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Nash problem for surface singularities is a topological problem ☆

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Abstract

We address Nash problem for surface singularities using *wedges*. We give a refinement of the characterisation in A. Reguera-López (2006) [32] of the image of the Nash map in terms of wedges. Our improvement consists in a characterisation of the bijectivity of the Nash mapping using wedges defined over the base field, which are convergent if the base field is $\mathbb{C}$, and whose generic arc has transverse lifting to the exceptional divisor. This improves the results of M. Lejeune-Jalabert and A. Reguera (2008) [16] for the surface case. In the way to do this we find a reformulation of Nash problem in terms of branched covers of normal surface singularities. As a corollary of this reformulation we prove that the image of the Nash mapping is characterised by the combinatorics of a resolution of the singularity, or, what is the same, by the topology of the abstract link of the singularity in the complex analytic case. Using these results we prove several reductions of the Nash problem, the most notable being that, if Nash problem is true for singularities having rational homology sphere links, then it is true in general.

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1. Introduction

Nash problem [20] was formulated in the sixties (but published later) in the attempt to understand the relation between the structure of resolution of singularities of an algebraic variety X over a field of characteristic 0 and the space of arcs (germs of algebroid curves) in the variety. He proved that the space of arcs centred at the singular locus (endowed with a infinite-dimensional algebraic variety structure) has finitely many irreducible components, and proposed to study the relation of these components with the essential irreducible components of the exceptional set a resolution of singularities. An irreducible component E of the exceptional divisor of a resolution of singularities

$$\pi : \tilde{X} \to X$$

is called essential, if given any other resolution

$$\pi' : \tilde{X}' \to X$$

the birational transform of E to $\tilde{X}'$ is an irreducible component of the exceptional divisor. Nash defined a mapping from the set of irreducible components of the space of arcs centred at the singular locus to the set of essential components of a resolution as follows: he assigns to each component Z of the space of arcs centred at the singular locus the unique component of the exceptional divisor which meets the lifting of a generic arc of Z to the resolution. Nash established the injectivity of this mapping and asked whether it is bijective. He viewed as a plausible fact that Nash mapping is bijective in the surface case, and also proposed to study the higher-dimensional case.

Nash gave an affirmative answer to his problem in the case of A_k-singularities. Since then, there has been much progress showing an affirmative answer to the problem for many classes of singularities: toric singularities of arbitrary dimension, quasi-ordinary singularities, certain infinite families of non-normal threefolds, minimal surface singularities, sandwiched surface singularities, quotient surface singularities, and other classes of surface singularities defined in terms of the combinatorics of the minimal resolution (see [5,14,12,13,17,19,22–24,28,26,27,30, 31]). However, Ishii and Kollar showed in [14] a 4-dimensional example with non-bijective Nash mapping. Now the general problem has turned into characterising the class of singularities with bijective Nash mapping. Besides Nash problem, the study of arc spaces is interesting because it lays the foundations for motivic integration and because the study of its geometric properties reveals properties of the underlying varieties (see papers of Denef, Loeser, de Fernex, Ein, Ishii, Lazarsfeld, Mustata, Yasuda and others).

Nash problem seems different in nature in the surface case than in the higher-dimensional case, since birational geometry in dimension 2 is much easier than in higher dimension. For example the essential components are the irreducible components of the exceptional divisor of a minimal resolution of singularities. Although Nash problem is known for many classes of surfaces it is not yet known in general for the surface case. Even for the case of the rational double points E_6, E_7 and E_8 the proof has been obtained only very recently: see [28] for the E_6 case and [22] for the quotient surface singularities, which includes the rational double points and uses essentially the developments of this paper. Recently, and also using in a essential way the results of this paper, a proof of the bijectivity of the Nash mapping for surfaces has been obtained by M. Pe Pereira and the author [4].

From now on we shall concentrate in the surface case, we let (X, O) be a normal surface singularity defined over a field of characteristic 0, and $\mathcal{X}_\infty$ denotes the space of arcs through the singular point.

Let us explain the approach to Nash problem based on wedges, due to M. Lejeune-Jalabert [15]. Let E_u be an essential component of a surface singularity (X, O). Denote by N_{E_u} the set of arcs whose lifting meets E_u. The space of arcs centred at the singular point splits as the union of the N_{E_u}'s. It is known (Remark 2.3 of [32]) that E_v is *not* in the image of the Nash map if and only if N_{E_v} is in the Zariski closure of N_{E_u} for a different essential component E_u. If Curve Selection Lemma were true in $\mathcal{X}_\infty$ then, for any arc

$$\gamma : \operatorname{Spec}\big(\mathbb{K}[[t]]\big) \to (X, O)$$

in N_{E_v} there should exist a curve in $\mathcal{X}_\infty$ with special point γ and generic point an arc on N_{E_u}. To give a curve in $\mathcal{X}_\infty$ amounts to give a morphism

$$\alpha : \operatorname{Spec}\big(\mathbb{K}[[t, s]]\big) \to (X, O)$$

mapping the s-axis $V(t)$ to O (we see it as a "family of arcs parametrised by s"). Such a morphism is called a *wedge*. The lifting to $\tilde{X}$ of a generic arc in N_{E_v} is transverse to E_v at a smooth point of E, and if a wedge α has special arc equal to γ and generic arc in N_{E_u} it is clear that the rational lifting

$$\pi^{-1} \circ \alpha : \operatorname{Spec}\big(\mathbb{K}[[t, s]]\big) \to \tilde{X}$$

has an indetermination point at the origin, and hence there is no morphism lifting α to $\tilde{X}$. In [15], M. Lejeune-Jalabert proposes to attack Nash problem by studying the problem of lifting wedges whose special arc is a transverse arc through an essential component of (X, O).

Since Curve Selection Lemma in not known to hold in the space of arcs A. Reguera [32] introduced K-wedges, which are wedges

$$\alpha : \operatorname{Spec}\big(K[[t, s]]\big) \to (X, O)$$

defined over a field extension K of $\mathbb{K}$ and proved the following characterisation: an essential component E_v is in the image of the Nash map if and only if any K-wedge whose special arc is *the generic point* of N_{E_v} and whose generic arc is centred at the singular point O of X, admits a lifting to the resolution. However the field of definition K of the involved wedges has infinite transcendence degree over $\mathbb{K}$ and, hence, it is not easy to work with them. Building on this result, A. Reguera [32] and M. Lejeune-Jalabert [16, Proposition 2.9] proved a sufficient condition for a divisor E_v to be in the image of the Nash map based on wedges defined over the base field: it is enough to check that there is a very dense collection of closed points of E_v such that any $\mathbb{K}$-wedge whose special arc is transverse to E_v through any of these points lifts to $\tilde{X}$. A very dense set is a set which intersects any countable intersection of dense open subsets. The results of [32] and [16] hold in any dimension.

We say that a component E_u of the exceptional divisor is *adjacent* to E_v if N_{E_v} is contained in the Zariski closure of N_{E_u}. The first main result of this article is a characterisation of all the possible adjacencies between essential components of the exceptional divisor of a resolution in terms of existence of wedges defined over the base field. We are even able to show that the generic arc of the wedge satisfies certain genericity conditions. For example we can show that the generic arc is transverse to E_u.

We denote by Δ_{E_u} the set of arcs in N_{E_u} whose lifting is not transverse to E_u through a non-singular point of E. We prove (see Theorem 33 for a more precise version):

Theorem A. *Let (X, O) be a normal surface singularity defined over an uncountable algebraically closed field $\mathbb{K}$ of characteristic 0. Let E_u, E_v be different essential irreducible components of the exceptional divisor E of a resolution. We consider E with its reduced structure. Let $\mathcal{Z}$ be a proper Zariski closed subset of N_{E_u}. Equivalent are*:

(1) *The component E_u is adjacent to E_v.*
(2) *There exists a $\mathbb{K}$-wedge whose special arc has transverse lifting to E_v at a non-singular point of E and with generic arc belonging to N_{E_u}.*
(3) *There exists a $\mathbb{K}$-wedge whose special arc has transverse lifting to E_v at a non-singular point of E and with generic arc belonging to $N_{E_u} \setminus \mathcal{Z}$.*

If the base field is $\mathbb{C}$ the following conditions are also equivalent:

(a) *Given* any *convergent arc γ whose lifting is transverse to E_v at a non-singular point of E there exists a convergent $\mathbb{C}$-wedge with special arc γ and generic arc belonging to N_{E_u}.*
(b) *Given* any *convergent arc γ whose lifting is transverse to E_v at a non-singular point of E there exists a convergent $\mathbb{C}$-wedge with special arc γ and generic arc having lifting to the resolution transverse to E_u through a non-singular point of E.*

An immediate Corollary characterises the image of the Nash map in terms of $\mathbb{K}$-wedges:

Corollary B. *Let (X, O) be a normal surface singularity defined over an uncountable algebraically closed field $\mathbb{K}$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor. Equivalent are*:

(1) *The component E_v is in the image of the Nash map.*
(2) *It does not exist a different component E_u and a $\mathbb{K}$-wedge whose special arc has transverse lifting to E_v at a non-singular point of E and with generic arc belonging to N_{E_u}.*

If the base field is $\mathbb{C}$ the following condition is also equivalent:

(a) *There exists a convergent arc γ whose lifting is transverse to E_v at a non-singular point of E such that there is no convergent $\mathbb{C}$-wedge whose special arc is γ and whose generic arc has a lifting to the resolution transverse to E_u through a non-singular point of E, for a different component E_u of the exceptional divisor.*

Our result is based on the Curve Selection Lemma of A. Reguera (see Theorem 4.1 and Corollary 4.8 of [32]), and improves the statement corresponding to surfaces of Corollary 2.5 of [16] in the following sense:

- In order to prove that E_v is not in the image of the Nash map it is sufficient to exhibit a *single* wedge defined over the base field realising an adjacency.

- If $\mathbb{K} = \mathbb{C}$, in order to prove that E_v is in the image of the Nash map it is sufficient to find a *single* transverse convergent arc which cannot be the special arc of a wedge realising an adjacency. This has turned out to be essential for the proof of the bijectivity of the Nash mapping for quotient surface singularities [22].
- The convergence and the condition that the wedge avoids a proper closed subset $\mathcal{Z}$ of N_{E_u} is also very useful in practise since it allows to work geometrically with wedges whose generic arc has also a transverse lifting. See [22] for an application of these ideas.

Our condition on the wedges is more precise that the liftability, since, when a component E_v is not in the image of the Nash map, we want to keep track of the responsible adjacencies. However, we prove also an improvement of the result of [16] in terms of the original condition of lifting wedges of [15]:

Theorem C. *Let (X, O) be a normal surface singularity defined over $\mathbb{C}$. Let E_v be any essential irreducible component of the exceptional divisor of a resolution of singularities. If there exists a convergent arc γ whose lifting is transverse to E_v such that any $\mathbb{C}$-wedge having γ as special arc lifts to $\tilde{X}$, then the component E_v is in the image of the Nash map.*

The ideas of the proofs and the plan of the paper are as follows:

If the Zariski closure of N_{E_u} contains N_{E_v} we can use Corollary 4.8 of [32] to obtain a K-wedge (with K an infinite transcendence degree extension of $\mathbb{K}$) whose special arc is *the generic point* of N_{E_v} and whose generic arc lifts to E_u. After this we can follow a specialisation procedure similar to the one in [16] to obtain a $\mathbb{K}$-wedge whose special arc has transverse lifting to E_v and with generic arc belonging to N_{E_u}. What we will actually do is to produce a slight improvement of Corollary 4.8 of [32] which, after the above mentioned specialisation procedure, gives $\mathbb{K}$-wedges with the properties required in Theorem A(2) and (3). The improvement of Corollary 4.8 of [32] is needed because applying it directly we do not obtain wedges avoiding the proper closed subset $\mathcal{Z}$ of N_{E_u}. This is done in Section 3.

Most of the technique introduced in this paper is devoted to the proof of the converse statement. Let γ be an arc whose lifting to $\tilde{X}$ is transverse to E_v. A $\mathbb{K}$-wedge whose special arc is γ and with generic arc belonging to N_{E_u} (with E_u another component of the exceptional divisor) will be called a *wedge realising an adjacency* from E_u to γ.

In Section 4 we use an approximation theorem to replace a $\mathbb{K}$-wedge realising an adjacency by an algebraic one with the same property with respect to another transverse arc γ', and retaining genericity conditions.

In Section 5, using Stein factorisation, we "compactify" the wedge and factorise it through a finite covering of normal surface singularities "realising an adjacency from E_u to γ'" (see Definitions 19 and 20). We prove that there exists a wedge realising an adjacency from E_u to γ' if and only if there exists a finite covering realising an adjacency from E_u to γ'.

In Section 6 we work in the complex analytic category. We use a topological argument and a suitable change of complex structures in order to prove that given two convergent arcs γ and γ' on a complex analytic normal surface singularity, there exists a finite covering realising an adjacency from E_u to γ if and only if there exists a finite covering realising an adjacency from E_u to γ'. This technique allows to move wedges in a very flexible way, and is the key to the following surprising result (see Theorem 34, and Corollary 37):

Theorem D. *The set of adjacencies between exceptional divisors of a normal surface singularity is a combinatorial property of the singularity: it only depends on the dual weighted graph of the minimal good resolution. In the complex analytic case this means that the set of adjacencies only depends on the topological type of the singularity, and not on the complex structure.*

In Section 7 we use Lefschetz principle to reduce the existence of a finite covering realising an adjacency from E_u to a γ' to the analogue statement in the complex analytic case. This allows to finish the proof the main results of the paper.

Finally, in Section 8 we use the fact that Nash problem is topological to study and compare the adjacency structure of different singularities: we prove reductions of Nash problem for singularities with symmetries in the dual weighted graph of the minimal good resolution (see Proposition 39 and Corollary 40). We prove results comparing the adjacency structure of different singularities (see Corollary 41). We also reduce the Nash problem in the following sense: we introduce *extremal graphs*, which is a subclass of the class of dual graphs with only rational vertices and no loops (see Definition 45), and *extremal rational homology spheres*, which are the plumbing 3-manifolds associated with extremal graphs.

Corollary E. *If the Nash mapping is bijective for normal surface singularities whose minimal good resolution graph is extremal then it is bijective in general. Equivalently, if the Nash mapping is bijective for all complex analytic normal surface singularities having extremal $\mathbb{Q}$-homology sphere links then it is bijective for any normal surface singularity.*

The last corollary improves Proposition 4.2 of [16], which reduces Nash problem for surfaces to the class of surfaces having only rational vertices in their resolutions, and makes essential use of Theorem D.

2. Terminology

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. Let X be a normal algebraic surface over $\mathbb{K}$ with a singularity at a point O. Since we will be interested in the germ (X, O) we may assume X to be embedded in $\mathbb{K}^N$, O being the origin. Denote by I the defining ideal of X and let $(F_1, \dots, F_r)$ be generators of I. If the coordinates of $\mathbb{K}^N$ are $x_1, \dots, x_N$ then each F_i is a polynomial in these variables.

Unless we state the contrary we denote by $\pi : \tilde{X} \to X$ an Hironaka resolution of the singularity of X at O (obtained by a blowing-up at an ideal whose zero set is the singular locus [9]). Let $E = \bigcup_{u=1}^{s} E_u$ be the decomposition in irreducible components of E.

Given a field K containing the base field $\mathbb{K}$, a K-arc in (X, O) is a scheme morphism

$$\gamma : \mathrm{Spec}\big(K[[t]]\big) \to (X, O).$$

It is determined by the formal power series $x_i \circ \gamma(t) \in K[[t]]$ for $1 \leqslant i \leqslant n$ (the coordinate series). A $\mathbb{K}$-arc is algebraic if its coordinate series are algebraic power series: there exists a polynomial $P_i \in \mathbb{K}[t, x]$ such that $P_i(t, x_i \circ \gamma(t)) = 0$ for any $i \leqslant N$. In the case that $\mathbb{K} = K = \mathbb{C}$, a $\mathbb{C}$-arc is convergent if its coordinate series are convergent.

Denote by $\mathcal{X}_n$ the set of n-jets in X centred at the origin, i.e. the scheme representing the contravariant functor from schemes to sets which, for any field K, assigns to $\mathrm{Spec}(K)$ the set of morphisms

$$\mathrm{Spec}\big(K[[t]]/\big(t^{n+1}\big)\big) \to (X, O)$$

sending the only point to the origin O. There are obvious truncation morphisms $\mathcal{X}_n \to \mathcal{X}_m$ for any $n \geqslant m$, which form a projective system. Denote by $\mathcal{X}_\infty$ be arc space of (X, O), with its infinite-dimensional scheme structure given by the limit of the projective system described above. The Zariski topology of the arc space is the projective limit of the Zariski topologies of the n-jets. There is a natural bijection between the set of K-arcs and the K-valued points of $\mathcal{X}_\infty$. See [14] for a more detailed exposition on arc spaces.

Denote by N_{E_u} the subspace of formal arcs whose lifting to $\tilde{X}$ sends the special point into E_u; these arcs are called *arcs through E_u*.

Definition 1. Let p be a regular point of E, that is $\mathcal{O}_{E,p}$ is a regular local ring. Let γ be a K-arc for a given field K such that we have $\gamma(0) = p$. Let h be a reduced equation for E at $(\tilde{X}, p)$. The arc γ is *transverse* to E if and only if the first derivative $(h \circ \gamma)'(0)$ does not vanish.

Let $\bar{N}_{E_u}$ be its Zariski closure in $\mathcal{X}_\infty$ and $\dot{N}_{E_u}$ the subset of arcs in N_{E_u} whose lifting meets E_u transversely at a non-singular point of E; these arcs are called *transverse arcs through E_u*.

Given any algebraic variety X a K-wedge in X is a scheme morphism

$$\alpha : \mathrm{Spec}\big(K[[t, s]]\big) \to X.$$

Denote the space of wedges by $\mathcal{X}_{\infty,\infty}$. As the space of arcs it is a scheme [16].

