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Contributions of non-extremum critical points to the semi-classical trace formula

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Abstract

We study the semi-classical trace formula at a critical energy level for a h-pseudo-differential operator on $\mathbb{R}^n$ whose principal symbol has a totally degenerate critical point for that energy. We compute the contribution to the trace formula of isolated non-extremum critical points under a condition of "real principal type". The new contribution to the trace formula is valid for all time in a compact subset of $\mathbb{R}$ but the result is modest since we have restrictions on the dimension.

Keywords: Semi-classical analysis; Trace formula; Degenerate oscillatory integrals

1. Introduction

The semi-classical trace formula for a self-adjoint h-pseudo-differential operator P_h, or more generally h-admissible (see [19]), studies the asymptotic behavior, as h tends to 0, of the spectral function

$$\gamma(E,h,\varphi) = \sum_{|\lambda_j(h)-E|\leqslant\varepsilon} \varphi\left(\frac{\lambda_j(h)-E}{h}\right), \tag{1}$$

where the $\lambda_j(h)$ are the eigenvalues of P_h. Here we suppose that the spectrum is discrete in $[E-\varepsilon, E+\varepsilon]$, some sufficient conditions for this will be given below. If p_0

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is the principal symbol of P_h we recall that an energy E is regular when $\nabla p_0(x,\xi)\neq 0$ on the energy surface $\Sigma_E = \{(x,\xi)\in T^*\mathbb{R}^n/p_0(x,\xi)=E\}$ and critical when it is not regular.

It is well known that the asymptotics of (1), as h tends to 0, is closely related to the closed trajectories of the Hamiltonian flow of p on the surface Σ_E, i.e.

$$\lim_{h\to 0}\gamma(E,h,\varphi)\rightleftharpoons\{(t,x,\xi)\in\mathbb{R}\times\Sigma_E/\Phi_t(x,\xi)=(x,\xi)\},$$

where

$$\Phi_t = \exp(tH_{p_0}) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n,$$

and $H_{p_0} = \partial_\xi p_0.\partial_x - \partial_x p_0.\partial_\xi$ is the Hamiltonian vector field attached to p_0.

When E is a regular energy a non-exhaustive list of references concerning this subject is Gutzwiller [11], Balian and Bloch [1] for the physic literature and for a mathematical point of view Helffer and Robert [12], Brummelhuis and Uribe [3], Paul and Uribe [17], and more recently Combescure et al. [8], Petkov and Popov [18], and Charbonnel and Popov [7].

When one drops the assumption that E be a regular value, the behavior of (1) will depend on the nature of the singularities of p on Σ_E which can be complicated. The semi-classical trace formula for a non-degenerate critical energy, that is such that the critical-set $\mathfrak{C}(p_0) = \{(x,\xi)\in T^*\mathbb{R}^n/dp_0(x,\xi)=0\}$ is a compact C^∞ manifold with a Hessian d^2p_0 transversely non-degenerate along this manifold has been studied first by Brummelhuis et al. [2]. They studied this question for quite general operators but for some "small times", that is they have supposed that the support of $\hat{\varphi}$ is contained in such a small neighborhood of the origin that the only period of the linearized flow in $\mathrm{supp}(\hat{\varphi})$ is 0. Later, Khuat-Duy [15,16] has obtained the contributions of the non-zero periods of the linearized flow with the assumption that $\mathrm{supp}(\hat{\varphi})$ is compact, but for Schrödinger operators with symbol $\xi^2+V(x)$ and a non-degenerate potential V. Our contribution to this subject was to compute the contributions of the non-zero periods of the linearized flow for some more general operators, always with $\hat{\varphi}$ of compact support and under some geometrical assumptions on the flow (see [4] or [5]). Finally, in [6] we have obtained the contributions to the semi-classical trace formula of totally degenerate extremum and the objective of this article is to obtain a generalization when one drop the extremum condition.

Basically, the asymptotics of (1) can be expressed in terms of oscillatory integrals whose phases are related to the flow of p_0 on Σ_E. When (x_0,ξ_0) is a critical point of p_0, it is well known that the relation $\mathrm{Ker}(d_{x,\xi}\Phi_t(x_0,\xi_0)-\mathrm{Id})\neq\{0\}$ leads to the study of degenerate oscillatory integrals. Here we examine the case of a totally degenerate energy, that is such that the Hessian matrix at our critical point is zero. Hence, the linearized flow for such a critical point satisfies $d_{x,\xi}\Phi_t(x_0,\xi_0)=\mathrm{Id}$, for all $t\in\mathbb{R}$ and the oscillatory integrals we have to consider are totally degenerate.

The core of the proof lies in establishing suitable normal forms for our phase functions and in a generalization of the stationary phase formula for these normal forms. The construction proposed for the normal forms is independent of the dimension but the asymptotic expansion of the related oscillatory integrals depends on the dimension and on the order of the singularity at the critical point. This explains why the main result is stated with a restriction on the dimension.

