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Advances in Mathematics 230 (2012) 1272–1321

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Orthogonal polynomials in the normal matrix model with a cubic potential

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Available online 23 April 2012

Communicated by Nikolai Makarov

Abstract

We consider the normal matrix model with a cubic potential. The model is ill-defined, and in order to regularize it, Elbau and Felder introduced a model with a cut-off and corresponding system of orthogonal polynomials with respect to a varying exponential weight on the *cut-off region* on the complex plane. In the present paper we show how to define orthogonal polynomials on a specially chosen *system of infinite contours* on the complex plane, without any cut-off, which satisfy the same recurrence algebraic identity that is asymptotically valid for the orthogonal polynomials of Elbau and Felder.

The main goal of this paper is to develop the Riemann–Hilbert (RH) approach to the orthogonal polynomials under consideration and to obtain their asymptotic behavior on the complex plane as the degree n of the polynomial goes to infinity. As the first step in the RH approach, we introduce an auxiliary vector equilibrium problem for a pair of measures (μ_1, μ_2) on the complex plane. We then formulate a 3×3 matrix valued RH problem for the orthogonal polynomials in hand, and we apply the nonlinear steepest descent method of Deift–Zhou to the asymptotic analysis of the RH problem. The central steps in our study are a sequence of transformations of the RH problem, based on the equilibrium vector measure (μ_1, μ_2) , and the construction of a global parametrix.

The main result of this paper is a derivation of the large n asymptotics of the orthogonal polynomials on the whole complex plane. We prove that the distribution of zeros of the orthogonal polynomials converges to the measure μ_1 , the first component of the equilibrium measure. We also obtain analytical results for the

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measure μ_1 relating it to the distribution of eigenvalues in the normal matrix model which is uniform in a domain bounded by a simple closed curve.

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Keywords: Normal matrix model; Orthogonal polynomials; Vector equilibrium problem; Riemann–Hilbert problem; Steepest descent analysis

1. Introduction

The normal matrix model is a probability measure on the space of $n \times n$ normal matrices M of the form

$$\frac{1}{Z_n} \exp(-n\text{Tr}\mathcal{V}(M)) dM. \tag{1.1}$$

A typical form for \mathcal{V} is

$$\mathcal{V}(M) = \frac{1}{t_0} (MM^* - V(M) - \bar{V}(M^*)), \quad t_0 > 0, \tag{1.2}$$

where V is a polynomial and \bar{V} is the polynomial obtained from V by conjugating the coefficients. In this case, the model may alternatively be defined on general $n \times n$ complex matrices M , see e.g. [34]. In this paper we study in particular (1.1) and (1.2) with a cubic potential

$$V(M) = \frac{t_3}{3} M^3, \quad t_3 > 0. \tag{1.3}$$

The main feature of the normal matrix model is that the eigenvalues of M fill out a bounded two-dimensional domain Ω as $n \rightarrow \infty$ with a uniform density. Wiegmann and Zabrodin [32] showed that if

$$V(M) = \sum_{k=1}^{\infty} \frac{t_k}{k} M^k \tag{1.4}$$

then $\Omega = \Omega(t_0; t_1, t_2, \dots)$ is such that

$$t_0 = \frac{1}{\pi} \text{area}(\Omega), \quad t_k = -\frac{1}{\pi} \iint_{\mathbb{C} \setminus \Omega} \frac{dA(z)}{z^k}, \quad k = 1, 2, 3, \dots \tag{1.5}$$

where dA denotes the two-dimensional Lebesgue measure in the complex plane. The relations (1.5) characterize the domain Ω by means of its area πt_0 and its exterior harmonic moments t_k for $k \geq 1$ (it is assumed that $0 \in \Omega$, and the integrals in (1.5) need to be regularized for $k \leq 2$). An important fact, first shown in [22], is that the boundary of Ω as a function of $t_0 > 0$ (which is seen as a time parameter) evolves according to the model of Laplacian growth. The Laplacian growth is unstable and singularities such as boundary cusps may appear in finite time. See also [27,30,33,34] for related work and surveys.

However, from a mathematical point of view, the model (1.1) and (1.2) is not well-defined if V is a polynomial of degree ≥ 3 , since then the integral

$$Z_n = \int \exp(-n\text{Tr}\mathcal{V}(M)) dM \tag{1.6}$$

diverges, and one cannot normalize the measure (1.1) to make it a probability measure. To make the model well-defined, Elbau and Felder [17,18] propose to use a cut-off procedure. Instead of considering all normal matrices M , they restrict to normal matrices with spectrum in a fixed bounded domain D . The integral (1.6) restricted to all such matrices is convergent and the model is well-defined. As it is the case for unitary random matrices, the eigenvalues of M are then distributed according to a determinantal point process with a kernel that is built out of orthogonal polynomials with respect to the scalar product

$$\langle f, g \rangle_D = \iint_D f(z) \overline{g(z)} e^{-n\mathcal{V}(z)} dA(z) \tag{1.7}$$

(which depends on n), with

$$\mathcal{V}(z) = \frac{1}{t_0} \left(|z|^2 - V(z) - \overline{V(z)} \right), \tag{1.8}$$

see [17]. For each n , we have the sequence $(P_{k,n})_{k=0}^\infty$ of monic polynomials (i.e., $P_{k,n}(z) = z^k + \dots$) such that

$$\langle P_{k,n}, P_{j,n} \rangle_D = h_{k,n} \delta_{j,k},$$

and then the correlation kernel for the determinantal point process is

$$K_n(w, z) = e^{-\frac{n}{2}(\mathcal{V}(w)+\mathcal{V}(z))} \sum_{k=0}^{n-1} \frac{P_{k,n}(z) \overline{P_{k,n}(w)}}{h_{k,n}}.$$

Elbau and Felder [17,18] prove that for a polynomial V as in (1.4) with $t_1 = 0$, $|t_2| < 1$, and for t_0 small enough it is possible to find a suitable domain D such that indeed the eigenvalues of the normal matrix model with cut-off D accumulate on a domain Ω as $n \rightarrow \infty$. The domain Ω is characterized by (1.5) and so in particular evolves according to Laplacian growth in the time parameter t_0 . Note that the cut-off approach works fine for t_0 small enough but fails to capture important features of the normal matrix model such as the formation of cusp singularities at a critical value of t_0 .

Elbau [17] also discusses the zeros of the orthogonal polynomials $P_{n,n}$ as $n \rightarrow \infty$. For the cubic potential (1.3) and again for t_0 sufficiently small, he shows that these zeros accumulate on a starlike set

$$\Sigma_1 = [0, x^*] \cup [0, \omega x^*] \cup [0, \omega^2 x^*], \quad \omega = e^{2\pi i/3}, \tag{1.9}$$

for some explicit value of $x^* > 0$. The set Σ_1 is contained in Ω . The limiting distribution of zeros is a probability measure μ_1^* on Σ_1 satisfying

$$\int \log |z - \zeta| d\mu_1^*(\zeta) = \frac{1}{\pi t_0} \iint_\Omega \log |z - \zeta| dA(\zeta), \quad z \in \mathbb{C} \setminus \Omega. \tag{1.10}$$

For general polynomial V and t_0 small enough, Elbau conjectures that the zeros of the orthogonal polynomials accumulate on a tree-like set strictly contained in Ω with a limiting distribution whose logarithmic potentials outside of Ω agrees with that of the normalized Lebesgue measure on Ω .

In this paper we want to analyze the orthogonal polynomials for the normal matrix model without making a cut-off. We cannot use the scalar product

$$\langle f, g \rangle = \iint_{\mathbb{C}} f(z)\overline{g(z)}e^{-n\mathcal{V}(z)}dA(z) \tag{1.11}$$

defined on \mathbb{C} since the integral (1.11) diverges if f and g are polynomials and V is a polynomial of degree ≥ 3 . Our approach is to replace (1.11) by a Hermitian form defined on polynomials that is a priori not given by any integral, but that should satisfy the relevant algebraic properties of the scalar product (1.11). We define and classify these Hermitian forms. See the next section for precise statements.

It turns out that there is more than one possibility for such a Hermitian form. We conjecture that for any polynomial V it is possible to choose the Hermitian form in such a way that the corresponding orthogonal polynomials have the same asymptotic behavior as the orthogonal polynomials in the cut-off approach of Elbau and Felder. That is, for small values of t_0 the zeros of the polynomials accumulate on a tree like set Σ_1 with a limiting distribution μ_1^* satisfying (1.10).

We are able to establish this for the cubic case (1.3) and this is the main result of the paper. We recover the same set Σ_1 as in (1.9) and also the domain Ω that evolves according to the Laplacian growth, as we will show. In our approach we do not have to restrict to t_0 sufficiently small. We can actually take any t_0 up to the critical time $t_{0,crit}$. For this value the endpoints of Σ_1 come to the boundary of Ω . Then three cusps are formed on the boundary and the Laplacian growth breaks down. An asymptotic analysis at the critical time would involve the Painlevé I equation, which is what we see in our model and that we will address in a future paper.

We note that Ameur, Hedenmalm and Makarov [2,21] do not use a hard cut-off as in [18, 19], but instead a soft cut-off where $\mathcal{V}(z)$ tends to $+\infty$ as $z \rightarrow \infty$ in all directions of the complex plane. This approach includes cases (1.8) with a polynomial V of degree ≥ 3 only if V is modified outside of the domain Ω .

2. Statement of results

2.1. Hermitian forms

We propose to consider orthogonal polynomials with respect to Hermitian forms that share the algebraic properties of the scalar product (1.11) with $\mathcal{V}(z)$ given by (1.8). The Hermitian form is a sesquilinear form $\langle \cdot, \cdot \rangle$ defined on the vector space of complex polynomials in one variable that satisfies the Hermitian condition

$$\langle f, g \rangle = \overline{\langle g, f \rangle}. \tag{2.1}$$

We use the convention that $\langle \cdot, \cdot \rangle$ is linear in the first argument and conjugate-linear in the second.

To see what kind of condition we want to put on the Hermitian form we look at the scalar product (1.7) on the cut-off region D . From the complex version of Green’s theorem we obtain for polynomials f and g

$$\begin{aligned} \frac{t_0}{2i} \oint_{\partial D} f(z)\overline{g(z)}e^{-n\mathcal{V}(z)}dz &= t_0 \iint_D \frac{\partial}{\partial \bar{z}} \left[f(z)\overline{g(z)}e^{-n\mathcal{V}(z)} \right] dA(z) \\ &= t_0 \iint_D f(z)\overline{g'(z)}e^{-n\mathcal{V}(z)}dA(z) \end{aligned}$$

$$\begin{aligned}
 & -n \iint_D z f(z) \overline{g(z)} e^{-n\mathcal{V}(z)} dA(z) \\
 & + n \iint_D f(z) \overline{V'(z)g(z)} e^{-n\mathcal{V}(z)} dA(z),
 \end{aligned} \tag{2.2}$$

which can be written as

$$t_0 \langle f, g' \rangle_D - n \langle zf, g \rangle_D + n \langle f, V'g \rangle_D = \frac{t_0}{2i} \oint_{\partial D} f(z) \overline{g(z)} e^{-n\mathcal{V}(z)} dz. \tag{2.3}$$

The cut-off approach works if the domain D is chosen such that the effect of the boundary ∂D becomes small when n is large. This means that the boundary integral in (2.3) is exponentially small compared to the other terms in (2.3) in the large n limit.

Inspired by (2.3) our idea is to ignore the right-hand side of (2.3) and require that the Hermitian form satisfies the identity

$$t_0 \langle f, g' \rangle - n \langle zf, g \rangle + n \langle f, V'g \rangle = 0 \tag{2.4}$$

for all polynomials f and g . We call (2.4) the structure relation. Combining (2.4) with (2.1) we also have the dual structure relation

$$t_0 \langle f', g \rangle - n \langle f, zg \rangle + n \langle V'f, g \rangle = 0. \tag{2.5}$$

We do not impose the condition that the Hermitian form is positive definite, and therefore it may not be a scalar product.

Our first result is a classification of Hermitian forms satisfying (2.1) and (2.4). Sesquilinear forms that satisfy (2.4) can be found as follows. We assume V is a polynomial and

$$d = \deg V - 1.$$

There are $d + 1$ directions in the complex plane, given by $\arg z = \theta_j, j = 0, 1, \dots, d$, such that

$$\arg(-V(z)) \rightarrow 0 \text{ as } |z| \rightarrow \infty \text{ with } \arg z = \theta_j.$$

We choose the directions in such a way that

$$\theta_j = \theta_0 + \frac{2\pi j}{d+1}, \quad j = 0, 1, \dots, d+1. \tag{2.6}$$

Let Γ_j for $j = 0, \dots, d$, be a simple smooth curve in the complex plane, starting at infinity in the direction $\arg z = \theta_j$ and ending at infinity in the direction $\arg z = \theta_{j+1}$. See Fig. 1 for possible contours Γ_j in case $d = 2$ and $t_3 > 0$. Then define for polynomials f and g , and for $j, k = 0, 1, \dots, d$,

$$\Phi_{j,k}(f, g) = \int_{\Gamma_j} dz \int_{\overline{\Gamma}_k} dw f(z) \overline{g(w)} e^{-\frac{n}{i}(wz - V(z) - \overline{V}(w))}, \tag{2.7}$$

where \overline{g} is the polynomial obtained from g by conjugating the coefficients. The integrals in (2.7) are convergent because of our choice of contours.

It is an easy exercise, based on integration by parts, that the sesquilinear forms $\Phi_{j,k}(\cdot, \cdot)$ indeed satisfy (2.4) and (2.5). Then so does any linear combination. We prove the following.

Theorem 2.1. *For every $n \in \mathbb{N}, t_0 > 0$, and polynomial V of degree $\deg V = d + 1$, the real vector space of Hermitian forms satisfying (2.1) and (2.4) is d^2 dimensional.*

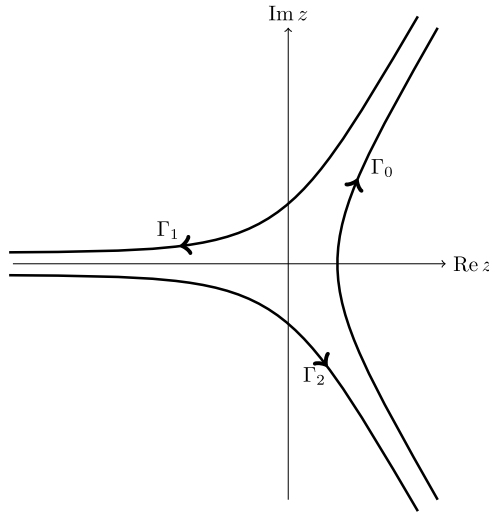


Fig. 1. Contours $\Gamma_0, \Gamma_1, \Gamma_2$ for the case of a cubic potential ($d = 2$).

Every such Hermitian form can be uniquely represented as

$$\langle f, g \rangle = \sum_{j=0}^d \sum_{k=0}^d C_{j,k} \Phi_{j,k}(f, g), \tag{2.8}$$

where $\Phi_{j,k}$ is given by (2.7) and where $C = (C_{j,k})_{j,k=0}^d$ is a Hermitian matrix having zero row and column sums.

The proof of Theorem 2.1 is in Section 3.

Now the following obvious question arises. Given a polynomial V , is there a Hermitian matrix $C = (C_{j,k})$ such that the Hermitian form (2.8) captures the main features of the normal matrix model? We answer this question for the cubic case.

2.2. The cubic potential and rotational symmetry

We consider from now on the case

$$V(z) = \frac{t_3}{3} z^3, \quad t_3 > 0. \tag{2.9}$$

Then $d = \deg V - 1 = 2$ and so by Theorem 2.1 the space of Hermitian forms (2.8) is four-dimensional. For the case (2.9) it is natural to require an additional rotational symmetry

$$\langle f(\omega z), g(\omega z) \rangle = \langle f, g \rangle, \quad \omega = e^{2\pi i/3}. \tag{2.10}$$

The condition (2.10) corresponds to the fact that $V(\omega z) = V(z)$ for the cubic potential (2.9) so that the integral in (1.11) is invariant under the change of variables $z \mapsto \omega z$. Thus (2.10) is specific for the cubic potential and will have to be modified for other potentials.

One may verify from the definition (2.7) with V given by (2.9) that

$$\Phi_{j,k}(f(\omega \cdot), g(\omega \cdot)) = \Phi_{j+1,k+1}(f, g),$$

where the indices are taken modulo 3. Thus

$$\sum_{j=0}^2 \sum_{k=0}^2 C_{j,k} \Phi_{j,k}(f(\omega \cdot), g(\omega \cdot)) = \sum_{j=0}^2 \sum_{k=0}^2 C_{j-1,k-1} \Phi_{j,k}(f, g),$$

where again the indices are taken modulo 3 so that for example $C_{-1,-1} = C_{2,2}$. It follows that (2.8) satisfies the symmetry condition (2.10) if and only if

$$C_{j,k} = C_{j-1,k-1}, \quad (\text{indices modulo } 3). \tag{2.11}$$

The condition (2.11) means that $C = (C_{j,k})$ is a circulant matrix. As dictated by Theorem 2.1, we also require that C is Hermitian with zero row and column sums.

The real vector space of circulant Hermitian matrices of size 3×3 with zero row and column sums is two-dimensional. A basis is given by the two matrices

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}. \tag{2.12}$$

It turns out that we are able to do asymptotic analysis on the orthogonal polynomials only if we choose for C a multiple of the second basis matrix in (2.12). However, we have no a priori reason to prefer this matrix above the other one, or above a linear combination of the two. It is only because of our ability to do large n asymptotics that we choose

$$C = \frac{1}{2\pi i} \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \tag{2.13}$$

This leads to the following definition of the Hermitian form and the corresponding orthogonal polynomials.

Definition 2.2. Given $n, t_0, t_3 > 0$ we define the Hermitian form $\langle \cdot, \cdot \rangle$ on the vector space of polynomials by

$$\langle f, g \rangle = \frac{1}{2\pi i} \sum_{j=0}^2 \sum_{k=0}^2 \epsilon_{j,k} \int_{\Gamma_j} dz \int_{\bar{\Gamma}_k} dw f(z) \bar{g}(w) e^{-\frac{n}{t_0} (wz - \frac{t_3}{3} (w^3 + z^3))} \tag{2.14}$$

where

$$(\epsilon_{j,k})_{j,k=0}^2 = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}. \tag{2.15}$$

Note that the Hermitian form (2.14) depends on n even though we do not emphasize this in the notation. For each n , we denote by $(P_{k,n})_{k=0,1,2,\dots}$ the sequence of monic orthogonal polynomials for the Hermitian form (2.14). That is, $P_{k,n}(z) = z^k + \dots$ is a polynomial of degree k such that

$$\langle P_{k,n}, z^j \rangle = 0, \quad \text{for } j = 0, 1, \dots, k - 1. \tag{2.16}$$

2.3. Existence of orthogonal polynomials $P_{n,n}$

The Hermitian form $\langle \cdot, \cdot \rangle$ may not be positive definite, and therefore the existence and uniqueness of the orthogonal polynomials is not guaranteed. This will be our next main result. We focus on the diagonal polynomials $P_{n,n}$ and for these polynomials we also determine the limiting behavior of the zeros.