A K-wedge in (X, O) is a K-wedge in X

$$\alpha : \mathrm{Spec}\big(K[[t, s]]\big) \to (X, O)$$

which sends the point $V(t)$ to O. Its coordinate series are formal power series in the variables t, s. A $\mathbb{K}$-wedge is said to be algebraic if all the coordinate series are algebraic (that is, satisfy a polynomial equation $P_i(t, s, x_i \circ \alpha(t, s)) = 0$ for $P_i \in \mathbb{K}[t, s, x]$). In the case that $\mathbb{K} = K = \mathbb{C}$, a $\mathbb{C}$-wedge is convergent if its coordinate series are convergent.

Given a K-wedge, we call the K-arc $\alpha_s(t) := \alpha(t, 0)$ the *special* K-arc associated to it, and the $K((s))$-arc α_g (same coordinate series than α, but viewed in $K((s))[[t]]$) the *generic* $K((s))$-arc associated to it.

We will denote the space of wedges in (X, O) by $\mathcal{X}^{Sing}_{\infty,\infty}$. It is a subscheme of $\mathcal{X}_{\infty,\infty}$. There is a natural bijection between the set of K-wedges in (X, O) and the K-valued points of $\mathcal{X}^{Sing}_{\infty,\infty}$ (see [16]).

Definition 2. A K-wedge in (X, O) realises an adjacency from E_u to E_v if its generic arc belongs to N_{E_u} and its special arc belongs to $\dot{N}_{E_v}$ (it is transverse to E_v).

We will be able to produce wedges for which the generic arc satisfies any finite amount of genericity conditions. This can be formulated as follows:

Definition 3. Let $\mathcal{Z}$ be a proper closed subset of $\bar{N}_{E_u}$. A K-wedge realises an adjacency from E_u to E_v avoiding $\mathcal{Z}$ if it realises an adjacency from E_u to E_v and its generic arc does not belong to $\mathcal{Z}$.

A very natural class of wedges are those realising an adjacency and for which the generic arc is transverse to E_u. For this we define the proper closed subset $\Delta_{E_u} \subset \bar{N}_{E_u}$ to be the Zariski closure of $N_{E_u} \setminus \dot{N}_{E_u}$. Any wedge realising an adjacency from E_u to E_v and avoiding Δ_{E_u} has the desired properties.

Remark 4. A $\mathbb{K}$-wedge α realising an adjacency from E_u to E_v is a wedge with special arc in $\dot{N}_{E_v}$ which does not lift to $\tilde{X}$ (that is, the rational map $\pi^{-1} \circ \alpha$ is not defined at the origin of $\mathrm{Spec}(\mathbb{K}[[t,s]])$). This makes the link with the original wedge problem of [15].

3. Existence of formal wedges over the base field

The aim of this section is to prove that (1) implies (3) in Theorem A. We follow closely the method of [32] and [16], which consists in two steps: finding a wedge defined over a field extension $\mathbb{K} \subset K$, and perform a specialisation procedure to deduce the existence of wedges over the base field. In [32] and [16] wedges that do not lift to the resolution are produced. Our aim is to produce wedges realising adjacencies avoiding a closed subset $\mathcal{Z}$ in $\bar{N}_{E_u}$. Both conditions are related, but are not the same: a wedge realising an adjacency does not lift to the resolution, but a wedge which does not lift to the resolution does not necessarily realise an adjacency. The condition of avoiding a closed subset $\mathcal{Z}$ is not considered in [32] and [16].

3.1. An improved Curve Selection Lemma

Let E_v be an essential divisor. Suppose that there is an adjacency from E_u to E_v. Let z be the generic point corresponding to the irreducible closed subset $N_{E_v} \subset \mathcal{X}_\infty$, and k_z its residue field. Corollary 4.8 of [32] is a Curve Selection Lemma in arc spaces which implies (see the proof of Theorem 5.1 of [32]) that there exists a finite extension K of k_z and a K-wedge whose special arc is the generic point of N_{E_v} and whose generic arc belongs to N_{E_u}. Here we improve the Curve Selection Lemma so we can show that the generic arc of the wedge also satisfies certain genericity conditions. The proof we give here is a modification of the proof of Corollary 4.8 of [32], and is a Corollary of Theorem 4.1 of [32] as well. However we want to make several points more explicit for later use and we also want to prove the existence of wedges avoiding a proper closed subset $\mathcal{Z}$ of N_{E_u}. We shall follow closely [32].

Definition 5. (See [32].) An irreducible Zariski-closed subset N of $\mathcal{X}_\infty$ is generically stable if there exists an affine open subscheme U of $\mathcal{X}_\infty$ such that $N \cap U$ is non-empty and its defining ideal is the radical of a finitely generated ideal.

Let N be any irreducible closed subset of $\mathcal{X}_\infty$, denote by z its scheme-theoretic generic point and k_z its residue field. Notice that $\mathcal{X}_\infty$ is an affine scheme. Let R be its coordinate ring. There is a prime ideal $I \subset R$ defining the set N. Then k_z is the quotient field of R/I and the point z is the k_z-valued point

$$\mathrm{Spec}(k_z) \to \mathcal{X}_\infty$$

associated to the natural homomorphism given by the composition

$$R \to R/I \hookrightarrow k_z.$$

Recall that the arc space $\mathcal{X}_\infty$ is the scheme representing the contravariant functor which assigns to any affine scheme $\mathrm{Spec}(A)$ the set of A-arcs

$$\gamma : \mathrm{Spec}\bigl(A[[t]]\bigr) \to X$$

which send the closed subset $V(t)$ to the origin O of X. By the above discussion the generic point z of N is identified with a k_z-arc in $\mathcal{X}_\infty$. Similarly giving a K-wedge in X is the same as giving a morphism

$$\alpha : \mathrm{Spec}\bigl(K[[s]]\bigr) \to \mathcal{X}_\infty.$$

Here is our improved Curve Selection Lemma, whose proof is an adaptation of the proof of Corollary 4.8 of [32]:

Lemma 6. *Let N and N' be two irreducible closed subsets of $\mathcal{X}_\infty$ such that N is generically stable, contained in N' and different to it. Let $\mathcal{Z}$ be any other closed subset of $\mathcal{X}_\infty$ such that N' is not contained in it. Let z be the generic point of N and k_z its residue field. There exist a finite field extension $k_z \subset K$ and a K-wedge whose special arc is the generic point z and whose generic arc belongs to $N' \setminus \mathcal{Z}$.*

Proof. By the discussion preceding the lemma finding such a wedge is equivalent to finding a finite extension $k_z \subset K$ and a morphism

$$\alpha : \mathrm{Spec}\bigl(K[[s]]\bigr) \to N' \tag{1}$$

such that the image of the generic point falls in $N' \setminus \mathcal{Z}$ and, if $\mathcal{X}_\infty = \mathrm{Spec}(R)$ and if I is the ideal defining N in $\mathcal{X}_\infty$, the restriction of α to the closed point of $\mathrm{Spec}(K[[s]])$ is associated to the composition of the natural ring homomorphisms

$$R \to R/I \hookrightarrow k_z \hookrightarrow K. \tag{2}$$

By Corollary 4.6 (i) of [32], since N is generically stable the completion $\hat{\mathcal{O}}_{N',z}$ of the local ring $\mathcal{O}_{N',z}$ at the maximal ideal is a Noetherian ring of dimension at least 1 (the inclusion $N \subset N'$ is strict). The ring $\hat{\mathcal{O}}_{N',z}$ is equicharacteristic since it contains the base field $\mathbb{K}$. Since it is complete, by Theorem 28.3 of [18] it has a coefficient field. That is, there is an inclusion of a field L in $\hat{\mathcal{O}}_{N',z}$ such that composed by the quotient map by the maximal ideal gives an isomorphism to the residue field k_z. Therefore $\hat{\mathcal{O}}_{N',z}$ is a quotient $k_z[[X_1, \ldots, X_r]]/\mathcal{I}$, with r the minimal number of generators of the maximal ideal, and $\mathcal{I}$ an ideal in $k_z[[X_1, \ldots, X_r]]$.

Let J be the ideal of $\mathcal{Z} \cap N'$ in the coordinate ring of $\mathcal{O}_{N',z}$. Denote by $\hat{J}$ the ideal generated by J in $\hat{\mathcal{O}}_{N',z}$. Let d be the dimension of $\hat{\mathcal{O}}_{N',z}$. If $d = 1$ then $\sqrt{\hat{J}}$ equals the maximal ideal of $\hat{\mathcal{O}}_{N',z}$ because otherwise $\mathcal{Z}$ contains N'. If $d > 1$ and $\sqrt{\hat{J}}$ is different to the maximal ideal then, by Prime Avoidance (see Lemma 3.3 of [3]), there exists an element b_1 in $\hat{\mathcal{O}}_{N',z}$ and not belonging to any minimal prime containing $\hat{J}$, such that the dimension of $\mathcal{O}_{N',z}/(b_1)$ equals

$d-1$. Define $\hat{J}_1 = \hat{J} + (b_1)$. By construction no irreducible component of $\mathrm{Spec}(\mathcal{O}_{N',z}/(b_1))$ is contained in $V(\hat{J}_1)$ and hence the dimension of $V(\hat{J}_1)$ is at most $d-2$. We proceed inductively and construct $b_1, \ldots, b_{d-1}$ such that the dimension of $\mathrm{Spec}(\mathcal{O}_{N',z}/(b_1, \ldots, b_{d-1}))$ equals 1 and the dimension of $V(\hat{J}_{d-1})$ is at most 0, where $\hat{J}_{d-1}$ equals $\hat{J} + (b_1, \ldots, b_{d-1})$.

Since $\mathcal{O}_{N',z}/(b_1, \ldots, b_{d-1})$ is the quotient $k_z[[X_1, \ldots, X_r]]/\mathcal{I} + (b_1, \ldots, b_{d-1})$ we have that $\mathcal{C} = \mathrm{Spec}(\mathcal{O}_{N',z}/(b_1, \ldots, b_{d-1}))$ is an algebroid curve over k_z, and since $V(\hat{J}_{d-1})$ is at most 0-dimensional it contains at most the closed point of the curve. Let $\bar{k}_z$ be the algebraic closure of k_z. In the ring $\bar{k}_z[[X_1, \ldots, X_r]]$ the ideal $\sqrt{\mathcal{I} + (b_1, \ldots, b_{d-1})}$ can be expressed as the intersection of finitely many prime ideals $\mathcal{P}_1, \ldots, \mathcal{P}_m$, each of them corresponding to a geometrically irreducible component $\mathcal{C}_i$ of the curve $\mathcal{C}$. By Noetherianity each of the curves $\mathcal{C}_i$ is defined over a finite extension K of k_z.

The desired morphism α may be taken as the composition of the normalisation mapping

$$\alpha : \mathrm{Spec}\big(K[[s]]\big) \to \mathcal{C}_1$$

with the natural morphism

$$\mathcal{C}_1 \to N'. \qquad \square$$

3.2. Locally closed subsets of wedges realising adjacencies

Having a K-wedge realising an adjacency from E_u to E_v avoiding $\mathcal{Z}$ our aim is to find a wedge with the same properties, but defined over the base field. Here we follow [16]. The first step is to prove the following analog of Proposition 2.5 in [16]. The way of proving the proposition prepares the way to an approximation result for wedges needed in the next section.

Proposition 7. *Let E_u and E_v be two essential divisors of a resolution of X and $\mathcal{Z}$ be a proper closed subset of $\bar{N}_{E_u}$. Assume that there is a K-wedge α realising an adjacency from E_u to E_v avoiding $\mathcal{Z}$ and such that the image of the special point of the special arc α_s is the generic point of E_v. Then there is a locally closed subset Λ of the space of wedges $\mathcal{X}^{Sing}_{\infty,\infty}$ such that the K-point associated to α belongs to it and, for any field extension $\mathbb{K} \subset L$, any L-point of Λ corresponds to a L-wedge realising an adjacency from E_u to E_v avoiding $\mathcal{Z}$.*

Some discussion is needed before proving Proposition 7.

3.2.1. Power series expansions of arcs with transverse lifting

Here we give finitely many polynomial conditions ensuring that an arc is in N_{E_u} or in $\dot{N}_{E_u}$ for a certain component E_u of the exceptional divisor. The discussion is similar to that of Proposition 3.8 of [32].

Recall that $\mathbb{K}[X] = \mathbb{K}[x_1, \ldots, x_N]/I$ denotes the coordinate ring of X. By Main Theorem I of [9] there exists an ideal $\mathcal{J}$ in $\mathbb{K}[X]$ of X such that the resolution π is the blow-up with respect to $\mathcal{J}$ and such that its zero set is the origin. Consider polynomials $g_0, \ldots, g_s$ in $x_1, \ldots, x_N$ which generate $\mathcal{J}$. Let $\tilde{X}$ be the Zariski closure in $X \times \mathbb{P}^s$ of the graph of the mapping

$$(g_0 : \ldots : g_s) : X \setminus \{O\} \to \mathbb{P}^s.$$

The morphism $\pi : \tilde{X} \to X$ coincides with the restriction to $\tilde{X}$ of the projection of $X \times \mathbb{P}^s$ to the first factor. Consequently, the exceptional divisor E is an algebraic subset of $\mathbb{P}^s$. Denote by J_u the homogeneous ideal associated to the component E_u; let $h_{u,1}, \ldots, h_{u,k_u}$ be homogeneous polynomials generating J_u.

Consider indeterminates $A_{i,j}$, where $i \in \{1, \ldots, N\}$ and $j \in \mathbb{N}$, for each $i \leqslant N$ we consider the series

$$Y_i(t) := \sum_{j \in \mathbb{N}} A_{i,j} t^j$$

inside the ring $\mathbb{K}[A_{i,j}][[t]]$. The tuple

$$\Gamma(t) = \big(Y_1(t), \ldots, Y_N(t)\big)$$

is a $\mathbb{K}[A_{i,j}]$-point in the arc space of $\mathbb{K}^N$ which we call the universal arc. The coordinates of any K-arc are obtained by substituting the variables $A_{i,j}$ by values $a_{i,j} \in K$.

For any $l \in \{0, \ldots, s\}$, if we consider the power series expansion

$$g_l\big(Y_1(t), \ldots, Y_n(t)\big) := \sum_{k \in \mathbb{N}} Q_{l,k} t^k,$$

each coefficient $Q_{l,k}$ is a polynomial in $\mathbb{K}[A_{i,j}]$, for $i \in \{1, \ldots, N\}$ and $j \leqslant k$.

Definition 8. Given any K-arc

$$\gamma(t) = \left(\ldots, \sum_{j \in \mathbb{N}} a_{i,j} t^j, \ldots\right)$$

we define the first order of γ with respect to π to be the minimum $m(\gamma)$ of the orders in t of the power series $g_i(\gamma(t))$ for $i \in \{0, \ldots, s\}$.

Notice that $m(\gamma) = \min\{\mathrm{ord}(g \circ \gamma)\colon g \in \mathcal{J}\}$. Therefore the definition above does not depend on the choice of generators of $\mathcal{J}$.

A function $\theta : \mathcal{X}_\infty :\to \mathbb{Z}$ is *upper semicontinuous*, in the Zariski topology if $\theta^{-1}[a, +\infty)$ is Zariski-closed for any $a \in \mathbb{Z}$.

Consider the following system $S_1(M)$ of polynomial equations:

$$Q_{l,k}(A_{i,j}) = 0 \quad \text{for } l \in \{0, \ldots, s\} \text{ and } k < M. \tag{3}$$

By definition of $m(\gamma)$, the coefficients $a_{i,j}$ of the coordinate series of γ satisfy the system $S_1(M)$ for $M = m(\gamma)$, but not for $M = m(\gamma) + 1$. This leads to the following:

Remark 9. The first order of γ with respect to π is upper semicontinuous in the Zariski topology of $\mathcal{X}_\infty$.

Given any arc $\gamma \in \mathcal{X}_\infty$, the lifting

$$\tilde{\gamma} : \mathrm{Spec}\big(K[[t]]\big) \to \tilde{X}$$

of γ to $\tilde{X}$ has the coordinate expansion

$$\tilde{\gamma}(t) := \big(\big(x_1\big(\gamma(t)\big), \ldots, x_N\big(\gamma(t)\big)\big), \big(g_0\big(\gamma(t)\big) : \ldots : g_s\big(\gamma(t)\big)\big)\big) \in \mathbb{C}^N \times \mathbb{P}^s. \tag{4}$$

Dividing each $g_i(\gamma(t))$ by $t^{m(\gamma)}$, we can evaluate $\tilde{\gamma}$ at 0 and obtain:

$$\tilde{\gamma}(0) = \big((0, \ldots, 0), \big(Q_{0,m(\gamma)}(a_{i,j}) : \ldots : Q_{s,m(\gamma)}(a_{i,j})\big)\big), \tag{5}$$

where at least one of the coordinates $Q_{l,m(\gamma)}(a_{i,j})$ is different from 0.

The arc γ belongs to N_{E_u} if and only if its special point is sent inside E_u, and this happens precisely when

$$h_{u,e}\big(Q_{0,m(\gamma)}(a_{i,j}) : \ldots : Q_{s,m(\gamma)}(a_{i,j})\big) = 0. \tag{6}$$

Hence, for any M, we define the system $S_2(M)$ of polynomial equations with coefficients in $\mathbb{K}$ in the indeterminates $A_{i,j}$

$$h_{u,e}\big(Q_{0,M}(A_{i,j}) : \ldots : Q_{s,M}(A_{i,j})\big) = 0, \tag{7}$$

for $e \in \{1, \ldots, k_u\}$. Rephrasing the above discussion we have that γ belongs to N_{E_u} if and only if the coefficients of γ satisfy the system $S_2(m(\gamma))$.