2. Hypotheses and main result

Let $P_h = Op_h^{\mathrm{w}}(p(x,\xi,h))$ be a h-pseudo-differential operator, obtained by Weyl quantization, in the class of h-admissible operators with symbol $p(x,\xi,h) \sim \sum h^j p_j(x,\xi)$. This means that there exists sequences $p_j \in \Sigma_0^m(T^*\mathbb{R}^n)$ and $R_N(h)$ such that

$$P_h = \sum_{j<N} h^j p_j^{\mathrm{w}}(x, hD_x) + h^N R_N(h), \forall N \in \mathbb{N},$$

where $R_N(h)$ is a bounded family of operators on $L^2(\mathbb{R}^n)$, for $h \leqslant h_0$, and

$$\Sigma_0^m(T^*\mathbb{R}^n) = \{a : T^*\mathbb{R}^n \to \mathbb{C}, \sup|\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| < C_{\alpha,\beta} m(x,\xi), \forall \alpha, \beta \in \mathbb{N}^n\},$$

where m is a tempered weight on $T^*\mathbb{R}^n$. For a detailed exposition on h-admissible operators we refer to the book of Robert [19]. In particular, $p_0(x,\xi)$ is the principal symbol of P_h and $p_1(x,\xi)$ the sub-principal symbol. Let be $\Phi_t = \exp(tH_{p_0}) : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, the Hamiltonian flow of $H_{p_0} = \partial_\xi p_0.\partial_x - \partial_x p_0.\partial_\xi$.

We study here the asymptotics of the spectral function:

$$\gamma(E_{\mathrm{c}}, h) = \sum_{\lambda_j(h) \in [E_{\mathrm{c}}-\varepsilon, E_{\mathrm{c}}+\varepsilon]} \varphi\left(\frac{\lambda_j(h) - E_{\mathrm{c}}}{h}\right), \tag{2}$$

under hypotheses (H_1)–(H_4) given below.

(H_1) *The symbol of P_h is real and there exists $\varepsilon_0 > 0$ such that the set $p_0^{-1}([E_{\mathrm{c}} - \varepsilon_0, E_{\mathrm{c}} + \varepsilon_0])$ is compact in $T^*\mathbb{R}^n$.*

Remark 1. By Theorem 3.13 of [19] the spectrum $\sigma(P_h) \cap [E_{\mathrm{c}} - \varepsilon, E_{\mathrm{c}} + \varepsilon]$ is discrete and consists in a sequence $\lambda_1(h) \leqslant \lambda_2(h) \leqslant \cdots \leqslant \lambda_j(h)$ of eigenvalues of finite multiplicities, if ε and h are small enough.

To simplify notations we write $z = (x,\xi)$ for any point of the phase space.

(H_2) *On $\Sigma_{E_{\mathrm{c}}} = p_0^{-1}(\{E_{\mathrm{c}}\})$, p_0 has a unique critical point $z_0 = (x_0, \xi_0)$ and near z_0:*

$$p_0(z) = E_{\mathrm{c}} + \sum_{j=k}^{N} \mathfrak{p}_j(z) + \mathcal{O}(||(z - z_0)||^{N+1}), \;\; k > 2,$$

where the functions $\mathfrak{p}_j$ are homogeneous of degree j in $z - z_0$.

(H_3) *We have $\hat{\varphi} \in C_0^\infty(\mathbb{R})$.*

Since we are interested in the contribution of the fixed point z_0, to understand the new phenomenon it is suffices to study

$$\gamma_{z_0}(E_c, h) = \frac{1}{2\pi} \operatorname{Tr} \int_{\mathbb{R}} e^{i\frac{tE_c}{h}} \hat{\varphi}(t) \psi^w(x, hD_x) \exp\left(-\frac{i}{h} tP_h\right) \Theta(P_h)\, dt. \tag{3}$$

Here Θ is a function of localization near the critical energy surface Σ_{E_c} and $\psi \in C_0^\infty(T^*\mathbb{R}^n)$ has an appropriate support near z_0. Rigorous justifications are given in Section 3 for the introduction of $\Theta(P_h)$ and in Section 4 for $\psi^w(x, hD_x)$.

In [6] it was proven that:

Theorem 2. *Under hypotheses* (H_1)–(H_3) *and if* z_0 *is a local extremum of the principal symbol* p_0 *we have*

$$\gamma_{z_0}(E_c, h) \sim h^{\frac{2n}{k}-n} \left(\sum_{j=0}^{N} \Lambda_{j,k}(\varphi) h^{\frac{j}{k}} + \mathcal{O}(h^{\frac{N+1}{k}}) \right) \text{ as } h \to 0,$$

where the $\Lambda_{j,k}$ *are some distributions and the leading coefficient is given by*

$$\Lambda_{0,k}(\varphi) = \frac{1}{k} \left\langle \varphi(t + p_1(z_0)), t_{z_0}^{\frac{2n-k}{k}} \right\rangle \frac{1}{(2\pi)^n} \int_{\mathbb{S}^{2n-1}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta, \tag{4}$$

with $t_{z_0} = \max(t, 0)$ *if* z_0 *is a minimum and* $t_{z_0} = \max(-t, 0)$ *for a maximum.*

To obtain a generalization of Theorem 2, when the critical point z_0 is not a local extremum, we consider the classical hypothesis:

(H_4) *We have* $\nabla \mathfrak{p}_k \neq 0$ *on the set* $C_{\mathfrak{p}_k} = \{\theta \in \mathbb{S}^{2n-1} / \mathfrak{p}_k(\theta) = 0\}$.