Theorem 2.3. *Let $t_3 > 0$ and define*

$$t_{0,crit} = \frac{1}{8t_3^2}. \tag{2.17}$$

Then for every $t_0 \in (0, t_{0,crit})$ the orthogonal polynomials $P_{n,n}$ for the Hermitian form (2.14) exist if n is sufficiently large. In addition, the zeros of $P_{n,n}$ accumulate as $n \rightarrow \infty$ on the set

$$\Sigma_1 = \{z \in \mathbb{C} \mid z^3 \in [0, (x^*)^3]\} = \bigcup_{j=0}^2 [0, \omega^j x^*], \tag{2.18}$$

where $\omega = e^{2\pi i/3}$ and

$$x^* = \frac{3}{4t_3} \left(1 - \sqrt{1 - 8t_0 t_3^2}\right)^{2/3}. \tag{2.19}$$

Theorem 2.3 will follow from a strong asymptotic formula for the orthogonal polynomials, see Lemma 6.1.

2.4. Limiting zero distribution

For a polynomial P of degree n with zeros z_1, \dots, z_n in the complex plane, we use

$$\nu(P) = \frac{1}{n} \sum_{j=1}^n \delta_{z_j} \tag{2.20}$$

to denote the normalized zero counting measure.

The normalized zero counting measures $\nu(P_{n,n})$ of the orthogonal polynomials $P_{n,n}$ have a limit that we characterize in terms of the solution of a vector equilibrium problem from logarithmic potential theory. We use the following standard notation. For a measure μ , we define the logarithmic energy

$$I(\mu) = \iint \log \frac{1}{|x - y|} d\mu(x) d\mu(y) \tag{2.21}$$

and for two measures μ and ν , we define the mutual energy

$$I(\mu, \nu) = \iint \log \frac{1}{|x - y|} d\mu(x) d\nu(y). \tag{2.22}$$

In addition to the set Σ_1 from (2.18) we also need

$$\Sigma_2 = \{z \in \mathbb{C} \mid z^3 \in \mathbb{R}^-\} = \bigcup_{j=0}^2 [0, -\omega^j \infty). \tag{2.23}$$

Definition 2.4. Given $t_0, t_3 > 0$, we define the energy functional

$$E(\mu_1, \mu_2) = I(\mu_1) + I(\mu_2) - I(\mu_1, \mu_2) + \frac{1}{t_0} \int \left(\frac{2}{3\sqrt{t_3}} |z|^{3/2} - \frac{t_3}{3} z^3 \right) d\mu_1(z). \tag{2.24}$$

The vector equilibrium problem is to minimize (2.24) among all measures μ_1 and μ_2 such that

$$\int d\mu_1 = 1, \quad \text{supp}(\mu_1) \subset \Sigma_1, \quad \int d\mu_2 = \frac{1}{2}, \quad \text{supp}(\mu_2) \subset \Sigma_2, \tag{2.25}$$

where Σ_1 and Σ_2 are given by (2.18) and (2.23).

We prove the following.

Theorem 2.5. Let $t_3 > 0, 0 < t_0 \leq t_{0,crit}$ and let $x^* > 0$ be given by (2.19). Then there is a unique minimizer (μ_1^*, μ_2^*) for $E(\mu_1, \mu_2)$ among all vectors of measures (μ_1, μ_2) satisfying (2.25). If $t_0 < t_{0,crit}$ then the first component μ_1^* of the minimizer is the weak limit of the normalized zero counting measures of the polynomials $P_{n,n}$ as $n \rightarrow \infty$.

2.5. Two dimensional domain

We finally make the connection to the domain Ω that contains the eigenvalues in the normal matrix model, and that is characterized by the relations (1.5). In the cubic model that we are considering we are able to construct the domain Ω in terms of the solution (μ_1^*, μ_2^*) of the vector equilibrium problem for the energy functional (2.24) as follows.

Theorem 2.6. Let $t_3 > 0$ and $0 < t_0 \leq t_{0,crit}$. Let μ_1^* be the first component of the minimizer of the vector equilibrium problem, as described in Theorem 2.5. Then the equation

$$t_3 z^2 + t_0 \int \frac{d\mu_1^*(\zeta)}{z - \zeta} = \bar{z} \tag{2.26}$$

defines a simple curve $\partial\Omega$ that is the boundary of a domain Ω containing Σ_1 .

The domain Ω satisfies (1.5) and is such that

$$\int \frac{d\mu_1^*(\zeta)}{z - \zeta} = \frac{1}{\pi t_0} \iint_{\Omega} \frac{dA(\zeta)}{z - \zeta} \quad z \in \mathbb{C} \setminus \bar{\Omega}. \tag{2.27}$$

The identity (2.26) means that the left-hand side of (2.26) is the Schwarz function of $\partial\Omega$. See Fig. 2 for the domain Ω .

The proof of Theorem 2.6 is in Section 4. It follows from an analysis of the vector equilibrium problem and an associated three sheeted Riemann surface. This Riemann surface will also be important in the proofs of Theorems 2.3 and 2.5 that are based on a steepest descent analysis of a 3×3 matrix valued Riemann–Hilbert problem. The connection to the RH problem is explained in Section 5 and we refer to the discussion in Section 5.1 for motivation and connections with previous work. The actual steepest descent analysis is done in Section 6. It is quite involved and ultimately leads to a strong asymptotic formula for the polynomials $P_{n,n}$ as $n \rightarrow \infty$, see Lemma 6.1, from which the Theorems 2.3 and 2.5 will follow.

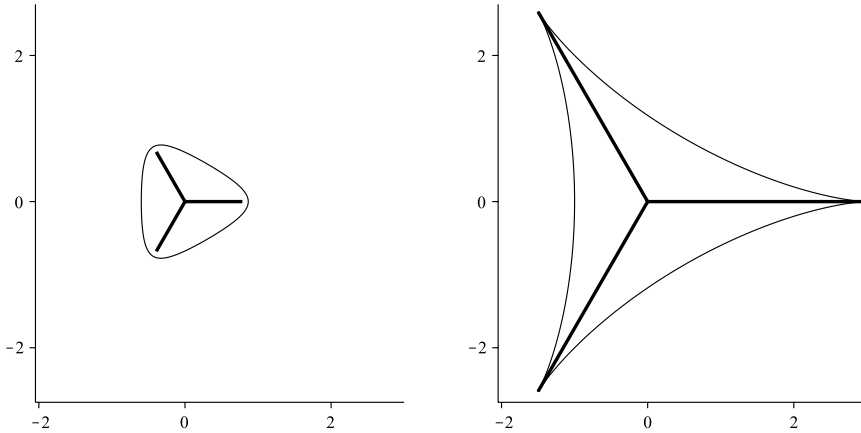


Fig. 2. The set Σ_1 where the zeros of $P_{n,n}$ accumulate and the boundary of the domain Ω for the values $t_0 = 1/2$ and $t_3 = 1/4$ (left) and for the critical values $t_0 = 2$ and $t_3 = 1/4$ (right).

3. Proof of Theorem 2.1

We have already seen that any linear combination (2.8) of the basic forms (2.7) gives a sesquilinear form that satisfies (2.4). We first show that any such sesquilinear form is characterized by a unique matrix $C = (C_{j,k})$ with zero row and column sums.

Lemma 3.1. *Any linear combination of the basic forms (2.7) can be written in the form (2.8) with a unique matrix $C = (C_{j,k})_{j,k=0}^d$ having zero row and column sums.*

Proof. The basic forms (2.7) are such that

$$\sum_{k=0}^d \Phi_{j,k}(f, g) = 0, \quad \text{for every } j$$

and

$$\sum_{j=0}^d \Phi_{j,k}(f, g) = 0, \quad \text{for every } k.$$

Thus one may add a constant to a row or to a column of C and obtain the same form (2.8). Hence we may assume that C has zero row sums

$$\sum_{k=0}^d C_{j,k} = 0, \quad \text{for every } j \tag{3.1}$$

and column sums

$$\sum_{j=0}^d C_{j,k} = 0, \quad \text{for every } k. \tag{3.2}$$

To show that the conditions (3.1) and (3.2) determine the sesquilinear form (2.8), we suppose that

$$\forall f, g : \langle f, g \rangle = \sum_{j=0}^d \sum_{k=0}^d C_{j,k} \Phi_{j,k}(f, g) = 0, \tag{3.3}$$

where $C = (C_{j,k})$ has zero row and column sums, and we prove that $C = O$.

Recall that V is a polynomial of degree $d + 1$ which we write as

$$V(z) = \frac{t_{d+1}}{d+1} z^{d+1} + V_1(z), \quad \deg V_1 \leq d. \tag{3.4}$$

The directions θ_j from (2.6) are such that

$$\frac{t_{d+1}}{d+1} z^{d+1} \in \mathbb{R}^-, \quad \arg z = \theta_j.$$

We deform the contour Γ_j in the definition (2.7) of $\Phi_{j,k}(f, g)$ so that it consists of the two rays $\arg z = \theta_j$ and $\arg z = \theta_{j+1}$. Then it follows from (2.7) and (3.3) that for every polynomial f ,

$$\begin{aligned} 0 &= \langle f, 1 \rangle \\ &= \sum_{j=0}^d \int_0^\infty \int_0^\infty dz f(z) e^{\frac{n}{t_0} V(z)} \sum_{k=0}^d (C_{j-1,k} - C_{j,k}) \int_{\Gamma_k} dwe^{-\frac{n}{t_0}(wz - \bar{V}(w))}, \end{aligned} \tag{3.5}$$

where $C_{-1,k} = C_{d,k}$. Making the substitution $z = xe^{i\theta_j}$ in the integral over z in (3.5) and using (3.4) we obtain

$$\sum_{j=0}^d \int_0^\infty f(e^{i\theta_j x}) e^{-cx^{d+1}} \phi_j(x) dx = 0, \tag{3.6}$$

where $c = \frac{n |t_{d+1}|}{t_0 (d+1)} > 0$, and

$$\phi_j(x) = e^{i\theta_j x} e^{\frac{n}{t_0} V_1(e^{i\theta_j x})} \sum_{k=0}^d (C_{j-1,k} - C_{j,k}) \int_{\Gamma_k} dwe^{-\frac{n}{t_0}(e^{i\theta_j} wx - \bar{V}(w))}. \tag{3.7}$$

Taking $f(x) = x^{(d+1)l+r}$ with non-negative integers l and r , we find from (2.6) and (3.6)

$$\sum_{j=0}^d \omega_{d+1}^{jr} \int_0^\infty x^{(d+1)l+r} e^{-cx^{d+1}} \phi_j(x) dx = 0, \quad \omega_{d+1} = e^{\frac{2\pi i}{d+1}}. \tag{3.8}$$

Since this holds for every $l = 0, 1, 2, \dots$, and since the functions ϕ_j grow at most as $O(e^{c_2 x^d})$ as $x \rightarrow +\infty$ for some $c_2 > 0$, we find from (3.8) that

$$\sum_{j=0}^d \omega_{d+1}^{jr} \phi_j(x) \equiv 0, \quad r = 0, 1, \dots, d$$

which in turn implies that

$$\phi_j(x) \equiv 0, \quad j = 0, \dots, d. \tag{3.9}$$

From (3.7) we know that the function $x \mapsto \phi_j(x)e^{-\frac{n}{t_0}V_1(e^{i\theta_j x})}$ is a linear combination of the entire functions

$$x \mapsto \int_{\overline{\Gamma}_k} dw e^{-\frac{n}{t_0}(e^{i\theta_j} wx - \overline{V}(w))}, \quad k = 0, \dots, d.$$

The only linear relation between these function is that they add up to zero. Because of (3.7) and (3.9) it follows that $C_{j-1,k} - C_{j,k}$ is independent of k . Summing over $k = 0, 1, \dots, d$ and using the fact that the matrix C has zero row sums, we find that

$$C_{j-1,k} - C_{j,k} = 0, \quad \text{for } j, k = 0, 1, \dots, d,$$

i.e., C has constant columns. Since the column sums are zero as well, we get $C = O$ as claimed.

Now we give the proof of Theorem 2.1.

Proof of Theorem 2.1. Using the rule that

$$\overline{\int_{\Gamma} f(z)dz} = \int_{\overline{\Gamma}} \overline{f(\overline{z})}dz$$

it is easy to obtain from (2.7) that

$$\Phi_{j,k}(f, g) = \overline{\Phi_{k,j}(g, f)}.$$

It follows that (2.8) with the normalization (3.1) and (3.2) satisfies the Hermitian condition (2.1) if and only if $C_{j,k} = \overline{C_{k,j}}$ for every j, k , that is, if and only if

$$C = C^* \quad (C \text{ is Hermitian matrix}).$$

By Lemma 3.1 any sesquilinear form (2.8) is characterized by a unique matrix C with zero row and column sums. The space of all $(d + 1) \times (d + 1)$ Hermitian matrices with the zero row/column sum property is isomorphic to the space of all $d \times d$ Hermitian matrices. Indeed, the restriction of C to the first d rows and columns provides an isomorphism. The dimension of this real vector space is d^2 . Hence the real dimension of the vector space of Hermitian forms satisfying (2.4) and (2.1) is at least d^2 .

To complete the proof we show that the dimension is at most d^2 . To that end, we consider the moments

$$\mu_{j,k} = \langle z^j, z^k \rangle$$

where $\langle \cdot, \cdot \rangle$ is a Hermitian form satisfying (2.4) and (2.1). Suppose $V(z) = \sum_{l=1}^{d+1} \frac{t_l}{t} z^l$ with $t_{d+1} \neq 0$. Then (2.4) with $f(z) = z^j$ and $g(z) = z^k$ implies that

$$kt_0\mu_{j,k-1} - n\mu_{j+1,k} + n \sum_{l=0}^d \overline{t_{l+1}}\mu_{j,k+l} = 0 \tag{3.10}$$

and by (2.1)

$$\mu_{j,k} = \overline{\mu_{k,j}}. \tag{3.11}$$

From (3.10) and (3.11) it is easy to see that all moments are determined by the moments $\mu_{j,k}$ with $j, k = 0, \dots, d - 1$. This block of moments yields a Hermitian matrix of size $d \times d$, which is determined by d^2 real parameters. Therefore the vector space of Hermitian forms satisfying (2.4) and (2.1) is at most d^2 dimensional.

4. Proof of Theorem 2.6

4.1. Vector equilibrium problem

We begin by analyzing the vector equilibrium problem for the energy functional (2.24). For every choice of $x^* > 0$ there is a unique minimizer (μ_1^*, μ_2^*) , since Σ_1 is compact. Both measures are symmetric under $2\pi/3$ rotations.

Given μ_1^* , the measure μ_2^* minimizes the functional

$$\mu_2 \mapsto I(\mu_2) - I(\mu_1^*, \mu_2)$$

among all measures on Σ_2 with $\int d\mu_2 = 1/2$. This means that μ_2^* is half of the balayage of μ_1^* onto Σ_2 . Hence μ_2^* always has full support, $\text{supp}(\mu_2^*) = \Sigma_2$, and it is characterized by the property that

$$2 \int \log |z - \zeta| d\mu_2^*(\zeta) = \int \log |z - \zeta| d\mu_1^*(\zeta), \quad z \in \Sigma_2. \tag{4.1}$$

Given μ_2^* , the measure μ_1^* minimizes the functional

$$\mu_1 \mapsto I(\mu_1) - I(\mu_1, \mu_2^*) + \frac{1}{t_0} \int \left(\frac{2}{3\sqrt{t_3}} |z|^{3/2} - \frac{t_3}{3} z^3 \right) d\mu_1(z)$$

among all probability measures on Σ_1 . Thus μ_1^* is a minimizer for an energy functional with external field, see [10,29], and it is characterized by the condition that there exists a constant $\ell \in \mathbb{R}$ such that

$$2 \int \log |z - \zeta| d\mu_1^*(\zeta) - \int \log |z - \zeta| d\mu_2^*(\zeta) - \frac{1}{t_0} \left(\frac{2}{3\sqrt{t_3}} |z|^{3/2} - \frac{t_3}{3} z^3 \right) \begin{cases} = \ell, & z \in \text{supp}(\mu_1^*) \\ \leq \ell, & z \in \Sigma_1 \setminus \text{supp}(\mu_1^*). \end{cases} \tag{4.2}$$

The support of μ_1^* consists of a finite union of intervals, in general.

The number x^* is at our disposal. We want to choose it an optimal way as described in the following lemma.

Lemma 4.1. *Let $0 < t_0 \leq t_{0,crit}$. Then there is a unique value for $x^* > 0$ such that*

- μ_1^* has full support, i.e., $\text{supp}(\mu_1^*) = \Sigma_1$, and
- the density of μ_1^* vanishes at the endpoints $\omega^j x^*$, $j = 0, 1, 2$.

If $t_0 < t_{0,crit}$ the density of μ_1^ vanishes like a square root at the endpoints, while for $t_0 = t_{0,crit}$ it vanishes with an exponent $3/2$.*

To prove Lemma 4.1 we first assume that we choose x^* satisfying the conditions in the lemma. From that assumption we will find explicit expressions for the measures, from which we can indeed check that the conditions are satisfied. The proof of Lemma 4.1 is given in Section 4.3. The proof will also give that x^* is given by (2.19).

Theorem 2.6 is proved in Section 4.4.

4.2. Riemann surface

The construction of the measures is based on the consideration of a three sheeted Riemann surface \mathcal{R} with sheets

$$\mathcal{R}_1 = \overline{\mathbb{C}} \setminus \Sigma_1, \quad \mathcal{R}_2 = \overline{\mathbb{C}} \setminus (\Sigma_1 \cup \Sigma_2), \quad \mathcal{R}_3 = \overline{\mathbb{C}} \setminus \Sigma_2. \tag{4.3}$$

For $k = 1, 2$, the sheet \mathcal{R}_k is connected to \mathcal{R}_{k+1} along Σ_k in the usual crosswise manner. Then \mathcal{R} is a compact Riemann surface of genus zero.

Define the Cauchy transforms of the measures μ_1^* and μ_2^*

$$F_k(z) = \int \frac{d\mu_k^*(\zeta)}{z - \zeta}, \quad z \in \mathbb{C} \setminus \Sigma_k, \quad k = 1, 2. \tag{4.4}$$

These functions have the symmetry property

$$F_k(\omega z) = \omega^2 F_k(z), \quad z \in \mathbb{C} \setminus \Sigma_k, \quad k = 1, 2.$$

The conditions (4.1) and (4.2) together with the symmetry properties, lead to the following relations for the Cauchy transforms

$$F_{1,+}(z) + F_{1,-}(z) - F_2(z) = \omega^{2j} \frac{1}{t_0} \left(\frac{1}{\sqrt{t_3}} |z|^{1/2} - t_3 |z|^2 \right), \quad z \in \Sigma_1, \tag{4.5}$$

$$F_{2,+}(z) + F_{2,-}(z) - F_1(z) = 0, \quad z \in \Sigma_2.$$

These functions are used to define the ξ -functions that play a major role. Throughout the paper we use the principal arguments of fractional powers, that is, with a branch cut along the negative real axis.