Let U be a Zariski open subset of E_u. Let $C := E_u \setminus U$. An argument similar to the one above shows that, for any $M \in \mathbb{N}$ there exists a system of finitely many polynomial equations $S_3(M)$ such that the lifting of an arc $\gamma \in \mathcal{X}_\infty$ meets C if and only if the coefficients of γ satisfy the system $S_3(m(\gamma))$.

In order to characterise transverse arcs we recall the proof of Lemma 2.3 of [16]. Let ν be the divisorial valuation associated to E_u. Suppose, after a possible reordering, that $n_0 := \nu(g_0) = \min(\nu(g_0), \ldots, \nu(g_s))$. There is a Zariski open subset U of E_u such that an arc γ has transverse lifting to E_u through a point of U if an only if $\mathrm{ord}_t(g_0(\gamma)) = n_0$. Moreover, for any arc having lifting through E_u we have $\mathrm{ord}_t(g_0(\gamma)) \geqslant n_0$. Rephrasing, if γ is an arc with lifting through E_u then γ lifts tranversely to E_u if and only if the following polynomial inequality is satisfied by the coefficients of γ:

$$Q_{0,n_0}(A_{i,j}) \neq 0. \tag{8}$$

After this discussion the following proposition is obvious:

Proposition 10. *For any essential divisor E_u and a natural number M we have*:

(1) *The set $N(E_u, M)$ of arcs with lifting through E_u and first order with respect to π equal to M is the set of arcs in $\mathcal{X}_\infty$ whose coefficients satisfy the systems $S_1(M)$, $S_2(M)$ and do not satisfy the system $S_1(M+1)$. This is a finite amount of polynomial equalities and inequalities in the coefficients of the arc.*
(2) *There is a Zariski open subset $U \subset E_u$ such that the set of arcs $\dot{N}(U, M)$ with transverse lifting through U and first order with respect to π equal to M is a locally closed subset of $\mathcal{X}_\infty$ defined by finitely many polynomial equalities and inequalities.*

3.2.2. Power series expansions of wedges realising adjacencies

Given indeterminates $B_{i,j,k}$ with $1 \leqslant i \leqslant N$ and $j, k \in \mathbb{N}$ the power series expansion

$$\Omega(t,s) := \left(\sum_{j,k\in\mathbb{N}} B_{1,j,k}t^j s^k, \dots, \sum_{j,k\in\mathbb{N}} B_{i,j,k}t^j s^k, \dots, \sum_{j,k\in\mathbb{N}} B_{N,j,k}t^j s^k\right).$$

is a $\mathbb{K}[B_{i,j,k}]$-point in the space of wedges in $\mathbb{K}^N$, which we call the universal wedge. A K-wedge α is obtained by substituting the variables $B_{i,j,k}$ by elements $b_{i,j,k} \in K$. Its associated generic arc $\alpha_g(t)$ has coordinate power series

$$\sum_{j\in\mathbb{N}} a_{i,j}t^j \tag{9}$$

with

$$a_{i,j} = \sum_{k\in\mathbb{N}} b_{i,j,k}s^k \in K[[s]]. \tag{10}$$

Definition 11. Let α be a K-wedge. We define the *first order $m(\alpha)$ of α with respect to π* to be equal to the first order $m(\alpha_g)$ of its generic arc with respect to π.

Definition 12. Let l be the minimal index such that $Q_{l,m(\alpha_g)}(a_{i,j}) \neq 0$. *The second order of α with respect to π* is the order of $Q_{l,m(\alpha_g)}$ in s, and is denoted by $R(\alpha)$.

Since $Q_{l,m(\alpha_g)}$ only depends on the coefficients $a_{i,j}$ when $j \leqslant m(\alpha_g)$, the coefficient of the term of order $R(\alpha)$ of $Q_{l,m(\alpha_g)}$ is obtained by substitution of the variables $B_{i,j,k}$ by the coefficients $b_{i,j,k}$ in a polynomial

$$P_\alpha(B_{i,j,k}) \tag{11}$$

in $\mathbb{K}[B_{i,j,k}]$ for $j \leqslant m(\alpha_g)$ and $k \leqslant R(\alpha)$. Observe that the polynomial P_α only depends on the numbers l, $m(\alpha_g)$ and $R(\alpha)$.

Definition 13. *The characteristic polynomial P_α of α with respect to π* is the polynomial in $\mathbb{K}[B_{i,j,k}]$ defined in (11).

Lemma 14. *Given a K-wedge α and a K'-wedge α' (with K and K' possibly different). Denote by $b'_{i,j,k}$ the coefficients of the coordinate series of α' and by $a'_{i,j}$ the coefficients of the coordinate series of α'_g. If the systems of equations $S_1(m(\alpha))$ (see (3)) and $S_2(m(\alpha))$ (see (7)) are satisfied by substituting $A_{i,j}$ by $a'_{i,j}$, and if we have*

$$P_\alpha\left(b'_{i,j,k}\right) \neq 0 \tag{12}$$

then the generic arc α'_g belongs to N_{E_u} and the first characteristic order of α' with respect to π equal to $m(\alpha)$.

Proof. Since the system of equations $S_1(m(\alpha))$ is satisfied by the $a'_{i,j}$'s the first characteristic order of α' is at least $m(\alpha)$. Since

$$P_\alpha\left(b'_{i,j,k}\right) \neq 0 \tag{13}$$

the system $S_1(m(\alpha)+1)$ is not satisfied by the $a'_{i,j}$'s, and thus the first characteristic order of α' equals $m(\alpha)$. Since the system $S_2(m(\alpha))$ is satisfied by the $a'_{i,j}$'s, by Proposition 10 the generic arc of α' belongs to N_{E_u}. $\square$

3.2.3. Wedges avoiding a closed subset

Let $\mathcal{Z}$ be a proper closed subset of $\bar{N}_{E_u}$. Let $\mathcal{I}$ be the ideal defining $\mathcal{Z}$ in $\mathbb{A}^N_{\mathbb{K}}$. Notice that the coordinate ring of the space of arcs in $\mathbb{A}^N_{\mathbb{K}}$ is the ring of polynomials in the indeterminates $A_{i,j}$ with coefficients in $\mathbb{K}$.

Let α be a K-wedge with coordinate series

$$\left(\sum_{j,k\in\mathbb{N}} b_{1,j,k}t^j s^k, \ldots, \sum_{j,k\in\mathbb{N}} b_{i,j,k}t^j s^k, \ldots, \sum_{j,k\in\mathbb{N}} b_{N,j,k}t^j s^k\right).$$

Its generic arc α_g is of the form (9), with the $a_{i,j}$ defined by formula (10). If α_g is a $K((s))$-point of $N_{E_u} \setminus \mathcal{Z}$ then there is a polynomial $H \in \mathcal{I}$ such that $H(\alpha_g) = H(a_{i,j})$ is a non-zero power series in $K[[s]]$. Since H depends on finitely many variables for being a polynomial, we have that there is a positive l_1 such that H depends only on $A_{i,j}$ for $j \leqslant l_1$. Let l_2 be the order in s of $H(a_{i,j})$. Clearly, there is a polynomial

$$\Phi_\alpha \in \mathbb{K}[B_{i,j,k}] \tag{14}$$

depending only on the variables $B_{i,j,k}$ for $i \leqslant N$, $j \leqslant l_1$ and $k \leqslant l_2$ such that the coefficient in s^{l_2} is a non-zero element of K obtained by substitution of $b_{i,j,k}$ into the polynomial Φ_α.

After this preparation we can prove Proposition 7.

Proof of Proposition 7. Let $m(\alpha_s)$ be the first order of the special arc of the wedge α with respect to π. By Proposition 10 the set $\dot{N}(U, m(\alpha_s))$ is a locally closed subset of $\mathcal{X}_\infty$. The mapping

$$\mathcal{S} : \mathcal{X}^{Sing}_{\infty,\infty} \to \mathcal{X}_\infty$$

assigning to any wedge its special arc is a morphism of infinite-dimensional algebraic varieties. Therefore $\Theta_1 := \mathcal{S}^{-1}(\dot{N}(U, m(\alpha_s)))$ is a locally closed subset of $\mathcal{X}^{Sing}_{\infty,\infty}$.

Let $m(\alpha)$ be the first order of the wedge α with respect to π. Since the generic arc α_g belongs to N_{E_u}, by Proposition 10 the systems $S_1(m(\alpha))$ and $S_2(m(\alpha))$ are satisfied by the coordinates $a_{i,j} = \sum_{k\in\mathbb{N}} b_{i,j,k}s^k$ of the generic arc α_g.

Let $P_\alpha \in \mathbb{K}[b_{i,j,k}]$ be the characteristic polynomial of α with respect to π. For any wedge

$$\alpha'(t,s) = \left(\sum_{j,k\in\mathbb{N}} b'_{1,j,k}t^j s^k, \ldots, \sum_{j,k\in\mathbb{N}} b'_{N,j,k}t^j s^k\right)$$

such that the coefficients of its generic arc satisfy system $S_1(m(\alpha))$ and such that the inequality

$$P_\alpha\left(b'_{1,j,k}\right) \neq 0 \tag{15}$$

holds, we have that the first order of α' with respect to π equal to $m(\alpha')$.

We conclude that a wedge α' has a generic arc belonging to N_{E_u} as long as the systems $S_1(m(\alpha))$ and $S_2(m(\alpha))$ are satisfied by the coordinates $a_{i,j} = \sum_{k\in\mathbb{N}} b_{i,j,k}s^k$ of the generic arc α'_g, and it is such that inequality (15) holds. We denote by Θ_2 the corresponding locally closed subset of $\mathcal{X}^{Sing}_{\infty,\infty}$.

The locally closed subset $\Lambda = \Theta_1 \cap \Theta_2$ has the desired properties. □

3.3. *Specialisation of wedges*

Let E_u, E_v be components of the exceptional divisor of the resolution π with E_v being essential. By Proposition 3.8 of [32] the set $\bar{N}_{E_v}$ is generically stable. Let $\mathcal{Z}$ be any proper closed subset of $\bar{N}_{E_u}$. By Lemma 6 we obtain a finite extension K of k_z and a K-wedge α whose special arc is the generic arc of N_{E_v} and whose generic arc belongs to $N_{E_u} \setminus \mathcal{Z}$. We follow a specialisation method of [16] to produce a wedge defined over the base field satisfying the same requirements.

Proposition 15. *Suppose that $\mathbb{K}$ is uncountable. Suppose that there is an adjacency from E_u to E_v, that is we have the inclusion $N_{E_v} \subset \bar{N}_{E_u}$. Let $\mathcal{Z}$ be any proper closed subset of $\bar{N}_{E_u}$. There exists a formal $\mathbb{K}$-wedge realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*

Proof. The proof follows the method of Section 2 of [16]. Let z be the generic point of $\bar{N}_{E_v}$ and k_z its residue field. We apply Lemma 6 and obtain a finite extension K of k_z and a K-wedge α whose special arc is the generic arc of N_{E_v} and whose generic arc is belongs to $N_{E_u} \setminus \mathcal{Z}$.

In the proof of Proposition 2.10 of [11] it is literally shown that, if $\mathbb{K}$ is uncountable, any prime ideal of the polynomial ring in countably many variables $(x_1, \ldots, x_n, \ldots)$ is contained in a maximal ideal of the form $(x_1 - a_1, \ldots, x_n - a_n, \ldots)$, with $a_i \in \mathbb{K}$ for any i. This implies that the set of closed points in the affine space $\operatorname{Spec}(\mathbb{K}[x_1, \ldots, x_n, \ldots])$ of countable dimension is the set of $\mathbb{K}$-valued points. Moreover, for any ideal I the set of $\mathbb{K}$-valued points form a dense subset in

$$\operatorname{Spec}\left(\mathbb{K}[x_1, \ldots, x_n, \ldots]/I\right).$$

Indeed, a basis of the Zariski topology of $\operatorname{Spec}(\mathbb{K}[x_1, \ldots, x_n, \ldots]/I)$ is given by open subsets of the form

$$\operatorname{Spec}\left(\mathbb{K}[x_1, \ldots, x_n, \ldots]/I\right) \setminus V(f),$$

being $V(f)$ the set of prime ideals containing a polynomial $f \in \mathbb{K}[x_1, \ldots, x_n, \ldots]$. But such an open subset is isomorphic to

$$\operatorname{Spec}\left([x_0, x_1, \ldots]/(x_0 f - 1) + I\right),$$

a Zariski closed subset of $\operatorname{Spec}([x_0, x_1, \ldots])$, which therefore contains a $\mathbb{K}$-valued point.

Since $\mathcal{X}_{\infty,\infty}$ is a Zariski closed subset in the affine space of countable dimension we deduce that its closed points are $\mathbb{K}$-valued points, and that they are dense. In Proposition 7 we have proved that there is a non-empty locally closed subset Λ of wedges realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$. The density of $\mathbb{K}$-valued points implies the existence of a $\mathbb{K}$-point in Λ, which corresponds to a $\mathbb{K}$-wedge with the desired properties. □

4. Approximation of wedges

The aim of this section is to approximate wedges realising adjacencies, avoiding closed sets and defined over the base field by wedges with the same properties, but defined by algebraic power series. The idea is to use the following approximation theorem, due to Becker, Denef, Lipshitz and van den Dries (Theorem 4.1 of [2]), which later was generalised by Popescu [29]. I thank G. Rond and H. Hauser for pointing me out these approximation theorems.

Theorem 16. *Let t, s and $y_1, \ldots, y_r$ be variables. Consider polynomials $G_1, \ldots, G_k \in \mathbb{K}[t, s, y_1, \ldots, y_r]$. Suppose that there exist formal power series $y_1(s), \ldots, y_{r'}(s)$ and $y_{r'+1}(t, s), \ldots, y_r(t, s)$ such that $G_i(t, s, y_1(s), \ldots, y_r(t, s)) = 0$ for any $i \leqslant k$. For any $L > 0$ there exist* algebraic *power series $y'_1(s), \ldots, y'_{r'}(s)$ and $y_{r'+1}(t, s), \ldots, y_r(t, s)$ such that $G_i(t, s, y'_1(s), \ldots, y'_r(t, s)) = 0$ for any $i \leqslant k$ and y'_j coincides with y_j up to order L.*

Proposition 17. *Given any $\mathbb{K}$-wedge α realising an adjacency from E_u to E_v, and any natural number L, there exists an algebraic wedge β realising an adjacency from E_u to E_v such that their power series expansion coincide up to order L. Moreover if $\mathcal{Z}$ is a proper closed subset of $\bar{N}_{E_u}$ and α avoids it then β can be chosen avoiding $\mathcal{Z}$.*

Proof. If the generic arc α_g does not belong to $\mathcal{Z}$ then let l_1, l_2 and Φ_α be the natural numbers and the polynomial introduced in Section 3.2.3. Otherwise set $l_1 = l_2 = 0$. Recall from Section 3.2.1 that $n_0 := \min(\nu(g_0), \ldots, \nu(g_s))$.

We define

$$m := \max\{m(\alpha_s), m(\alpha_g)\}$$

to be the maximum of the orders of the special and generic arcs with respect to π and assume

$$L > \max\{m, R(\alpha), l_1, l_2, n_0\}.$$

We write the wedge α according with the expression of its generic arc as

$$\alpha(t) := \left(\sum_{j\in\mathbb{N}} a_{1,j}(s)t^j, \ldots, \sum_{j\in\mathbb{N}} a_{N,j}(s)t^j\right),$$

with $a_{i,j}(s) = \sum_{k\in\mathbb{N}} b_{i,j,k}s^k$. We regroup this expansion in the following way:

$$\alpha(t,s) := \left(\sum_{j=1}^{m} a_{1,j}(s)t^j + c_1(t,s)t^{m+1}, \ldots, \sum_{j=1}^{m} a_{N,j}(s)t^j + c_N(t,s)t^{m+1}\right), \tag{16}$$

with $c_i(s,t) \in \mathbb{K}[[t,s]]$.

Consider variables $A_{i,j}$ and C_i for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant m$. Given any generator F_l of the ideal I of X we consider the polynomial

$$G_l(t, A_{i,j}, C_k) := F_l\left(\sum_{j=1}^{m} A_{1,j}t^j + C_1 t^{m+1}, \dots, \sum_{j=1}^{m} A_{N,j}t^j + C_N t^{m+1}\right)$$

in $\mathbb{K}[t, A_{i,j}, C_k]$.

The fact that the image of the wedge α lies in X is expressed by the system of equations

$$G_l\big(t, a_{i,j}(s), c_k(t,s)\big) = 0, \tag{17}$$

for $1 \leqslant l \leqslant r$. Moreover, since the wedge α realises an adjacency from E_u to E_v, the general arc α_g belongs to N_{E_u}, and hence the power series $a_{i,j}(s)$ satisfy the systems of polynomial equations $S_1(m(\alpha))$ (see (3)) and $S_2(m(\alpha))$ (see (7)).

By Theorem 16 there exist algebraic power series $a'_{i,j}(s)$ and $c'_i(t,s)$ coinciding respectively with $a_{i,j}(s)$ and $c_i(t,s)$ up to order L, for $1 \leqslant i \leqslant N$ and $1 \leqslant j \leqslant m$, such that they satisfy the systems of Eqs. (17), $S_1(m(\alpha))$ and $S_2(m(\alpha))$.

We claim that

$$\alpha'(t,s) := \left(\sum_{j=1}^{m} a'_{1,j}(s)t^j + c'_1(t,s)t^{m+1}, \dots, \sum_{j=1}^{m} a'_{N,j}(s)t^j + c'_N(t,s)t^{m+1}\right) \tag{18}$$

is the required algebraic wedge.