Remark 3. We would like to emphasize that, contrary to the case of a local extremum, the critical point z_0 is not necessarily isolated on Σ_{E_c}. This imposes to study the classical dynamic in a micro-local neighborhood of z_0. Moreover, by homogeneity (H_4) implies that $\nabla \mathfrak{p}_k \neq 0$ on the cone $\{(x, \xi) \neq 0 / \mathfrak{p}_k(x, \xi) = 0\}$. This allows to define the Liouville measure on the set $C_{\mathfrak{p}_k}$.

Then, the new contribution to the trace formula is given by:

Theorem 4. *Under hypotheses* (H_1)–(H_4) *and if* $k > 2n$ *we have*

$$\gamma_{z_0}(E_c, h) \sim h^{\frac{2n}{k}-n} \Lambda_{0,k}(\varphi) + \mathcal{O}(h^{\frac{2n+1}{k}-n} \log^2(h)) \text{ as } h \to 0,$$

where the leading coefficient is given by

$$\Lambda_{0,k}(\varphi) = \frac{1}{k}\Bigg(\left\langle |t|_{+}^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \frac{1}{(2\pi)^n}\int_{\mathbb{S}^{2n-1}\cap\{\mathfrak{p}_k\geqslant 0\}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta$$
$$+\left\langle |t|_{-}^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \frac{1}{(2\pi)^n}\int_{\mathbb{S}^{2n-1}\cap\{\mathfrak{p}_k\leqslant 0\}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta\Bigg). \tag{5}$$

If $k=2n$ *a similar result is*

$$\gamma_{z_0}(E_c,h) \sim h^{1-n}\log(h)\Lambda(\varphi) + \mathcal{O}(h^{1-n}) \text{ as } h\to 0,$$

with leading coefficient

$$\Lambda(\varphi) = \frac{1}{(2\pi)^n}\hat{\varphi}(0)\mathrm{LVol}(C_{\mathfrak{p}_k}), \tag{6}$$

where LVol *is the Liouville measure attached to* $\mathfrak{p}_k$ *restricted to the sphere*:

$$\mathrm{LVol}(C_{\mathfrak{p}_k}) = \int_{C_{\mathfrak{p}_k}} dL_{\mathfrak{p}_k}(\theta),$$

with $dL_{\mathfrak{p}_k}(\theta)\wedge d_\theta\mathfrak{p}_k(\theta) = d\theta$ *on* $C_{\mathfrak{p}_k}$.

Remark 5. In Eq. (5) and under the assumption on the dimension both terms are well defined since for the first term $|t|^{\frac{2n}{k}-1}$ is locally integrable and for the second term the integral is convergent.

3. Oscillatory representation

Let be $\varphi\in\mathscr{S}(\mathbb{R})$ with $\hat{\varphi}\in C_0^\infty(\mathbb{R})$, we recall that

$$\gamma(E_c,h) = \sum_{\lambda_j(h)\in I_\varepsilon} \varphi\left(\frac{\lambda_j(h)-E_c}{h}\right), I_\varepsilon = [E_c-\varepsilon, E_c+\varepsilon],$$

with $p_0^{-1}(I_{\varepsilon_0})$ compact in $T^*\mathbb{R}^n$. By Proposition 3.13 of [19] the spectrum of P_h is discrete in I_ε for $h>0$ small enough and $\varepsilon<\varepsilon_0$. Now, we localize near the critical energy E_c with a cut-off function $\Theta\in C_0^\infty(]E_c-\varepsilon, E_c+\varepsilon[)$, such that $\Theta=1$ near E_c and $0\leqslant\Theta\leqslant 1$ on $\mathbb{R}$. The associated decomposition is:

$$\gamma(E_c,h) = \gamma_1(E_c,h)+\gamma_2(E_c,h)$$

with

$$\gamma_1(E_c,h) = \sum_{\lambda_j(h)\in I_\varepsilon} (1-\Theta)(\lambda_j(h))\varphi\left(\frac{\lambda_j(h)-E_c}{h}\right), \tag{7}$$

$$\gamma_2(E_c,h) = \sum_{\lambda_j(h)\in I_\varepsilon} \Theta(\lambda_j(h))\varphi\left(\frac{\lambda_j(h)-E_c}{h}\right). \tag{8}$$

The asymptotic behavior of $\gamma_1(E_c,h)$ is classical and is given by:

Lemma 6. *$\gamma_1(E_c, h) = \mathcal{O}(h^\infty)$ as $h \to 0$.*

For a proof see e.g. [6].

Consequently, for the study of $\gamma(E_c, h)$ modulo $\mathcal{O}(h^\infty)$, we have only to consider the quantity $\gamma_2(E_c, h)$. By inversion of the Fourier transform we obtain the identity

$$\Theta(P_h)\varphi\left(\frac{P_h - E_c}{h}\right) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\frac{tE_c}{h}}\hat{\varphi}(t)\exp\left(-\frac{i}{h}tP_h\right)\Theta(P_h)\,dt.$$

Since the trace of the left-hand side is exactly $\gamma_2(E_c, h)$, we have

$$\gamma_2(E_c, h) = \frac{1}{2\pi}\mathrm{Tr}\int_{\mathbb{R}} e^{i\frac{tE_c}{h}}\hat{\varphi}(t)\exp\left(-\frac{i}{h}tP_h\right)\Theta(P_h)\,dt, \tag{9}$$

and with Lemma 6 this gives

$$\gamma(E_c, h) = \frac{1}{2\pi}\mathrm{Tr}\int_{\mathbb{R}} e^{i\frac{tE_c}{h}}\hat{\varphi}(t)\exp\left(-\frac{i}{h}tP_h\right)\Theta(P_h)\,dt + \mathcal{O}(h^\infty).$$