Definition 4.2. We define

$$\xi_1(z) = t_3 z^2 + t_0 F_1(z), \quad z \in \mathcal{R}_1, \tag{4.6}$$

$$\xi_2(z) = \begin{cases} \frac{1}{\sqrt{t_3}} z^{1/2} + t_0 (F_2(z) - F_1(z)), & z \in \mathcal{R}_2 \cap S_0, \\ -\frac{1}{\sqrt{t_3}} z^{1/2} + t_0 (F_2(z) - F_1(z)), & z \in \mathcal{R}_2 \cap (S_1 \cup S_2), \end{cases} \tag{4.7}$$

$$\xi_3(z) = \begin{cases} -\frac{1}{\sqrt{t_3}} z^{1/2} - t_0 F_2(z), & z \in \mathcal{R}_3 \cap S_0, \\ \frac{1}{\sqrt{t_3}} z^{1/2} - t_0 F_2(z), & z \in \mathcal{R}_3 \cap (S_1 \cup S_2) \end{cases} \tag{4.8}$$

where S_0, S_1, S_2 denote the sectors

$$\begin{aligned} S_0 &: -\pi/3 < \arg z < \pi/3, \\ S_1 &: \pi/3 < \arg z < \pi, \\ S_2 &: -\pi < \arg z < -\pi/3. \end{aligned} \tag{4.9}$$

Note that by (4.4) and (4.6) we can express the densities of the measure μ_1^* in terms of ξ_1 . Indeed, by (4.4) we have by the Sokhotski–Plemelj formula that

$$d\mu_1^*(x) = \frac{1}{2\pi i} (F_{1,-}(x) - F_{1,+}(x)) dx, \quad x \in \text{supp}(\mu_1^*),$$

which by (4.6) leads to

$$d\mu_1^*(x) = \frac{1}{2\pi i t_0} (\xi_{1,-}(x) - \xi_{1,+}(x)) dx, \tag{4.10}$$

$$= \frac{1}{2\pi i t_0} (\xi_{2,+}(x) - \xi_{1,+}(x)) dx, \quad x \in \text{supp}(\mu_1^*). \tag{4.11}$$

Similarly

$$d\mu_2^*(z) = \frac{1}{2\pi i t_0} \left(\pm \frac{2z^{1/2}}{\sqrt{t_3}} + \xi_{3,+}(z) - \xi_{2,+}(z) \right) dz, \quad z \in \text{supp}(\mu_2^*), \tag{4.12}$$

with appropriate choice of signs \pm and square roots.

Lemma 4.3. *Suppose that $\text{supp}(\mu_1^*) = \Sigma_1$ and that the density of μ_1^* vanishes at the endpoints $\omega^j x^*$ for $j = 0, 1, 2$. Then*

$$t_0 \leq t_{0,crit} = \frac{1}{8t_3^2} \tag{4.13}$$

and the following hold.

- (a) *The three functions ξ_1, ξ_2, ξ_3 given by (4.6)–(4.8) define a meromorphic function on the Riemann surface \mathcal{R} whose only poles are at the points at infinity.*
- (b) *The functions $\xi_j, j = 1, 2, 3$ are the three solutions of the cubic equation*

$$\xi^3 - t_3 z^2 \xi^2 - \left(t_0 t_3 + \frac{1}{t_3} \right) z \xi + z^3 + A = 0 \tag{4.14}$$

where

$$A = \frac{1 + 20t_0 t_3^2 - 8t_0^2 t_3^4 - (1 - 8t_0 t_3^2)^{3/2}}{32t_3^3}. \tag{4.15}$$

- (c) *The algebraic equation (4.14) has branch points at $z = \omega^j x^*, j = 0, 1, 2$, where x^* is given by (2.19) and nodes at the values $z = \omega^j \hat{x}, j = 0, 1, 2$, where*

$$\hat{x} = \frac{3 + \sqrt{1 - 8t_0 t_3^2}}{4t_3} > x^*. \tag{4.16}$$

The values $z = \omega^j x^$ and $z = \omega^j \hat{x}, j = 0, 1, 2$, are the only zeros of the discriminant of (4.14).*

- (d) *We have $\hat{x} \geq x^* > 0$ with equality only if $t_0 = t_{0,crit}$, and*

$$\xi_1(\omega^j \hat{x}) = \xi_2(\omega^j \hat{x}) = \omega^{2j} \hat{x}, \quad j = 0, 1, 2. \tag{4.17}$$

Proof. The conditions (4.5) imply that the three functions ξ_1, ξ_2, ξ_3 given in (4.6)–(4.8) define a meromorphic function on the Riemann surface. There is a double pole at infinity on the first sheet, and a simple pole at the other point at infinity. There are no poles at the endpoints $\omega^j x^*$ due to the assumption that the density of μ_1^* vanishes at these points. This proves part (a).

It follows that any symmetric function of the three ξ_j functions is a polynomial in z . From (4.6)–(4.8) and the fact that

$$F_1(z) = z^{-1} + O(z^{-4}), \quad F_2(z) = \frac{1}{2}z^{-1} + O(z^{-5/2}) \tag{4.18}$$

as $z \rightarrow \infty$, it is then easy to see that

$$\begin{cases} \xi_1(z) + \xi_2(z) + \xi_3(z) = t_3 z^2, \\ \xi_1(z)\xi_2(z) + \xi_1(z)\xi_3(z) + \xi_2(z)\xi_3(z) = -\left(t_0 t_3 + \frac{1}{t_3}\right)z, \\ \xi_1(z)\xi_2(z)\xi_3(z) = -z^3 - A, \end{cases} \tag{4.19}$$

where A is some real constant. Thus ξ_1, ξ_2, ξ_3 are indeed the solutions of the cubic equation (4.14). The algebraic equation has branch points at $z = \omega^j x^*$, $j = 0, 1, 2$, and no other finite branch points. This property allows us to determine the constant A .

The discriminant of (4.14) with respect to ξ is a cubic equation in $\zeta = z^3$ that we calculated with the help of Maple. The result is

$$\begin{aligned} &4t_3^3 \zeta^3 + (t_0^2 t_3^4 + 4A t_3^3 + 12t_0 t_3^2 - 8)\zeta^2 \\ &+ (4t_0^3 t_3^3 + 18A t_0 t_3^2 + 12t_0^2 t_3 - 36A + 12t_0 t_3^{-1} + 4t_3^{-3})\zeta - 27A^2. \end{aligned} \tag{4.20}$$

The discriminant (4.20) has a root for the value $\zeta^* = (x^*)^3$ that corresponds to the branch points $z = \omega^j x^*$. The fact that there are no other finite branch points implies that either ζ^* is a triple root of (4.20) or ζ^* is a simple root and (4.20) has a double root as well. The case of a triple root happens for the values (calculated with Maple)

$$t_0 = \frac{1}{8t_3^2}, \quad A = \frac{27}{256t_3^3}, \quad x^* = \frac{3}{4t_3}, \tag{4.21}$$

and this is the only possible combination of values with $t_0 > 0$ (one combination with $t_0 < 0$ also gives rise to a triple root). The values (4.21) are the values for the critical case.

We may from now on assume that (4.20) has one simple root $\zeta^* > 0$ and one double root. Since x^* is a branch point that connects the first and second sheets, we then have that $\xi_2(z) - \xi_1(z)$ vanishes as a square root as $z \rightarrow x^*$. For $x > x^*$, we have that $\xi_2(x) - \xi_1(x)$ is real, while according to (4.10) we must have $\xi_{2,+}(x) - \xi_{1,+}(x) \in i\mathbb{R}^+$ for $x \in [0, x^*]$, since the density of μ_1^* is positive on $[0, x^*]$. This implies that $\xi_2(x) - \xi_1(x) > 0$ for x in some interval $(x^*, x^* + \delta)$ to the right of x^* . The definitions (4.6) and (4.7) imply that $\xi_1(x) > \xi_2(x)$ for $x > x^*$ large enough. Therefore there is a value

$$\widehat{x} > x^* \tag{4.22}$$

such that $\xi_1(\widehat{x}) = \xi_2(\widehat{x})$. Then clearly $\widehat{\zeta} = (\widehat{x})^3$ is the double root of (4.20), and $z = \widehat{x}, \xi = \xi_1(\widehat{x})$ is a node of the spectral curve (4.14).

Because of the symmetry of the cubic equation (4.14) in the variables z and ξ , we can interchange the values of z and ξ , and we get that $z = \xi_1(\widehat{x}), \xi = \widehat{x}$ is also a node. Thus $(\xi_1(\widehat{x}))^3$ is also a double root of (4.20), which because of the uniqueness of the double root implies (4.17) for $j = 0$. Because of $2\pi/3$ rotational symmetry we also have (4.17) for $j = 1, 2$. This proves part (d) of the lemma.

Taking $\xi = z$ in (4.14) we find from (4.17) that $z = \widehat{x}$ is a double root of the quartic polynomial

$$2z^3 - t_3z^4 - \left(t_0t_3 + \frac{1}{t_3}\right)z^2 + A. \tag{4.23}$$

Thus \widehat{x} is one of the critical points of the polynomial (4.23), which are easily calculated to be $z = 0$ and the two solutions of $2t_3z^2 - 3z + \left(t_0t_3 + \frac{1}{t_3}\right) = 0$. Since $\widehat{x} > x^* > 0$ we discard $z = 0$, and find two possible values for \widehat{x}

$$\widehat{x}_1 = \frac{3 + \sqrt{1 - 8t_0t_3^2}}{4t_3}, \quad \widehat{x}_2 = \frac{3 - \sqrt{1 - 8t_0t_3^2}}{4t_3}. \tag{4.24}$$

This value must be real and so we conclude that $8t_0t_3^2 < 1$. This proves (4.13).

The corresponding values of A we now find by substituting $z = \widehat{x}_j$ into (4.23) and equating to 0. The results are

$$A_1 = \frac{1 + 20t_0t_3^2 - 8t_0^2t_3^4 - (1 - 8t_0t_3^2)^{3/2}}{32t_3^3},$$

$$A_2 = \frac{1 + 20t_0t_3^2 - 8t_0^2t_3^4 + (1 - 8t_0t_3^2)^{3/2}}{32t_3^3}.$$

For the value $A = A_j$, the simple root of (4.20) comes out as $\zeta^* = (x_j^*)^3$ with

$$x_1^* = \frac{3}{4t_3} \left(1 - \sqrt{1 - 8t_0t_3^2}\right)^{2/3}, \quad x_2^* = \frac{3}{4t_3} \left(1 + \sqrt{1 - 8t_0t_3^2}\right)^{2/3}, \tag{4.25}$$

which is again calculated with Maple. From (4.24) and (4.25) it follows that

$$x_1^* < \frac{3}{4t_3} < \widehat{x}_1, \quad x_2^* > \frac{3}{4t_3} > \widehat{x}_2.$$

The inequality (4.17) is only satisfied in the first case, and so $x^* = x_1^*$, $A = A_1$, and $\widehat{x} = \widehat{x}_1$. This proves the formulas (2.19), (4.15) and (4.16), which establishes the parts (b) and (c) and completes the proof of the lemma.

4.3. Proof of Lemma 4.1

We now prove Lemma 4.1 by reversing the arguments given above.

For $t_0 \leq t_{0,crit}$, we start from the spectral curve (4.14) with the value of A as in (4.15). The spectral curve defines a Riemann surface with sheet structure as in (4.3). This defines in particular the value of x^* . There is one solution of (4.14) that satisfies

$$\xi_1(z) = t_3z^2 + t_0z^{-1} + O(z^{-4}) \quad \text{as } z \rightarrow \infty$$

and ξ_1 is defined on the first sheet. The analytic continuations onto the second and third sheets are then denoted by ξ_2 and ξ_3 , respectively.

Then we define a measure μ_j^* on Σ_j for $j = 1, 2$, by the formulas (4.10) and (4.12). All parts of Σ_1 and Σ_2 are oriented away from the origin. The vector of measures (μ_1^*, μ_2^*) is the minimizer of the vector equilibrium problem, and μ_1^* satisfies the conditions of Lemma 4.1, since

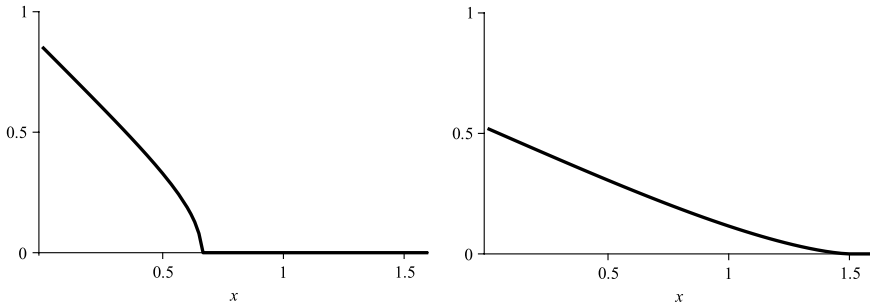


Fig. 3. Density of μ_1^* on $[0, x^*]$ for the values $t_0 = 1/4$ and $t_3 = 1/2$ (left) and for the critical values $t_0 = 1/2$ and $t_3 = 1/2$ (right). In the non-critical case the density behaves as $c(x^* - x)^{1/2}$ as $x \rightarrow x^*$, while in the critical case it vanishes like $c(x^* - x)^{3/2}$.

it is positive on $[0, x^*)$ and vanishes at x^* , see also Fig. 3 for the density of μ_1^* for a non-critical value t_0 and for the critical value $t_{0,crit}$. In a non-critical case the density of μ_1^* vanishes as a square root at x^* , while in the critical case it vanishes like $(x^* - x)^{3/2}$ as $x \rightarrow x^*$. See Fig. 3 for plots of the density of μ_1^* in the non-critical and critical cases.

4.4. Proof of Theorem 2.6

The spectral curve (4.14) with A given by (4.15) has genus zero. The curve has a remarkable rational parametrization

$$z = h(w) = rw + aw^{-2}, \quad \xi = h(w^{-1}) = aw^2 + rw^{-1} \tag{4.26}$$

with

$$r = \frac{\sqrt{1 - \sqrt{1 - 8t_0t_3^2}}}{2t_3}, \quad a = \frac{1 - \sqrt{1 - 8t_0t_3^2}}{4t_3}. \tag{4.27}$$

This can easily be checked by plugging (4.26)–(4.27) into (4.14). The calculations show that it only works if A has the value as in (4.15).

Assuming $t_0 < t_{0,crit}$ we see that r and a are both real and positive. We view $z = h(w)$ as a mapping from the w -plane to the z -plane. The mapping is conformal around infinity. For large enough ρ we have that the circle $|w| = \rho$ is mapped to a simple closed curve. This will continue to be the case if we decrease ρ until the circle $|w| = \rho$ contains a critical point, that is, a solution of

$$h'(w) = r - 2aw^{-3} = 0.$$

The critical points are on the circle with radius

$$\rho_{crit} = \left(\frac{2a}{r}\right)^{1/3} = \left(1 - \sqrt{1 - 8t_0t_3^2}\right)^{1/6} < 1.$$

Since $\rho_{crit} < 1$, we see that $h(|w| = 1)$ is therefore a simple closed curve which is smooth for $t_0 < t_{0,crit}$ and has three cusp points for the critical value $t_{0,crit}$. Let Ω be the bounded domain that is enclosed by $h(|w| = 1)$.

For $z \in \partial\Omega$, we have $z = h(w)$ for some w with $|w| = 1$. Then $\xi_1(z) = h(1/w)$, see (4.26), and so since $1/w = \bar{w}$, and since the coefficient of h are real numbers

$$\xi_1(z) = h(\bar{w}) = \overline{h(w)} = \bar{z}, \quad z \in \partial\Omega. \tag{4.28}$$

This shows that (2.26) indeed defines a curve $\partial\Omega$, by virtue of (4.6) and (4.4).

We compute the area of Ω by means of Green’s formula

$$\text{area}(\Omega) = \iint_{\Omega} dA = \frac{1}{2i} \oint_{\partial\Omega} \bar{z} dz = \frac{1}{2i} \oint_{\partial\Omega} \xi_1(z) dz.$$

Here we used (4.28). Now recall that ξ_1 is analytic in the exterior of Ω with

$$\xi_1(z) = t_3 z^2 + t_0 z^{-1} + O(z^{-4}), \quad \text{as } z \rightarrow \infty, \tag{4.29}$$

see (4.6) and (4.18). Thus we can move the contour to infinity and by doing so we only pick up a residue contribution at ∞ . This proves the formula for t_0 in (1.5).

The exterior harmonic moments (1.5) are computed in a similar way. We have by Green’s formula applied to the exterior domain

$$-\frac{1}{\pi} \iint_{\mathbb{C} \setminus \bar{\Omega}} \frac{dA(z)}{z^k} = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\bar{z}}{z^k} dz = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\xi_1(z)}{z^k} dz.$$

By contour deformation and using (4.29) we find that this is t_3 for $k = 3$ and 0 for other $k \geq 2$. Thus (1.5) follows.

We finally prove (2.27) with yet again similar arguments. For $z \in \mathbb{C} \setminus \bar{\Omega}$ we have by Green’s formula and (4.28)

$$\frac{1}{\pi} \iint_{\Omega} \frac{dA(\zeta)}{z - \zeta} = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\bar{\zeta}}{z - \zeta} d\zeta = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{\xi_1(\zeta)}{z - \zeta} d\zeta.$$

Moving the contour to infinity we pick up a residue contribution at $\zeta = z$, which is $\xi_1(z)$, and at $\zeta = \infty$, which is $-t_3 z^2$ because of (4.29). In total we find that

$$\frac{1}{\pi} \iint_{\Omega} \frac{dA(\zeta)}{z - \zeta} = \xi_1(z) - t_3 z^2$$

which is equal to $t_0 F_1(z)$ by (4.6). Then dividing by t_0 we obtain (2.27). This completes the proof of Theorem 2.6.

