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An explicit formula for the number of permutations with a given number of alternating runs ☆

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ABSTRACT

Let $R(n,k)$ denote the number of permutations of $\{1,2,\ldots,n\}$ with k alternating runs. In this paper we present an explicit formula for the numbers $R(n,k)$.

1. Introduction

Let $\mathcal{S}_n$ be the symmetric group of all permutations of $[n]$, where $[n]=\{1,2,\ldots,n\}$. Let $\pi=\pi(1)\pi(2)\cdots\pi(n)\in\mathcal{S}_n$. We say that π changes direction at position i if either $\pi(i-1)<\pi(i)>\pi(i+1)$, or $\pi(i-1)>\pi(i)<\pi(i+1)$, where $i\in\{2,3,\ldots,n-1\}$. We say that π has k *alternating runs* if there are $k-1$ indices i such that π changes direction at these positions. Let $R(n,k)$ denote the number of permutations in $\mathcal{S}_n$ with k alternating runs. André [1] was the first to study the alternating runs of permutations and he obtained the following recurrence

$$R(n,k)=kR(n-1,k)+2R(n-1,k-1)+(n-k)R(n-1,k-2) \tag{1}$$

for $n,k\geqslant 1$, where $R(1,0)=1$ and $R(1,k)=0$ for $k\geqslant 1$.

Let $R_n(x)=\sum_{k=1}^{n-1}R(n,k)x^k$. Then the recurrence (1) induces

$$R_{n+2}(x)=x(nx+2)R_{n+1}(x)+x\left(1-x^2\right)R'_{n+1}(x),$$

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with initial values $R_1(x) = 1$. The first few terms of $R_n(x)$'s are given as follows:

$$R_2(x) = 2x,$$
$$R_3(x) = 2x + 4x^2,$$
$$R_4(x) = 2x + 12x^2 + 10x^3,$$
$$R_5(x) = 2x + 28x^2 + 58x^3 + 32x^4.$$

For a permutation $\pi = \pi(1)\pi(2)\cdots\pi(n) \in \mathcal{S}_n$, we define a *descent* to be a position i such that $\pi(i) > \pi(i+1)$. Denote by $\operatorname{des}(\pi)$ the number of descents of π. Let

$$A_n(x) = \sum_{\pi\in\mathcal{S}_n} x^{\operatorname{des}(\pi)+1} = \sum_{k=1}^{n} A(n,k)x^k.$$

The polynomial $A_n(x)$ is called an *Eulerian polynomial*, while $A(n,k)$ is called an *Eulerian number*. The polynomial $R_n(x)$ is closely related to $A_n(x)$:

$$R_n(x) = \left(\frac{1+x}{2}\right)^{n-1}(1+w)^{n+1}A_n\left(\frac{1-w}{1+w}\right), \quad w = \sqrt{\frac{1-x}{1+x}}, \tag{2}$$

which was first established by David and Barton [10, p. 157–162] and then stated more concisely by Knuth [13, p. 605]. In a series of papers [5–7], Carlitz studied the generating functions for the numbers $R(n,k)$. In particular, Carlitz [5] proved that

$$\sum_{n=0}^{\infty}\left(1-x^2\right)^{-n/2}\frac{z^n}{n!}\sum_{k=0}^{n}R(n+1,k)x^{n-k} = \frac{1-x}{1+x}\left(\frac{\sqrt{1-x^2}+\sin z}{x-\cos z}\right)^2. \tag{3}$$

In [2], Bóna and Ehrenborg proved that $R_n(x)$ has the zero $x = -1$ with multiplicity $\lfloor n/2\rfloor - 1$.

Recently there has been much interest in obtaining explicit formula for the numbers $R(n,k)$. In [16], Stanley gave an exact formula for $R(n,k)$:

$$R(n,k) = \sum_{i=0}^{k}\frac{1}{2^{i-1}}(-1)^{k-i}z_{k-i}\sum_{\substack{r+2m\leqslant i\\ r\equiv i \bmod 2}}(-2)^m\binom{i-m}{(i+r)/2}\binom{n}{m}r^n,$$

where $z_0 = 2$ and $z_n = 4$ for $n \geqslant 1$. In [3], Canfield and Wilf showed that

$$R(n,k) = \frac{1}{2^{k-2}}k^n - \frac{1}{2^{k-4}}(k-1)^n + \psi_2(n,k)(k-2)^n + \cdots + \psi_{k-1}(n,k) \quad \text{for } n \geqslant 2,$$

in which each $\psi_i(n,k)$ is a polynomial in n whose degree in n is $\lfloor i/2\rfloor$.