Let be $U_h(t) = \exp(-\frac{it}{h}P_h)$, the evolution operator. For each integer N we can approximate $U_h(t)\Theta(P_h)$, modulo $\mathcal{O}(h^N)$, by a Fourier integral-operator, or FIO, depending on a parameter h. Let Λ be the Lagrangian manifold associated to the flow of p_0, i.e.

$$\Lambda = \{(t, \tau, x, \xi, y, \eta) \in T^*\mathbb{R} \times T^*\mathbb{R}^n \times T^*\mathbb{R}^n : \tau = p(x, \xi), (x, \xi) = \Phi_t(y, \eta)\}.$$

Theorem 7. *The operator $U_h(t)\Theta(P_h)$ is h-FIO associated to Λ, there exist $U^{(N)}_{\Theta,h}(t)$ with integral kernel in $I(\mathbb{R}^{2n+1}, \Lambda)$ and $R^{(N)}_h(t)$ bounded, with a L^2-norm uniformly bounded for $0 < h \leqslant 1$ and t in a compact subset of $\mathbb{R}$, such that $U_h(t)\Theta(P_h) = U^{(N)}_{\Theta,h}(t) + h^N R^{(N)}_h(t)$.*

We refer to Duistermaat [9] for a proof of this theorem.

Remark 8. By a theorem of Helffer and Robert, see e.g. [19, Theorem 3.11 and Remark 3.14], $\Theta(P_h)$ is an h-admissible operator with a symbol supported in $p_0^{-1}(I_\varepsilon)$. This allows us to consider only oscillatory-integrals with compact support.

For the control of the remainder, associated to $R^{(N)}_h(t)$, we use:

Corollary 9. *Let be $\Theta_1 \in C_0^\infty(\mathbb{R})$ such that $\Theta_1 = 1$ on $\mathrm{supp}(\Theta)$ and $\mathrm{supp}(\Theta_1) \subset I_\varepsilon$, then $\forall N \in \mathbb{N}$:*

$$\mathrm{Tr}\left(\Theta(P_h)\varphi\left(\frac{P_h - E_c}{h}\right)\right) = \frac{1}{2\pi}\mathrm{Tr}\int_{\mathbb{R}} \hat{\varphi}(t)e^{\frac{i}{h}tE_c}U^{(N)}_{\Theta,h}(t)\Theta_1(P_h)\,dt + \mathcal{O}(h^N).$$

For a proof, see e.g. [6].

If $(x_0, \xi_0) \in \Lambda$ and if $\varphi = \varphi(x, \theta) \in C^\infty(\mathbb{R}^k \times \mathbb{R}^N)$ parameterizes Λ in a sufficiently small neighborhood U of (x_0, ξ_0) then for each $u_h \in I(\mathbb{R}^k, \Lambda)$ and $\chi \in C_0^\infty(T^*\mathbb{R}^k)$, $\mathrm{supp}(\chi) \subset U$, there exists a sequence of amplitudes $a_j = a_j(x, \theta) \in C_0^\infty(\mathbb{R}^k \times \mathbb{R}^N)$ such that for all $L \in \mathbb{N}$:

$$\chi^{\mathrm{w}}(x, hD_x)u_h = \sum_{-d \leqslant j < L} h^j I(a_j e^{\frac{i}{h}\varphi}) + \mathcal{O}(h^L). \tag{10}$$

We will use this remark with the following result of Hörmander [14, Tome 4, Proposition 25.3.3]. Let be $(T, \tau, x_0, \xi_0, y_0, -\eta_0) \in \Lambda_{\mathrm{flow}}$, $\eta_0 \neq 0$, then near this point there exists, after perhaps a change of local coordinates in y near y_0, a function $S(t, x, \eta)$ such that

$$\phi(t, x, y, \eta) = S(t, x, \eta) - \langle y, \eta \rangle, \tag{11}$$

parameterizes Λ_{flow}. In particular this implies that

$$\{(t, \partial_t S(t, x, \eta), x, \partial_x S(t, x, \eta), \partial_\eta S(t, x, \eta), -\eta)\} \subset \Lambda_{\mathrm{flow}},$$

and that the function S is a generating function of the flow, i.e.

$$\Phi_t(\partial_\eta S(t, x, \eta), \eta) = (x, \partial_x S(t, x, \eta)). \tag{12}$$

Moreover, S satisfies the Hamilton–Jacobi equation

$$\begin{cases} \partial_t S(t, x, \eta) + p_0(x, \partial_x S(t, x, \eta)) = 0, \\ S(0, x, \xi) = \langle x, \xi \rangle. \end{cases}$$

Now, we apply this result with $(x_0, \xi_0) = (y_0, \eta_0)$, our unique fixed point of the flow on the energy surface Σ_{E_c}.