5. Preliminary steps towards the proofs of Theorems 2.3 and 2.5

5.1. Discussion

The orthogonality induced by the Hermitian form (2.14) is similar to the biorthogonality that plays a role in the two-matrix model, see [4]. The two-matrix model is a model for two random matrices with two potentials V and W on \mathbb{R} with sufficient increase at $\pm\infty$. The biorthogonal polynomials in this model are two sequences $(p_{j,n}(x))_j$, $\deg p_{j,n} = j$ and $(q_{k,n})_k$, $\deg q_{k,n} = k$ of monic polynomials satisfying

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{j,n}(x) q_{k,n}(y) e^{-n(V(x)+W(y)-\tau xy)} dx dy = h_k \delta_{j,k} \tag{5.1}$$

where τ is the coupling constant. Comparing (5.1) with (2.14) one sees that (2.14) is like the biorthogonality in the two matrix model with equal cubic potentials $V(x) = W(x) = -\frac{t_3}{3t_0}x^3$ and coupling constant $\tau = -\frac{1}{t_0}$. The main difference is that integrals in (5.1) are over the real line, while integrals in (2.14) are over combinations of the contours $\Gamma_j, j = 0, 1, 2$.

This does not play a role on a formal level. The main algebraic properties that are known for the biorthogonal polynomials, see e.g. [4–6], also hold for the orthogonal polynomials with respect to the Hermitian form (2.14). This includes the existence of differential and difference equations, and an integrable structure of τ -functions and Toda equations. However, analytic and asymptotic properties depend crucially on the precise definition of contours. For example, it is known that the biorthogonal polynomials characterized by (5.1) have real and simple zeros [16], which is not the case for the orthogonal polynomials for (2.14).

Recently [14,15,28], the biorthogonal polynomials $p_{n,n}$ from (5.1) with an even polynomial V and a quartic potential $W(y) = \frac{1}{4}y^4 + \frac{\alpha}{2}y^2$ were successfully analyzed in the large n limit by means of a steepest descent analysis of a Riemann–Hilbert problem. The Riemann–Hilbert problem was obtained earlier in [24] from a reformulation of the biorthogonality (5.1) as multiple orthogonality, since for multiple orthogonal polynomials a Riemann–Hilbert problem is known [31] as a generalization of the well-known Riemann–Hilbert problem for orthogonal polynomials [20]. Because of the formal similarity the same setup works in the case of orthogonal polynomials for the Hermitian form (2.14). We can reformulate the orthogonality as multiple orthogonality and it leads to a 3×3 matrix valued Riemann–Hilbert problem with jumps on the contours Γ_j . In order to prepare for the asymptotic analysis we first adjust the contours in a suitable way. This will be done in this section.

The systematic asymptotic analysis of Riemann–Hilbert problems is due to Deift and Zhou [13] who developed their steepest descent analysis first in the context of large time asymptotics of integrable systems. It was applied to orthogonal polynomials and random matrices in [8,11,12]. These papers also emphasized the use of equilibrium measures in the asymptotic analysis. The Riemann–Hilbert problem for orthogonal polynomials [20] is of size 2×2 . Extensions of the steepest descent analysis to larger size Riemann–Hilbert problems were first discussed in [9,26]. In the next section we will build on these and later works (see [23] for an overview) and apply the Deift/Zhou steepest descent analysis to the 3×3 matrix RH problem 5.4. A crucial role is played by the minimizer (μ_1^*, μ_2^*) of the vector equilibrium problem. This is inspired by [7,14,15] where the steepest descent analysis also depended crucially on a vector equilibrium problem.

5.2. Multiple orthogonality and Airy functions

The orthogonality with respect to the Hermitian form (2.14) is very similar to the biorthogonality that plays a role in the two-matrix model. We will use ideas that were developed for the asymptotic analysis of the two-matrix model with a quartic potential [14,15] and apply these to the orthogonal polynomials for the Hermitian form (2.14).

First of all we identify the orthogonal polynomials as multiple orthogonal polynomials. We define for $j = 0, 1, 2$ and $n \in \mathbb{N}$ the entire functions

$$w_{0,j,n}(z) = \frac{1}{2\pi i} \sum_{k=0}^2 \epsilon_{j,k} \int_{\Gamma_k} e^{-\frac{n}{t_0}(wz - \frac{t_3}{3}(w^3 + z^3))} dw, \tag{5.2}$$

$$w_{1,j,n}(z) = \frac{1}{2\pi i} \sum_{k=0}^2 \epsilon_{j,k} \int_{\bar{\Gamma}_k} w e^{-\frac{n}{t_0} (wz - \frac{t_3}{3}(w^3 + z^3))} dw. \tag{5.3}$$

Lemma 5.1. *The monic orthogonal polynomial $P_{n,n}$ is characterized by the properties*

$$\begin{aligned} \sum_{j=0}^2 \int_{\Gamma_j} P_{n,n}(z) z^k w_{0,j,n}(z) dz &= 0, \quad k = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \sum_{j=0}^2 \int_{\Gamma_j} P_{n,n}(z) z^k w_{1,j,n}(z) dz &= 0, \quad k = 0, 1, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \end{aligned} \tag{5.4}$$

Proof. This follows as in [24], but for convenience to the reader we present the argument. Recall that $P_{n,n}$ satisfies the orthogonality conditions (2.16) with $k = n$.

We prove that for every k ,

$$\langle z^k P_{n,n}(z), z^j \rangle = 0, \quad j = 0, \dots, n - 2k - 1. \tag{5.5}$$

For $k = 0$, the condition (5.5) reduces to the orthogonality condition (2.16).

Assume that (5.5) holds for certain $k \geq 0$. The structure relation (2.4) with cubic potential (2.9) gives

$$n \langle zf, g \rangle = t_0 \langle f, g' \rangle + nt_3 \langle f, z^2 g \rangle. \tag{5.6}$$

Taking $f(z) = z^k P_{n,n}(z)$ and $g(z) = z^j$ we find that both terms in the right-hand side of (5.6) vanish for $j + 2 \leq n - 2k - 1$, because of the induction hypothesis. Then the left-hand side vanishes as well, and this gives (5.5) with $k + 1$. Thus (5.5) follows by induction.

Taking $j = 0$ and $j = 1$ in (5.5), we find

$$\begin{aligned} \langle z^k P_{n,n}, 1 \rangle &= 0, \quad k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \langle z^k P_{n,n}, z \rangle &= 0, \quad k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \end{aligned} \tag{5.7}$$

and these conditions are equal to the conditions (5.4) because of the representation (2.14) of the Hermitian form and the definition (5.2) and (5.3) of the weight functions.

The conditions (5.4) in fact characterize the monic orthogonal polynomial since we can similarly show that the conditions (5.7) are in fact equivalent to the orthogonality conditions (2.16) (again by using the structure relation (5.6)).

The functions (5.2) and (5.3) can be expressed in terms of Airy functions and their derivatives. The classical Airy differential equation $y'' = zy$ has the three solutions

$$\begin{aligned} y_0(z) &= \text{Ai}(z) = \frac{1}{2\pi i} \int_{\Gamma_0} e^{\frac{1}{3}s^3 - zs} ds = -\frac{1}{2\pi i} \int_{\bar{\Gamma}_0} e^{\frac{1}{3}s^3 - zs} ds, \\ y_1(z) &= \omega \text{Ai}(\omega z) = \frac{1}{2\pi i} \int_{\Gamma_1} e^{\frac{1}{3}s^3 - zs} ds = -\frac{1}{2\pi i} \int_{\bar{\Gamma}_2} e^{\frac{1}{3}s^3 - zs} ds, \\ y_2(z) &= \omega^2 \text{Ai}(\omega^2 z) = \frac{1}{2\pi i} \int_{\Gamma_2} e^{\frac{1}{3}s^3 - zs} ds = -\frac{1}{2\pi i} \int_{\bar{\Gamma}_1} e^{\frac{1}{3}s^3 - zs} ds, \end{aligned} \tag{5.8}$$

that are related by the identity $y_0 + y_1 + y_2 = 0$. Then we get from (5.2) and (5.8) after the change of variables $w = \left(\frac{t_0}{nt_3}\right)^{1/3} s$,

$$\begin{aligned} w_{0,0,n}(z) &= d_n(y_2(c_n z) - y_1(c_n z))e^{\frac{nt_3}{3t_0} z^3}, \\ w_{0,1,n}(z) &= d_n(y_1(c_n z) - y_0(c_n z))e^{\frac{nt_3}{3t_0} z^3}, \\ w_{0,2,n}(z) &= d_n(y_0(c_n z) - y_2(c_n z))e^{\frac{nt_3}{3t_0} z^3}, \end{aligned} \tag{5.9}$$

with constants

$$c_n = \frac{n^{2/3}}{t_0^{2/3} t_3^{1/3}} > 0, \quad d_n = \left(\frac{t_0}{nt_3}\right)^{1/3}. \tag{5.10}$$

Similarly from (5.3) and (5.8)

$$\begin{aligned} w_{1,0,n}(z) &= -d_n^2(y_2'(c_n z) - y_1'(c_n z))e^{\frac{nt_3}{3t_0} z^3}, \\ w_{1,1,n}(z) &= -d_n^2(y_1'(c_n z) - y_0'(c_n z))e^{\frac{nt_3}{3t_0} z^3}, \\ w_{1,2,n}(z) &= -d_n^2(y_0'(c_n z) - y_2'(c_n z))e^{\frac{nt_3}{3t_0} z^3}. \end{aligned} \tag{5.11}$$

5.3. Riemann–Hilbert problem

Observe that there is no complex conjugation in the multiple orthogonality conditions (5.4). Following [31] we then find a characterization of $P_{n,n}$ in terms of a 3×3 matrix valued Riemann–Hilbert (RH) problem. We assume that the contours Γ_j are disjoint, and have the orientation as shown in Fig. 1. The orientation induces a + and – side on Γ_j , where the + side (– side) is on the left (right) while traversing the contour according to its orientation.

RH problem 5.2. Let $\Gamma = \bigcup_{j=0}^2 \Gamma_j$. We look for $Y : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{3 \times 3}$ satisfying

- Y is analytic in $\mathbb{C} \setminus \Gamma$,
- $Y_+(z) = Y_-(z)J_Y(z)$ for $z \in \Gamma$ with jump matrix J_Y given by

$$J_Y = \begin{pmatrix} 1 & w_{0,j,n} & w_{1,j,n} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{on } \Gamma_j, \tag{5.12}$$

(here Y_+ and Y_- denote the limiting values of Y , when approaching Γ on the + side and – side, respectively),

- $Y(z) = (I + O(1/z))\text{diag}(z^n, z^{-\lceil n/2 \rceil}, z^{-\lfloor n/2 \rfloor})$ as $z \rightarrow \infty$.

The jump condition (5.12) has to be adjusted in the case of overlapping contours.

Provided that the orthogonal polynomial $P_{n,n}$ uniquely exists, the RH problem 5.2 has a unique solution, see [31]. The first row of Y is given by

$$\begin{aligned} Y_{1,1}(z) &= P_{n,n}(z), \\ Y_{1,2}(z) &= \frac{1}{2\pi i} \sum_{j=0}^2 \int_{\Gamma_j} \frac{P_{n,n}(x)w_{0,j,n}(x)}{x - z} dx, \end{aligned}$$

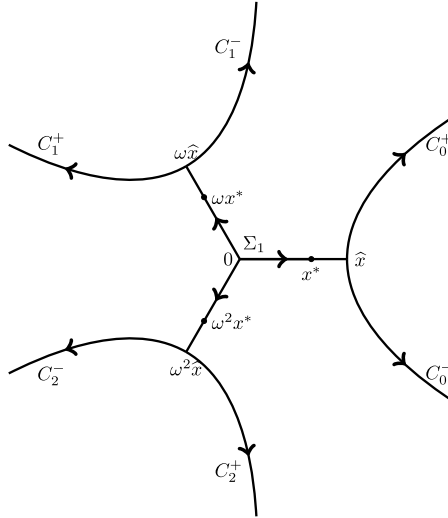


Fig. 4. Deformed contours $\Gamma_0, \Gamma_1, \Gamma_2$. All contours are oriented away from 0.

$$Y_{1,3}(z) = \frac{1}{2\pi i} \sum_{j=0}^2 \int_{\Gamma_j} \frac{P_{n,n}(x)w_{1,j,n}(x)}{x - z} dx.$$

The remaining two rows of Y are built in a similar way out of certain polynomials of degrees $n - 1$ (one of which is proportional to $P_{n-1,n}$, if this orthogonal polynomial exists).

5.4. Deformation of contours

In order to prepare for the steepest descent analysis of the RH problem, we first adjust the contours Γ_j . Because the weights (5.9), (5.11) in the multiple orthogonality conditions (5.4) are entire functions, we have the freedom to make arbitrary deformation for each contour Γ_j , as long as the contours start and end at the same asymptotic angles at infinity. We are going to deform the contours such that their union contains the set $\Sigma_1 = \bigcup_{j=0}^2 [0, \omega^j x^*]$ where the zeros of the polynomials are going to accumulate. The deformed contours are shown in Fig. 4.

We recall that $\hat{x} > x^*$, see (4.16). We deform Γ_0 to contain the intervals $[0, \omega^2 \hat{x}]$ and $[0, \omega \hat{x}]$ and we continue Γ_0 with an unbounded contour from $\omega \hat{x}$ to infinity at asymptotic angle $5\pi/12$, and its mirror image in the real axis which goes from $\omega^2 \hat{x}$ to infinity at angle $-5\pi/12$. The number $5\pi/12$ could be replaced with any number between $\pi/3$ and $\pi/2$. We obtain Γ_1 and Γ_2 by rotating Γ_0 over angles $2\pi/3$ and $-2\pi/3$, respectively.

It will be convenient to give new names to the unbounded contours. We define

$$\begin{aligned} C_0^+ &= (\Gamma_1 \cap S_0) \setminus [0, \hat{x}], & C_0^- &= (\Gamma_2 \cap S_0) \setminus [0, \hat{x}], \\ C_1^+ &= (\Gamma_2 \cap S_1) \setminus [0, \omega \hat{x}], & C_1^- &= (\Gamma_0 \cap S_1) \setminus [0, \omega \hat{x}], \\ C_2^+ &= (\Gamma_0 \cap S_2) \setminus [0, \omega^2 \hat{x}], & C_2^- &= (\Gamma_1 \cap S_2) \setminus [0, \omega^2 \hat{x}], \end{aligned} \tag{5.13}$$

and

$$C_j = C_j^+ \cup C_j^-, \quad j = 0, 1, 2. \tag{5.14}$$

Then, see also Fig. 4,

$$\begin{aligned} \Gamma_0 &= C_2^+ \cup [0, \omega^2 \widehat{x}] \cup [0, \omega \widehat{x}] \cup C_1^-, \\ \Gamma_1 &= C_0^+ \cup [0, \widehat{x}] \cup [0, \omega^2 \widehat{x}] \cup C_2^-, \\ \Gamma_2 &= C_1^+ \cup [0, \omega \widehat{x}] \cup [0, \widehat{x}] \cup C_0^-, \end{aligned} \tag{5.15}$$

and

$$\Gamma = \bigcup_{j=0}^2 \left([0, \omega^j \widehat{x}] \cup C_j \right). \tag{5.16}$$

We redefine the orientation on the contours such that all parts are from now on oriented away from 0 and towards ∞ , as shown in Fig. 4. We have a freedom in the choice of the precise location of the contours C_j and we make use of this freedom later on.

After this deformation of contours and partial reversion of orientation we find new expressions for the weights. Indeed, for $x \in [0, \widehat{x}]$, we find by (5.9)

$$-w_{0,1,n}(x) + w_{0,2,n}(x) = d_n \left(-(y_1(c_n x) - y_0(c_n x)) + (y_0(c_n x) - y_2(c_n x)) \right) e^{\frac{nt_3}{3i_0} x^3}.$$

Using the identity $y_0 + y_1 + y_2 = 0$ and $y_0 = \text{Ai}$, this reduces to

$$-w_{0,1,n}(x) + w_{0,2,n}(x) = 3d_n \text{Ai}(c_n x) e^{\frac{nt_3}{3i_0} x^3}. \tag{5.17}$$

Similarly we get from (5.11)

$$-w_{1,1,n}(x) + w_{1,2,n}(x) = -3d_n^2 \text{Ai}'(c_n x) e^{\frac{nt_3}{3i_0} x^3}. \tag{5.18}$$

It is thanks to our choice (2.13) for the matrix C , that we obtain the Airy function Ai and its derivative in (5.17) and (5.18), and not some other solution of the Airy differential equation. The Airy function Ai is a special solution of $y'' = zy$ because of its asymptotic behavior, see [1, formulas 10.4.59, 10.4.61],

$$\begin{aligned} \text{Ai}(z) &= \frac{z^{-1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} (1 + O(z^{-3/2})), \\ \text{Ai}'(z) &= -\frac{z^{1/4}}{2\sqrt{\pi}} e^{-\frac{2}{3}z^{3/2}} (1 + O(z^{-3/2})), \end{aligned} \tag{5.19}$$

as $z \rightarrow \infty$, $-\pi < \arg z < \pi$, which is decaying for z going to infinity on the positive real axis. Any other linearly independent solution of the Airy differential equation increases on the positive real axis. We use (5.19) in the steepest descent analysis of the RH problem, and this explains why we chose the matrix C as we did in (2.13).

We now redefine the weights on the new contour Γ by combining the contributions of overlapping contours. We also rescale the weights by dropping the irrelevant constant prefactor $3d_n$ in (5.17) and $-3d_n^2$ in (5.18).

Definition 5.3. Let $\Gamma = \bigcup_{j=0}^2 ([0, \omega_j \widehat{x}] \cup C_j)$ be as in Fig. 4. We define two functions $w_{0,n}$ and $w_{1,n}$ on Γ by first defining them in the sector $S_0 : |\arg z| < \pi/3$ by putting

$$\begin{cases} w_{0,n}(x) = \text{Ai}(c_n x) e^{\frac{nt_3}{3t_0} x^3}, \\ w_{1,n}(x) = \text{Ai}'(c_n x) e^{\frac{nt_3}{3t_0} x^3}, \end{cases} \quad \text{for } x \in [0, \widehat{x}], \tag{5.20}$$

$$\begin{cases} w_{0,n}(z) = \frac{1}{3} (y_0(c_n z) - y_1(c_n z)) e^{\frac{nt_3}{3t_0} z^3}, \\ w_{1,n}(z) = \frac{1}{3} (y'_0(c_n z) - y'_1(c_n z)) e^{\frac{nt_3}{3t_0} z^3}, \end{cases} \quad \text{for } z \in C_0^+, \tag{5.21}$$

$$\begin{cases} w_{0,n}(z) = \frac{1}{3} (y_0(c_n z) - y_2(c_n z)) e^{\frac{nt_3}{3t_0} z^3}, \\ w_{1,n}(z) = \frac{1}{3} (y'_0(c_n z) - y'_2(c_n z)) e^{\frac{nt_3}{3t_0} z^3}, \end{cases} \quad \text{for } z \in C_0^-, \tag{5.22}$$

and then by extending to the parts of Γ in the other sectors by the property

$$\begin{cases} w_{0,n}(\omega z) = \omega^2 w_{0,n}(z), \\ w_{1,n}(\omega z) = \omega w_{1,n}(z), \end{cases} \quad \text{for } z \in \Gamma. \tag{5.23}$$

With the definition (5.20) we have by (5.19) that for $x \in (0, \widehat{x}]$,

$$\begin{cases} w_{0,n}(x) = \frac{(c_n x)^{-1/4}}{2\sqrt{\pi}} \exp\left(-\frac{n}{t_0} \left(\frac{2}{3\sqrt{t_3}} x^{3/2} - \frac{t_3}{3} x^3\right)\right) (1 + O(n^{-1})), \\ w_{1,n}(x) = -\frac{(c_n x)^{1/4}}{2\sqrt{\pi}} \exp\left(-\frac{n}{t_0} \left(\frac{2}{3\sqrt{t_3}} x^{3/2} - \frac{t_3}{3} x^3\right)\right) (1 + O(n^{-1})), \end{cases} \tag{5.24}$$

as $n \rightarrow \infty$. The exponential part in the asymptotic behavior (5.24) of the weight functions is reflected in the term

$$\frac{1}{t_0} \int \left(\frac{2}{3\sqrt{t_3}} |x|^{3/2} - \frac{t_3}{3} x^3\right) d\mu_1(x)$$

that appears in the energy functional (2.24).

