An explicit formula for the number of permutations with a given number of alternating runs

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1. Introduction

Let \( S_n \) be the symmetric group of all permutations of \([n]\), where \([n] = \{1, 2, \ldots, n\}\). Let \( \pi = \pi(1)\pi(2) \cdots \pi(n) \in S_n \). We say that \( \pi \) changes direction at position \( i \) if either \( \pi(i-1) < \pi(i) > \pi(i+1) \), or \( \pi(i-1) > \pi(i) < \pi(i+1) \), where \( i \in \{2, 3, \ldots, n-1\} \). We say that \( \pi \) has \( k \) alternating runs if there are \( k-1 \) indices \( i \) such that \( \pi \) changes direction at these positions. Let \( R(n,k) \) denote the number of permutations in \( S_n \) with \( k \) alternating runs. André [1] was the first to study the alternating runs of permutations and he obtained the following recurrence

\[
R(n,k) = kR(n-1,k) + 2R(n-1,k-1) + (n-k)R(n-1,k-2)
\]

for \( n,k \geq 1 \), where \( R(1,0) = 1 \) and \( R(1,k) = 0 \) for \( k \geq 1 \).

Let \( R_n(x) = \sum_{k=1}^{n-1} R(n,k)x^k \). Then the recurrence (1) induces

\[
R_{n+2}(x) = x(nx+2)R_{n+1}(x) + x(1-x^2)R'_{n+1}(x).
\]

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with initial values $R_1(x) = 1$. The first few terms of $R_n(x)$’s are given as follows:

\[
\begin{align*}
R_2(x) & = 2x, \\
R_3(x) & = 2x + 4x^2, \\
R_4(x) & = 2x + 12x^2 + 10x^3, \\
R_5(x) & = 2x + 28x^2 + 58x^3 + 32x^4.
\end{align*}
\]

For a permutation $\pi = \pi(1)\pi(2) \cdots \pi(n) \in S_n$, we define a descent to be a position $i$ such that $\pi(i) > \pi(i + 1)$. Denote by $\text{des}(\pi)$ the number of descents of $\pi$. Let

\[
A_n(x) = \sum_{\pi \in S_n} x^{\text{des}(\pi)} = \sum_{k=1}^{n} A(n, k)x^k.
\]

The polynomial $A_n(x)$ is called an Eulerian polynomial, while $A(n, k)$ is called an Eulerian number. The polynomial $R_n(x)$ is closely related to $A_n(x)$:

\[
R_n(x) = \frac{\left(1 + x \right)^{n-1}}{2} \left(1 + w\right)^{n+1} A_n \left(\frac{1 - w}{1 + w}\right), \quad w = \sqrt{\frac{1 - x}{1 + x}},
\]

which was first established by David and Barton [10, p. 157–162] and then stated more concisely by Knuth [13, p. 605]. In a series of papers [5–7], Carlitz studied the generating functions for the numbers $R(n, k)$. In particular, Carlitz [5] proved that

\[
\sum_{n=0}^{\infty} (1 - x^2)^{-n/2} z^n \sum_{k=0}^{n} R(n + 1, k)x^{n-k} = \frac{1 - x}{1 + x} \left(\frac{\sqrt{1 - x^2} + \sin z}{x - \cos z}\right)^2.
\]

In [2], Bóna and Ehrenborg proved that $R_n(x)$ has the zero $x = -1$ with multiplicity $\lfloor n/2 \rfloor - 1$.

Recently there has been much interest in obtaining explicit formula for the numbers $R(n, k)$. In [16], Stanley gave an exact formula for $R(n, k)$:

\[
R(n, k) = \sum_{i=0}^{k} \frac{1}{2i-1} (-1)^{k-i} z_{2i} \sum_{r \equiv i \mod 2} \sum_{m \in \mathbb{Z}} (-2)^m \left( \binom{i-m}{i+r/2} \right) \left( \binom{n}{m} \right) r^n,
\]

where $z_0 = 2$ and $z_n = 4$ for $n \geq 1$. In [3], Canfield and Wilf showed that

\[
R(n, k) = \frac{1}{2^{k-2}} k^n - \frac{1}{2^{k-4}} (k - 1)^n + \psi_2(n, k)(k - 2)^n + \cdots + \psi_{k-1}(n, k) \quad \text{for} \ n \geq 2,
\]

in which each $\psi_i(n, k)$ is a polynomial in $n$ whose degree in $n$ is $\lfloor i/2 \rfloor$.