Remark 10. If $\xi_0 = 0$ we can replace the operator P_h by $e^{\frac{i}{h}\langle x, \xi_1 \rangle} P_h e^{-\frac{i}{h}\langle x, \xi_1 \rangle}$ with $\xi_1 \neq 0$. This will not change the spectrum since this new operator has the symbol $p(x, \xi - \xi_1, h)$ and the critical point is now (x_0, ξ_1) with $\xi_1 \neq 0$.

Consequently, the localized trace $\gamma_2(E_c, h)$, defined in Eq. (9), can be written for all $N \in \mathbb{N}$ and modulo $\mathcal{O}(h^N)$ as

$$\gamma_2(E_c, h) = \sum_{j < N} (2\pi h)^{-d+j} \int_{\mathbb{R} \times \mathbb{R}^{2n}} e^{\frac{i}{h}(S(t,x,\xi) - \langle x, \xi \rangle + tE_c)} a_j(t, x, \xi) \hat{\varphi}(t)\, dt\, dx\, d\xi. \tag{13}$$

To obtain the right power $-d$ of h occurring here we apply results of Duistermaat [9] (following here Hörmander for the FIO, see [13, Tome 4], for example) concerning the order. An h-pseudo-differential operator obtained by Weyl quantization

$$(2\pi h)^{-\frac{N}{2}} \int_{\mathbb{R}^N} a\left(\frac{x+y}{2}, \xi\right) e^{\frac{i}{h}\langle x-y, \xi\rangle}\, d\xi$$

is of order 0 w.r.t. $1/h$. Now since the order of $U_h(t)\Theta(P_h)$ is $-\frac{1}{4}$, we find that

$$\psi^{\mathrm{w}}(x, hD_x)U_h(t)\Theta(P_h) \sim \sum_{j<N} (2\pi h)^{-n+j} \int_{\mathbb{R}^n} a_j(t, x, y, \eta) e^{\frac{i}{h}(S(t,x,\eta)-\langle y,\eta\rangle)}\, dy. \tag{14}$$

Multiplying Eq. (14) by $\hat{\varphi}(t)e^{\frac{i}{h}tE_{\mathrm{c}}}$ and passing to the trace we find Eq. (13) with $d = n$ and where we write again $a_j(t, x, \eta)$ for $a_j(t, x, x, \eta)$.

To each element u_h of $I(\mathbb{R}^k, \Lambda)$ we can associate a principal symbol $e^{\frac{i}{h}S}\sigma_{\mathrm{princ}}(u_h)$, where S is a function on Λ such that $\xi\, dx = dS$ on Λ. In fact, if $u_h = I(ae^{\frac{i}{h}\varphi})$ then we have $S = S_\varphi = \varphi \circ i_\varphi^{-1}$ and $\sigma_{\mathrm{princ}}(u_h)$ is a section of $|\Lambda|^{\frac{1}{2}} \otimes M(\Lambda)$, where $M(\Lambda)$ is the Maslov vector-bundle of Λ and $|\Lambda|^{\frac{1}{2}}$ the bundle of half-densities on Λ. The half-density of the propagator $U_h(t)$ can be easily expressed in the global coordinates (t, y, η) on Λ_{flow}. If p_1 is the sub-principal symbol of P_h, then this half-density is given by

$$\exp\left(i \int_0^t p_1(\Phi_s(y, -\eta))\, ds\right) |\, dt\, dy\, d\eta|^{\frac{1}{2}}. \tag{15}$$

This expression is related to the resolution of the first transport equation for the propagator, for a proof we refer to Duistermaat and Hörmander [10].

4. Classical dynamic near the equilibrium

A critical points of the phase function of (13) satisfies the equations

$$\begin{cases} E_{\mathrm{c}} = -\partial_t S(t, x, \xi), \\ x = \partial_\xi S(t, x, \xi), \\ \xi = \partial_x S(t, x, \xi), \end{cases} \Leftrightarrow \begin{cases} p_0(x, \xi) = E_{\mathrm{c}}, \\ \Phi_t(x, \xi) = (x, \xi), \end{cases}$$

where the right-hand side defines a closed trajectory of the flow inside $\Sigma_{E_{\mathrm{c}}}$. Since we are interested in the contribution of the critical point, we choose a function $\psi \in C_0^\infty(T^*\mathbb{R}^n)$, with $\psi = 1$ near z_0, hence

$$\begin{aligned} \gamma_2(E_{\mathrm{c}}, h) = & \frac{1}{2\pi} \mathrm{Tr} \int_{\mathbb{R}} e^{i\frac{tE_{\mathrm{c}}}{h}} \hat{\varphi}(t) \psi^{\mathrm{w}}(x, hD_x) \exp\left(-\frac{i}{h} tP_h\right) \Theta(P_h)\, dt \\ & + \frac{1}{2\pi} \mathrm{Tr} \int_{\mathbb{R}} e^{i\frac{tE_{\mathrm{c}}}{h}} \hat{\varphi}(t)(1 - \psi^{\mathrm{w}}(x, hD_x)) \exp\left(-\frac{i}{h} tP_h\right) \Theta(P_h)\, dt. \end{aligned}$$

Under the additional hypothesis of having a clean flow, the asymptotics of the second term is given by the semi-classical trace formula on a regular level. We also observe that the contribution of the first term is micro-local. Hence this allows to introduce local coordinates near z_0. To separate the contribution of z_0 from other closed trajectories we use the following result on the classical dynamic.