5.5. Riemann–Hilbert problem after deformation

After the contour deformation we find that the multiple orthogonality conditions (5.4) satisfied by the monic orthogonal polynomial can also be stated as

$$\begin{aligned} \int_{\Gamma} P_{n,n}(z) z^k w_{0,n}(z) dz &= 0, \quad k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1, \\ \int_{\Gamma} P_{n,n}(z) z^k w_{1,n}(z) dz &= 0, \quad k = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor - 1. \end{aligned} \tag{5.25}$$

The corresponding RH problem is as follows.

RH problem 5.4. We look for $Y : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{3 \times 3}$ satisfying

- $Y : \mathbb{C} \setminus \Gamma \rightarrow \mathbb{C}^{3 \times 3}$ is analytic,

- $Y_+(z) = Y_-(z) \begin{pmatrix} 1 & w_{0,n}(z) & w_{1,n}(z) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ for $z \in \Gamma$
- $Y(z) = (I + O(1/z)) \text{diag}(z^n, z^{-\lceil n/2 \rceil}, z^{-\lfloor n/2 \rfloor})$ as $z \rightarrow \infty$.

The RH problems 5.2 and 5.4 are equivalent. Also RH problem 5.4 has a solution if and only if the monic multiple orthogonal polynomial uniquely exists and in that case one has

$$Y_{11}(z) = P_{n,n}(z), \tag{5.26}$$

as before.

6. Riemann–Hilbert steepest descent analysis and the proofs of Theorems 2.3 and 2.5

In this section we perform an asymptotic analysis of the RH problem 5.4. For convenience we assume that n is even, although this is not essential. The asymptotic analysis will lead to the proof of the following result.

Lemma 6.1. *Let $t_3 > 0$ and $0 < t_0 < t_{0,crit}$. Then for n large enough the orthogonal polynomial $P_{n,n}$ exists and satisfies*

$$P_{n,n}(z) = M_{1,1}(z)e^{ng_1(z)}(1 + O(1/n)), \quad \text{as } n \rightarrow \infty \tag{6.1}$$

uniformly for z in compact subsets of $\overline{\mathbb{C}} \setminus \Sigma_1$. Here g_1 is defined by

$$g_1(z) = \int \log(z - s)d\mu_1^*(s), \quad z \in \mathbb{C} \setminus \Sigma_1,$$

see also (6.16) below, and $M_{1,1}$ is an analytic function with no zeros in $\mathbb{C} \setminus \Sigma_1$.

We write $M_{1,1}$ since this function will arise as the (1, 1) entry of a matrix-valued function M , see Section 6.6.

6.1. First transformation

We start from the RH problem 5.4 for Y with n even. The first transformation $Y \mapsto X$ will have a different form in each of the three sectors (4.9).

We recall the Airy functions $y_j, j = 0, 1, 2$ from (5.8). We also introduce

$$\begin{aligned} y_3(z) &= 2\pi i(\omega^2 \text{Ai}(\omega z) - \omega \text{Ai}(\omega^2 z)), \\ y_4(z) &= \omega y_3(\omega z), \\ y_5(z) &= \omega^2 y_3(\omega^2 z). \end{aligned} \tag{6.2}$$

These solutions of the Airy differential equation are chosen such that the Wronskians $W(y_j, y_k) = y_j y'_k - y'_j y_k$ satisfy

$$\begin{aligned} W(y_j, y_{j+3}) &= 1, \quad j = 0, 1, 2, \\ W(y_3, y_4) &= W(y_3, y_5) = W(y_4, y_5) = 0. \end{aligned} \tag{6.3}$$

We obviously have from (6.2) that $y'_3(0) = 0$. Hence, by the uniqueness of the solution of the initial value problem for the Airy equation at $z = 0$, we obtain that $y_3(\omega z) = y_3(z)$. This implies that $y_4(z) = \omega y_3(z)$ and $y_5(z) = \omega^2 y_3(z)$.

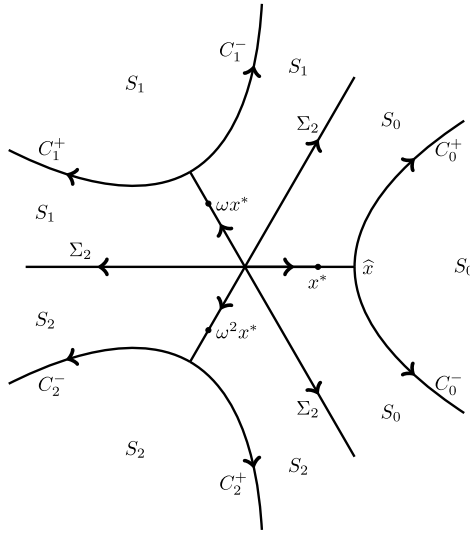


Fig. 5. Contour Γ_X for the RH problem for X and the sectors S_0, S_1 and S_2 .

We also recall the Wronskian relations [1, formulas 10.4.11–10.4.13]

$$W(y_0, y_1) = W(y_1, y_2) = W(y_2, y_0) = -\frac{1}{2\pi i}. \tag{6.4}$$

Definition 6.2. We define

$$\tilde{X}(z) = Y(z) \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & y'_3(c_n z) & -y'_0(c_n z) \\ 0 & -y_3(c_n z) & y_0(c_n z) \end{pmatrix}, & z \in S_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & y'_5(c_n z) & -y'_2(c_n z) \\ 0 & -y_5(c_n z) & y_2(c_n z) \end{pmatrix}, & z \in S_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & y'_4(c_n z) & -y'_1(c_n z) \\ 0 & -y_4(c_n z) & y_1(c_n z) \end{pmatrix}, & z \in S_2, \end{cases} \tag{6.5}$$

and

$$X(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & (2\pi)^{-1/2} c_n^{-1/4} & 0 \\ 0 & 0 & i(2\pi)^{-1/2} c_n^{1/4} \end{pmatrix} \tilde{X}(z) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2\pi i \end{pmatrix}, \tag{6.6}$$

where c_n is the constant given by (5.10).

Since the formula (6.5) is different in the three sectors, we will have that X is discontinuous on the boundary of the sectors, which are the rays $\arg z = \pm\pi/3, \pi$ that form the contour Σ_2 given in (2.23). Thus X is defined and analytic in $\mathbb{C} \setminus \Gamma_X$ where $\Gamma_X = \Gamma \cup \Sigma_2$, see Fig. 5, with

a jump $X_+ = X_- J_X$ on Γ_X . The jump matrix on the intervals $(0, \omega^j \widehat{x}]$ simplifies to

$$J_X(x) = \begin{pmatrix} 1 & e^{\frac{nt_3}{3t_0} x^3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad x \in \bigcup_j [0, \omega^j \widehat{x}],$$

as a consequence of the definitions (6.5) and (6.6) and the Wronskian relation (6.3).

The jump matrices on the new contours Σ_2 are piecewise constant, since they contain Wronskians $W(y_i, y_j)$ of solutions of the Airy equation. Indeed, we have by (6.5), (6.3) and (6.4),

$$\begin{aligned} J_{\widehat{X}}(z) &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & y_0(c_n z) & y'_0(c_n z) \\ 0 & y_3(c_n z) & y'_3(c_n z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & y'_5(c_n z) & -y'_2(c_n z) \\ 0 & -y_5(c_n z) & y_2(c_n z) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & W(y_0, y_5) & W(y_2, y_0) \\ 0 & W(y_3, y_5) & W(y_2, y_3) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & -\frac{1}{2\pi i} \\ 0 & 0 & \omega \end{pmatrix}, \quad \text{for } \arg z = \pi/3 \end{aligned}$$

and then by (6.6)

$$J_X(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 1 \\ 0 & 0 & \omega \end{pmatrix}, \quad \text{for } \arg z = \pi/3.$$

The jump matrix J_X turns out to be exactly the same for $\arg z = -\pi/3$ and $\arg z = \pi$.

On the remaining parts of Γ_X (the unbounded contours C_j), the jump matrix takes the form

$$J_X(z) = \begin{pmatrix} 1 & \alpha e^{\frac{nt_3}{3t_0} z^3} & \beta e^{\frac{nt_3}{3t_0} z^3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

for certain constants α and β , that again come from Wronskians of solutions of the Airy equation. Let

$$y_6(z) = \frac{1}{3}(y_2(z) - y_1(z)), \quad y_7(z) = \omega y_6(\omega z), \quad y_8(z) = \omega^2 y_6(\omega^2 z).$$

Then the constants turn out to be

$$\begin{cases} \alpha = W(-y_8, y_3) = \frac{1}{2} + \frac{1}{6}i\sqrt{3}, \\ \beta = 2\pi i W(-y_8, y_0) = -\frac{1}{3}, \end{cases} \quad \text{on } C_0^+,$$

$$\begin{cases} \alpha = W(y_7, y_3) = \frac{1}{2} - \frac{1}{6}i\sqrt{3}, \\ \beta = 2\pi i W(y_7, y_0) = \frac{1}{3}, \end{cases} \quad \text{on } C_0^-,$$

with the same expressions on the other unbounded parts.

Thus we obtain the following RH problem for X .

RH problem 6.3. The matrix-valued function X defined by (6.5) and (6.6) is the solution of the following RH problem.

- X is analytic in $\mathbb{C} \setminus \Gamma_X$,
- $X_+ = X_- J_X$ on Γ_X with

$$J_X(z) = \begin{cases} \begin{pmatrix} 1 & e^{\frac{n\tau_3}{3i_0} z^3} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } z \in \bigcup_j (0, \omega_j \hat{x}], \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 1 \\ 0 & 0 & \omega \end{pmatrix}, & \text{for } z \in \Sigma_2, \\ \begin{pmatrix} 1 & \left(\frac{1}{2} \pm \frac{1}{6} i \sqrt{3}\right) e^{\frac{n\tau_3}{3i_0} z^3} & \mp \frac{1}{3} e^{\frac{n\tau_3}{3i_0} z^3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & \text{for } z \in \bigcup_j C_j^\pm. \end{cases} \tag{6.7}$$

- As $z \rightarrow \infty$ we have

$$X(z) = (I + O(z^{-1}))A(z) \\ \times \begin{cases} \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} e^{\frac{2n}{3i_0 \sqrt{t_3}} z^{3/2}} & 0 \\ 0 & 0 & z^{-n/2} e^{-\frac{2n}{3i_0 \sqrt{t_3}} z^{3/2}} \end{pmatrix}, & z \in S_0, \\ \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} e^{-\frac{2n}{3i_0 \sqrt{t_3}} z^{3/2}} & 0 \\ 0 & 0 & z^{-n/2} e^{\frac{2n}{3i_0 \sqrt{t_3}} z^{3/2}} \end{pmatrix}, & z \in S_1 \cup S_2, \end{cases} \tag{6.8}$$

with

$$A(z) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & z^{1/4} & 0 \\ 0 & 0 & z^{-1/4} \end{pmatrix} \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ 0 & -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, & \text{for } z \in S_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \end{pmatrix}, & \text{for } z \in S_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{i}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix}, & \text{for } z \in S_2. \end{cases} \tag{6.9}$$

Remark 6.4. The asymptotic condition (6.8) for X comes from the asymptotic condition in the RH problem 5.4 for Y combined with the asymptotic behavior (5.19) of the Airy functions. The condition (6.8), however, is not valid uniformly as $z \rightarrow \infty$ for z close to Σ_2 . More precisely, (6.8) holds uniformly as $z \rightarrow \infty$ with

$$\arg z \in (-\pi + \varepsilon, -\pi/3 - \varepsilon) \cup (-\pi/3 + \varepsilon, \pi/3 - \varepsilon) \cup (\pi/3 + \varepsilon, \pi - \varepsilon)$$

for some $\varepsilon > 0$, but not for $\varepsilon = 0$. This is due to the fact that the functions y_3, y_5 and y_4 appearing in the second column of the transformation (6.5) (which are dominant solutions of the Airy equation in the respective sectors used in (6.5)) are not the recessive solutions in the neighboring sectors.

This non-uniformity is a minor issue that will be resolved more or less automatically during the transformations of the steepest descent analysis, see also [14,15]. To fully justify the analysis it suffices to supplement the RH problem for X with the following condition that controls the behavior as $z \rightarrow \infty$ near Σ_2 .

- As $z \rightarrow \infty$ we have

$$X(z) = O(1)A(z) \times \begin{cases} \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} e^{\frac{2n}{3t_0\sqrt{t_3}}z^{3/2}} & 0 \\ 0 & 0 & z^{-n/2} e^{-\frac{2n}{3t_0\sqrt{t_3}}z^{3/2}} \end{pmatrix}, & z \in S_0, \\ \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} e^{-\frac{2n}{3t_0\sqrt{t_3}}z^{3/2}} & 0 \\ 0 & 0 & z^{-n/2} e^{\frac{2n}{3t_0\sqrt{t_3}}z^{3/2}} \end{pmatrix}, & z \in S_1 \cup S_2, \end{cases} \tag{6.10}$$

with $O(1)$ term that is uniform in z .

Remark 6.5. If we put

$$\Phi(z) = X(z) \begin{pmatrix} e^{\frac{nt_3}{3t_0}z^3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then it is easy to verify that Φ satisfies a RH problem with piecewise constant jumps. Hence Φ satisfies a differential equation

$$\frac{d}{dz} \Phi(z) = C(z) \Phi(z)$$

with a polynomial coefficient matrix $C(z)$. In addition there are also differential equations with respect to t_0 and t_3 , as well as a difference equation in n . We will not elaborate on this here. Instead we are going to make a different transformation.

6.2. Second transformation

In the next transformation the factors $e^{\pm \frac{2n}{3t_0\sqrt{t_3}}z^{3/2}}$ are removed from the asymptotic condition (6.8) in the RH problem for X . The transformation also has the effect of eliminating the $(1, 3)$ entry $\mp \frac{1}{3} e^{\frac{nt_0}{3t_3}z^3}$ from the jump matrix (6.7) on the unbounded contours C_j .

Definition 6.6. We define

$$\tilde{V}(z) = X(z) \begin{pmatrix} 1 & 0 & \frac{1}{3} e^{\frac{nt_3}{3i_0} z^3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.11}$$

for z in the domain bounded by $C_0 \cup C_1 \cup C_2$ (this is the unbounded domain containing Σ_2 in its interior, see Fig. 5),

$$\tilde{V}(z) = X(z) \quad \text{elsewhere,} \tag{6.12}$$

and

$$V(z) = \tilde{V}(z) \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\frac{2n}{3i_0\sqrt{t_3}} z^{3/2}} & 0 \\ 0 & 0 & e^{\frac{2}{3} \frac{n}{i_0\sqrt{t_3}} z^{3/2}} \end{pmatrix} & \text{in } S_0, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\frac{2n}{3i_0\sqrt{t_3}} z^{3/2}} & 0 \\ 0 & 0 & e^{-\frac{2}{3} \frac{n}{i_0\sqrt{t_3}} z^{3/2}} \end{pmatrix} & \text{in } S_1 \cup S_2. \end{cases} \tag{6.13}$$

By straightforward calculations based on the RH problem 6.3 for X and the definitions (6.11)–(6.13) we find that V is the solution of the following RH problem.

RH problem 6.7. The matrix valued function $V : \mathbb{C} \setminus \Gamma_V \rightarrow \mathbb{C}^{3 \times 3}$ satisfies

- V is analytic in $\mathbb{C} \setminus \Gamma_V$ where $\Gamma_V = \Gamma_X$.
- $V_+ = V_- J_V$ on Γ_V with

$$J_V(z) = \begin{cases} \begin{pmatrix} 1 & e^{-\frac{n}{i_0} \left(\frac{2}{3\sqrt{t_3}} |z|^{3/2} - \frac{t_3}{3} z^3 \right)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_j (0, \omega^j \hat{x}], \\ \begin{pmatrix} 1 & 0 & \left(\frac{1}{2} - \frac{1}{6} i\sqrt{3} \right) e^{-\frac{n}{i_0} \left(\frac{2}{3\sqrt{t_3}} z^{3/2} - \frac{t_3}{3} z^3 \right)} \\ 0 & \omega^2 e^{\frac{4n}{3i_0\sqrt{t_3}} z^{3/2}} & 1 \\ 0 & 0 & \omega e^{-\frac{4n}{3i_0\sqrt{t_3}} z^{3/2}} \end{pmatrix}, & \arg z = \pi/3, \\ \begin{pmatrix} 1 & 0 & \left(\frac{1}{2} - \frac{1}{6} i\sqrt{3} \right) e^{\frac{n}{i_0} \left(\frac{2}{3\sqrt{t_3}} z^{3/2} + \frac{t_3}{3} z^3 \right)} \\ 0 & \omega^2 e^{-\frac{4n}{3i_0\sqrt{t_3}} z^{3/2}} & 1 \\ 0 & 0 & \omega e^{\frac{4n}{3i_0\sqrt{t_3}} z^{3/2}} \end{pmatrix}, & \arg z = -\pi/3 \text{ or } \arg z = \pi, \end{cases} \tag{6.14}$$

and

$$J_V(z) = \begin{cases} \begin{pmatrix} 1 & \left(\frac{1}{2} \pm \frac{1}{6}i\sqrt{3}\right) e^{-\frac{n}{40}\left(\frac{2}{3\sqrt{3}}z^{3/2} - \frac{t_3}{3}z^3\right)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in C_0^\pm, \\ \begin{pmatrix} 1 & \left(\frac{1}{2} \pm \frac{1}{6}i\sqrt{3}\right) e^{\frac{n}{40}\left(\frac{2}{3\sqrt{3}}z^{3/2} + \frac{t_3}{3}z^3\right)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in C_1^\pm \cup C_2^\pm. \end{cases} \tag{6.15}$$

- $V(z) = (I + O(1/z))A(z) \begin{pmatrix} z^n & 0 & 0 \\ 0 & z^{-n/2} & 0 \\ 0 & 0 & z^{-n/2} \end{pmatrix}$ as $z \rightarrow \infty$, where $A(z)$ is given by (6.9).