In Section 3, we express the polynomials $R_n(x)$ in terms of the derivative polynomials $P_n(x)$ defined by Hoffman [12]:

\[
P_n(\tan \theta) = \frac{d^n}{d\theta^n} \tan \theta.
\]

2. Derivative polynomials

Let $D$ denote the differential operator $d/d\theta$. Set $x = \tan \theta$. Then $D(x^n) = nx^{n-1}(1 + x^2)$ for $n \geq 1$. Thus $D^n(x)$ is a polynomial in $x$. Let $P_n(x) = D^n(x)$. Then $P_0(x) = x$ and $P_{n+1}(x) = (1 + x^2)P_n'(x)$. Clearly, $\deg P_n(x) = n + 1$. The first few terms can be computed directly as follows:

\[
\begin{align*}
P_1(x) & = 1 + x^2, \\
P_2(x) & = 2x + 2x^3,
\end{align*}
\]
\[ P_3(x) = 2 + 8x^2 + 6x^4, \]
\[ P_4(x) = 16x + 40x^3 + 24x^5. \]

Let \( P_n(x) = \sum_{k=0}^{n+1} p(n, k)x^k \). It is easy to verify that
\[ p(n, k) = (k + 1)p(n - 1, k + 1) + (k - 1)p(n - 1, k - 1). \]

Note that \( P_n(-x) = (-1)^{n+1}P_n(x) \). Thus we have the following expression:
\[ P_n(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n - 2k + 1)x^{n-2k+1}. \]  (4)

The study of properties of the polynomials \( P_n(x) \), initiated in [12], is presently very active (see [9,14]). For example, let
\[ \tan^k(t) = \sum_{n=k}^{\infty} T(n, k) \frac{t^n}{n!}. \]
The numbers \( T(n, k) \) are called the tangent numbers of order \( k \) (see [4, p. 428]). Cvijović [9, Theorem 1] obtained the following formula:
\[ P_n(x) = T(n, 1) + \sum_{k=1}^{n+1} \frac{1}{k} T(n + 1, k)x^k. \]

In the following discussion, we will present an explicit expression for the numbers \( p(n, k) \). The Stirling numbers of the second kind \( \{n\ \mid k\} \) count the number of ways to partition an \( n \)-set into \( k \) blocks. They can be calculated using the following explicit formula:
\[ \{n\ \mid k\} = \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} r^n. \]

The Eulerian polynomial \( A_n(x) \) admits several expansions in terms of different polynomial bases. One representative example is the Frobenius formula:
\[ A_n(x) = \sum_{i=1}^{n} i! \binom{n}{i} x^i (1 - x)^{n-i} \]  (5)
(see [8, Theorem 14.4]). Setting \( x = (y - 1)/(y + 1) \) in (5) and then multiplying both sides by \( (y + 1)^{n+1} \), we get
\[ \sum_{k=1}^{n} A(n, k)(y - 1)^k(y + 1)^{n-k+1} = (y + 1) \sum_{i=1}^{n} i! \binom{n}{i} 2^{n-i}(y - 1)^i. \]

Let \( a_n(y) = (y + 1) \sum_{i=1}^{n} i! \binom{n}{i} 2^{n-i}(y - 1)^i. \)  (6)

The first few terms of \( a_n(y) \)'s are given as follows:
\[ a_1(y) = -1 + y^2, \]
\[ a_2(y) = -2y + 2y^3, \]
\[ a_3(y) = 2 - 8y^2 + 6y^4, \]
\[ a_4(y) = 16y - 40y^3 + 24y^5. \]
Then from [11, Theorem 5], we have

$$a_n(y) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^k p(n, n - 2k + 1) y^{n-2k+1}. \quad (7)$$

Therefore, combining (6) and (7), we get the following result.

**Proposition 1.** For $n \geq 1$ and $0 \leq k \leq \lfloor (n+1)/2 \rfloor$, we have

$$p(n, n - 2k + 1) = (-1)^k \sum_{i=1}^{n} i! \left[ \begin{array}{c} n \\ i \end{array} \right] (-2)^{n-i} \left[ \left( \begin{array}{c} i \\ n-2k \end{array} \right) - \left( \begin{array}{c} i \\ n-2k+1 \end{array} \right) \right]. \quad (8)$$

3. Explicit formulas

In this section, we present an explicit formula for the numbers $R(n, k)$.