We expand α' as

$$\alpha'(t,s) = \left(\sum_{i,j} b'_{i,j,1}t^i s^j, \dots, \sum_{i,j} b'_{i,j,N}t^i s^j\right).$$

The wedge α' is algebraic since it is a polynomial combination of algebraic series and its image lies in X since it satisfies the system of Eqs. (17). The systems of equations $S_1(m(\alpha))$ and $S_2(m(\alpha))$ are satisfied for the coefficients of α'_g by construction.

As α and α' coincide up to order L we have $b_{i,j,k} = b'_{i,j,k}$ for $i \leqslant N$ and $j \leqslant L$. Since the characteristic polynomial of α with respect to π only involves variables $b_{i,j,k}$ with $i \leqslant N$, $j \leqslant m(\alpha)$, $k \leqslant R(\alpha)$, and $L > \max\{m(\alpha), R(\alpha)\}$ we have

$$P_\alpha\big(b'_{i,j,k}\big) = P_\alpha(b_{i,j,k}) \neq 0.$$

Thus, since the system $S_1(m(\alpha))$ is satisfied by the coefficients of α'_g, the first order of α' with respect to π is equal to $m(\alpha)$. Hence, since the system $S_2(m(\alpha))$ is satisfied the generic arc of α' belongs to N_{E_u} (see Proposition 10).

Let H, l_1, l_2 and Φ be as in Section 3.2.3. Since the generic arc α'_g is a $K((s))$-arc, the evaluation $H(\alpha'_g)$ is a power series belonging to $K((s))$. Its coefficient in l_2 is given by the substitution in Φ_α of the variables $B_{i,j,k}$ by the coefficients $b'_{i,j,k}$. Since Φ_α only depends on the variables $B_{i,j,k}$ for $i \leqslant N$ and $j,k \leqslant L$ and we have the equality $b_{i,j,k} = b'_{i,j,k}$ in this range of indexes we find that the coefficient is non-zero as with the case of $H(\alpha_g)$. This shows that α'_g does not belong to $\mathcal{Z}$.

We know that α'_s belongs to $\dot{N}_v$ because it coincides with α_s up to order $L > n_0$, the arc α_s belongs to $\dot{N}_v$, and we have chosen L big enough so that any perturbation of α_s which keeps the Taylor expansion fixed up to order L belongs to $\dot{N}_v$. $\square$

5. Algebraic wedges and finite morphisms

We need to refine at this point the definition of wedges realising adjacencies:

Definition 18. Let γ be a $\mathbb{K}$-arc in $\dot{N}_{E_v}$ and $\mathcal{Z}$ a proper closed subset of $\bar{N}_{E_u}$. A $\mathbb{K}$-wedge α realises an adjacency from E_u to γ (avoiding $\mathcal{Z}$) if its generic arc belongs to N_{E_u} (and not to $\mathcal{Z}$) and its special arc coincides with γ.

Along this section

$$\pi : \tilde{X} \to X$$

denotes the minimal good resolution of the singularity at O.

A technical difficulty in dealing with wedges is that they are not proper morphisms. In the following definition we remedy this difficulty by enlarging the domain of a wedge. The idea is to consider a finite morphism

$$\psi : (Z, Q) \to (X, O)$$

of normal surface singularities such that there is a resolution

$$\rho_2 : W \to (Z, Q)$$

and a point Q' of the exceptional divisor such that the wedge α is the composition of the mapping $\psi \circ \rho_2$ with a local chart

$$\beta : \left(\mathbb{K}^2, O\right) \to \left(W, Q'\right)$$

of W centred at Q'.

The following diagram explains several of the mappings appearing in the following definition. Dashed arrows means birational maps which are not defined everywhere.

$$\begin{tikzcd}
 & \tilde{Z} \arrow[d, "\rho_2"] \arrow[dr, "\pi^{-1}\circ\rho\circ\psi"] & \\
\operatorname{Spec}\mathbb{K}[[t,s]] \arrow[r, "\beta"] & W \arrow[r, dotted] \arrow[d, "\rho_1"] & \tilde{X} \arrow[d, "\pi"] \\
 & (Z, Q) \arrow[r, "\psi"] \arrow[ur, dotted] & (X, O)
\end{tikzcd} \tag{19}$$

Definition 19. Let γ be an algebraic $\mathbb{K}$-arc in $\dot{N}_{E_v}$. A morphism (in the category of algebraic varieties)

$$\psi : (Z, Q) \to (X, O)$$

of germs of normal surface singularities realises an adjacency from E_u to γ if there exist a resolution of singularities

$$\rho : \tilde{Z} \to (Z, Q),$$

of the singularity of Z at Q, with exceptional divisor $F := \rho^{-1}(0)$ and an algebraic $\mathbb{K}$-arc

$$\varphi : \mathrm{Spec}\big(\mathbb{K}[[t]]\big) \to \tilde{Z}$$

satisfying the equality $\psi \circ \rho \circ \varphi = \gamma$ and the following additional properties:

(1) The morphism ρ resolves the indeterminacy of the rational map $\pi^{-1} \circ \psi$; that is, the rational map $\tilde{\psi} := \pi^{-1} \circ \psi \circ \rho$ is a morphism.
(2) There is an irreducible component F_0 of F such that:
   (a) The image $\tilde{\psi}(F_0)$ is contained in E_u.
   (b) The resolution ρ can be factored into $\rho_2 \circ \rho_1$, where

$$\rho_1 : \tilde{Z} \to W$$

   is a sequence of contractions of (-1)-curves which does not collapse F_0, and

$$\rho_2 : W \to (Z, Q)$$

   is another resolution of singularities. The arc $\rho_1 \circ \varphi$ sends the special point into a point Q' of F_0 in which F_0 has a smooth branch L. Moreover $\rho_1 \circ \varphi$ is transverse to L at Q'.

Let $\mathcal{Z}$ be a proper closed subset of $\bar{N}_{E_u}$. We say that the morphism ψ *realises an adjacency from E_u to γ avoiding $\mathcal{Z}$* if there exist a resolution of singularities ρ, an algebraic $\mathbb{K}$-arc φ, a factorisation $\rho = \rho_2 \circ \rho_1$, an irreducible component F_0 of the exceptional divisor of ρ, a smooth branch germ (L, Q') into $\rho_1(F_0)$ with the properties predicted above, and such that there exists a morphism

$$\beta : \mathrm{Spec}\big(\mathbb{K}[[t, s]]\big) \to \big(W, Q'\big)$$

such that:

(i) The $\mathbb{K}$-arc $\beta(t, 0)$ coincides with $\rho_1 \circ \varphi(t)$.
(ii) The restriction $\beta(0, s)$ is a formal parametrisation of the germ (L, Q').
(iii) The composition $\psi \circ \rho_2 \circ \beta$ is a $\mathbb{K}$-wedge whose generic arc does not belong to $\mathcal{Z}$.

Definition 20. A morphism (in the category of algebraic varieties)

$$\psi : (Z, Q) \to (X, O)$$

of normal surface singularities realises an adjacency from E_u to E_v (avoiding an proper closed subset $\mathcal{Z}$ of $\bar{N}_{E_u}$) if there exists an algebraic $\mathbb{K}$-arc γ in $\dot{N}_{E_v}$ for which the morphism realises an adjacency from E_u to γ (avoiding $\mathcal{Z}$).

Definition 21. Suppose $\mathbb{K} = \mathbb{C}$.

(1) Let γ be a convergent $\mathbb{C}$-arc in $\dot{N}_{E_v}$. An analytic mapping germ

$$\psi : (Z, Q) \to (X, O)$$

of normal surface singularities realises an adjacency from E_u to γ (avoiding a proper closed subset $\mathcal{Z}$ of $\bar{N}_{E_u}$) if it satisfies the conditions of Definition 19 in the complex analytic category.

(2) An analytic mapping germ

$$\psi : (Z, Q) \to (X, O)$$

of normal surface singularities realises an adjacency from E_u to E_v (avoiding an proper closed subset $\mathcal{Z}$ of $\bar{N}_{E_u}$) if there exists a convergent $\mathbb{C}$-arc γ in $\dot{N}_{E_v}$ for which the mapping realises an adjacency from E_u to γ (avoiding $\mathcal{Z}$).

Proposition 22. *Let γ be an algebraic $\mathbb{K}$-arc in $\dot{N}_{E_v}$ and $\mathcal{Z}$ a proper closed subset of $\bar{N}_{E_u}$. The following holds*:

(1) *If there exists a morphism realising an adjacency from E_u to γ (avoiding $\mathcal{Z}$) then there exists a $\mathbb{K}$-wedge realising an adjacency from E_u to γ (avoiding $\mathcal{Z}$).*
(2) *If there exists an algebraic $\mathbb{K}$-wedge realising an adjacency from E_u to γ (avoiding $\mathcal{Z}$) then there exists a* finite *morphism realising an adjacency from E_u to γ (avoiding $\mathcal{Z}$).*

Proof. Let

$$\psi : (Z, Q) \to (X, O)$$

be a morphism realising an adjacency from E_u to γ.

Recall that given a point Q' in an algebraic variety W, the formal neighbourhood $(\widehat{W, Q'})$ of Q' in W is, by definition, the scheme $\operatorname{Spec}(\hat{\mathcal{O}}_{W,Q'})$, where $\hat{\mathcal{O}}_{W,Q'}$ is the completion of the local ring $\mathcal{O}_{W,Q'}$ with respect to the topology induced by the powers of the maximal ideal.

As W is smooth and $\rho_1 \circ \varphi$ is transverse to L at Q' (Definition 19(2)(b)), there exists a scheme isomorphism

$$\beta : \operatorname{Spec}\bigl(\mathbb{K}[[t, s]]\bigr) \to \bigl(\widehat{W, Q'}\bigr),$$

such that the $\mathbb{K}$-arc $\beta(t, 0)$ coincides with $\rho_1 \circ \varphi(t)$, and $\beta(0, s)$ is a formal parametrisation of (L, Q'). The mapping

$$\alpha := \psi \circ \rho_2 \circ \beta : \operatorname{Spec}\bigl(\mathbb{K}[[t, s]]\bigr) \to (X, O)$$

is a wedge since $\rho_2(0,s) = Q$ for any s. Its generic arc belongs to N_{E_u} since $\tilde{\psi}(F_0)$ is contained in E_u, and its special arc is $\psi \circ \rho_2 \circ \beta(t,0)$. We have

$$\psi \circ \rho_2 \circ \beta(t,0) = \psi \circ \rho_2 \circ \rho_1 \circ \varphi(t) = \psi \circ \rho \circ \varphi = \gamma.$$

Thus α is a wedge realising an adjacency from E_u to γ.

If the morphism ψ also avoids $\mathcal{Z}$ we take as β the morphism predicted in Definition 19. This proves (1).

Let

$$\alpha : \mathrm{Spec}\big(\mathbb{K}[[t,s]]\big) \to (X,O) \subset \mathbb{K}^N$$

be an algebraic $\mathbb{K}$-wedge realising an adjacency from E_u to γ.

As α is algebraic, for any i, there exists a polynomial $G_i \in \mathbb{K}[t,s,x_i]$ such that

$$G_i\big(t,s,x_i(\alpha)(t,s)\big) = 0.$$

The polynomials $G_1,\ldots,G_N$ define an algebraic set Y_1 in the affine space $\mathbb{K}^2 \times \mathbb{K}^N$. Let

$$\alpha' : \mathrm{Spec}\big(\mathbb{K}[[t,s]]\big) \to \mathbb{K}^2 \times \mathbb{K}^N$$

be the graph of α (that is $\alpha'(t,s) := (t,s,\alpha(t,s))$). Let Y_2 be the (2-dimensional) irreducible component of Y_1 containing the image of the graph of the mapping α. Denote by $n : Y_3 \to Y_2$ the normalisation. The mapping α' clearly admits a lifting

$$\beta : \mathrm{Spec}\big(\mathbb{K}[[t,s]]\big) \to Y_3,$$

which is a formal isomorphism at the origin O' of $\mathrm{Spec}(\mathbb{K}[[t,s]])$. Thus, we may see $\mathrm{Spec}(\mathbb{K}[[t,s]])$ as the formal neighbourhood of Y_3 at the smooth point $\beta(O')$.

Denote by pr_2 the projection of $\mathbb{K}^2 \times \mathbb{K}^N$ to the second factor. We consider projective completions $\bar{X}$ and $\bar{Y}_3$ of X and Y_3 respectively such that the mapping

$$\mathrm{pr}_2|_{Y_2} \circ n : Y_3 \to X$$

extends to a projective morphism

$$\phi : \bar{Y}_3 \to \bar{X}.$$

Taking normalisations if needed we can assume $\bar{X}$ and $\bar{Y}_3$ to be normal. By construction we have the equality

$$\phi \circ \beta = \alpha. \tag{20}$$

We quote for further use Stein Factorisation Theorem. See Corollary III.11.5 of [8].

Theorem 23. *Let $f : X \to Y$ be a projective morphism of Noetherian schemes. Then one can factor f into $g \circ f'$, where $f' : X \to Y'$ is a projective morphism with connected fibres and $g : Y' \to Y$ is a finite morphism.*

Observe that if X and Y are normal algebraic varieties of the same dimension and f is dominant then the generic fibre of f' is of dimension 0, and since it is connected it must be a point. Thus f' is generically an isomorphism, and since it is also projective it is a proper birational morphism.

Using Stein factorisation and the remark above we can factor ϕ into $\psi \circ \sigma$, where

$$\sigma : \bar{Y}_3 \to Z$$

is a proper birational morphism, the variety Z is a normal projective surface, and

$$\psi : Z \to \bar{X}$$

is a finite morphism.

Denote by Q the point $\sigma(\beta(O'))$. The restriction

$$\psi : (Z, Q) \to (X, O) \tag{21}$$

is a finite morphism of normal surface singularities. Let

$$\rho : \tilde{Z} \to (Z, Q)$$

be the minimal resolution of singularities such that

(i) it resolves the indeterminacy of the rational map $\pi^{-1} \circ \psi$ (that is, the rational map $\tilde{\psi} := \pi^{-1} \circ \psi \circ \rho$ is a morphism),
(ii) it dominates σ.

Let

$$\sigma' : W \to \bar{Y}_3$$

be the minimal resolution of singularities of $\bar{Y}_3$. The composition $\rho_2 := \sigma \circ \sigma'$ is a resolution of singularities of (Z, Q) and the resolution ρ factors into $\rho_2 \circ \rho_1$, where ρ_1 is a sequence of contractions of (-1)-curves. As $\bar{Y}_3$ is smooth at the point $\beta(O')$ there is a unique point $Q' \in W$ such that $\sigma'(Q') = \beta(O')$, and the mapping β admits an isomorphic lifting

$$\beta' : \mathrm{Spec}(\mathbb{K}[[t, s]]) \to \big(W, Q'\big). \tag{22}$$

Since we have the equalities

$$\psi \circ \rho_2 \circ \beta' = \psi \circ \sigma \circ \sigma' \circ \beta' = \psi \circ \sigma \circ \beta = \phi \circ \beta = \alpha \tag{23}$$

and ψ is finite we have that $\rho_2 \circ \beta'$ transforms the s-axis into Q. Thus the image L of the s-axis by β' belongs to the exceptional divisor of ρ_2. Let F_0' be the irreducible component of the exceptional divisor of ρ_2 containing the smooth branch L. Denote by F_0 the strict transform of F_0' by ρ_1. By construction ρ_1 does not collapse F_0. As α realises an adjacency from E_u to γ, it sends the special point of its general arc into E_u. This implies that the divisor F_0 is transformed into E_u

by $\tilde{\psi}$. On the other hand the equalities (23) imply that the arc φ predicted in Definition 19(2) is the lifting of the special arc of β' to $\tilde{Z}$.

If the wedge α avoids $\mathcal{Z}$ then the lifting β' defined in (22) is the additional morphism predicted in Definition 19. □

6. Moving wedges

In this section we assume that the base field $\mathbb{K}$ is the field of complex numbers $\mathbb{C}$ and we will work in the complex analytic category. Thus (X, O) denotes a complex analytic normal surface singularity. Since (X, O) has an isolated singularity at the origin, a result of Artin [1] shows that it is formally isomorphic with a germ of algebraic surface at a point. Then, by a theorem of Hironaka and Rossi [10], we know that the formal isomorphism comes in fact from a local analytic one. Thus, when we need it we will freely assume that (X, O) is an algebraic surface germ. It is worth to remind, for intuition, that in the category of normal complex analytic spaces, finite analytic morphisms correspond topologically to branched coverings.

We develop a technique that allows, given a $\mathbb{C}$-wedge realising an adjacency from E_u to a particular $\mathbb{C}$-arc of $\dot{N}_{E_v}$, to construct wedges realising adjacencies from E_u to *any* convergent $\mathbb{C}$-arc in $\dot{N}_{E_v}$. The technique does not allow to ensure that if the original wedge avoids a proper closed subset $\mathcal{Z}$ of $\bar{N}_{E_u}$ then the newly constructed wedge also enjoys this property. However, if the subset $\mathcal{Z}$ equals the set of non-transverse arcs Δ_{E_u} we will prove that the property of avoiding it is preserved.

Definition 24. Given any irreducible component E_u of the exceptional divisor we define the proper closed subset $\Delta_{E_u} \subset N_{E_u}$ consisting of the union of the set of arcs whose lifting to the resolution sends the special point to a singular point of E on E_u with the set of arcs whose lifting is non-transverse to E_u. In other words, the set Δ_{E_u} is the complement in E_u of the set of arcs whose lifting to the resolution is transversal to E_u at a smooth point of E.