In (6.14) we used the convention that $\arg z^{3/2} = \frac{3}{2} \arg z$, which means that $z^{3/2} = -i|z|^{3/2}$ for $\arg z = \pi$.

The entries $e^{\pm \frac{4n}{30\sqrt{3}}z^{3/2}}$ in the jump matrix (6.14) on Σ_2 are oscillatory. Later they will be turned into exponentially decaying entries by opening up unbounded lenses around Σ_2 . This indeed leads to exponentially decaying entries because of the upper triangularity of the jump matrices.

6.3. Third transformation

In the third transformation we make use of the minimizer (μ_1^*, μ_2^*) of the vector equilibrium problem. Associated with the measures μ_1^* and μ_2^* are the g -functions

$$g_k(z) = \int \log(z - s) d\mu_k^*(s), \quad k = 1, 2, \tag{6.16}$$

with appropriately chosen branches of the logarithm. We choose the branches in such a way that g_1 is defined and analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \mathbb{R}^-)$ with $g_1(x)$ real for x real and $x > x^*$. Then we have the symmetry

$$\begin{cases} g_1(\omega z) = g_1(z) + 2\pi i/3, \\ g_1(\omega^2 z) = g_1(z) - 2\pi i/3, \end{cases} \quad z \in S_0. \tag{6.17}$$

The branches of the logarithm in the definition (6.16) of g_2 are chosen such that g_2 is defined and analytic in $\mathbb{C} \setminus \Sigma_2 = S_0 \cup S_1 \cup S_2$, and $g_2(x)$ is real for real $x > 0$ with the symmetry

$$\begin{cases} g_2(\omega z) = g_2(z) + \pi i/3, \\ g_2(\omega^2 z) = g_2(z) - \pi i/3, \end{cases} \quad z \in S_0. \tag{6.18}$$

The conditions (4.1) and (4.2) lead to the following properties of the g -functions. We emphasize that all contours are oriented away from 0 and towards ∞ , so that for $\arg z = \pi$, we have that $g_{2,+}(z)$ is the limiting value from the lower half-plane.

Lemma 6.8. (a) *We have, with the constant ℓ from (4.2),*

$$g_{1,+}(z) + g_{1,-}(z) - g_2(z) = \begin{cases} \frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3 + \ell, & z \in [0, x^*], \\ -\frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3 + \ell + \pi i, & z \in [0, \omega x^*], \\ -\frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3 + \ell - \pi i, & z \in [0, \omega^2 x^*]. \end{cases} \tag{6.19}$$

(b) *We have*

$$\begin{aligned} g_{2,+}(z) + g_{2,-}(z) - g_1(z) &= 0, \quad \arg z = \pm\pi/3, \\ g_{2,+}(z) + g_{2,-}(z) - g_{1,\pm}(z) &= \pm\pi i, \quad \arg z = \pi. \end{aligned} \tag{6.20}$$

Proof. (a) The equality of the real parts follows from (4.2), since $\text{supp}(\mu_1^*) = \Sigma_1 = \bigcup_j [0, \omega^j x^*]$. Both sides of (6.19) are real for $z \in [0, x^*]$ and so the identity (6.19) holds on $[0, x^*]$. The identity on the other intervals, then follows from the symmetry (6.17).

(b) The identities in (6.20) follow in a similar way from (4.1).

With the g -functions and the constant ℓ we make the next transformation.

Definition 6.9. We define

$$U(z) = \begin{pmatrix} e^{-n\ell} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V(z) \begin{pmatrix} e^{-n(g_1(z)-\ell)} & 0 & 0 \\ 0 & e^{n(g_1(z)-g_2(z))} & 0 \\ 0 & 0 & e^{ng_2(z)} \end{pmatrix}. \tag{6.21}$$

It is our next task to state the RH problem for U . It will be convenient to write the jump matrices in terms of the two functions φ_1 and φ_2 defined as follows. Recall that ξ_1, ξ_2 and ξ_3 are defined in (4.6)–(4.8).

Definition 6.10. We define $\varphi_1 : \mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2) \rightarrow \mathbb{C}$ by

$$\varphi_1(z) = \frac{1}{2t_0} \int_{\omega^j x^*}^z (\xi_1(s) - \xi_2(s)) ds, \quad z \in S_j \setminus [0, \omega^j x^*], \quad j = 0, 1, 2, \tag{6.22}$$

with integration along a path lying in $S_j \setminus [0, \omega^j x^*]$ if $z \in S_j \setminus [0, \omega^j x^*]$ for $j = 0, 1, 2$, and $\varphi_2 : \mathbb{C} \setminus \{z \in \mathbb{C} \mid z^3 \in \mathbb{R}\} \rightarrow \mathbb{C}$ by

$$\varphi_2(z) = \frac{1}{2t_0} \int_0^z (\xi_2(s) - \xi_3(s)) ds \begin{cases} -\pi i/6, & 0 < \arg z < \pi/3, \\ +\pi i/6, & \pi/3 < \arg z < 2\pi/3, \\ +\pi i/3, & 2\pi/3 < \arg z < \pi, \\ +\pi i/6, & -\pi/3 < \arg z < 0, \\ -\pi i/6, & -2\pi/3 < \arg z < -\pi/3, \\ -\pi i/3, & -\pi < \arg z < -2\pi/3, \end{cases} \tag{6.23}$$

with integration along a path in the sector $k\pi/3 < \arg s < (k+1)\pi/3$ if z lies in that sector, for $k = -3, \dots, 2$.

The basic properties of φ_1 and φ_2 that connect them with the g -functions are collected in the next lemma.

Lemma 6.11. (a) For $z \in \Sigma_1$, we have

$$g_{1,+}(z) - g_{1,-}(z) = \pm 2\varphi_{1,\pm}(z). \tag{6.24}$$

(b) For $z \in \mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$ we have

$$2g_1(z) - g_2(z) - \ell = 2\varphi_1(z) \pm \frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3 + \begin{cases} 0 & \text{in } S_0, \\ \pi i & \text{in } S_1, \\ -\pi i & \text{in } S_2, \end{cases} \tag{6.25}$$

with $+$ in S_0 and $-$ in $S_1 \cup S_2$.

(c) For $z \in \Sigma_2$, we have

$$\begin{aligned} g_{2,+}(z) - g_{2,-}(z) &= 2\varphi_{2,+}(z) \pm \frac{4}{3t_0\sqrt{t_3}}z^{3/2} \\ &= -2\varphi_{2,-}(z) \pm \frac{4}{3t_0\sqrt{t_3}}z^{3/2}, \end{aligned} \tag{6.26}$$

where we use $+$ for $\arg z = \pi/3$ and $-$ for $\arg z = -\pi/3$ or $\arg z = \pi$.

(d) For $z \in \Sigma_2$, we have

$$\begin{aligned} &g_{1,-}(z) + g_{2,+}(z) - \ell \\ &= \begin{cases} 2\varphi_{1,-}(z) + \frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3, & \arg z = \pi/3, \\ 2\varphi_{1,-}(z) - \frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3, & \arg z = \pi, \\ 2\varphi_{1,-}(z) - \frac{2}{3t_0\sqrt{t_3}}z^{3/2} - \frac{t_3}{3t_0}z^3 - \pi i, & \arg z = -\pi/3. \end{cases} \end{aligned} \tag{6.27}$$

(e) We have for $z \in \Sigma_1$,

$$2\varphi_{1,+}(z) + 2\varphi_{2,+}(z) = 2\varphi_{2,-}(z) + \begin{cases} 0, & \text{for } z \in [0, x^*], \\ \pi i, & \text{for } z \in [0, \omega x^*], \\ -\pi i, & \text{for } z \in [0, \omega^2 x^*] \end{cases} \tag{6.28}$$

and for $z \in \Sigma_2$,

$$2\varphi_{1,+}(z) + 2\varphi_{2,+}(z) = 2\varphi_{1,-}(z) - \begin{cases} \pi i, & \text{for } \arg z = \pm\pi/3, \\ 2\pi i, & \text{for } \arg z = \pi. \end{cases} \tag{6.29}$$

Proof. By (4.4) and (6.16) we have that $g'_1 = F_1$ and $g'_2 = F_2$, so that by (4.6)–(4.8) we can express the derivatives of the left-hand sides of (6.24)–(6.26) in terms of the ξ -functions. This yields

$$\begin{aligned} g'_{1,+}(z) - g'_{1,-}(z) &= \frac{1}{t_0}(\xi_{1,+}(z) - \xi_{1,-}(z)), \quad z \in \Sigma_1, \\ 2g'_1(z) - g'_2(z) &= \frac{1}{t_0}(\xi_1(z) - \xi_2(z)) \pm \frac{2z^{1/2}}{t_0\sqrt{t_3}} - \frac{t_3}{t_0}z^2, \\ g'_{2,+}(z) - g'_{2,-}(z) &= \frac{1}{t_0}(\xi_{3,-}(z) - \xi_{3,+}(z)) \pm \frac{2z^{1/2}}{t_0\sqrt{t_3}}, \quad z \in \Sigma_2, \end{aligned}$$

with the same conventions on \pm signs as in parts (b) or (c) of the lemma. In view of the definitions (6.22) and (6.23) we have

$$2\varphi'_j(z) = \frac{1}{t_0}(\xi_j(z) - \xi_{j+1}(z)), \quad j = 1, 2. \tag{6.30}$$

Using also that $\xi_{1,\pm} = \xi_{2,\mp}$ on Σ_1 and $\xi_{3,\pm} = \xi_{2,\mp}$ on Σ_2 , we then easily check that the derivatives of the two sides in (6.24)–(6.26) agree on the respective contours or regions. It follows that (6.24)–(6.26) hold up to a possible constant of integration.

Both sides of (6.24) vanish for $z = \omega^j x^*$, $j = 0, 1, 2$, while the equality in (6.25) for $z = \omega^j x^*$, $j = 0, 1, 2$ follows from the identities (6.19). Hence the equalities in parts (a) and (b) follow.

To establish the equality in part (c) we examine the behavior as $z \rightarrow 0$. Because of the symmetry (6.18) we find that there exists a constant γ_2 such that g_2 has the following limits at the origin

$$\lim_{z \rightarrow 0} g_2(z) = \begin{cases} \gamma_2, & z \in S_0, \\ \gamma_2 + \pi i/3, & z \in S_1, \\ \gamma_2 - \pi i/3, & z \in S_2. \end{cases} \tag{6.31}$$

From (6.31) we find the limits of $g_{2,+}(z) - g_{2,-}(z)$ as $z \rightarrow 0$ on the three half-rays in Σ_2 . These values correspond to the limits of $\pm 2\varphi_{2,\pm}(z)$ as $z \rightarrow 0$ that we obtain from the definition (6.23) and part (c) follows.

Using (6.20) we find for $z \in \Sigma_2$ that

$$g_{1,-}(z) + g_{2,+}(z) - \ell = \begin{cases} 2g_1(z) - g_{2,-}(z) - \ell, & \arg z = \pm\pi/3, \\ 2g_{1,-}(z) - g_{2,-}(z) - \ell - \pi i, & \arg z = \pi. \end{cases} \tag{6.32}$$

Then (6.27) follows by letting z in (6.25) go to the minus-side of Σ_2 , and inserting this into (6.32).

Because of (6.30) we have

$$2(\varphi'_1(z) + \varphi'_2(z)) = \frac{1}{t_0}(\xi_1(z) - \xi_3(z)).$$

Using $\xi_{1,+} = \xi_{2,-}$ on Σ_1 and $\xi_{3,+} = \xi_{2,-}$ on Σ_2 , we then find that the derivatives of the two sides of (6.28) and (6.29) agree. To prove the two equalities we again examine the behavior as $z \rightarrow 0$. From (6.31) and (6.20) we find the limits of g_1 at the origin,

$$\lim_{z \rightarrow 0} g_1(z) = \begin{cases} 2\gamma_2 + \pi i/3, & 0 < \arg z < 2\pi/3, \\ 2\gamma_2 + \pi i, & 2\pi/3 < \arg z < \pi, \\ 2\gamma_2 - \pi i/3, & -2\pi/3 < \arg z < 0, \\ 2\gamma_2 - \pi i, & -\pi < \arg z < -2\pi/3, \end{cases} \tag{6.33}$$

where γ_2 is the constant from (6.31). Using (6.19) we also find

$$\ell = 3\gamma_2.$$

Combining (6.31) and (6.33) with (6.24) and (6.26) we obtain the limits of φ_1 and φ_2 at the origin.

$$\lim_{z \rightarrow 0} 2\varphi_1(z) = \begin{cases} 2\pi i/3, & \text{if } k\pi/3 < \arg z < (k+1)\pi/3 \text{ with } k \text{ even,} \\ -2\pi i/3, & \text{if } k\pi/3 < \arg z < (k+1)\pi/3 \text{ with } k \text{ odd,} \end{cases} \tag{6.34}$$

$$\lim_{z \rightarrow 0} 2\varphi_2(z) = \begin{cases} -\pi i/3, & 0 < \arg z < \pi/3 \text{ or } -2\pi/3 < \arg z < -\pi/3, \\ \pi i/3, & -\pi/3 < \arg z < 0 \text{ or } \pi/3 < \arg z < 2\pi/3, \\ 2\pi i/3, & 2\pi/3 < \arg z < \pi, \\ -2\pi i/3, & -\pi < \arg z < -2\pi/3. \end{cases} \tag{6.35}$$

The values in (6.35) also follow from the definition (6.23) of φ_2 . From (6.34) and (6.35) we check that the left- and right-hand sides of (6.28) and (6.29) have the same values as $z \rightarrow 0$ and the equalities (6.28) and (6.29) follow.

Then we have the following.

RH problem 6.12. U is the solution of the following RH problem.

- U is analytic in $\mathbb{C} \setminus \Gamma_U$, where $\Gamma_U = \Gamma_V$,
- $U_+ = U_- J_U$ on Γ_U with

$$J_U(z) = \begin{cases} \begin{pmatrix} e^{-2n\varphi_{1,+}(z)} & 1 & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_j [\omega^j x^*, \omega^j \hat{x}], \\ \begin{pmatrix} 1 & 0 & \left(\frac{1}{2} - \frac{1}{6}i\sqrt{3}\right) e^{2n\varphi_{1,-}(z)} \\ 0 & \omega^2 e^{-2n\varphi_{2,+}(z)} & 1 \\ 0 & 0 & \omega e^{-2n\varphi_{2,-}(z)} \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & \left(\frac{1}{2} \pm \frac{1}{6}i\sqrt{3}\right) e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_j C_j^\pm. \end{cases} \tag{6.36}$$

- $U(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$, where $A(z)$ is given by (6.9).

Proof. From (6.14) on $\bigcup_j (0, \omega^j \hat{x}]$ and the definition (6.21) we obtain by the fact that g_2 is analytic, that J_U on $\bigcup_j [0, \omega^j \hat{x}]$ takes the form

$$\begin{pmatrix} e^{-n(g_{1,+} - g_{1,-})} & e^{n(g_{1,+} + g_{1,-} - g_2 - \ell - \frac{1}{i_0} (\frac{2}{3\sqrt{3}} |z|^{3/2} - \frac{t_3}{3} z^3))} & 0 \\ 0 & e^{n(g_{1,+} - g_{1,-})} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.37}$$

The (1, 2) entry of (6.37) is equal to 1 on Σ_1 because of (6.19) (recall that n is even by our assumption at the beginning of Section 6.1, so that the terms $\pm\pi i$ that are in (6.19) on $[0, \omega x^*]$ and $[0, \omega^2 x^*]$ do not matter). The nontrivial diagonal entries in (6.37) reduce on Σ_1 to the ones as given in (6.36) on Σ_1 because of the identities (6.24). On $\bigcup_j [\omega^j x^*, \omega^j \hat{x}]$ the diagonal entries in (6.37) are equal to 1, since $g_{1,+} = g_{1,-}$ is analytic. The (1, 2) entry is $e^{2n\varphi_1}$ because of the identity (6.25).

For the jump matrix J_U on Σ_2 we obtain from (6.14), (6.21) and the fact that e^{ng_1} is analytic on Σ_2

$$\begin{pmatrix} 1 & 0 & \left(\frac{1}{2} - \frac{1}{6}i\sqrt{3}\right) e^{n\left(g_{1,-} + g_{2,+} - \ell + \frac{1}{i_0} \left(\mp \frac{2}{3\sqrt{3}} z^{3/2} + \frac{i_3}{3} z^3\right)\right)} \\ 0 & \omega^2 e^{-n\left(g_{2,+} + g_{2,-} \pm \frac{4}{3i_0\sqrt{3}} z^{3/2}\right)} & 0 \\ 0 & 0 & \omega e^{n\left(g_{2,+} + g_{2,-} \pm \frac{4}{3i_0\sqrt{3}} z^{3/2}\right)} \end{pmatrix} \tag{6.38}$$

with $+$ for $\arg z = \pi/3$ and $-$ for $\arg z = -\pi/3$ or $\arg z = \pi$. The nonconstant entries in (6.38) are simplified using (6.26) and (6.27) which leads to the expressions given in (6.36). Again we use the fact that n is even so that the term $\pm\pi i$ in (6.27) on $\arg z = -\pi/3$ does not contribute.

For $z \in \bigcup_j C_j^\pm$ we find from (6.15) and (6.21) that the jump matrix J_U is equal to

$$\begin{pmatrix} 1 & \left(\frac{1}{2} \pm \frac{1}{6}i\sqrt{3}\right) e^{\frac{n}{i_0} \left(\mp \frac{2}{3\sqrt{3}} z^{3/2} + \frac{i_3}{3} z^3\right) + n(2g_1(z) - g_2(z) - \ell)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Here we simplify the (1, 2) entry by means of (6.25) and thereby obtain the jump matrix that is given in (6.36) on $\bigcup_j C_j^\pm$.

The asymptotic condition in the RH problem 6.12 follows from the asymptotic condition in the RH problem 6.7 for V , the transformation (6.21) and the fact that

$$g_1(z) = \log z + O(z^{-3}), \quad g_2(z) = \frac{1}{2} \log z + O(z^{-3/2})$$

as $z \rightarrow \infty$.

The jump matrices J_U in the RH problem 6.12 for U have a good form for the opening of lenses around Σ_1 and Σ_2 . Indeed, the expressions $\varphi_{j,\pm}(z)$, $j = 1, 2$, that appear in the non-trivial diagonal entries in the jump matrices are purely imaginary and the diagonal entries $e^{-2n\varphi_{j,\pm}(z)}$ are highly oscillatory if n is large. The opening of lenses will turn these entries into exponentially decaying off-diagonal entries.