Lemma 11. *For all $T>0$ there exists a neighborhood U_T of the critical point such that $\Phi_t(z)\neq z$ for all $z\in U_T\backslash\{z_0\}$ and for all $t\in]\text{-}T,0[\cup]0,T[$.*

Proof. Since z_0 is a degenerate critical point we have $dH_{p_0}(z_0)=0$. Hence, for all $\varepsilon>0$ we can find a neighborhood U of z_0 such that

$$||dH_{p_0}(z)||\leqslant\varepsilon, \quad \forall z\in U.$$

By a theorem of Yorke [21] we obtain that any closed trajectory in U has a period T_0 that satisfies $T_0\geqslant 2\pi\varepsilon^{-1}$. Thus for any T we can choose ε_T, and then U_T, such that $T_0>T$. □

With $\mathrm{supp}(\hat{\varphi})$ compact we can choose ψ such that Lemma 11 holds on $\mathrm{supp}(\psi)$ for all $t\in\mathrm{supp}(\hat{\varphi})$. Hence, on the support of ψ there is two contributions:

(1) Points $(t,x,\xi)=(0,x,\xi)$ for $(x,\xi)\in\Sigma_{E_c}$.
(2) Points $(t,x,\xi)=(t,z_0)$ for $t\in\mathrm{supp}(\hat{\varphi})$.

The first contribution is non-singular for $(x,\xi)\neq z_0$ and can be treated, again, by the regular trace formula. Now, we restrict our attention to the second contribution and, since z_0 is totally degenerate, we obtain

$$d\Phi_t(z_0)=\exp(0)=\mathrm{Id},\ \forall t. \tag{16}$$

The next homogeneous components of the flow are given, with (H_2), by

$$d^j\Phi_t(z_0)=0,\forall t,\forall j\in\{2,\dots,k-2\}. \tag{17}$$

To obtain the next non-zero term of the Taylor expansion of the flow, we will use the following technical result:

Lemma 12. *Let be z_0 an equilibrium of the C^∞ vector field X and Φ_t the flow of X. Then for all $m\in\mathbb{N}^*$, there exists a polynomial map P_m, vector valued and of degree at most m, such that*

$$d^m\Phi_t(z_0)(z^m)=d\Phi_t(z_0)\int_0^t d\Phi_{-s}(z_0)P_m(d\Phi_s(z_0)(z),\dots,d^{m-1}\Phi_s(z_0)(z^{m-1}))\,ds.$$

For a proof we refer to [4] or [5].

Since $d\Phi_t(z_0) = \mathrm{Id}$, for all t, the first non-zero term of the Taylor expansion of the flow is given by

$$d^{k-1}\Phi_t(z_0)(z^{k-1}) = \int_0^t d^{k-1}H_{p_0}(z_0)(z^{k-1})\,ds = td^{k-1}H_{\mathfrak{p}_k}(z_0)(z^{k-1}), \tag{18}$$

where the last identity is obtained using that $d^{k-1}H_{p_0}(z_0) = d^{k-1}H_{\mathfrak{p}_k}(z_0)$. Moreover, with $d^2p(z_0) = 0$, for the next term Lemma 12 gives

$$d^k\Phi_t(z_0)(z^k) = \int_0^t d^kH_{p_0}(z_0)(z^k)\,ds = td^kH_{\mathfrak{p}_{k+1}}(z_0)(z^k).$$

Remark 13. For higher order derivatives $d^j\Phi_t(z_0)$, with $j>k$, there is two different kind of terms, namely

$$\int_0^t d^jH_{p_0}(z_0)(z^j)\,ds = td^jH_{p_0}(z_0)(z^j) = \mathcal{O}(||z||^j),$$

and terms involving powers of t, for example we have:

$$\int_0^t d^{j+2-k}H_{p_0}(z_0)(z^{j+1-k}, d^{k-1}\Phi_t(z_0)(z^{k-1}))$$
$$= \frac{t^2}{2}d^{j+2-k}H_{p_0}(z_0)(z^{j+1-k}, d^{k-1}H_{\mathfrak{p}_k}(z_0)(z^{k-1})).$$

This term is simultaneously $\mathcal{O}(t^2)$ and $\mathcal{O}(||z||^j)$ near $(0, z_0)$. A similar result holds for other terms, which are $\mathcal{O}(t^d)$ for $d\geqslant 2$, by an easy recurrence.

Lemma 14. *Near z_0, here supposed to be 0 to simplify, we have*

$$S(t,x,\xi) - \langle x,\xi\rangle + tE_c = -t(\mathfrak{p}_k(x,\xi) + R_{k+1}(x,\xi) + tG_{k+1}(t,x,\xi)), \tag{19}$$

where $R_{k+1}(x,\xi) = \mathcal{O}(||(x,\xi)||^{k+1})$ and $G_{k+1}(t,x,\xi) = \mathcal{O}(||(x,\xi)||^{k+1})$, uniformly with respect to t.