We need to remind some common notations on dual graphs and the plumbing construction. Given a good resolution

$$\pi : \tilde{X} \to (X, O)$$

of a normal surface singularity, the dual graph $\mathcal{G}$ of the resolution is constructed as follows: there is a vertex for each irreducible component E_i of E, with a genus weight (which is equal to the genus of E_i), and a self-intersection weight (which is equal to the self intersection of E_i in $\tilde{X}$). Two vertices are joined by one edge for each intersection point of their corresponding exceptional divisors. In this article a *weighted graph* is a doubly weighted graph as above. Let r be the number of vertices of a weighted graph $\mathcal{G}$. The incidence matrix of $\mathcal{G}$ is the symmetric square $r \times r$ matrix with the following entries: enumerate the vertices of $\mathcal{G}$, the (i, i)-entry is the self-intersection weight of the i-th vertex; if $i \neq j$ the (i, j) entry is the number of edges between the i-th and j-th vertices. It is well known [6] that a weighted graph is associated to a good resolution of a normal surface singularity if and only if its incidence matrix is negative-definite. In that case we say that the graph is a negative definite weighted graph. Another well-known fact is that an adequate neighbourhood of E in $\tilde{X}$ is diffeomorphic to the plumbing 4-manifold associated to the dual graph of the resolution [21].

The main technical difficulty is solved in:

Proposition 25. *If there exists a convergent $\mathbb{C}$-arc $\gamma \in \dot{N}_{E_v}$ and a finite analytic mapping realising an adjacency from E_u to γ (avoiding Δ_{E_u}), then, for any other convergent $\mathbb{C}$-arc $\gamma' \in \dot{N}_{E_v}$ there exists a finite analytic mapping realising an adjacency from E_u to γ' (avoiding Δ_{E_u}).*

The result which allows to move wedges is:

Proposition 26. *If there exists a $\mathbb{C}$-arc $\gamma \in \dot{N}_{E_v}$ and a $\mathbb{C}$-wedge realising and adjacency from E_u to γ (avoiding Δ_{E_u}), then, for any convergent $\mathbb{C}$-arc $\gamma' \in \dot{N}_{E_v}$ there exists a $\mathbb{C}$-wedge realising an adjacency from E_u to γ' (avoiding Δ_{E_u}).*

Proof. Let us assume Proposition 25. By Proposition 17 there exists an algebraic $\mathbb{C}$-wedge α realising an adjacency from E_u to E_v (avoiding Δ_{E_u}) . If γ'' denotes the special arc of α then α realises an adjacency from E_u to γ'' (avoiding Δ_{E_u}). By Proposition 22(2) there exists a finite morphism realising an adjacency from E_u to γ'' (avoiding Δ_{E_u}). By Proposition 25 for any convergent $\gamma' \in \dot{N}_{E_v}$ there exists a finite morphism realising an adjacency from E_u to γ' (avoiding Δ_{E_u}) . Finally, by Proposition 22(1) for any convergent $\gamma' \in \dot{N}_{E_v}$ there exists a $\mathbb{C}$-wedge realising an adjacency from E_u to γ' (avoiding Δ_{E_u}). $\square$

Proof of Proposition 25. The proof is somewhat involved but the idea is simple: let

$$\psi : (Z, Q) \to (X, O)$$

be a finite analytic morphism of normal surface singularities realising an adjacency from E_u to γ. Since γ and γ' are arcs with transverse lifting through the same essential component there exists a self-homeomorphism Φ from of (X, O) transforming the arc γ into the arc γ'. The composition of $\Phi \circ \psi$ is a topological branched cover. Since the conditions of the definition of finite analytic morphisms realising an adjacency are of a topological nature, if one could put another analytic structure on (Z, Q) so that $\Phi \circ \psi$ is holomorphic, then $\Phi \circ \psi$ would be a finite analytic mapping realising an adjacency from E_u to γ'.

The proof has three parts, the first is a construction of the self-homeomorphism which allows to change the complex structure of (Z, Q) so that the composition becomes holomorphic, the second consists in constructing the new complex structure, and the third consists in checking that our construction gives, in fact, a finite analytic mapping realising an adjacency from E_u to γ'.

PART I.

We will construct the homeomorphism at the level of the resolution $\tilde{X}$.

Let

$$\psi : (Z, Q) \to (X, O)$$

be a finite analytic morphism of normal surface singularities realising an adjacency from E_u to γ. Let

$$\rho : \tilde{Z} \to (Z, Q)$$

be the resolution of singularities predicted in Definition 19, and $\tilde{\psi} : \tilde{Z} \to \tilde{X}$ the lifting of ψ. By Stein factorisation $\tilde{\psi}$ factors into $\tilde{\psi}' \circ \rho'$, where

$$\tilde{\psi}' : Z' \to \tilde{X}$$

is a finite analytic mapping, with Z' a normal surface, and

$$\rho' : \tilde{Z} \to Z'$$

a resolution of singularities. Moreover the resolution ρ factors into $\rho'' \circ \rho'$, where

$$\rho'' : Z' \to (Z, Q)$$

is a bimeromorphic morphism.

Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be liftings of γ and γ' to $\tilde{X}$. We shall assume the inequality $\tilde{\gamma}(0) \neq \tilde{\gamma}'(0)$. We do not loose generality because if we have the equality $\tilde{\gamma}(0) = \tilde{\gamma}'(0)$ we may choose a third arc $\gamma'' \in \dot{N}_{E_v}$ such that $\tilde{\gamma}''(0) \neq \tilde{\gamma}(0) = \tilde{\gamma}'(0)$ and apply the procedure below twice.

Let Δ be the branching locus of $\tilde{\psi}'$, and Δ' be the union of the irreducible components of Δ not contained in the exceptional divisor E. Since $\tilde{X}$ is diffeomorphic to the plumbing 4-manifold whose plumbing graph is the dual graph of the resolution, possibly having to shrink it to a smaller tubular neighbourhood of the exceptional divisor of π we may assume:

(1) There are neighbourhoods $U_1, \ldots, U_m$ of each singular point of the exceptional divisor in $\tilde{X}$ which contain all the components of Δ' which meet the corresponding singular point.
(2) The difference $\tilde{X} \setminus \bigcup_{j=1}^{m} U_j$ splits in one connected component $\tilde{X}_v$ for each irreducible component E_v of E. In addition, for any v we have that $\hat{E}_v := E_v \setminus \bigcup_{j=1}^{m} U_j$ is the deletion of several disks in E_v, and we have a smooth product structure $\tilde{X}_v = \hat{E}_v \times D$, where D is a disk.
(3) Consider the set $\{p_1, \ldots, p_s\}$ of meeting points of Δ' with E which are not singularities of E. Given any point p_i contained in a divisor E_u, there exists a disk D_i around p_i in $\hat{E}_u$ such that all the components of Δ' meeting p_i are contained in $D_i \times D$, and any component of Δ' meeting $D_i \times D$ meet E at p_i. In addition the closures of the disks $D_1, \ldots, D_s$ are two-by-two disjoint and do not meet the boundary of $\hat{E}_v$.
(4) The images of $\tilde{\gamma}$ and $\tilde{\gamma}'$ are fibre disks in $\tilde{X}_v$ by the projection to $\hat{E}_v$. Either there is a disk $D_{\tilde{\gamma}}$ in $\hat{E}_v$ around $\tilde{\gamma}(0)$ with closure disjoint to the closure of any D_i and to the boundary of $\hat{E}_v$, or $\tilde{\gamma}$ is the fibre of a point in $\{p_1, \ldots, p_s\}$ by the projection to $\hat{E}_v$. In the later case we define the disk $D_{\tilde{\gamma}}$ to be the equal to D_j, for $\tilde{\gamma}(0) = p_j$. The same holds for $\tilde{\gamma}'$. The closures of the two disks $D_{\tilde{\gamma}}$ and $D_{\tilde{\gamma}'}$ are disjoint and disjoint to the boundary of $\hat{E}_v$.

Denote by $\mathcal{D}$ an open region of $\hat{E}_v$ diffeomorphic to a disk such that the closure of the disks $D_{\tilde{\gamma}}$ and $D_{\tilde{\gamma}'}$ are contained in $\mathcal{D}$ and the closure of $\mathcal{D}$ is disjoint to the boundary of $\hat{E}_v$ and to the closures of the disks D_i different from $D_{\tilde{\gamma}}$ and $D_{\tilde{\gamma}'}$.

The construction of the homeomorphism is achieved in the following lemma.

Lemma 27. *There exists a diffeomorphism*

$$\eta : \tilde{X} \to \tilde{X}$$

such that

(i) *it leaves E invariant,*
(ii) *the restriction of η to $\tilde{X} \setminus (\mathcal{D} \times D)$ is the identity,*

(iii) *we have* $\eta(D_{\tilde{\gamma}} \times D) = D_{\tilde{\gamma}'} \times D$ *and* $\eta(D_{\tilde{\gamma}'} \times D) = D_{\tilde{\gamma}} \times D$. *Moreover the restrictions*

$$\eta : D_{\tilde{\gamma}} \times D \to D_{\tilde{\gamma}'} \times D,$$
$$\eta : D_{\tilde{\gamma}'} \times D \to D_{\tilde{\gamma}} \times D$$

are biholomorphisms such that we have the equalities $\eta \circ \tilde{\gamma} = \tilde{\gamma}'$ *and* $\eta \circ \tilde{\gamma}' = \tilde{\gamma}$.

Proof. The diffeomorphism can be constructed as follows: define first a biholomorphism

$$\xi : \bar{D}_{\tilde{\gamma}} \to \bar{D}_{\tilde{\gamma}'}$$

taking $\tilde{\gamma}(0)$ to $\tilde{\gamma}'(0)$. Now define

$$\eta' : D_{\tilde{\gamma}} \times D \to D_{\tilde{\gamma}'} \times D$$

by $\eta'(x, y) := (\xi(x), y)$. The composition $\eta' \circ \tilde{\gamma}$ is a re-parametrisation of $\tilde{\gamma}'$: in fact there exists a biholomorphic mapping $\zeta : D \to D$ such that we have $(id, \zeta) \circ \eta' \circ \tilde{\gamma} = \tilde{\gamma}'$. Let $\bar{\xi} : \mathcal{D} \to \mathcal{D}$ be any diffeomorphism whose restrictions to $\bar{D}_{\tilde{\gamma}}$ and $\bar{D}_{\tilde{\gamma}'}$ coincides with ξ and ξ^{-1} respectively, and which is the identity in a neighbourhood of the boundary of $\mathcal{D}$. Consider a family of diffeomorphisms Ψ_x of D parametrised over $\mathcal{D}$ such that

- if $x \in D_{\tilde{\gamma}}$ then Ψ_x coincides with ζ,
- if $x \in D_{\tilde{\gamma}'}$ then Ψ_x coincides with ζ^{-1},
- if x belongs to a certain neighbourhood of the boundary of $\mathcal{D}$ then Ψ_x is the identity.

The family Ψ_x can be constructed as follows: observe that both ζ and ζ^{-1} are self-diffeomorphisms of the disk D preserving the orientation. The group of self-difeomorphisms of a disk preserving the orientation is connected. Take disjoint compact disks $D'_{\tilde{\gamma}}$ and $D'_{\tilde{\gamma}'}$ contained in $\mathcal{D}$ and containing a neighbourhood of $D_{\tilde{\gamma}}$ and $D_{\tilde{\gamma}'}$ respectively. We define Ψ_x to be equal to the identity if x is outside $D'_{\tilde{\gamma}} \cup D'_{\tilde{\gamma}'}$, to be equal to ζ if x is in $D_{\tilde{\gamma}}$, to be equal to ζ^{-1} if x is in $D_{\tilde{\gamma}'}$. If x belongs to $D'_{\tilde{\gamma}} \setminus D_{\tilde{\gamma}}$ then Ψ_x interpolates from ζ to the identity and if x belongs to $D'_{\tilde{\gamma}'} \setminus D_{\tilde{\gamma}'}$ then Ψ_x interpolates from ζ^{-1} to the identity.

We define

$$\eta(x, y) := \bigl(\bar{\xi}(x), \Psi_x(y)\bigr),$$

if $(x, y) \in \mathcal{D} \times D$, and $\eta(p) = p$ if p is outside $\mathcal{D} \times D$. $\square$

The lemma defines the homeomorphism at the level of the resolution. Since it leaves E invariant it may be push down to a self-homeomorphism of

$$\nu : (X, O) \to (X, O).$$

PART II.

In this part we define a new holomorphic structure in (Z, Q) which makes the composition

$$\nu \circ \pi : (Z, Q) \to (X, O)$$

an analytic morphism. Since in the process we need to deal often with the same topological space with different complex structures we will denote an analytic set by a pair $(Y, \mathcal{B})$, where Y denotes the underlying topological space and $\mathcal{B}$ the complex structure.

The original complex structure of the surface germ (Z, Q) is denoted by $\mathcal{D}_1$.

Lemma 28. *There exists a unique complex structure $\mathcal{D}_2$ in the germ of topological space (Z, Q) such that*

$$\nu \circ \psi : (Z, Q, \mathcal{D}_2) \to (X, O)$$

is a finite analytic morphism

Proof. The new complex structure is constructed first at the level of the resolution of (Z, Q) and pushed down later. The finiteness of $\nu \circ \psi$ is a topological property and holds because ψ is finite.

STEP 1. The mapping

$$\eta \circ \tilde{\psi}' : Z' \to \tilde{X} \tag{24}$$

is a topological branched covering, which is a local diffeomorphism outside $\eta(\Delta)$, but it is not holomorphic. We are going to change the analytic structure in Z' so the mapping $\eta \circ \tilde{\psi}'$ becomes holomorphic. We endow $Z' \setminus (\tilde{\psi}')^{-1}(\Delta)$ with the unique complex structure making the restriction $\eta \circ \tilde{\psi}'|_{Z' \setminus (\tilde{\psi}')^{-1}(\Delta)}$ holomorphic. This restriction becomes automatically a finite and étale analytic morphism with the new complex structure. Notice that by construction of η the branch locus $\eta(\Delta)$ of $\eta \circ \tilde{\psi}'$ is an analytic subset. Now we apply the following theorem of Grauert and Remmert [7]:

Theorem 29 *(Grauert, Remmert). Let A be a normal analytic space, let $B \subset A$ be a closed analytic subset such that $A \setminus B$ is dense in A. Let $f : U \to A \setminus B$ be a finite and etale analytic morphism. Then there exists a finite analytic mapping $\bar{f} : V \to A$ from a normal analytic space V extending f. Moreover V is unique up to isomorphism.*

Thus, the theorem above, there exists a unique normal complex analytic set Z'' containing $Z' \setminus (\tilde{\psi}')^{-1}(\Delta)$ for which $\eta \circ \tilde{\psi}'|_{Z' \setminus (\tilde{\psi}')^{-1}(\Delta)}$ extends to a finite analytic mapping

$$\tilde{\psi}'' : Z'' \to \tilde{X} \tag{25}$$

Since a branched covering is determined as a topological space by its restriction over the complement of the branching locus we have that Z' and Z'' coincide as topological spaces and that $\tilde{\psi}''$ equals $\eta \circ \tilde{\psi}'$. Let $\mathcal{A}_1$ denote the original complex structure of Z' and $\mathcal{A}_2$ denote the complex structure of Z''. We will denote Z' by $(Z', \mathcal{A}_1)$ and Z'' by $(Z', \mathcal{A}_2)$.

STEP 2. The singularities of $(Z', \mathcal{A}_1)$ are over the singular points of the discriminant. Since these points are mapped by $\tilde{\psi}'$ to points where the mapping η is biholomorphic, the singularities of $(Z', \mathcal{A}_2)$ are analytically equivalent to those of $(Z', \mathcal{A}_1)$. Let

$$\rho_{\min} : (Z_{\min}, \mathcal{B}_1) \to \big(Z', \mathcal{A}_1\big)$$

be the minimal resolution of singularities (we denote by $\mathcal{B}_1$ the complex structure of $Z_{\min}$). If we change the analytic structure $\mathcal{A}_1$ by $\mathcal{A}_2$ then the mapping $\rho_{\min}$ is holomorphic over neighbourhoods of the singularities of $(Z', \mathcal{A}_2)$, and is a non-necessarily holomorphic diffeomorphism outside these neighbourhoods. Thus there is a unique complex structure $\mathcal{B}_2$ in $Z_{\min}$ making

$$\rho_{\min} : (Z_{\min}, \mathcal{B}_2) \to \big(Z', \mathcal{A}_2\big)$$

holomorphic.

STEP 3. Let $\mathcal{C}_1$ denote the complex structure of the resolution $\tilde{Z}$ of $(Z', \mathcal{A}_1)$. The mapping

$$\rho' : (\tilde{Z}, \mathcal{C}_1) \to \big(Z', \mathcal{A}_1\big)$$

can be factored into $\rho_{\min} \circ \rho'''$, where $\rho''' : \tilde{Z} \to Z_{\min}$ is a chain of blowing-ups at points. We claim that there is a unique complex structure $\mathcal{C}_2$ on $\tilde{Z}$ which makes

$$\rho''' : (\tilde{Z}, \mathcal{C}_2) \to (Z_{\min}, \mathcal{B}_2)$$

analytic. Indeed, as ρ''' is a composition of blowing-ups at points, by induction, it is sufficient to consider the case in which ρ''' is a single blow-up at a point p. The topological space obtained by blowing up a surface at a smooth point does not depend on the complex structure of the surface. Thus, the complex structure on the blow-up of $(Z_{\min}, \mathcal{B}_2)$ at p is the required complex structure of $\tilde{Z}$.

STEP 4. The mapping

$$\rho' : (\tilde{Z}, \mathcal{C}_2) \to \big(Z', \mathcal{A}_2\big)$$

is analytic for being a composition of analytic mappings.

By Stein factorisation the composition

$$\pi \circ \eta \circ \tilde{\psi}' \circ \rho' : (\tilde{Z}, \mathcal{C}_2) \to (X, O)$$

factors into $\phi \circ \sigma$ where

$$\phi : (Z_2, Q_2) \to (X, O)$$

is a finite analytic mapping of normal surface singularities, and

$$\sigma : (\tilde{Z}, \mathcal{C}_2) \to (Z_2, Q_2)$$

is a resolution of singularities. From the topological viewpoint the mapping σ consists in collapsing the exceptional divisor F to a point. Hence, the singularity (Z_2, Q_2) is topologically equivalent to (Z, Q): the germ (Z_2, Q_2) may be viewed as the germ of topological space (Z, Q) with a different analytic structure, and the mapping σ coincides with ρ. By construction the mapping ϕ is equal to the composition $\nu \circ \psi$. This ends the proof of the lemma.