The other non-constant entries in the jump matrix J_U have a factor $e^{2n\varphi_1}$ or $e^{2n\varphi_{1,-}}$. These are exponentially decaying provided that $\operatorname{Re}\varphi_1 < 0$ on $\bigcup_j ((\omega^j x^*, \omega^j \widehat{x}) \cup C_j)$ and $\operatorname{Re}\varphi_{1,-} < 0$ on Σ_2 . The first of these inequalities indeed holds, but the second one is not satisfied on the part of Σ_2 in a neighborhood of the origin. Fortunately, we can remove the (1, 3) entry in the jump matrix (6.36) on Σ_2 by a transformation in a larger domain that is similar to the global opening of lenses in [3]. We will do this in the next subsection. The transformation will not affect the jump matrices on Σ_1 and $\bigcup_j [\omega^j x^*, \omega^j \widehat{x}]$.

6.4. Fourth transformation

The transformation $U \mapsto T$ is based on the following property of the set where $\operatorname{Re}\varphi_1 < 0$. Recall that φ_1 , see (6.22), is defined and analytic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$. From (6.24) and (6.16) it follows that

$$\varphi_{1,\pm}(z) = \pm\pi i \mu_1^*([z, \omega^j x^*]), \quad \text{for } z \in [0, \omega^j x^*], \quad j = 0, 1, 2, \tag{6.39}$$

so that $\operatorname{Re}\varphi_{1,\pm} = 0$ on Σ_1 . On Σ_2 we have the identities (6.29) with $\varphi_{2,+}$ purely imaginary. Therefore $\operatorname{Re}\varphi_1$ extends to a continuous function on \mathbb{C} , which we also denote by $\operatorname{Re}\varphi_1$. With this

understanding we define the domain D as follows

$$D = \{z \in \mathbb{C} \mid \operatorname{Re}\varphi_1(z) < 0\}. \tag{6.40}$$

Since $\operatorname{Re}\varphi_1$ is harmonic in $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$, we have that the boundary ∂D consists of a finite number of analytic arcs that start and end on $\Sigma_1 \cup \Sigma_2$ or at infinity.

From (6.39) it follows that $\pm i\varphi_{1,\pm}$ is real and strictly increasing with a positive derivative along each interval $[0, \omega^j x^*]$ in Σ_1 . Then by the Cauchy–Riemann equations, it follows that the closed set $\mathbb{C} \setminus D$ contains the intervals $[0, \omega^j x^*]$, $j = 0, 1, 2$ in its interior. Note that 0 is also an interior point, since μ_1^* has a positive density at 0. Near the branch point x^* we have

$$\varphi_1(z) = -c(z - x^*)^{3/2} \left(1 + O\left((z - x^*)^{-1}\right) \right) \quad \text{as } z \rightarrow x^*$$

with a positive constant $c > 0$. This follows from the fact that μ_1^* vanishes as a square root at x^* . Thus $\operatorname{Re}\varphi_1 < 0$ immediately to the right of x^* . It also follows that $x^* \in \partial D$ and ∂D makes angles $\pm 2\pi/3$ with $[0, x^*]$. The same holds at $\omega^j x^*$ because of the rotational symmetry.

Since

$$2\varphi_1(z) = \frac{t_3}{3t_0} z^3 + O(z^{3/2}) \quad \text{as } z \rightarrow \infty,$$

which follows for example from (6.25), the domain D is unbounded and extends to infinity in the three directions

$$|\arg z - (2k - 1)\pi/3| < \pi/6, \quad k = 0, 1, 2,$$

around the directions $\pm\pi/3, \pi$ of Σ_2 . The unbounded parts of ∂D then extend to infinity at angles $\pm\pi/6, \pm\pi/2$ and $\pm 5\pi/6$. From what we now know about D it follows that ∂D has three unbounded parts. One of them intersects the positive real line and tends to infinity at angles $\pm\pi/6$. The other two are obtained by rotation over angles $\pm 2\pi/3$.

The analytic arcs in ∂D that start at the branch points $\omega^j x^*$ then necessarily end at points on Σ_2 . The conclusion is that we have a situation as shown in Fig. 6. In particular the domain D is connected.

We also find that $\operatorname{Re}\varphi_1$ takes on a negative minimum on the interval (x^*, ∞) which by (6.22) and parts (c) and (d) of Lemma 4.3 is taken at \widehat{x} . Thus $\widehat{x} \in D$, and by symmetry also $\omega^j \widehat{x} \in D$ for $j = 1, 2$. We may then also assume that the unbounded contours C_j from (5.13) and (5.14) (see also Fig. 4) satisfy

$$C_j \subset D \cap S_j, \quad j = 0, 1, 2,$$

and that they extend to infinity at asymptotic directions $-\pi/3 + \theta_0$ and $\pi/3 - \theta_0$ (for C_0), $\pi/3 + \theta_0$ and $\pi - \theta_0$ (for C_1) and $-\pi + \theta_0$ and $-\pi/3 - \theta_0$ (for C_2), where $\theta_0 \in (0, \pi/6)$ is some fixed angle. The above choice makes sure that

$$\operatorname{Re}\varphi_1(z) < 0, \quad z \in C_j, \quad j = 0, 1, 2, \tag{6.41}$$

and, with positive constants $c_1, c_2 > 0$,

$$\operatorname{Re}\varphi_1(z) \leq -c_1|z|^3 + c_2, \tag{6.42}$$

for z in the domain bounded by $C_0 \cup C_1 \cup C_2$.

After these preparations we define the transformation $U \mapsto T$ as follows.

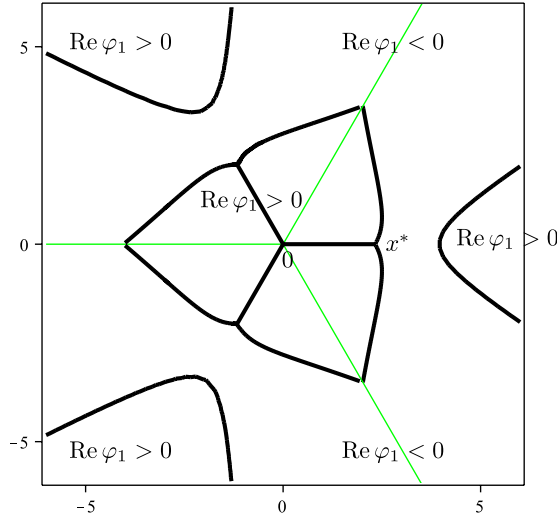


Fig. 6. Curves $\text{Re } \varphi_1(z) = 0$ and the domains where $\text{Re } \varphi_1(z) < 0$ and $\text{Re } \varphi_1(z) > 0$. The figure is for the values $t_0 = 1.8, t_3 = 0.25$.

Definition 6.13. We define \tilde{T} as

$$\tilde{T}(z) = U(z)$$

for z outside the domain bounded by $C_0 \cup C_1 \cup C_2$, and for z inside this domain,

$$\begin{aligned} \tilde{T}(z) &= U(z) \begin{pmatrix} 1 & \left((-1)^k \frac{1}{2} - \frac{1}{6} i \sqrt{3} \right) e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ for } \arg z \in \left(\frac{k\pi}{3}, \frac{(k+1)\pi}{3} \right) \\ & \text{ and } k = -3, \dots, 2. \end{aligned} \tag{6.43}$$

Then T is defined as

$$T(z) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tilde{T}(z) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{6.44}$$

The RH problem for T is the following.

RH problem 6.14. T is the solution of the following RH problem.

- T is analytic in $\mathbb{C} \setminus \Gamma_T$, where $\Gamma_T = \Gamma_U$,
- $T_+ = T_- J_T$ on Γ_T with

$$J_T(z) = \begin{cases} \begin{pmatrix} e^{-2n\varphi_{1,+}(z)} & 1 & 0 \\ 0 & e^{-2n\varphi_{1,-}(z)} & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 e^{-2n\varphi_{2,+}(z)} & 1 \\ 0 & 0 & \omega e^{-2n\varphi_{2,-}(z)} \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 [\omega^j x^*, \omega^j \hat{x}], \\ \begin{pmatrix} 1 & \frac{1}{2} e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 C_j. \end{cases} \tag{6.45}$$

• $T(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$, where $A(z)$ is given by (6.9).

Proof. From (6.42) and (6.43) we get that $\tilde{T}(z) = U(z) \left(I + O\left(e^{-c_1|z|^3} \right) \right)$ as $z \rightarrow \infty$, which by the asymptotic condition in the RH problem 6.12 for U leads to

$$\tilde{T}(z) = (I + O(1/z))A(z)$$

as $z \rightarrow \infty$. The transformation (6.44) does not affect this asymptotic behavior, because of the special form (6.9) of $A(z)$.

We next verify the jump matrices (6.45). We write

$$\alpha = \frac{1}{2} + \frac{1}{6}i\sqrt{3}. \tag{6.46}$$

For the jump matrix $J_{\tilde{T}} = (\tilde{T}_-)^{-1} \tilde{T}_+$ on Σ_1 we obtain by (6.36) and (6.43)

$$\begin{aligned} J_{\tilde{T}} &= \begin{pmatrix} 1 & \alpha e^{2n\varphi_{1,-}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} J_U \begin{pmatrix} 1 & \bar{\alpha} e^{2n\varphi_{1,+}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{-2n\varphi_{1,+}} & 1 + \alpha + \bar{\alpha} & 0 \\ 0 & e^{-2n\varphi_{1,-}} & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{-2n\varphi_{1,+}} & 2 & 0 \\ 0 & e^{-2n\varphi_{1,-}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

The transformation (6.44) has the effect of dividing the (1, 2) entry in $J_{\tilde{T}}$ by 2 and we obtain the jump matrix J_T on Σ_1 as given in (6.45). Similar calculations lead to the jump matrices J_T on $\bigcup_j [\omega^j x^*, \omega^j \hat{x}]$ and on $\bigcup_j C_j^\pm$.

For the jump matrix $J_{\tilde{T}}$ on Σ_2 we obtain by (6.36) and (6.43)

$$\begin{aligned} J_{\tilde{T}} &= \begin{pmatrix} 1 & -\bar{\alpha} e^{2n\varphi_{1,-}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} J_U \begin{pmatrix} 1 & -\alpha e^{2n\varphi_{1,+}} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -\alpha e^{2n\varphi_{1,+}} - \bar{\alpha} \omega^2 e^{2n(\varphi_{1,-} - \varphi_{2,+})} & 0 \\ 0 & \omega^2 e^{-2n\varphi_{2,+}} & 1 \\ 0 & 0 & \omega e^{-2n\varphi_{2,-}} \end{pmatrix}. \end{aligned} \tag{6.47}$$

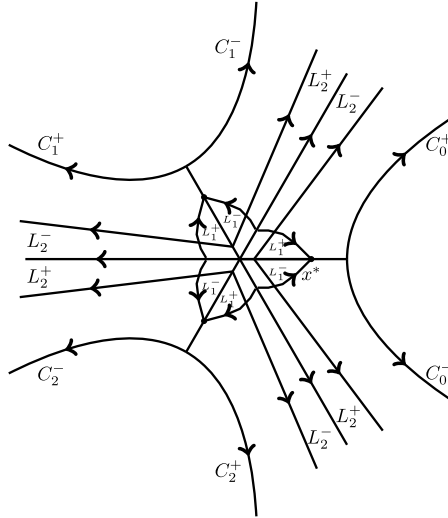


Fig. 7. Lenses L_1 and L_2 around Σ_1 and Σ_2 and the contours Σ_j in the RH problem for S .

By (6.29) we have $e^{2n(\varphi_{1,-} - \varphi_{2,+})} = e^{2n\varphi_{1,+}}$ on Σ_2 , and so the (1, 2) entry in (6.47) reduces to

$$(-\alpha - \bar{\alpha}\omega^2)e^{2n\varphi_{1,+}} = 0,$$

because of the value (6.46) we have for α . Thus the (1, 2) entry in (6.47) vanishes. The transformation (6.44) then gives $J_T = J_{\bar{T}}$ and it is equal to the jump matrix (6.45) on Σ_2 .

Since $\text{Re}\varphi_1 < 0$ on $\bigcup_j ((\omega^j x^*, \omega^j \hat{x}^*] \cup C_j^\pm)$ we see from (6.45) that the jump matrices J_T on these parts of Γ_T tend to the identity matrix as $n \rightarrow \infty$. What remains are oscillatory jump matrices on Σ_1 and Σ_2 .

6.5. Fifth transformation

We open up lenses around the three intervals in Σ_1 and around Σ_2 . We use

$$L_1 = L_1^+ \cup L_1^- \tag{6.48}$$

to denote the lens around Σ_1 where L_1^+ (L_1^-) is the part that lies on the + side (− side) of Σ_1 .

The boundary ∂L_1 of L_1 meets Σ_1 at the endpoints $\omega^j x^*$, $j = 0, 1, 2$ but not at 0. The boundary meets Σ_2 at points at a positive distance $\delta_1 > 0$ from the origin, see Fig. 7, and we choose L_1 such that $\partial L_1 \setminus \{\omega^j x^*\}$ is contained in the region where $\text{Re}\varphi_1 > 0$, see also Figs. 6 and 7.

Similarly we write

$$L_2 = L_2^+ \cup L_2^- \tag{6.49}$$

for the lens around Σ_2 . The boundary ∂L_2 lies in the region where $\text{Re}\varphi_2 > 0$ and they are bounded by infinite rays that meet Σ_1 at points at a positive distance $\delta_2 > 0$ from the origin. The rays have an asymptotic angles $\pm\pi/3 \pm \varepsilon$, $\pi \pm \varepsilon$ for some small $\varepsilon > 0$, see also Fig. 7. We can indeed open the lens around Σ_2 this way, since μ_2^* has a positive density, also at 0.

Note that the lenses L_1 and L_2 have non-empty intersection. More precisely, $L_1^+ \cap L_1^- \neq \emptyset$ and $L_1^- \cap L_2^+ \neq \emptyset$.

Definition 6.15. We define

$$S(z) = T(z)$$

for z outside of the lenses L_1 and L_2 . For z inside the lenses we define

$$S(z) = T(z) \times \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ \mp e^{-2n\varphi_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in L_1^\pm \setminus L_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \mp \omega^\mp e^{-2n\varphi_2(z)} & 1 \end{pmatrix}, & z \in L_2^\pm \setminus L_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ \mp e^{-2n\varphi_1(z)} & 1 & 0 \\ \omega^\mp e^{-2n(\varphi_1(z)+\varphi_2(z))} & \pm \omega^\pm e^{-2n\varphi_2(z)} & 1 \end{pmatrix}, & z \in L_1^\pm \cap L_2^\mp, \end{cases} \quad (6.50)$$

where $\omega^+ = \omega$ and $\omega^- = \omega^{-1} = \omega^2$.

Then S satisfies a RH problem on a union of contours Γ_S that consists of Γ_T and the lips ∂L_1 and ∂L_2 of the lenses. We continue to follow the convention that the contours are oriented away from the origin, and towards infinity. For ∂L_1 and ∂L_2 this means that each part in ∂L_1 is oriented towards a branch point $\omega^j x^*$, and each part in ∂L_2 is oriented towards infinity, as indicated in Fig. 7.

RH problem 6.16. • S is analytic in $\mathbb{C} \setminus \Gamma_S$,

- $S_+ = S_- J_S$ on Γ_S with J_S given as follows on Γ_T ,

$$J_S(z) = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, & z \in \Sigma_2, \\ \begin{pmatrix} 1 & e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 [\omega^j x^*, \omega^j \widehat{x}], \\ \begin{pmatrix} 1 & \frac{1}{2} e^{2n\varphi_1(z)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \bigcup_{j=0}^2 C_j, \end{cases} \quad (6.51)$$

and on the lips of the lenses by

$$J_S(z) = \begin{cases} \begin{pmatrix} 1 & 0 & 0 \\ e^{-2n\varphi_1(z)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \partial L_1 \setminus L_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega^\mp e^{-2n\varphi_2(z)} & 1 \end{pmatrix}, & z \in \partial L_2^\pm \setminus L_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ e^{-2n\varphi_1(z)} & 1 & 0 \\ \pm e^{-2n(\varphi_1(z)+\varphi_2(z))} & 0 & 1 \end{pmatrix}, & z \in \partial L_1^\pm \cap L_2, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega^\pm e^{-2n(\varphi_1(z)+\varphi_2(z))} & \omega^\mp e^{-2n\varphi_2(z)} & 1 \end{pmatrix}, & z \in \partial L_2^\pm \cap L_1. \end{cases} \tag{6.52}$$

• $S(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$ where $A(z)$ is given by (6.9).

Proof. The jump matrices J_S arise from combining the jump condition (6.45) for T with the definition (6.50). As an example we show how the jump matrix (6.52) on $\partial L_1^+ \cap L_2$ arises.

To compute the jump matrix on $\partial L_1^+ \cap L_2$ we need the definition (6.50) of S in the parts $L_1^+ \cap L_2$ (which is on the $-$ side of $\partial L_1^+ \cap L_2$) and $L_2^- \setminus L_1$ (which is on the right). Since T has no jump we find from (6.50) that,

$$\begin{aligned} J_S(z) &= \begin{pmatrix} 1 & 0 & 0 \\ -e^{-2n\varphi_1(z)} & 1 & 0 \\ \omega^2 e^{-2n(\varphi_1(z)+\varphi_2(z))} & \omega e^{-2n\varphi_2(z)} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega e^{-2n\varphi_2(z)} & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ e^{-2n\varphi_1(z)} & 1 & 0 \\ e^{-2n(\varphi_1(z)+\varphi_2(z))} & -\omega e^{-2n\varphi_2(z)} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega e^{-2n\varphi_2(z)} & 1 \end{pmatrix}, \end{aligned}$$

since $1 + \omega + \omega^2 = 0$. This then easily reduces to the jump matrix J_S as given in (6.52) on $\partial L_1^+ \cap L_2$. The other jumps follow in a similar way.

The asymptotic condition for S outside of the lens L_2 follows trivially from the asymptotic condition in the RH problem 6.14 for T . Inside L_2 , we note that by (6.26) we have

$$\varphi_2(z) = \pm \frac{2}{3t_0\sqrt{t_3}} z^{3/2} + O(\log z) \quad \text{as } z \rightarrow \infty, z \in L_2,$$

where the \pm signs are such that $\text{Re}\varphi_2(z) > 0$ for z large enough in L_2 . Then it follows from (6.50) that the asymptotic condition is also valid in the lens L_2 .

We are now in a situation where all jump matrices J_S tend to the identity matrix as $n \rightarrow \infty$, except for those on Σ_1 and Σ_2 , which are constant. We already saw this for the jumps on $\bigcup_j ((\omega^j x^*, \omega^j \hat{x}) \cup C_j)$, see (6.51), since $\text{Re}\varphi_1 < 0$ on these parts of the contour. Also L_1 is contained in the region where $\text{Re}\varphi_1 > 0$, so that the entries $e^{-2n\varphi_1(z)}$ that appear in (6.52) are indeed decaying as $n \rightarrow \infty$.