Proof. By Taylor we obtain

$$\Phi_t(x,\xi) = (x,\xi) + \frac{1}{(k-1)!}d^{k-1}\Phi_t(0)(z^{k-1}) + \mathcal{O}(||z||^k).$$

We search our local generating function as

$$S(t,x,\xi) = -tE_c + \langle x,\xi\rangle + \sum_{j=3}^{N} S(t,x,\xi) + \mathcal{O}(||(x,\xi)||^{N+1}),$$

where the S_j are time dependant and homogeneous of degree j w.r.t. (x,ξ). With the implicit relation $\Phi_t(\partial_\xi S(t,x,\xi),\xi) = (x, \partial_x S(t,x,\xi))$, we have

$$S(t,x,\xi) = -tE_c + \langle x,\xi\rangle + S_k(t,x,\xi) + \mathcal{O}(||(x,\xi)||^{k+1}),$$

where S_k is homogeneous of degree k w.r.t. (x, ξ). If J is the matrix of the usual symplectic form, comparing terms of same degree gives

$$J\nabla S_k(t, x, \xi) = \frac{1}{(k-1)!} d^{k-1}\Phi_t(0)((x, \xi)^{k-1}).$$

By homogeneity and with Eq. (18) we obtain

$$S_k(t, x, \xi) = \frac{1}{k!}\langle (x, \xi), tJd^{k-1}H_{\mathfrak{p}_k}(x, \xi)^{k-1}\rangle = -t\mathfrak{p}_k(x, \xi),$$

$$S(t, x, \xi) = -tE_{\mathrm{c}} + \langle x, \xi\rangle - t\mathfrak{p}_k(x, \xi) + \mathcal{O}(||(x, \xi)||^{k+1}).$$

As concern the remainder, we first observe that $S(0, x, \xi) = \langle x, \xi\rangle$. Hence, we can write

$$S(t, x, \xi) - \langle x, \xi\rangle = tF(t, x, \xi),$$

with F smooth in a neighborhood of $(x, \xi) = 0$. Now, the Hamilton–Jacobi equation imposes that $F(0, x, \xi) = -p_0(x, \xi)$ and we obtain

$$R_{k+1}(x, \xi) = p_0(x, \xi) - E_{\mathrm{c}} - \mathfrak{p}_k(x, \xi) = \mathcal{O}(||(x, \xi)||^{k+1}).$$

Finally, the time-dependant remainder can be written as

$$S(t, x, \xi) - S(0, x, \xi) - t\partial_t S(0, x, \xi) = \mathcal{O}(t^2),$$

since by construction this term is of order $\mathcal{O}(||(x, \xi)||^{k+1})$ we get the desired result when t is in a compact subset of $\mathbb{R}$. $\square$

5. Normal forms of the phase function

Since the contribution we study is local, we can work with some coordinates and identify locally $T^*\mathbb{R}^n$ with $\mathbb{R}^{2n}$ near the critical point. We define

$$\Psi(t, z) = \Psi(t, x, \xi) = S(t, x, \xi) - \langle x, \xi\rangle + tE_{\mathrm{c}}, z = (x, \xi) \in \mathbb{R}^{2n}. \tag{20}$$

Lemma 15. *If P_h satisfies conditions (H_2) and (H_4) then, in a neighborhood of $(t, z) = (0, z_0)$, there exists local coordinates χ such that*

$$\Psi(t, z) \simeq -\chi_0\chi_1^k, \text{ in all directions where } p_k(\theta) > 0,$$

$$\Psi(t, z) \simeq +\chi_0\chi_1^k, \text{ in all directions where } p_k(\theta) < 0,$$

$$\Psi(t, z) \simeq \chi_0\chi_1^k\chi_2, \text{ near any point where } p_k(\theta) = 0.$$

Proof. We can here assume that z_0 is the origin and we use polar coordinates $z = (r, \theta)$, $\theta \in \mathbb{S}^{2n-1}(\mathbb{R})$. With Lemma 14, near the critical point we have

$$\Psi(t,z) \simeq -tr^k(\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t, r\theta)),$$

where $\mathfrak{p}_k(\theta)$ is the restriction of $\mathfrak{p}_k$ on $\mathbb{S}^{2n-1}$.

If $\mathfrak{p}_k(\theta_0) \neq 0$ we define new coordinates

$$(\chi_0, \chi_2, \dots, \chi_{2n})(t, r, \theta) = (t, \theta_1, \dots, \theta_{2n-1}),$$

$$\chi_1(t, r, \theta) = r|\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t, r\theta)|^{\frac{1}{k}}.$$

In these coordinates the phase becomes $-\chi_0\chi_1^k$ if $\mathfrak{p}_k(\theta_0)$ is positive (resp. $\chi_0\chi_1^k$ for a negative value). Near θ_0, we have

$$\frac{\partial \chi_1}{\partial r}(t, 0, \theta) = |\mathfrak{p}_k(\theta)|^{\frac{1}{k}} \neq 0, \forall t,$$

hence, the corresponding Jacobian satisfies $|J\chi|(t, 0, \theta) = |\mathfrak{p}_k(\theta)|^{\frac{1}{k}} \neq 0$.