Denote by $\mathcal{D}_1$ and $\mathcal{D}_2$ the analytic structures of (Z, Q) and (Z_2, Q_2). We denote the germs (Z, Q) and (Z_2, Q_2) respectively by $(Z, Q, \mathcal{D}_1)$ and $(Z, Q, \mathcal{D}_2)$. The mapping σ coincides then with

$$\rho : (\tilde{Z}, \mathcal{C}_2) \to (Z, Q, \mathcal{D}_2). \tag{26}$$

Part III.

Lemma 30. *The finite analytic morphism*

$$\nu \circ \psi : (Z, Q, \mathcal{D}_2) \to (X, O)$$

realises an adjacency from E_u to γ' and avoids Δ_{E_u} if ψ avoids Δ_{E_u}.

Proof. The resolution ρ defined in (26) resolves the indeterminacy of $\pi^{-1} \circ \nu \circ \psi$, since, by construction, this rational map coincides with the morphism $\eta \circ \tilde{\psi} = \eta \circ \tilde{\psi}' \circ \rho'$. The exceptional divisor of ρ is F, and, since η leaves each irreducible component of E invariant, we have that $\eta \circ \tilde{\psi}(F_0)$ is a subset of E_u.

Let

$$\varphi : (\mathbb{C}, O) \to (\tilde{Z}, \mathcal{C}_1)$$

be the arc predicted in Definition 19. Since $\tilde{\psi}' \circ \rho' \circ \varphi$ has image in an open set in which η is biholomorphic, the arc φ is also holomorphic with respect to the complex structure of $\mathcal{C}_2$. By construction we have the equality $\nu \circ \psi \circ \rho \circ \varphi = \gamma'$. Let

$$\rho_1 : (\tilde{Z}, \mathcal{C}_1) \to (W, \mathcal{E}_1)$$

be the sequence of contractions of (-1)-curves predicted in Definition 19. The same sequence of contractions in $(\tilde{Z}, \mathcal{C}_2)$ gives rise to a unique new complex structure $\mathcal{E}_2$ in W, which makes

$$\rho_1 : (\tilde{Z}, \mathcal{C}_2) \to (W, \mathcal{E}_2)$$

holomorphic. By construction ρ_1 does not collapse F_0, the arc $\rho_1 \circ \varphi$ sends the special point into a point Q' of F_0 in which F_0 has a smooth branch L and moreover $\rho_1 \circ \varphi$ is transverse to L at Q'. Thus the covering $\nu \circ \psi$ realises an adjacency from E_u to γ' (see Definition 19).

Let us prove now that ψ avoids Δ_{E_u} if and only if $\nu \circ \psi$ also avoids it. Let $\rho = \rho_2 \circ \rho_1$ be the factorisation of ρ predicted in Definition 19. The situation of Definition 19 in the complex analytic setting (Definition 21) makes clear the existence of a small neighbourhood Ω of Q' in W such that the only point of indetermination of the meromorphic map $\bar{\psi} := \pi^{-1} \circ \psi \circ \rho_2$ in Ω is Q'.

Claim. *We have that ψ avoids Δ_{E_u} if and only if Ω can be taken small enough so that for any $Q'' \in \Omega \setminus \{Q\}$ there exists a germ (D, Q'') of smooth real 2-dimensional subvariety of W which is transverse to the smooth branch L at Q' and such that the restriction of*

$$\bar{\psi} : (D, Q'') \to \tilde{X}$$

is transverse to E_u at Q'' in the smooth category.

The claim implies that avoiding Δ_{E_u} only depends of the restriction of the lifting $\bar{\psi}$ to an open subset $\Omega' \subset W$ containing a small punctured neighbourhood of Q' in L. Since η is a diffeomorphism the lifting $\overline{\nu \circ \psi}$ corresponding to $\nu \circ \psi$ satisfies the transversality condition of the claim if and only if $\bar{\psi}$ satisfies it. Thus, if the claim is true we have that ψ avoids Δ_{E_u} if and only if $\nu \circ \psi$ does, and the proof of the lemma is complete.

Let us prove the claim. Consider $(\mathbb{C}^2, O)$ with local coordinates (s, t). There is a local analytic diffeomorphism

$$\beta : (\mathbb{C}^2, O) \to (W, Q') \tag{27}$$

such that $\beta(s, 0)$ parametrises (L, Q') and $\beta(0, t)$ equals $\rho_1 \circ \varphi(t)$.

If ψ avoids Δ_{E_u}, by Definition 21 (that is Definition 19 in the complex analytic setting), it is possible to choose β in such a way that for any $s_0 \neq 0$ and small enough the arc $\psi \circ \rho_2 \circ \beta(s_0, t)$ has a transverse lifting to E_u. Since the lifting is equal to $\bar{\psi} \circ \beta(s_0, t)$ we have proved the "only if" part.

If there is a 2-dimensional germ (D, Q'') of smooth subvariety of W which is transverse to the smooth branch L at Q' and such that the restriction of

$$\bar{\psi} : (D, Q'') \to \tilde{X}$$

is transverse to E_u at Q'' in the smooth category then an easy manipulation with tangent spaces and tangent maps shows the transversality to E_u of the restriction of $\bar{\psi}$ to any other 2-dimensional germ of smooth subvariety of W which is transverse to the smooth branch L at Q''. This easily implies that for any holomorphic arc

$$\theta : (\mathbb{C}, O) \to (W, Q'')$$

transverse to L the lifting to $\tilde{X}$ of $\psi \circ \rho_2 \circ \theta$ is transverse to E_u. This shows the "if" part. □

The last lemma completes the proof of Proposition 25. □

The same technique leads to Proposition 32. First we need the following definition:

Definition 31. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be weighted graphs. An *isomorphism*

$$\varphi : \mathcal{G}_1 \to \mathcal{G}_2$$

is given by a bijection φ_1 from the set of vertices of $\mathcal{G}_1$ to the set of vertices of $\mathcal{G}_2$ and a bijection φ_2 from the set of edges of $\mathcal{G}_1$ to the set of edges of $\mathcal{G}_2$ such that, an edge $e \in \mathcal{G}_1$ connects two vertices v_1 and v_2 if and only if the edge $\varphi_2(e)$ connects $\varphi_1(v_1)$ and $\varphi_1(v_2)$. Moreover the weight of any vertex v_1 coincides with the weight of $\varphi_1(v_1)$.

Proposition 32. *Let $\kappa : \mathcal{G}_1 \to \mathcal{G}_2$ be an isomorphism between the weighted graphs of the minimal good resolution of two surface singularities (X_1, O) and (X_2, O). Let E_u and E_v be two exceptional divisors of the minimal good resolution of (X_1, O). If there is a finite analytic mapping realising an adjacency from E_u to E_v (avoiding Δ_{E_u}) then there is a finite analytic mapping realising an adjacency from $E_{\kappa(u)}$ to $E_{\kappa(v)}$ (avoiding $\Delta_{E_{\kappa(u)}}$).*

Proof. Let γ be a convergent $\mathbb{C}$-arc in $\dot{N}_{E_v}$ such that there exists a finite analytic mapping

$$\psi : (Z, Q) \to (X_1, O)$$

realising an adjacency from E_u to γ. Let γ' be any convergent $\mathbb{C}$-arc in $\dot{N}_{E_{\kappa(v)}}$. Let

$$\pi_i : \tilde{X}_i \to (X_i, O)$$

be the minimal good resolutions of X_i for $i = 1, 2$. Let

$$\rho : \tilde{Z} \to (Z, Q)$$

be the resolution of singularities predicted in Definition 19, and $\tilde{\psi} : \tilde{Z} \to \tilde{X}_1$ the lifting of ψ. By Stein factorisation $\tilde{\psi}$ factors into $\tilde{\psi}' \circ \rho'$, where

$$\tilde{\psi}' : Z' \to \tilde{X}_1$$

is a finite mapping (that is, a branched covering) with Z' a normal surface.

Let Δ' be the union of the irreducible components of the branching locus of $\tilde{\psi}'$ not contained in the exceptional divisor of π_1. Let $\tilde{\gamma}$ and $\tilde{\gamma}'$ be liftings of γ and γ' to $\tilde{X}_1$ and $\tilde{X}_2$.

Since $\tilde{X}_1$ and $\tilde{X}_2$ are diffeomorphic to the plumbing 4-manifolds of the dual graph of the resolutions, which are isomorphic by the isomorphism κ, possibly having to shrink them to a smaller tubular neighbourhood of the exceptional divisor of π_i we may assume:

(1) For $i = 1, 2$ there are neighbourhoods $U_1^i, \dots, U_m^i$ of each singular point of the exceptional divisor in $\tilde{X}$ which, if $i = 1$, contain all the components of Δ' which meet the corresponding singular point.
(2) For $i = 1, 2$ the difference $\tilde{X}_i \setminus \cup_{j=1}^m U_j$ splits in one connected component $\tilde{X}_i[v]$ for each irreducible component E_v^i of $\pi_i^{-1}(O)$. In addition, for any v we have that $\hat{E}_v^i := E_v^i \setminus \bigcup_{j=1}^m U_j^i$ is the deletion of several disks in E_v^i, and we have a smooth product structure $\tilde{X}^i[v] = \hat{E}_v^i \times D$, where D is a disk.
(3) Consider the set $\{p_1^1, \dots, p_s^1\}$ of meeting points of Δ' with $\pi_1^{-1}(O)$ which are not singularities of $\pi^{-1}(O)$. Given any point p_i^1 contained in a divisor E_u^1, there exists a disk D_i^1 around p_i^1 in $\hat{E}_u^1$ such that all the components of Δ' meeting p_i^1 are contained in $D_i^1 \times D$, and any component of Δ' meeting $D_i^1 \times D$ meet $\pi_1^{-1}(O)$ at p_i^1. In addition the closures of the disks $D_1^1, \dots, D_s^1$ are two-by-two disjoint and do not meet the boundary of $\hat{E}_v^1$.
(4) Given any component E_u^1 of $\pi_1^{-1}(O)$ let $\{p_{i(1)}^1, \dots, p_{i(l_u)}^1\}$ be the subset formed by the points in $\{p_1^1, \dots, p_s^1\}$ belonging to E_u^1. Choose points

$$\{p_{i(1)}^2, \dots, p_{i(l_u)}^2\}$$

in $\hat{E}_\kappa^2(u)$ and disks $D_{i(1)}^2, \dots, D_{i(l_u)}^1$ with pairwise different closures not meeting the boundary of $\hat{E}_{\kappa(u)}^2$.
(5) The images of $\tilde{\gamma}$ and $\tilde{\gamma}'$ are fibre disks in $\tilde{X}^1[v]$ and $\tilde{X}^2[\kappa(v)]$ by the projection to $\hat{E}_v^1$ and $\hat{E}_{\kappa(v)}^2$ respectively. Either there is a disk $D_{\tilde{\gamma}}$ in $\hat{E}_v^1$ around $\tilde{\gamma}(0)$ with closure disjoint to the closure of any D_i^1 and to the boundary of $\hat{E}_v^1$, or $\tilde{\gamma}$ is the fibre of a point in $\{p_1^1, \dots, p_s^1\}$ by the projection to $\hat{E}_v^1$. In the latter case we define the disk $D_{\tilde{\gamma}}^1$ to be equal to D_j^1, for $\tilde{\gamma}(0) = p_j^1$, we choose p_j^2 to be equal to $\tilde{\gamma}'(0)$, and define the disk $D_{\tilde{\gamma}}^2$ to be equal to D_j^2.

Otherwise we define $D^2_{\tilde{\gamma}}$ to be a disk in $\hat{E}^2_\kappa(v)$ centred at $\tilde{\gamma}'(0)$ and with closure disjoint to the closure of the D^2_i's and disjoint to the boundary of $\hat{E}^2_{\kappa(v)}$.

Choose biholomorphisms from U^1_i to U^2_i for any i, from $D^1_i \times D$ to $D^2_i \times D$, and from $D^1_{\tilde{\gamma}} \times D$ to $D^2_{\tilde{\gamma}'} \times D$ such that, using the methods of the proof of Lemma 27 we may extend to a diffeomorphism

$$\eta : \tilde{X}_1 \to \tilde{X}_2$$

with the following properties:

(i) We have $\eta(\pi_1^{-1}(0)) = \pi_2^{-1}(0)$, $\eta(E_u) = E_{\kappa(u)}$, and $\eta(E_v) = E_{\kappa(v)}$.
(ii) The restriction of η to a neighbourhood of $\Delta' \cup \mathrm{Sing}(\pi_1^{-1}(0)) \cup \tilde{\gamma}(\mathbb{C}, 0)$ is biholomorphic.
(iii) We have $\eta \circ \tilde{\gamma} = \tilde{\gamma}'$.

As (by properties (i) and (ii)), the branching locus of $\eta \circ \psi$ is an analytic subset of $\tilde{X}_2$, as in the proof of Lemma 28 we can change the complex structure in Z' so that the mapping $\eta \circ \psi'$ becomes holomorphic. The construction of the branched cover realising an adjacency from $E_{\kappa(u)}$ to γ' follows now word by word the analogous constructions in the proofs of Lemmata 28 and 30. The fact that if the original finite analytic mapping avoids Δ_{E_u} then the constructed one avoids $\Delta_{E_{\kappa(u)}}$ also follows like the analogous statement in Lemma 30. □

7. On Nash and wedge problems

In this section we prove several of the main results of the paper. Up to now, due to the use of topology, several results are valid only for complex singularities. We will use Lefschetz principle to transfer the results to singularities defined over algebraically closed fields of characteristic equal to 0. For this we need some preparations.

7.1. Stability of the dual graph of the minimal good resolution by base change

Let (X, O) be a normal surface singularity defined over an algebraically closed field $\mathbb{K}_1$ of characteristic 0. Let

$$\pi_1 : \tilde{X} \to (X, O)$$

be a good resolution of singularities. Consider a field extension $\mathbb{K}_1 \subset \mathbb{K}_2$, where $\mathbb{K}_2$ is also algebraically closed. The mapping

$$\pi_2 : \tilde{X} \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2) \to (X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$$

obtained from π_1 by base change is a good resolution of singularities. Indeed, smoothness and the normal crossings property of the exceptional divisor are preserved by base change. Moreover the dual graph of these two good resolutions is the same: for being $\mathbb{K}_1$ algebraically closed, the base change of each irreducible components of the exceptional divisor of π_1 gives an irreducible component of the exceptional divisor of π_2, and this assignment is obviously surjective. The genus of each irreducible component is clearly preserved (take as definition the dimension of the first cohomology group of the structure sheaf), and the self-intersection as well for being the first Chern number of the normal bundle of the irreducible component in the ambient surface.

Since the dual graph of the minimal good resolution is obtained from the dual graph of any good resolution by a purely combinatorial procedure, it is clear that the dual graph of the minimal good resolution is stable by base change.

7.2. Stability of the irreducible components of the arc space by base change

If $\mathcal{X}_n$ denotes the space of n-jets of (X, O), then $\mathcal{X}_n \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$ is the space of n-jets of $(X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$. Indeed, observe that the equations on the coefficients of the truncated arcs defining the space of n-jets of (X, O) define the space of n-jets on $(X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$, since the equations that define (X, O) in the affine space over $\mathbb{K}_1$ are the same that the equations that define $(X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$ in the affine space over $\mathbb{K}_2$. As above, since $\mathbb{K}_1$ is algebraically closed, base change induces a bijection between the irreducible components of the space of n-jets of (X, O), and the irreducible components of the space of n-jets of $(X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$. Taking projective limits we obtain that base change induces a bijection between the irreducible components of the space of arcs of (X, O), and the irreducible components of the space of arcs of $(X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$.

Let (X, O) and

$$\pi_1 : \tilde{X} \to (X, O)$$

be as above and denote by E the exceptional divisor of π_1. Let $E = \bigcup_{i=1}^{r} E_i$ be the decomposition in irreducible components. Then

$$E \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2) = \bigcup_{i=1}^{r} E_i \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$$

is a decomposition in irreducible components of the exceptional divisor of π_2. We have the equality

$$\bar{N}_{E_u \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)} = \bar{N}_{E_u} \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2).$$

This follows because the $\mathbb{K}_1$-arcs through (X, O) with lifting (transverse) to E_u can be seen as arcs in $(X, O) \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$ with lifting (transverse) to $E_u \times_{\mathrm{Spec}(\mathbb{K}_1)} \mathrm{Spec}(\mathbb{K}_2)$.

7.3. Main results

The following theorem contains Theorem A stated at the introduction.

Theorem 33. *Let (X, O) be a normal surface singularity defined over a uncountable algebraically closed field $\mathbb{K}$ of characteristic 0. Let E_v be an essential irreducible component of the exceptional divisor of a resolution. Equivalent are*:

(1) *The set N_{E_v} is in the Zariski closure of N_{E_u}, with E_u another component of the exceptional divisor.*

(2) *Given any proper closed subset $\mathcal{Z} \subset \bar{N}_{E_u}$ there exists an algebraic $\mathbb{K}$-wedge realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*
(3) *There exists a formal $\mathbb{K}$-wedge avoiding $\mathcal{Z}$ realising an adjacency from E_u to E_v.*
(4) *Given any proper closed subset $\mathcal{Z} \subset \bar{N}_{E_u}$ there exists a finite morphism realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$.*

If the base field is $\mathbb{C}$ the following further conditions are equivalent to those above:

(a) *Given* any *convergent arc $\gamma \in \dot{N}_{E_v}$ there exists a convergent $\mathbb{C}$-wedge realising an adjacency from E_u to γ and avoiding Δ_{E_u} (that is, with generic arc transverse to E_u at a smooth point of E).*
(b) *Given* any *convergent arc $\gamma \in \dot{N}_{E_v}$ there exists a convergent $\mathbb{C}$-wedge realising an adjacency from E_u to γ.*
(c) *Given* any *convergent arc $\gamma \in \dot{N}_{E_v}$ there exists a finite morphism realising an adjacency from E_u to γ and avoiding Δ_{E_u}.*

A component N_{E_v} is in the image of the Nash map if and only if N_{E_v} is *not* in the Zariski closure of N_{E_u} for a different irreducible component E_u of the exceptional divisor. The negation of the statements of the previous theorem characterises the image of the Nash map in terms of wedges defined over the base field, and in particular gives Corollary B stated at the introduction.