In Section 3, we express the polynomials $R_n(x)$ in terms of the *derivative polynomials* $P_n(x)$ defined by Hoffman [12]:

$$P_n(\tan\theta) = \frac{d^n}{d\theta^n}\tan\theta.$$

2. Derivative polynomials

Let D denote the differential operator $d/d\theta$. Set $x = \tan\theta$. Then $D(x^n) = nx^{n-1}(1+x^2)$ for $n \geqslant 1$. Thus $D^n(x)$ is a polynomial in x. Let $P_n(x) = D^n(x)$. Then $P_0(x) = x$ and $P_{n+1}(x) = (1+x^2)P_n'(x)$. Clearly, $\deg P_n(x) = n+1$. The first few terms can be computed directly as follows:

$$P_1(x) = 1 + x^2,$$
$$P_2(x) = 2x + 2x^3,$$

$$P_3(x) = 2 + 8x^2 + 6x^4,$$
$$P_4(x) = 16x + 40x^3 + 24x^5.$$

Let $P_n(x) = \sum_{k=0}^{n+1} p(n,k)x^k$. It is easy to verify that

$$p(n,k) = (k+1)p(n-1,k+1) + (k-1)p(n-1,k-1).$$

Note that $P_n(-x) = (-1)^{n+1}P_n(x)$. Thus we have the following expression:

$$P_n(x) = \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n-2k+1)x^{n-2k+1}. \tag{4}$$

The study of properties of the polynomials $P_n(x)$, initiated in [12], is presently very active (see [9,14]). For example, let

$$\tan^k(t) = \sum_{n=k}^{\infty} T(n,k)\frac{t^n}{n!}.$$

The numbers $T(n,k)$ are called *the tangent numbers of order k* (see [4, p. 428]). Cvijović [9, Theorem 1] obtained the following formula:

$$P_n(x) = T(n,1) + \sum_{k=1}^{n+1} \frac{1}{k} T(n+1,k)x^k.$$

In the following discussion, we will present an explicit expression for the numbers $p(n,k)$. The *Stirling numbers of the second kind* $\left\{ {n \atop k} \right\}$ count the number of ways to partition an n-set into k blocks. They can be calculated using the following explicit formula:

$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r} r^n.$$

The Eulerian polynomial $A_n(x)$ admits several expansions in terms of different polynomial bases. One representative example is the *Frobenius formula*:

$$A_n(x) = \sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} x^i (1-x)^{n-i} \tag{5}$$

(see [8, Theorem 14.4]). Setting $x = (y-1)/(y+1)$ in (5) and then multiplying both sides by $(y+1)^{n+1}$, we get

$$\sum_{k=1}^{n} A(n,k)(y-1)^k(y+1)^{n-k+1} = (y+1)\sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} 2^{n-i}(y-1)^i.$$

Let

$$a_n(y) = (y+1)\sum_{i=1}^{n} i! \left\{ {n \atop i} \right\} 2^{n-i}(y-1)^i. \tag{6}$$

The first few terms of $a_n(y)$'s are given as follows:

$$a_1(y) = -1 + y^2,$$
$$a_2(y) = -2y + 2y^3,$$
$$a_3(y) = 2 - 8y^2 + 6y^4,$$
$$a_4(y) = 16y - 40y^3 + 24y^5.$$

Then from [11, Theorem 5], we have

$$a_n(y)=\sum_{k=0}^{\lfloor (n+1)/2\rfloor}(-1)^k p(n,n-2k+1)y^{n-2k+1}. \tag{7}$$

Therefore, combining (6) and (7), we get the following result.

Proposition 1. *For $n\geqslant 1$ and $0\leqslant k\leqslant \lfloor (n+1)/2\rfloor$, we have*

$$p(n,n-2k+1)=(-1)^k\sum_{i\geqslant 1} i!\left\{ {n \atop i} \right\}(-2)^{n-i}\left[\binom{i}{n-2k}-\binom{i}{n-2k+1}\right].$$

3. Explicit formulas

In this section, we present an explicit formula for the numbers $R(n,k)$.