Regarding φ_2 , we note that by (6.16) and (6.26) we have that $\varphi_{2,\pm}(z)$ is purely imaginary for $z \in \Sigma_2$, with

$$\begin{aligned} i\varphi_{2,+}(z) &= -i\varphi_{2,-}(z) \\ &= \pi\mu_2^*([0, z]) + \frac{2}{3t_0\sqrt{t_3}}|z|^{3/2} + \begin{cases} -\pi/6, & \arg z = \pm\pi/3, \\ \pi/3, & \arg z = \pi. \end{cases} \end{aligned} \tag{6.53}$$

Thus $i\varphi_{2,+}$ and $-i\varphi_{2,-}$ are strictly increasing along each of the rays in Σ_2 . Then by the Cauchy–Riemann equations, we have that $\operatorname{Re}\varphi_2 > 0$ both to the left and to the right of Σ_2 . We may (and do) assume that the lips of the lens L_2 is contained in the region where $\operatorname{Re}\varphi_2 > 0$ and it indeed follows that the jump matrices (6.52) tend to the identity matrix as $n \rightarrow \infty$.

Remark 6.17. The asymptotic condition in the RH problem 6.16 for S is valid uniformly as $z \rightarrow \infty$ in any direction in the complex plane. This is in contrast to what happens in the RH problems for X, V, U and T , where the asymptotic condition is not valid uniformly in the directions $\arg z = \pm\pi/3, \pi$ that correspond to Σ_2 , see Remark 6.4.

The term $e^{-2n\varphi_2(z)}$ that appears in the transformation (6.50) for $z \in L_2^\pm \setminus L_1$ is small for z away from Σ_2 , but becomes of order 1 as z approaches Σ_2 . On Σ_2 a certain cancellation takes place, which results in the asymptotic condition being valid uniformly. One can establish this rigorously by looking back at the transformations $Y \mapsto X \mapsto V \mapsto U \mapsto T \mapsto S$ near Σ_2 , but we will not go into that here.

If we do not want to rely on this detailed analysis, then we can only guarantee at this stage that the asymptotic condition in the RH problem 6.16 for S is valid uniformly as $z \rightarrow \infty$ with

$$\arg z \in (-\pi + \varepsilon, -\pi/3 - \varepsilon) \cup (-\pi/3 + \varepsilon, \pi/3 - \varepsilon) \cup (\pi/3 + \varepsilon, \pi - \varepsilon)$$

and that

$$S(z) = O(1)A(z) \quad \text{uniformly as } z \rightarrow \infty. \tag{6.54}$$

The latter condition is a consequence of (6.10) in Remark 6.4 and the transformations leading from X to S .

6.6. Global parametrix

The global parametrix M is a solution of the following model RH problem, which we obtain from the RH problem 6.16 for S by dropping the jump matrices that tend to the identity matrix as $n \rightarrow \infty$.

RH problem 6.18. • M is analytic in $\mathbb{C} \setminus \Gamma_M$ where $\Gamma_M = \Sigma_1 \cup \Sigma_2$,

- $M_+ = M_- J_M$ on Γ_M with

$$J_M(z) = \begin{cases} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & z \in \Sigma_1, \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, & z \in \Sigma_2, \end{cases}$$

- $M(z) = (I + O(1/z))A(z)$ as $z \rightarrow \infty$, where $A(z)$ is given by (6.9).
- $M(z) = O((z - \omega^j x^*)^{-1/4})$ as $z \rightarrow \omega^j x^*$ for $j = 0, 1, 2$.
- $M(z)$ remains bounded as $z \rightarrow 0$.

Note that the product of the jump matrices around 0 is the identity matrix and so we can indeed require that M remains bounded near the origin. Also observe that the asymptotic condition is compatible with the jumps on the unbounded contours, since

$$A_+ = A_- J_M \quad \text{on } \Sigma_2, \tag{6.55}$$

which may be checked directly from the definition (6.9) of $A(z)$.

To construct M we use a meromorphic differential Ω on the Riemann surface \mathcal{R} , which we specify by its poles and residues, see also [15,25,28]. We require that Ω has simple poles at the branch points $\omega^j x^*$, $j = 0, 1, 2$ and at ∞_2 (the point at infinity that is common to the second and third sheets) with

$$\text{Res}(\Omega, \omega^j x^*) = -\frac{1}{2}, \quad \text{Res}(\Omega, \infty_2) = \frac{3}{2}$$

and Ω is holomorphic elsewhere. The residues add up to 0, and therefore the meromorphic differential exists and it is also unique since the genus is zero.

We use ∞_1 as base point for integration of Ω and define

$$u_j(z) = \int_{\infty_1}^z \Omega, \quad z \in \mathcal{R}_j, \quad j = 1, 2, 3, \tag{6.56}$$

where the path of integration is chosen according to the following rules:

- The path for $u_1(z)$ stays on the first sheet.
- The path for $u_2(z)$ starts on the first sheet and passes once through Σ_1 to go to the second sheet and then stays on the second sheet.
- The path for $u_3(z)$ starts on the first sheet, passes once through Σ_1 to go to the second sheet, and then passes once through Σ_2 to go the third sheet and then stays on the third sheet.
- All passages from one sheet to the next go via the $-$ -side on the upper sheet to the $+$ -side on the lower sheet.

Then the functions u_j are well-defined, u_1 is analytic on $\mathbb{C} \setminus \Sigma_1$, u_2 is analytic on $\mathbb{C} \setminus (\Sigma_1 \cup \Sigma_2)$, and u_3 is analytic on $\mathbb{C} \setminus \Sigma_2$. The functions satisfy

$$\begin{aligned} u_{3,+} &= u_{3,-} & u_{1,-} &= u_{2,+} & u_{1,+} &= u_{2,-} \pm \pi i, & \text{on } \Sigma_1, \\ u_{1,+} &= u_{1,-}, & u_{2,-} &= u_{3,+}, & u_{2,+} &= u_{3,-} \pm \pi i, & \text{on } \Sigma_2. \end{aligned}$$

We put

$$v_j = e^{u_j} \tag{6.57}$$

and then we have

$$(v_1 \ v_2 \ v_3)_+ = (v_1 \ v_2 \ v_3)_- J_M \quad \text{on } \Sigma_1 \cup \Sigma_2,$$

Since $v_1(z) = 1 + O(1/z)$, $v_2(z) = O(z^{-3/4})$, $v_3(z) = O(z^{-3/4})$ as $z \rightarrow \infty$, the vector (v_1, v_2, v_3) satisfies the conditions for the first row of M .

The vector space of holomorphic functions on $\mathcal{R} \setminus \{\infty_2\}$ with at most a double pole at ∞_2 has dimension 3. Let $f^{(1)} \equiv 1, f^{(2)}, f^{(3)}$ be a basis of this vector space. We use $f_j^{(i)}$ to denote the restriction of $f^{(i)}$ to the sheet \mathcal{R}_j . Then it is easy to see that

$$B = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 f_1^{(2)} & v_2 f_2^{(2)} & v_3 f_3^{(2)} \\ v_1 f_1^{(3)} & v_2 f_2^{(3)} & v_3 f_3^{(3)} \end{pmatrix}$$

is defined and analytic in $\mathbb{C} \setminus \Gamma_M$ with jump $B_+ = B_- J_M$. Also $B(z) = O(z^{1/4})$ as $z \rightarrow \infty$. It then easily follows that $\det B \equiv \text{const}$ and the constant is non-zero, since the functions $f^{(1)} \equiv 1, f^{(2)}, f^{(3)}$ are independent.

We already noted that $A_+ = A_- J_M$ on Σ_2 . Since A and B have the same jumps on Σ_2 , it follows that BA^{-1} is analytic in $\mathbb{C} \setminus \Sigma_1$, and therefore has a Laurent expansion at infinity. Since both $B(z) = O(z^{1/4})$ and $A(z) = O(z^{1/4})$ as $z \rightarrow \infty$, we have

$$B(z)A^{-1}(z) = C + O(z^{-1}) \quad \text{as } z \rightarrow \infty$$

for some constant matrix C . Since $\det B \equiv \text{const} \neq 0$ and $\det A \equiv 1$, we see that $\det C \neq 0$, and so C is invertible. Then

$$M(z) = C^{-1}B(z)$$

satisfies

$$M(z) = (I + O(z^{-1}))A(z) \quad \text{as } z \rightarrow \infty.$$

It also satisfies the jump condition $M_+ = M_- J_M$, it is bounded at 0 and it has at most fourth root singularities at the branch points. Thus M is the global parametrix we are looking for.

Remark 6.19. From the above construction it follows that

$$M_{1,1}(z) = v_1(z) = e^{\mu_1(z)}, \quad z \in \mathbb{C} \setminus \Sigma_1,$$

which in particular means that $M_{1,1}(z) \neq 0$ for $z \in \mathbb{C} \setminus \Sigma_1$.

6.7. Local parametrices

The local parametrix P is defined in the disks

$$D(\omega^j x^*, \delta) = \{z \in \mathbb{C} \mid |z - \omega^j x^*| < \delta\},$$

where $\delta > 0$ is taken sufficiently small.

RH problem 6.20. • P is continuous on $(\bigcup_j D(\omega^j x^*, \delta)) \setminus \Gamma_S$ and analytic in $(\bigcup_j D(\omega^j x^*, \delta)) \setminus \Gamma_S$,

- $P_+ = P_- J_S$ on $\Gamma_S \cap \bigcup_j D(\omega^j x^*, \delta)$, where J_S is as in (6.51) and (6.52)
- $P(z)$ matches with the global parametrix $M(z)$ in the sense that

$$P(z) = M(z) \left(I + O(n^{-1}) \right) \quad \text{as } n \rightarrow \infty, \tag{6.58}$$

uniformly for $z \in \bigcup_j \partial D(\omega^j x^*, \delta)$.

In the noncritical case $t_0 < t_{0,crit}$ the density of μ_1 vanishes as a square root at $\omega^j x^*$, for $j = 0, 1, 2$, which means that we can build the local parametrix P out of Airy functions in a small disk around each of these endpoints. The construction is done in a standard way and we do not give details here.

6.8. Sixth transformation

Now we make the final transformation. We choose a small disk $D(\omega^j, x^*, \delta)$ around each of the branch points, and define

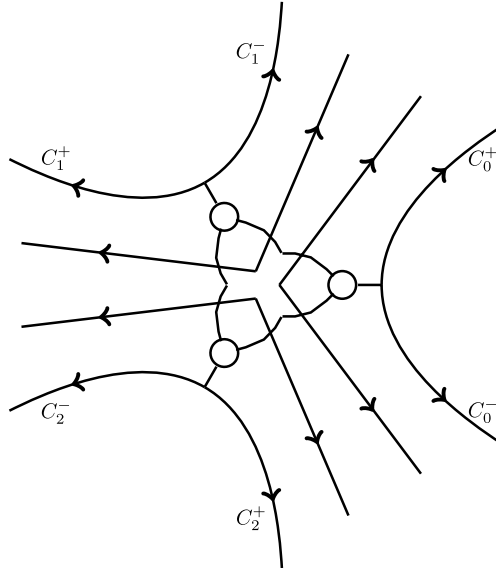


Fig. 8. Contour Γ_R for the RH problem for R .

Definition 6.21. We define

$$R(z) = \begin{cases} S(z)P(z)^{-1} & \text{in the disks around the endpoints } \omega^j x^*, j = 0, 1, 2, \\ S(z)M(z)^{-1} & \text{outside of the disks.} \end{cases} \tag{6.59}$$

Then R is defined and analytic outside of the union of Γ_S with the circles around the branch points. Since the jump matrices of S and M agree on $\Sigma_1 \cup \Sigma_2$, we see that R has an analytic continuation across Σ_2 and across the part of Σ_1 that is outside of the three disks. Similarly, the jump matrices on S and P agree inside the disks, and so R also has analytic continuation inside the three disks. It follows that R is analytic in $\mathbb{C} \setminus \Gamma_R$, where Γ_R is the contour shown in Fig. 8. It consists of the part of $\Sigma_S \setminus (\Sigma_1 \cup \Sigma_2)$ that is outside of the three circles, together with the three circles around the branch points.

The RH problem for R is thus as follows.

RH problem 6.22. • R is analytic in $\mathbb{C} \setminus \Gamma_R$,

- $R_+ = R_- J_R$ on Γ_R where

$$J_R(z) = \begin{cases} M(z)^{-1} P(z) & \text{for } z \text{ on the circles,} \\ M(z)^{-1} J_S(z) M(z) & \text{elsewhere on } \Gamma_R. \end{cases} \tag{6.60}$$

- $R(z) = I + O(z^{-1})$ as $z \rightarrow \infty$.

Remark 6.23. Following Remarks 6.4 and 6.17 we note that the asymptotic condition in the RH problem 6.22 is valid uniformly as $z \rightarrow \infty$ with

$$\arg z \in (-\pi + \varepsilon, -\pi/3 - \varepsilon) \cup (-\pi/3 + \varepsilon, \pi/3 - \varepsilon) \cup (\pi/3 + \varepsilon, \pi - \varepsilon)$$

for any $\varepsilon > 0$. From Remark 6.17 and the transformation (6.59) it also follows that $R(z) = O(1)$ as $z \rightarrow \infty$ uniformly. Since the jump matrix $J_R(z)$ for $z \in \Gamma_R$ tends to the identity matrix

as $z \rightarrow \infty$, see also (6.62) below, it is then easy to show that the asymptotic condition $R(z) = I + O(z^{-1})$ as $z \rightarrow \infty$ is in fact valid uniformly.

All jump matrices J_R in the RH problem 6.22 tend to the identity matrix as $n \rightarrow \infty$. Indeed, because of the matching condition (6.58) we have

$$J_R(z) = I + O(n^{-1}), \quad \text{for } z \in \partial D(\omega^j x^*, \delta) \tag{6.61}$$

while the jump matrices on the remaining parts of Γ_R are exponentially close to

$$J_R(z) = I + O(e^{-cn|z|^3}), \quad \text{elsewhere on } \Gamma_R. \tag{6.62}$$

We conclude two things from (6.61) and (6.62). The first thing is that the RH problem 6.22 has a solution if n is large enough. This indeed follows from (6.61) and (6.62) since for jump matrices J_R close enough to the identity matrix the RH problem 6.22 has a unique solution which can be written down as a Neumann series. Since the transformations

$$Y \mapsto X \mapsto V \mapsto U \mapsto T \mapsto S \mapsto R \tag{6.63}$$

are invertible it then also follows that the solution Y to the RH problem 5.4 uniquely exists. So in particular the (1, 1) entry exists, which is the orthogonal polynomial $P_{n,n}$.

The second thing we obtain from (6.61) and (6.62) is that the solution R of the RH problem 6.22 not only exists but is also close to the identity matrix as $n \rightarrow \infty$. There is an estimate

$$R(z) = I + O\left(\frac{1}{n(1+|z|)}\right) \quad \text{as } n \rightarrow \infty, \tag{6.64}$$

uniformly for $z \in \mathbb{C} \setminus \Gamma_R$. The estimate (6.64) completes the steepest descent analysis of the RH problem.

6.9. Proof of Lemma 6.1

The proof of Lemma 6.1 now follows by unraveling the transformations (6.63) and then use (6.64) to see the effect on the orthogonal polynomial, since

$$P_{n,n}(z) = Y_{1,1}(z),$$

see (5.26).

Proof. From the definitions (6.5), (6.6), (6.11)–(6.13), (6.21), (6.43) and (6.44) we easily see that

$$\begin{aligned} P_{n,n}(z) &= X_{1,1}(z) = V_{1,1}(z) = U_{1,1}(z)e^{ng_1(z)} \\ &= T_{1,1}(z)e^{ng_1(z)}, \quad z \in \mathbb{C} \setminus \Sigma_1. \end{aligned} \tag{6.65}$$

Using (6.65) and (6.50) we find

$$P_{n,n}(z) = S_{1,1}(z)e^{ng_1(z)}, \quad z \in \mathbb{C} \setminus L_1. \tag{6.66}$$

For z outside of the disks $D(\omega^j x^*, \delta)$ we have $S = RM$ by (6.59) and so by (6.64)

$$S_{1,1}(z) = (1 + O(1/n))M_{1,1}(z) + O(1/n)$$

which, since $M_{1,1}(z)$ is analytic with no zeros in $\overline{\mathbb{C}} \setminus \Sigma_1$, see Remark 6.19, can also be written as

$$S_{1,1}(z) = (1 + O(1/n))M_{1,1}(z), \quad z \in \mathbb{C} \setminus \left(L_1 \cup \bigcup_j D(\omega^j x^*, \delta) \right),$$

and the O -term is uniform. Inserting this into (6.66) we obtain (6.1) uniformly for $z \in \mathbb{C} \setminus \left(L_1 \cup \bigcup_j D(\omega^j x^*, \delta) \right)$. The lens L_1 around Σ_1 and the disks $D(\omega^j x^*, \delta)$ around the branch points can be made as small as we like. It follows that (6.1) holds uniformly for z in compact subsets of $\overline{\mathbb{C}} \setminus \Sigma_1$ and the lemma is proved.

6.10. Proofs of Theorems 2.3 and 2.5

Proof. For $t_0, t_3 > 0$ with $t_0 < t_{0,crit}$ we proved that the orthogonal polynomial $P_{n,n}$ exists for large enough n and satisfies (6.1). Since $M_{1,1}(z) \neq 0$ for $z \in \mathbb{C} \setminus \Sigma_1$, the zeros of $P_{n,n}$ accumulate on Σ_1 as $n \rightarrow \infty$, and this proves Theorem 2.3.

From (6.1) and (6.16) it follows that, uniformly for $z \in \mathbb{C} \setminus \Sigma_1$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log |P_{n,n}(z)| = \text{Reg}_1(z) = \int \log |z - s| d\mu_1^*(s).$$

Standard arguments from logarithmic potential theory, see e.g. [29, Theorem III.4.1] then imply that μ_1^* is the weak limit of the normalized zero counting measures of the polynomials $P_{n,n}$ as $n \rightarrow \infty$. By construction, μ_1^* is the first component of the minimizer of the vector equilibrium problem, see also Lemma 4.1, and Theorem 2.5 follows.

Acknowledgments

The first author is supported in part by the National Science Foundation (NSF) Grants DMS-0652005 and DMS-0969254.

The second author is supported in part by FWO-Flanders projects G.0427.09 and G.0641.11 by K.U. Leuven research grant OT/08/33, by the Belgian Interuniversity Attraction Pole P06/02, and by grant MTM2008-06689-C02-01 of the Spanish Ministry of Science and Innovation.

This work was started when both authors were visiting the Mathematical Sciences Research Institute in Berkeley in the fall of 2010 during the program ‘‘Random Matrices, Interacting Particle Systems and Integrable Systems’’. We thank the MSRI for its hospitality and wonderful research conditions.

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