Proof of Theorem 33. By Proposition 3.8 of [32] the set $\bar{N}_{E_v}$ is generically stable, and hence, if N_{E_v} is in the Zariski closure of N_{E_u} then, given any proper closed subset $\mathcal{Z}$ of $\bar{N}_{E_u}$, by Lemma 6 there exists a K-wedge whose special arc is the generic point of N_{E_v}, and whose general arc belongs to $N_{E_u} \setminus \mathcal{Z}$. Thus, by Proposition 15, there exists a formal $\mathbb{K}$-wedge realising an adjacency from E_u to E_v and avoiding $\mathcal{Z}$. By Proposition 17 we find an algebraic $\mathbb{K}$-wedge with the same properties. Thus (1) implies (2).

Obviously (2) implies (3). By Proposition 22 we have that (2) implies (4) and that (4) implies (3). Proposition 17 together with Proposition 22 shows that (3) implies (4). To prove the first set of equivalences we only need to show that (4) implies (1).

Let

$$\psi : (Z, Q) \to (X, O) \tag{28}$$

be a finite morphism realising an adjacency from E_u to E_v. In order to finish the proof we have to show that N_{E_v} is in the Zariski closure of N_{E_u}. It is well known that any morphism of algebraic varieties over a field of characteristic 0 is defined over a subfield $\mathbb{K}_1 \subset \mathbb{K}$ which is a finitely generated extension of $\mathbb{Q}$ (just observe that there are only finitely many coefficients involved in the equations of the varieties and of the morphism). Explicitly this means that there is a finite morphism

$$\psi' : \left(Z', Q'\right) \to \left(X', O'\right) \tag{29}$$

defined over $\mathbb{K}_1$ such that the morphism (28) is obtained by pull-back of (29) by

$$\mathrm{Spec}(\mathbb{K}) \to \mathrm{Spec}(\mathbb{K}_1). \tag{30}$$

Here ψ', (Z', Q') and (X', O') are the varieties and the morphism defined over $\mathbb{K}_1$ by the same equations than ψ, (Z, Q) and (X, O).

Let γ be the algebraic $\mathbb{K}$-arc such that ψ realises an adjacency from E_u to γ. The algebraicity of γ implies that there are polynomials $F_i \in \mathbb{K}[t, x_i]$ for $1 \leqslant i \leqslant N$ such that we have the equalities

$$F_i\big(t, x_i \circ \gamma(t)\big) = 0$$

in $\mathbb{K}[[t]]$. Let $\mathbb{K}_2$ be the finitely generated extension of $\mathbb{K}_1$ obtained adjoining the coefficients of the polynomials $F_1, \ldots, F_N$. Let $\bar{\mathbb{K}}_2$ be the algebraic closure of $\mathbb{K}_2$. For any $i \leqslant N$ we define the algebraic curve $C_i \subset \bar{\mathbb{K}}_2^2$ by the equation

$$F_i(t, x) = 0.$$

The formal arc $(t, x_i \circ \gamma(t))$ parametrises a smooth formal branch L of C_i at $(0, 0)$. Thus, the polynomial F_i admits a factor $G_i \in \bar{\mathbb{K}}_2[[t, x_i]]$ whose linear part is of the form $at + bx_i$ with $b \neq 0$ such that

$$G_i\big(t, x_i \circ \gamma(t)\big) = 0.$$

The last equation and the form of the linear part allows to prove by an easy inductive calculation that the coefficients of $x_i \circ \gamma(t)$ belong to $\bar{\mathbb{K}}_2$. We have proved that γ is a $\bar{\mathbb{K}}_2$-arc.

Since $\mathbb{K}_2$ is finitely generated over $\mathbb{Q}$ it admits an embedding in $\mathbb{C}$, and also its algebraic closure $\bar{\mathbb{K}}_2$. We redefine $\mathbb{K}_1$ to be equal to the algebraic closure $\bar{\mathbb{K}}_2$. Let

$$\pi' : \tilde{X}' \to X' \tag{31}$$

be the minimally good resolution of the singularity of X' at O'. Let

$$E' := \bigcup_u E'_u$$

be a decomposition of the exceptional divisor in irreducible components. The resolution $\pi : \tilde{X} \to X$ and the divisor E_u are the respective pull-backs of π' and E'_u by (30). It is clear that the arc γ belongs to $\dot{N}_{E_v}$ as $\mathbb{K}$-arc if and only if it belongs to $\dot{N}_{E'_v}$ as $\mathbb{K}_1$-arc.

On the other hand, at the beginning of this section we have observed that E_u is the base change of E'_u under (30), and that N_{E_u} is the base change of $N_{E'_u}$ under (30). Thus N_{E_v} is in the Zariski closure of N_{E_u} if and only if $N_{E'_v}$ is in the Zariski closure of $N_{E'_u}$.

Since $\mathbb{K}_1$ is the algebraic closure of a finitely generated extension of $\mathbb{Q}$, it admits an embedding into $\mathbb{C}$ which gives rise to a morphism

$$\operatorname{Spec}(\mathbb{C}) \to \operatorname{Spec}(\mathbb{K}_1). \tag{32}$$

Let

$$\psi'' : \big(Z'', Q''\big) \to \big(X'', O''\big) \tag{33}$$

and

$$\pi'' : \tilde{X}'' \to X'' \tag{34}$$

be the base change of (29) and (31) by (32). Let

$$E'' := \bigcup_u E''_u$$

be a decomposition of the exceptional divisor of π'' in irreducible components. As before $N_{E''_v}$ is in the Zariski closure of $N_{E''_u}$ if and only if $N_{E'_v}$ is in the Zariski closure of $N_{E'_u}$.

The mapping (33) is a finite morphism defined over $\mathbb{C}$ defining an adjacency from E''_u to the algebraic arc γ which, viewed as a $\mathbb{C}$-arc, belongs to $\dot{N}_{E''_v}$. By Proposition 22(1) there exists a $\mathbb{C}$-wedge realising an adjacency from E''_u to γ. Hence, by Proposition 26, for any convergent $\mathbb{C}$-arc γ' in $\dot{N}_{E''_v}$ there exists a $\mathbb{C}$-wedge realising an adjacency from E_u to γ'.

A $\mathbb{C}$-wedge realising an adjacency from E_u to γ' corresponds to a $\mathbb{C}$-arc in $\mathcal{X}_\infty$, taking the generic point into $N_{E''_u}$ and the special point γ'. Hence every convergent γ' in $\dot{N}_{E''_v}$ belongs to the Zariski-closure of $N_{E''_u}$. As, by Artin's Approximation Theorem, any formal arc admits a convergent approximation coinciding with the original one up to any fixed order, the set of convergent $\mathbb{C}$-arcs in $\dot{N}_{E''_v}$ is Zariski dense in $\dot{N}_{E''_v}$. As $\dot{N}_{E''_v}$ is Zariski dense in $\bar{N}_{E''_v}$ we have that $N_{E''_v}$ is in the Zariski closure of $N_{E''_u}$.

Assume now that the base field is $\mathbb{C}$. Proposition 26 shows that (2) implies (a), which clearly implies (b). Since the set of convergent transverse arcs through E_v is dense in N_{E_v} we have that (b) implies (1). The conditions (c) and (d) are also equivalent by a similar reasoning replacing Proposition 26 by Proposition 25. □

Now we can prove that the essentiality of the divisor is a combinatorial property, that is, a property of the vertex of the dual graph that is preserved by isomorphisms of graphs. In particular we have the following:

Theorem 34. *Let $\kappa : \mathcal{G}_1 \to \mathcal{G}_2$ be an isomorphism between the weighted graphs of the minimal good resolution of two normal surface singularities (X_1, O) and (X_2, O) defined over uncountable algebraically closed fields $\mathbb{K}_1$ and $\mathbb{K}_2$. Let E_u and E_v be two exceptional divisors of the minimal good resolution of (X_1, O).*

(1) *There is a finite analytic mapping realising an adjacency from E_u to E_v if and only if there is a finite analytic mapping realising an adjacency from $E_{\kappa(u)}$ to $E_{\kappa(v)}$.*
(2) *There is a $\mathbb{K}_1$-wedge realising an adjacency from E_u to E_v if and only if there is a $\mathbb{K}_2$-wedge realising an adjacency from $E_{\kappa(u)}$ to $E_{\kappa(v)}$.*
(3) *The set N_{E_v} is in the Zariski closure of N_{E_u} if and only if the set $N_{E_{\kappa(v)}}$ is in the Zariski closure of $N_{E_{\kappa(u)}}$.*

Proof. By Theorem 33 the three statements are equivalent. As in the proof of Theorem 33 we reduce the first statement to the case in which $\mathbb{K}_1 = \mathbb{K}_2 = \mathbb{C}$. Then the theorem follows from Proposition 32. □

Definition 35. Let (X, O) be a normal surface singularity defined over an algebraically closed field $\mathbb{K}$. The *adjacency graph* of (X, O) for a resolution is the directed graph whose vertices correspond in a bijective manner with the irreducible components of the exceptional divisor of the resolution of (X, O) and has an arrow from the vertex corresponding to E_u to the vertex corresponding to E_v if and only if N_{E_v} is in the Zariski closure of N_{E_u}. An arrow is called *trivial* if a sequence of contractions of rational curves with self-intersection -1 collapses E_v into E_u (in this case it is clear that N_{E_v} is in the Zariski closure of N_{E_u}).

Remark 36. The Nash mapping is bijective if and only if the adjacency graph contains only trivial arrows.

The previous theorem may be read as follows:

Corollary 37. *Let (X, O) be a normal surface singularity defined over an algebraically closed field $\mathbb{K}$ of characteristic* 0 (*without the uncountability hypothesis*)*. The minimal resolution graph of (X, O) determines the adjacency graph of any resolution. In the case of complex analytic singularities the topology of the abstract link determines the adjacency graph of any resolution. Hence the bijectivity of the Nash mapping is a topological property of the singularity.*

Proof. The result is a corollary of Theorem 34 if $\mathbb{K}$ is uncountable. For the countable case it is enough to observe that the resolution and adjacency graphs are preserved by base change, as it is easily derived from the paragraphs at the beginning of this section. □

7.4. The problem of lifting wedges

In the rest of the section we link our results with the problem of lifting wedges as studied in [15,32], and [16]. Here we work over the complex numbers. We remind some terminology:

Let $\mathbb{K} \subset K$ be a field extension. A K-wedge α lifts to $\tilde{X}$ if the rational map $\pi^{-1} \circ \alpha$ is a morphism. A K-wedge is centred at the generic point of $\bar{N}_{E_i}$ if its special arc is the generic point of $\bar{N}_{E_i}$. The resolution $\pi : \tilde{X} \to X$ satisfies the property of lifting wedges centred at the generic point of $\bar{N}_{E_v}$ if any such wedge lifts to $\tilde{X}$. By Theorem 5.1 of [32] this equivalent to the fact that E_v is in the image of the Nash map.

In Proposition 2.9 of [16] a sufficient condition for a resolution to have the property of lifting wedges centred at the generic point of $\bar{N}_{E_v}$ is given in the following way: it is sufficient to check that there exists a very dense collection of closed points of E_v such that any $\mathbb{K}$-wedge whose special arc is transverse to E_v through any of the points of the collection lifts to $\tilde{X}$ (a very dense set is a set which intersects any countable intersection of dense open subsets). We improve this result for normal complex surface singularities in the following theorem by proving that it is sufficient that there exists a single, convergent transverse arc to E_v such that any $\mathbb{C}$-wedge having it as special arc lifts. A use of Lefschetz principle as above shows that the assumption that the base field is $\mathbb{C}$ is harmless.

Corollary 38. *Let (X, O) be a normal surface singularity defined over $\mathbb{C}$. Let*

$$\pi : \tilde{X} \to X$$

be a resolution of singularities and E_v any essential irreducible component of the exceptional divisor. If there exists a convergent arc $\gamma \in \dot{N}_{E_v}$ such that any $\mathbb{K}$-wedge having γ as special arc *lifts to $\tilde{X}$, then any resolution of $\tilde{X}$ has the property of lifting wedges centred at the generic point of $\bar{N}_{E_v}$ (or equivalently, the component E_v is in the image of the Nash map).*

Proof. By Theorem 33 the component E_v is in the image of the Nash map if and only if it does not exist any formal $\mathbb{C}$-wedge realising an adjacency from E_u to E_v, for E_u a component of the exceptional divisor of π different from E_v. If such a $\mathbb{C}$-wedge exists, by Proposition 26 there exists a $\mathbb{C}$-wedge realising an adjacency from E_u to γ. Since such a wedge has γ as special arc and does not lift to $\tilde{X}$ we obtain a contradiction. □

8. Applications

In what follows we establish a few relations between the resolution graph and the adjacency graph. As both graphs are combinatorial we can work over $\mathbb{C}$ in the proofs without losing generality.

We start with a result for graphs with symmetries. We deal with dual graphs of good resolutions of singularities. Here vertices correspond to irreducible components of the exceptional divisor and are weighted by self-intersection and genus. An *essential vertex* is a vertex which corresponds to an essential component of a resolution. Essentiality of a vertex is a combinatorial property of a graph by Theorem 34.

Proposition 39. *If φ is an automorphism of a graph $\mathcal{G}$ not fixing an essential vertex u then there is no adjacency from u to $\varphi(u)$.*

Proof. Let $\{u_0 = u, \ldots, u_{d-1}\}$ be the orbit of u by φ; we name its points so that $u_{i+1} = \varphi(u_i)$ for any index i taken modulo d. Let

$$\pi : \tilde{X} \to (X, O)$$

be a resolution of a normal surface singularity with graph $\mathcal{G}$. For each vertex u_i we denote by E_{u_i} the corresponding irreducible component of the exceptional divisor. As essentiality is a combinatorial property of the graph by Theorem 34, since E_{u_0} is essential, every E_{u_i} is essential.

Suppose we have an adjacency from $u_0 = u$ to $u_1 = \varphi(u)$. This means that we have the inclusion

$$\bar{N}_{E_{u_0}} \supset \bar{N}_{E_{u_1}}.$$

By Theorem 34 we have an adjacency from u_i to u_{i+1} for any i, taking the indexes modulo d, and, hence, we have the chain of inclusions

$$\bar{N}_{E_{u_1}} \supset \bar{N}_{E_{u_2}} \supset \cdots \supset \bar{N}_{E_{u_0}}.$$

We conclude the equality $\bar{N}_{E_{u_0}} = \bar{N}_{E_{u_1}}$.

On the other hand, by Corollary 3.9 of [32] the sets $\dot{N}_{E_{u_0}}$ and $\dot{N}_{E_{u_1}}$ are Zariski open in the irreducible set $\bar{N}_{E_{u_0}} = \bar{N}_{E_{u_1}}$. Since $\dot{N}_{E_{u_0}}$ and $\dot{N}_{E_{u_1}}$ are disjoint we obtain a contradiction. □

As a curiosity we obtain an affirmative answer for Nash problem for very symmetric graphs:

Corollary 40. *If the automorphism group of a graph $\mathcal{G}$ acts transitively, then its adjacency graph contains only trivial arrows. In other words, Nash mapping is bijective for singularities having this kind of resolution graph.*

Examples of these graphs are graphs given by those negative definite graphs having the shape of any regular polygon or polyhedron with the same weight on each vertex.

The following corollary is related to results of C. Plénat (see [24, Proposition 3.1 and Corollary 3.4]).

Corollary 41. *The following statements hold*:

- *Let $\mathcal{G}_1$ be a graph contained in a negative definite weighted graph $\mathcal{G}_2$. If the adjacency graph of $\mathcal{G}_1$ has an arrow from u to v then the adjacency graph of $\mathcal{G}_2$ has also an arrow from u to v.*
- *Let $\mathcal{G}_2$ be a graph obtained from a negative definite weighted $\mathcal{G}_1$ by decreasing the self-intersection weights of its vertices* (*the graph $\mathcal{G}_2$ is automatically negative definite*)*. If the adjacency graph of $\mathcal{G}_2$ has an arrow from u to v then the adjacency graph of $\mathcal{G}_1$ has also an arrow from u to v.*

Proof. We start by the first assertion. Let (X_1, O_1) and (X_2, O_2) be normal surface singularities whose resolutions

$$\pi_i : \tilde{X}_i \to (X_i, O_i)$$

have graphs $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively. After suitable choice of representatives, the manifold $\tilde{X}_1$ is obtained by plumbing according to the graph $\mathcal{G}_1$. Since the graph $\mathcal{G}_2$ contains $\mathcal{G}_1$, the smooth manifold $\tilde{X}_2$ may be obtained from $\tilde{X}_1$ by adding:

- for each vertex of $\mathcal{G}_2 \setminus \mathcal{G}_1$, one disk bundle over a surface of genus prescribed by the genus weight of the vertex, such that the self-intersection of the only middle homology class is prescribed by the self-intersection weight of the vertex,
- and one plumbing identification for each edge e of $\mathcal{G}_2 \setminus \mathcal{G}_1$.

