d}\lambda^0(\partial F_\varepsilon \cap \cdot)/2$. Thus, the zeroth fractal curvature measure does not exist. Using the same function η it follows analogously that the first fractal curvature measure does not exist, which completes the proof. □

Proof of Corollary 2.18. Items (i) and (ii) of Corollary 2.18 are immediate consequences of Theorem 2.17. Corollary 2.18(iii) is going to be deduced from Theorem 2.12(iii). We let $\widetilde{\pi}$ and π respectively denote the code maps from Σ^∞ to E and F^n and observe that $\widetilde{\pi} = g_n^{-1} \circ \pi$. Further, we let ξ denote the geometric potential function associated with F^n. By Property (B′) we see that $\xi - \zeta = \psi - \psi \circ \sigma$, where $\psi := -\ln g_n' \circ \widetilde{\pi}$. By definition we have that $g_n'(x) = \left(\widetilde{g}(x)(\mathrm{e}^{\delta an} - 1) + 1\right)^{-1/\delta}$ for $x \in [-1, \infty)$. Thus, $\psi(\Sigma^\infty) = -\ln g_n' \circ \widetilde{\pi}(\Sigma^\infty) \subseteq [0, an]$. We now show that Eq. (2.3) from Theorem 2.12 is satisfied.

$$\sum_{i=0}^{n} \mathrm{e}^{-\delta a i} \nu_{-\delta\zeta} \circ \psi^{-1}([ai, ai+t)) = \sum_{i=0}^{n} \mathrm{e}^{-\delta a i} \widetilde{\nu} \circ \widetilde{g}^{-1}\left(\left[\frac{\mathrm{e}^{\delta a i}-1}{\mathrm{e}^{\delta a n}-1}, \frac{\mathrm{e}^{\delta a i+\delta t}-1}{\mathrm{e}^{\delta a n}-1}\right)\right)$$
$$= \sum_{i=0}^{n-1} \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{\delta a n}-1} = \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{\delta a}-1} \sum_{i=0}^{n} \mathrm{e}^{-\delta a i} \widetilde{\nu} \circ \widetilde{g}^{-1}\left(\left[\frac{\mathrm{e}^{\delta a i}-1}{\mathrm{e}^{\delta a n}-1}, \frac{\mathrm{e}^{\delta a(i+1)}-1}{\mathrm{e}^{\delta a n}-1}\right)\right)$$
$$= \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{\delta a}-1} \sum_{i=0}^{n} \mathrm{e}^{-\delta a i} \nu_{-\delta\zeta} \circ \psi^{-1}([ai, a(i+1)))$$

holds for all $t \in [0, a)$, which completes the proof. □

Proof of Theorem 2.22. For ease of notation, we assume without loss of generality that $\{0, 1\} \subset F \subset [0, 1]$. Let ξ denote the geometric potential function associated with Q and let π denote the code map from Σ^∞ to F. By Corollary 6.1.4 in [23] the eigenfunction $h_{-\delta\xi}$ of the Perron–Frobenius operator $\mathcal{L}_{-\delta\xi}$ possesses a real analytic extension to an open neighbourhood of X in $\mathbb{R}$. Denote this extension by h and define $\widetilde{\psi} := \delta^{-1} \ln h$. Since ξ is lattice, there exist $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ such that

$$\xi - \zeta = \psi - \psi \circ \sigma$$

and such that ζ is a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be the maximal real number for which $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. The function $\widetilde{\psi}$ satisfies the equation

$$\widetilde{\psi} \circ \pi = \psi + \delta^{-1} \ln h_{-\delta\zeta},$$

since h satisfies

$$h \circ \pi = h_{-\delta\xi} = \frac{\mathrm{d}\mu_{-\delta\xi}}{\mathrm{d}\nu_{-\delta\xi}} = \frac{\mathrm{d}\mu_{-\delta\zeta}}{\mathrm{e}^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta}} = \mathrm{e}^{\delta\psi} h_{-\delta\zeta}.$$