Setting $x=\cos 2\theta$ and replacing z by $2z$ in (3), we get

$$\sum_{n=0}^{\infty}(\sin 2\theta)^{-n}2^n\frac{z^n}{n!}\sum_{k=0}^{n}R(n+1,k)\cos^{n-k}2\theta=\tan^2\theta\cot^2(\theta-z).$$

Thus by replacing z by $-z$, we obtain

$$\sum_{n=0}^{\infty}(\sin 2\theta)^{-n}2^n(-1)^n\frac{z^n}{n!}\sum_{k=0}^{n}R(n+1,k)\cos^{n-k}2\theta=\tan^2\theta\cot^2(\theta+z). \tag{8}$$

By Taylor's theorem, we have

$$\cot^2(\theta+z)=\sum_{n=0}^{\infty}D^n\left(\cot^2\theta\right)\frac{z^n}{n!}.$$

Let $y=\cot\theta$. Then $D(y)=-1-y^2$ and $D(y^n)=-ny^{n-1}(1+y^2)$. Put $C_n(y)=D^n(y)$. Then $C_{n+1}(y)=-(1+y^2)C_n'(y)$ for $n\geqslant 0$. Clearly, $C_n(y)=(-1)^nP_n(y)$. Note that $D^2(y)=-D(y^2)$. Then $D^n(y^2)=(-1)^nP_{n+1}(y)$. Therefore, we have

$$\cot^2(\theta+z)=\sum_{n=0}^{\infty}(-1)^nP_{n+1}(\cot\theta)\frac{z^n}{n!}. \tag{9}$$

We now present the main result of this paper.

Theorem 2. *For $n\geqslant 2$, we have*

$$R_n(x)=\left(\frac{x+1}{2}\right)^{n-1}\left(\frac{x-1}{x+1}\right)^{\frac{1}{2}(n+1)}P_n\left(\sqrt{\frac{x+1}{x-1}}\right). \tag{10}$$

Proof. Substituting (9) into (8), we get

$$\sum_{n=0}^{\infty}(\sin 2\theta)^{-n}2^n(-1)^n\frac{z^n}{n!}\sum_{k=0}^{n}R(n+1,k)\cos^{n-k}2\theta=\tan^2\theta\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{n!}\sum_{k=0}^{n+2}p(n+1,k)\cot^k\theta.$$

Equating the coefficients of $(-1)^nz^n/n!$, we obtain

$$(\sin 2\theta)^{-n}2^n\sum_{k=0}^{n}R(n+1,k)\cos^{n-k}2\theta=\tan^2\theta\sum_{k=0}^{n+2}p(n+1,k)\cot^k\theta. \tag{11}$$

Replacing n by $n-1$ in (11), we have

$$\sum_{k=0}^{n-1} R(n,k)\cos^{n-k-1} 2\theta = \frac{(1+\cos 2\theta)^{n-1}}{2^{n-1}}\left(\frac{1-\cos 2\theta}{1+\cos 2\theta}\right)^{\frac{1}{2}(n+1)} \sum_{k=0}^{n+1} p(n,k)\left(\frac{1+\cos 2\theta}{1-\cos 2\theta}\right)^{\frac{1}{2}k}.$$

Consequently, replacing $\cos 2\theta$ by x, we get

$$\sum_{k=0}^{n-1} R(n,k)x^{n-k-1} = \left(\frac{1+x}{2}\right)^{n-1}\left(\frac{1-x}{1+x}\right)^{\frac{1}{2}(n+1)} P_n\left(\sqrt{\frac{1+x}{1-x}}\right). \tag{12}$$

Replacing x by $1/x$ in (12) and then multiplying both sides by x^{n-1}, we obtain the desired result. □

Combining (4) and (10), we get

$$R_n(x) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n-2k+1)(x+1)^{n-k-1}(x-1)^k. \tag{13}$$

Note that $n - \lfloor (n+1)/2 \rfloor = \lfloor n/2 \rfloor$. It follows from (13) that $R_n(x)$ is divisible by $(x+1)^{\lfloor n/2 \rfloor - 1}$. Denote by $E(n,k,s)$ the coefficients of x^s in $(x+1)^{n-k-1}(x-1)^k$. It is easy to verify that

$$E(n,k,s) = \sum_{j=0}^{\min(k,s)} (-1)^{k-j}\binom{n-k-1}{s-j}\binom{k}{j}.$$

Consequently, by (13), we obtain the following result.

Corollary 3. *For $n \geqslant 2$ and $1 \leqslant s \leqslant n-1$, we have*

$$R(n,s) = \frac{1}{2^{n-1}} \sum_{k=0}^{\lfloor (n+1)/2 \rfloor} p(n, n-2k+1)E(n,k,s).$$

It should be noted that the numbers $R(n,s)/2$ appear as A008970 in [15].

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