Therefore we have an inclusion

$$\varphi : \tilde{X}_1 \hookrightarrow \tilde{X}_2.$$

We choose a complex structure on $\tilde{X}_1$ such that φ becomes holomorphic, and change the complex structure of (X_1, O_1) accordingly.

By the hypothesis, Theorem 33 and Corollary 37, there is a $\mathbb{C}$-wedge in X_1 realising an adjacency from E_u to E_v. By Proposition 17 there exists a convergent $\mathbb{C}$-wedge

$$\alpha : (\mathbb{C}^2, O) \to X_1$$

realising the same adjacency. Let

$$\sigma : W \to (\mathbb{C}^2, O)$$

be the chain of blow-ups at points giving the minimal resolution of indeterminacy of $\pi_1^{-1} \circ \alpha$, and

$$\tilde{\alpha} : W \to \tilde{X}_1$$

the lifting of α. The mapping

$$\beta := \pi_2 \circ \varphi \circ \tilde{\alpha} \circ \sigma^{-1} : (\mathbb{C}^2, O) \to X_2$$

is a well-defined analytic mapping and a $\mathbb{C}$-wedge in X_2 realising the adjacency from E_u to E_v. This proves (1).

Now we deduce (2) from (1). For each vertex v let a_v be the difference of the self-intersection weight of the vertex v in $\mathcal{G}_1$ and in $\mathcal{G}_2$. Construct a graph $\mathcal{G}_3$ as follows: for each vertex v of $\mathcal{G}_2$ add a_v rational vertices (vertices with genus weight equal to 0) with self-intersection weight equal to -1 to $\mathcal{G}_2$ and edges from each of them to v. The graph $\mathcal{G}_1$ is obtained from $\mathcal{G}_3$ by deleting the vertices of $\mathcal{G}_3 \setminus \mathcal{G}_2$ (together with their attaching edges); the weights are obtained by increasing the self-intersection weight of the vertices in one unit for each deleted adjacent vertex (this is the combinatorial counterpart of the blow-down operation). Since the adjacency between components does not depend on the chosen resolution of singularities, and the second operation corresponds to collapsing (-1)-smooth rational curves, given $u, v \in \mathcal{G}_1$ there is an arrow from u to v in the adjacency graph of $\mathcal{G}_1$ if and only if there is an arrow from u to v in the adjacency graph of $\mathcal{G}_3$. As $\mathcal{G}_2$ is a subgraph of $\mathcal{G}_3$, (2) follows from (1). □

Our aim now is to improve the following result of M. Lejeune-Jalabert and A. Reguera [16].

Proposition 42 *(M. Lejeune-Jalabert, A. Reguera). The following assertions hold*:

(1) *Assume there is an arrow from a vertex u to a vertex v in the adjacency graph of (X, O). Then there is a path joining the vertices u and v in the minimal resolution graph such that all the vertices appearing in the path, with the possible exception of u, correspond to rational irreducible components of the exceptional divisor. In particular, if E_v is not rational then it is in the image of the Nash map.*
(2) *If the Nash mapping is bijective for all graphs containing only rational curves, then it is bijective in general.*

From now on we only work with singularities having resolutions where all the irreducible components of the exceptional divisor are rational.

Let $\mathcal{G}$ be a finite weighted graph with no edges starting and ending at the same vertex, and such that there don't exist two different edges connecting the same pair of vertices. It is clear that any normal surface singularity has a good resolution whose associated dual graph has these properties.

The realisation $Z(\mathcal{G})$ of $\mathcal{G}$ is the topological space constructed as follows: take a disjoint union of 3-dimensional balls of radius 1 indexed over the vertices of $\mathcal{G}$, and copies of the interval [0, 1] indexed over the edges of $\mathcal{G}$. For each edge of $\mathcal{G}$, identify the ends of the corresponding interval with points of the boundary of the balls corresponding to the vertices which are joined by the edge. Do the identifications such that no point in the boundary of a ball is the identification point of two different edges. It is clear that the topological type of $Z(\mathcal{G})$ only depends on $\mathcal{G}$. The realisation $Z(\mathcal{G})$ has the homotopy type of a wedge of circumferences. Given a map of graphs $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ mapping vertices to vertices, edges to edges and respecting the incidence of the edges, its realisation is a continuous map

$$t(\alpha) : Z(\mathcal{G}) \to Z'(\mathcal{G})$$

mapping the ball of a vertex v homeomorphically to the ball of the vertex $\alpha(v)$ and the interval of an edge e homeomorphically to the interval of the edge $\alpha(e)$. The topological conjugation class of the realisation of a mapping of graphs only depends on the mapping of graphs.

A *loop* in $\mathcal{G}$ is a sequence of vertices and edges $\{v_1, e_1, v_2, e_2, \dots, v_r, e_r\}$ such that v_i is joined by e_i to v_{i+1} for any $i < r$ and v_r is joined to v_0 by e_r. Notice that a loop is determined by its set of edges $\{e_1, \dots, e_r\}$. A loop in $\mathcal{G}$ is simple if there are no vertices repeated in the sequence. The finiteness of $\mathcal{G}$ implies that there are finitely many simple loops $l_1, \dots, l_k$. Since there are no edges starting and ending at the same point each simple loop contains at least two vertices. For each simple loop l_i there is a simple closed curve β_i in $Z(\mathcal{G})$ (in other words, an embedding of $\mathbb{S}^1$ into $Z(\mathcal{G})$) going, through the balls and intervals corresponding to the vertices and edges l_i, in the order prescribed by l_i. The free homotopy class of β_i in $Z(\mathcal{G})$ is determined by l_i, and the homology class of $[\beta_i] \in H_1(Z(\mathcal{G}), \mathbb{Z})$ determines l_i. A *trivial loop* is a loop inducing the trivial element in the fundamental group.

We say that a loop $l = \{e_1, e_2, \dots, e_r\}$ *contains a simple loop* $l_i := \{f_1, f_2, \dots, f_s\}$ if:

(1) There is an increasing function

$$\theta : \{1, \dots, s\} \to \{1, \dots, r\}$$

such that $f_i = e_{\alpha(i)}$ and for any $i \leqslant s$ the sub-loop

$$\{e_{\alpha(i)}, e_{\alpha(i)+1}, \dots, e_{\alpha(i+1)}\}$$

is trivial (for the case $i = s$ consider the indexes modulo s).

(2) The loop $\{e_{\alpha(1)}, \dots, e_{\alpha(r)}\}$ is not a sub-loop of a trivial sub-loop of l.

Given any finite covering $\varphi : Z' \to Z(\mathcal{G})$ we associate a finite graph $\mathcal{G}(Z')$ to Z' as follows: a point of Z' is 1-dimensional if it has a neighbourhood of topological dimension equal to 1. Take a vertex v' for each connected component $C_{v'}$ of the complement in Z' of the set of 1-dimensional points (intuitively take a vertex for each "3-dimensional ball"). The component $C_{v'}$ is mapped under φ to a 3-dimensional ball corresponding to a vertex v. Give to v' the weight of v. Join two vertices with an edge if they are connected in Z' with an path homeomorphic to $[0, 1]$ such that all the points in $(0, 1)$ are 1-dimensional. It is clear that Z' is homeomorphic to the realisation $Z(\mathcal{G}(Z'))$, and that there is a map

$$\alpha : \mathcal{G}(Z') \to \mathcal{G} \tag{35}$$

whose realisation is topologically conjugate to φ. A map of graphs is a finite covering if its realisation is a finite covering.

Lemma 43. *For any $M \in \mathbb{N}$ there exists a finite covering $\varphi : Z' \to Z(\mathcal{G})$ such that any non-trivial loop in $\mathcal{G}(Z')$ has at least M vertices.*

Proof. Let $\{l_1, \ldots, l_k\}$ be the simple loops of $\mathcal{G}$.

Consider any simple loop $l_i = \{v_1, e_1, \ldots, v_r, e_r\}$ for $i \leqslant k$. Take M disjoint copies $\mathcal{G}^1, \ldots, \mathcal{G}^M$ of $\mathcal{G}$. Let $l_i^j = \{v_1^j, e_1^j, \ldots, v_r^j, e_r^j\}$ the loop corresponding to l_i in $\mathcal{G}^j$. In the disjoint union

$$\coprod_{j=1}^{r} \mathcal{G}^j$$

we change the attaching points of the edges e_r^j for any j as follows: the edge e_r^j joins now the vertices v_r^j and v_1^{j+1} if $j < M$ and e_r^M joins v_r^m and v_1^1. The new graph is connected and denoted by $\mathcal{G}_i$ Let

$$\alpha_i : \mathcal{G}_i \to \mathcal{G}$$

be defined as $\alpha_i(v_j^i) := v_j$ and $\alpha_i(e_j^i) := e_j$. The realisation

$$t(\alpha_i) : Z(\mathcal{G}_i) \to Z(\mathcal{G})$$

is a covering of degree M. Let $l := \{v'_1, e'_1, \ldots, v'_r, e'_r\}$ be any loop in $\mathcal{G}_i$. By construction, if its image under α_i contains the simple loop l_i then it contains M copies of it.

For any i the projection to the i-th factor

$$\mathrm{pr}_i : Z' := Z(\mathcal{G}_1) \times_{Z(\mathcal{G})} \cdots \times_{Z(\mathcal{G})} Z(\mathcal{G}_k)$$

of the fibre product of the coverings $t(\alpha_i)$ is a covering. Define

$$\varphi : Z' \to Z(\mathcal{G})$$

by $\varphi := \alpha_1 \circ \mathrm{pr}_1 = \cdots = \alpha_k \circ \mathrm{pr}_k$. Let α and β_i be the mappings from $\mathcal{G}(Z')$ to $\mathcal{G}$ and $\mathcal{G}_i$ whose respective realisations are φ and pr_i for any $i \leqslant k$.

We claim that the finite covering

$$\alpha : \mathcal{G}(Z') \to \mathcal{G}$$

is the one we need.

Indeed, let $l := \{v'_1, e'_1, \ldots, v'_r, e'_r\}$ be any non-trivial loop in $\mathcal{G}(Z')$. Its image $\alpha(l)$ is a non-trivial loop in $\mathcal{G}$, since a covering induces an injection of fundamental groups. Thus $\alpha(l)$ contains a simple loop l_i. As $\alpha(l)$ equals $\alpha_i(\beta_i(l))$, and $\beta_i(l)$ is a non-trivial loop in $\mathcal{G}_i$ whose image under α_i contains l_i, the loop $\alpha(l)$ contains M-copies of l_i. Hence the loop l has at least M-vertices. $\square$

Proposition 44. *Let $\mathcal{G}$ be a negative definite weighted graph with rational vertices. Assume that there is an adjacency from a vertex u to a vertex v. Then there exist a negative-definite weighted graph $\mathcal{G}'$ without loops, with rational vertices and with an adjacency from a vertex u' to a vertex v', and a mapping of graphs $\alpha : \mathcal{G}' \to \mathcal{G}$ such that $\alpha(u') = u$ and $\alpha(v') = v$. In addition α can be factorised as $\alpha = \beta \circ \iota$ where ι is the inclusion of $\mathcal{G}'$ in a graph $\mathcal{G}''$ and β is a finite covering from $\mathcal{G}''$ to $\mathcal{G}$.*

Proof. Let

$$\pi : \tilde{X} \to (X, O)$$

be a resolution of a complex analytic normal surface singularity with graph equal to $\mathcal{G}$. Let $\pi^{-1}(0) = E = \bigcup_{u=1}^{r} E_u$ be a decomposition in irreducible components of the exceptional divisor. There is a neighbourhood U of O in X such that $\tilde{X} := \pi^{-1}(U)$ admits a deformation retract

$$r : \tilde{X} \to E.$$

We choose X to be equal to this neighbourhood U.

By Theorem 33 and Proposition 17 there exists a convergent $\mathbb{C}$-wedge

$$\alpha : (\mathbb{C}^2, O) \to X$$

realising an adjacency from E_u to E_v. Let

$$\sigma : W \to \mathbb{C}^2$$

be the composition of blow-ups at points giving the minimal resolution of indeterminacy of $\pi^{-1} \circ \alpha$. Let

$$\tilde{\alpha} : W \to \tilde{X}$$

be the lifting of α. Let K be the number of irreducible components of the exceptional divisor F of σ.

By Lemma 43 there exists a finite covering of graphs

$$\beta : \mathcal{G}'' \to \mathcal{G}$$

such that any loop in $\mathcal{G}''$ has at least $K + 1$-vertices. Now we will construct from it a finite morphism of singularities.

We construct a topological space B as follows: take the disjoint union of a copy B_i of $\mathbb{P}^1$ for each irreducible component of E_i of E. Let $g_i : B_i \to E_i$ be a homeomorphism for each i. If two components E_i and E_j meet at a common point x in E then add a copy of the interval [0, 1] and identify the point 0 with $g_i^{-1}(x)$ and 1 with $g_j^{-1}(x)$. The mapping

$$c : B \to E$$

which contracts the copies of intervals to a point and whose restriction to B_i coincides with g_i is a homotopy equivalence. Moreover the space B is naturally included in the realisation $Z(\mathcal{G})$ (let E_i' be the boundary of the 3-ball corresponding to the vertex associated to E_i) and the inclusion

$$\iota : B \to Z(\mathcal{G})$$

induces an isomorphism of fundamental groups. Therefore we have an isomorphism of fundamental groups

$$\iota_* \circ (c_*)^{-1} \circ r_* : \pi_1(\tilde{X}) \to \pi_1\big(Z(\mathcal{G})\big).$$

The covering β induces an injection of fundamental groups

$$t(\beta)_* : \pi_1\big(Z\big(\mathcal{G}''\big)\big) \to \pi_1\big(Z(\mathcal{G})\big).$$

Let

$$\tilde{\phi} : \tilde{X}'' \to \tilde{X}$$

be the finite covering corresponding to the subgroup $(\iota_* \circ (c_*)^{-1} \circ r_*)^{-1}(\pi_1(Z(\mathcal{G}'')))$. We give to $\tilde{X}''$ the unique complex structure making $\tilde{\phi}$ holomorphic. It is easy to check that $\tilde{X}''$ is the plumbing manifold associated to the graph $\mathcal{G}''$. As a consequence $E'' := \tilde{\phi}^{-1}(E)$ is connected.

By Stein Factorisation Theorem there exists a proper modification

$$\rho : \tilde{X}'' \to X''$$

and a finite morphism

$$\phi : X'' \to X,$$

with X'' normal such that $\phi \circ \rho = \pi \circ \tilde{\rho}$. As $(\pi \circ \tilde{\rho})^{-1}(O) = E''$ is connected, there is a unique point O'' mapped to O by ϕ. Therefore ϕ is a finite analytic mapping of normal surface singularities and the graph $\mathcal{G}''$ is negative definite.

Since W is simply connected there exists a lifting

$$\tilde{\alpha}'' : W \to \tilde{X}''$$

which is holomorphic. Let E' be the image of F by $\tilde{\alpha}''$. Since F has K irreducible components, the set E' is a connected union of at most K irreducible components. Let $\mathcal{G}'$ be the subgraph of $\mathcal{G}''$ corresponding to these components. Since each simple loop of $\mathcal{G}''$ has at least $K+1$ vertices, the graph $\mathcal{G}'$ has no loops. The graph $\mathcal{G}'$ is negative definite for being a subgraph of $\mathcal{G}''$. Thus, by Grauert's Contraction Theorem [6] there exists a birational morphism

$$\kappa : \tilde{X}' \to \big(X', O'\big)$$

which contracts E' giving rise to a normal surface singularity. The mapping

$$\kappa \circ \tilde{\alpha}'' : \big(\mathbb{C}^2, O\big) \to \big(X', O'\big)$$

defines a wedge. Let u' and v' be the vertices of $\mathcal{G}'$ such that the general arc of the wedge sends the special point into $E'_{u'}$ and the special arc sends the special point into $E'_{v'}$. Then $\kappa \circ \tilde{\alpha}''$ realises an adjacency from $E'_{u'}$ to $E'_{v'}$. $\square$

Let us finish giving a name to the class of graphs to which we have reduced Nash problem.

Definition 45. A negative definite weighted graph is *extremal* if it has only rational vertices, no loops, and it is such that if we increase the weight of any vertex then the resulting graph is either not negative definite, or the number of trivial arrows in the adjacency graph increases. A $\mathbb{Q}$-homology sphere is *extremal* if it is the boundary of the plumbing 4-manifold of an extremal graph.

Corollary 46. *If the Nash map is bijective for all singularities whose resolution graph is extremal then it is bijective in general. Equivalently, if the Nash mapping is bijective for all complex analytic normal surface singularities having extremal $\mathbb{Q}$-homology sphere links then it is bijective in general.*

Proof. Proposition 42 reduces the class of singularities having only rational exceptional divisors. Proposition 44 allows to reduce to the class of singularities whose exceptional divisor is a tree of rational curves: an easy combinatorial argument shows that if $\varphi : \mathcal{G}_1 \to \mathcal{G}_2$ is a finite covering of graphs a vertex $v \in \mathcal{G}_1$ can be collapsed after a finite sequence of -1-vertices contractions if and only if $\varphi(v)$ can be collapsed in the same way.

The reduction to the extremal case is made using Corollary 41. □

Many of the classes of normal surface singularities for which Nash Problem has been settled in the affirmative up to now have links which are in general non-extremal rational homology spheres, for example toric singularities, minimal surface singularities and sandwiched surface singularities. On the other hand the rational double points D_n, E_6, E_7 and E_8 are extremal and Nash question has been specially difficult for them. In fact, it has not been settled until very recently, see [25,28,22]. In the last paper [22] Nash mapping is proved to be bijective for any quotient surface singularity. In fact the most difficult case coincides with the cases of quotient surface singularities having extremal rational homology sphere links. These are the rational double points D_n, E_6, E_7 and E_8.

In a new recent paper M. Pe Pereira and the author [4] have settled the bijectivity of the Nash mapping for surfaces using essentially several of the results of this paper.

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