We define the function $\widetilde{g}\colon [0, 1] \to \mathbb{R}$ by $\widetilde{g}(x) := \int_0^x \mathrm{e}^{\widetilde{\psi}(y)}\, \mathrm{d}y/A$ for $x \in [0, 1]$, where $A := \int_0^1 \mathrm{e}^{\widetilde{\psi}(y)}\, \mathrm{d}y$. As $\widetilde{\psi}$ is analytic, the Fundamental Theorem of Calculus implies that $\widetilde{\psi} - \ln A = \ln \widetilde{g}'$. Moreover, the analyticity of $\widetilde{\psi}$ implies that $\widetilde{\psi}$ is bounded on $[0, 1]$. Therefore, $\widetilde{g}'$ is bounded away from both 0 and ∞ and thus $\widetilde{g}$ is invertible. Note that $\widetilde{g}([0, 1]) = [0, 1]$, set $g := \widetilde{g}^{-1}\colon [0, 1] \to [0, 1]$ and extend g to an analytic function on an open neighbourhood $\mathcal{U}$ of $[0, 1]$ such that $|g'| > 0$ on $\mathcal{U}$. Define

$$R_i := g^{-1} \circ Q_i \circ g \quad \text{for } i \in \{1, \ldots, N\} \quad \text{and} \quad E := g^{-1}(F) \subseteq [0, 1].$$

Then setting $\widetilde{\pi} := g^{-1} \circ \pi$, we have for $x \in \Sigma^\infty$ that

$$\begin{aligned} -\ln |R'_{x_1}(\widetilde{\pi}\sigma x)| &= -\ln \widetilde{g}'(Q_{x_1} g\widetilde{\pi}\sigma x) - \ln |Q'_{x_1}(g\widetilde{\pi}\sigma x)| + \ln \widetilde{g}'(g\widetilde{\pi}\sigma x) \\ &= -\widetilde{\psi}(\pi x) + \ln A + \xi(x) + \widetilde{\psi}(\pi\sigma x) - \ln A \\ &= -\psi(x) - \delta^{-1} \ln(h_{-\delta\zeta}(x)) + \xi(x) + \psi(\sigma x) + \delta^{-1} \ln(h_{-\delta\zeta}(\sigma x)) \\ &= \zeta(x) - \delta^{-1} \ln\left(\frac{h_{-\delta\zeta}(x)}{h_{-\delta\zeta}(\sigma x)}\right). \end{aligned}$$

Since the range of ζ is contained in the group $a\mathbb{Z}$ and ξ and ψ are bounded on Σ^∞, ζ in fact takes a finite number of values. Moreover, ζ is continuous which implies that there exists an $M \in \mathbb{N}$ such that ζ is constant on each $[\omega]$ for $\omega \in \Sigma^M$. This clearly implies that $\mathcal{L}^n_{-\delta\zeta} 1$ is constant on $[\omega]$ for all $\omega \in \Sigma^M$ and all $n \in \mathbb{N}$, where 1 denotes the constant 1-function. Thus, Eq. (3.2) implies that also $h_{-\delta\zeta}$ is constant on cylinder sets of length M. This can be seen by considering $|h_{-\delta\zeta}(x) - h_{-\delta\zeta}(y)|$ for x, y lying in the same cylinder of length M and applying the triangle inequality. Therefore, $x \mapsto -\ln|R'_{x_1}(\widetilde{\pi}\sigma x)|$ is constant on cylinder sets of length $M+1$. Hence, for $\omega \in \Sigma^{M+1}$ and $i \in \{1, \ldots, N\}$ there exists a $c^i_\omega \in \mathbb{R}$ such that $R'_i(\widetilde{\pi}x) = c^i_\omega$ for all $x \in [\omega]$. Since for each $\omega \in \Sigma^{M+1}$ the set $\{\widetilde{\pi}x \mid x \in [\omega]\}$ has accumulation points and is compact and the map R'_i is analytic by construction, it follows that R'_i is constant on its domain of definition. Therefore, the maps $R_1, \ldots, R_N$ are similarities. From the fact that $Q_1, \ldots, Q_N$ are contractions and g' is differentiable and bounded away from 0, one can deduce that there exists an iterate $\widetilde{R}$ of $R := \{R_1, \ldots, R_N\}$ which solely consists of contractions. The system $\widetilde{R}$ satisfies the OSC with open set $(0, 1) = g^{-1}(0, 1)$. Therefore, the unique nonempty compact invariant set of $\widetilde{R}$ is a self-similar set. It coincides with $E := g^{-1}(F)$, since $R_i(g^{-1}F) = g^{-1}Q_i g(g^{-1}F) = g^{-1}F$ for each $i \in \{1, \ldots, N\}$. □

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