Setting $x = \cos 2\theta$ and replacing $z$ by $2z$ in (3), we get

$$\sum_{n=0}^{\infty} (\sin 2\theta)^{-n} 2^n \frac{z^n}{n!} \sum_{k=0}^{n} R(n+1, k) \cos^{n-k} 2\theta = \tan^2 \theta \cot (\theta - z). \quad (9)$$

Thus by replacing $z$ by $-z$, we obtain

$$\sum_{n=0}^{\infty} (\sin 2\theta)^{-n} 2^n (-1)^n \frac{z^n}{n!} \sum_{k=0}^{n} R(n+1, k) \cos^{n-k} 2\theta = \tan^2 \theta \cot (\theta + z). \quad (10)$$

By Taylor’s theorem, we have

$$\cot^2(\theta + z) = \sum_{n=0}^{\infty} D^n(\cot^2 \theta) \frac{z^n}{n!}. \quad (11)$$

Let $y = \cot \theta$. Then $D(y) = -1 - y^2$ and $D(y^n) = -ny^{n-1}(1+y^2)$. Put $C_n(y) = D^n(y)$. Then $C_{n+1}(y) = -(1+y^2)C_n(y)$ for $n \geq 0$. Clearly, $C_n(\cot \theta) = (-1)^n P_n(\cot \theta)$. Note that $D^2(y) = -D(y^2)$. Then $D^n(y^2) = (-1)^n P_{n+1}(\cot \theta)$. Therefore, we have

$$\cot^2(\theta + z) = \sum_{n=0}^{\infty} (-1)^n P_{n+1}(\cot \theta) \frac{z^n}{n!}. \quad (12)$$

We now present the main result of this paper.

**Theorem 2.** For $n \geq 2$, we have

$$R_n(x) = \left( \frac{x+1}{2} \right)^{n-1} \left( \frac{x-1}{x+1} \right)^{\lfloor (n+1)/2 \rfloor} P_n\left( \sqrt{\frac{x+1}{x-1}} \right). \quad (13)$$

**Proof.** Substituting (9) into (8), we get

$$\sum_{n=0}^{\infty} (\sin 2\theta)^{-n} 2^n (-1)^n \frac{z^n}{n!} \sum_{k=0}^{n} R(n+1, k) \cos^{n-k} 2\theta = \tan^2 \theta \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \sum_{k=0}^{n+2} p(n+1, k) \cot^k \theta. \quad (14)$$

Equating the coefficients of $(-1)^n z^n/n!$, we obtain

$$(\sin 2\theta)^{-n} 2^n \sum_{k=0}^{n} R(n+1, k) \cos^{n-k} 2\theta = \tan^2 \theta \sum_{k=0}^{n+2} p(n+1, k) \cot^k \theta. \quad (15)$$

Therefore, we have

$$a_n(y) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} (-1)^k p(n, n - 2k + 1) y^{n-2k+1}. \quad (6)$$

**Proof.** Substituting (9) into (8), we get

$$\sum_{n=0}^{\infty} (\sin 2\theta)^{-n} 2^n (-1)^n \frac{z^n}{n!} \sum_{k=0}^{n} R(n+1, k) \cos^{n-k} 2\theta = \tan^2 \theta \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!} \sum_{k=0}^{n+2} p(n+1, k) \cot^k \theta. \quad (14)$$

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Replacing $n$ by $n - 1$ in (11), we have
\[
\sum_{k=0}^{n-1} R(n, k) \cos^{n-k-1} 2\theta = \frac{(1 + \cos 2\theta)^{n-1}}{2^{n-1}} \left( \frac{1 - \cos 2\theta}{1 + \cos 2\theta} \right)^{\frac{1}{2}(n+1)} \sum_{k=0}^{n-1} p(n, k) \left( \frac{1 + \cos 2\theta}{1 - \cos 2\theta} \right)^{\frac{1}{2}k}.
\]
Consequently, replacing $\cos 2\theta$ by $x$, we get
\[
\sum_{k=0}^{n-1} R(n, k) x^{n-k-1} = \left( \frac{1 + x}{2} \right)^{n-1} \left( \frac{1 - x}{1 + x} \right)^{\frac{1}{2}(n+1)} P_n \left( \frac{\sqrt{1+x}}{\sqrt{1-x}} \right).
\] (12)
Replacing $x$ by $1/x$ in (12) and then multiplying both sides by $x^{n-1}$, we obtain the desired result. \qedsymbol

Combining (4) and (10), we get
\[
R_n(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n - 2k + 1) (x + 1)^{n-k-1} (x - 1)^k.
\] (13)
Note that $n - \lfloor (n+1)/2 \rfloor = \lfloor n/2 \rfloor$. It follows from (13) that $R_n(x)$ is divisible by $(x + 1)^{\lfloor n/2 \rfloor - 1}$. Denote by $E(n, k, s)$ the coefficients of $x^s$ in $(x + 1)^{n-k-1} (x - 1)^k$. It is easy to verify that
\[
E(n, k, s) = \sum_{j=0}^{\min(k, s)} (-1)^{k-j} \binom{n-k-1}{s-j} \binom{k}{j}.
\]
Consequently, by (13), we obtain the following result.

**Corollary 3.** For $n \geq 2$ and $1 \leq s \leq n - 1$, we have
\[
R(n, s) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n - 2k + 1) E(n, k, s).
\]
It should be noted that the numbers $R(n, s)/2$ appear as A008970 in [15].

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**References**