Now, let θ_0 be such that $\mathfrak{p}_k(\theta_0) = 0$. Up to a permutation, we can suppose that $\partial_{\theta_1}\mathfrak{p}_k(\theta_0) \neq 0$. We choose here the new coordinates

$$(\chi_0, \chi_1, \chi_3, \dots, \chi_{2n})(t, r, \theta) = (t, r, \theta_2, \dots, \theta_{2n-1}),$$

$$\chi_2(t, r, \theta) = \mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t, r\theta).$$

Since we have $|J\chi|(t, 0, \theta_0) = |\partial_{\theta_1}\mathfrak{p}_k(\theta_0)| \neq 0$, lemma follows. $\square$

In order to use these normal forms we introduce an adapted partition of unity on $\mathbb{S}^{2n-1}$. We choose functions $\Omega_j(\theta)$, with compact supports, such that

$$\{\theta \in \mathbb{S}^{2n-1} / \mathfrak{p}_k(\theta) = 0\} \subset \bigcup_j \operatorname{supp}(\Omega_j),$$

so that normal forms of Lemma 15 exist inside $\operatorname{supp}(\Omega_i)$. Since $\mathbb{S}^{2n-1}$ is compact this set of functions can be chosen finite and we obtain a partition of unity with $\Omega_0 = 1 - \sum \Omega_i$. The support of Ω_0 might be not connected and we define Ω_0^+, with $\mathfrak{p}_k(\theta) > 0$ on $\operatorname{supp}(\Omega_0^+)$, and similarly we define Ω_0^- where $\mathfrak{p}_k < 0$, so that $\Omega_0 = \Omega_0^+ + \Omega_0^-$.

If we accordingly split up our oscillatory-integral we obtain

$$\begin{aligned} I_+(\lambda) &= \int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{2n-1}} e^{\frac{i}{h}\Psi(t,r,\theta)} \Omega_0^+(\theta) a(t, r\theta) r^{2n-1}\, dt\, dr\, d\theta \\ &= \int_{\mathbb{R}\times\mathbb{R}_+} e^{-\frac{i}{h}\chi_0\chi_1^k} A_0^+(\chi_0, \chi_1)\, d\chi_0\, d\chi_1, \end{aligned}$$

for the directions where $\mathfrak{p}_k(\theta) > 0$ and

$$\begin{aligned} I_-(\lambda) &= \int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{2n-1}} e^{\frac{i}{h}\Psi(t,r,\theta)}\Omega_0^-(\theta)a(t,r\theta)r^{2n-1}\,dt\,dr\,d\theta \\ &= \int_{\mathbb{R}\times\mathbb{R}_+} e^{\frac{i}{h}\chi_0\chi_1^k}A_0^-(\chi_0,\chi_1)\,d\chi_0\,d\chi_1, \end{aligned}$$

for the directions where $\mathfrak{p}_k(\theta) < 0$. Similarly, the contribution of the neighborhood of the set $\{\mathfrak{p}_k(\theta) = 0\}$ is given by

$$\begin{aligned} I_j(\lambda) &= \int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{2n-1}} e^{\frac{i}{h}\Psi(t,r,\theta)}\Omega_j(\theta)a(t,r\theta)r^{2n-1}\,dt\,dr\,d\theta \\ &= \int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{R}} e^{\frac{i}{h}\chi_0\chi_1^k\chi_2}A_j(\chi_0,\chi_1,\chi_2)\,d\chi_0\,d\chi_1\,d\chi_2. \end{aligned}$$

The associated new amplitudes are, respectively, given by

$$A_0^{\pm}(\chi_0,\chi_1) = \int \chi^*(\Omega_0^{\pm}(\theta)a(t,r\theta)r^{2n-1}|J\chi|)d\chi_2\ldots d\chi_{2n}, \tag{21}$$

$$A_j(\chi_0,\chi_1,\chi_2) = \int \chi^*(\Omega_j(\theta)a(t,r\theta)r^{2n-1}|J\chi|)d\chi_3\ldots d\chi_{2n}. \tag{22}$$

Remark 16. Since $\chi_1(t,r,\theta) = r|\mathfrak{p}_k(\theta) + rR_{k+1}(\theta) + tG_{k+1}(t,r\theta)|^{\frac{1}{k}}$, our new amplitude satisfies $A_0^{\pm}(\chi_0,\chi_1) = \mathcal{O}(\chi_1^{2n-1})$, near $\chi_1 = 0$. A similar argument shows that $A_i(\chi_0,\chi_1,\chi_2) = \mathcal{O}(\chi_1^{2n-1})$, near $\chi_1 = 0$. These facts will play a major role in Lemmas 17, 19 and 21.

We end this section with lemmas on asymptotics of oscillatory integrals with phases as in Lemma 15.

Lemma 17. *There exists a sequence $(c_j)_j$ of distributions, whose support is contained in the set $\{\chi_1 = 0\}$, such that for all function $a \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R})$:*

$$\int_0^\infty \left(\int_{\mathbb{R}} e^{i\lambda\chi_0\chi_1^k}a(\chi_0,\chi_1)d\chi_0 \right) d\chi_1 \sim \sum_{j=0}^{\infty} \lambda^{-\frac{j+1}{k}}c_j(a), \tag{23}$$

asymptotically for $\lambda \to \infty$, where

$$c_j = \frac{1}{k}\frac{1}{j!}(\mathscr{F}(x_-^{\frac{j+1-k}{k}})(\chi_0) \otimes \delta_0^{(j)}(\chi_1)), x_- = \max(-x,0).$$

We refer to [6] for a proof of this lemma.

Remark 18. A similar result holds for a phase $-\chi_0\chi_1^k$ if we replace terms x_- by x_+ in Lemma 17.

Lemma 19. *If $k>2n$ we have*

$$\int_0^\infty \left(\int_{\mathbb{R}^2} e^{i\lambda\chi_0\chi_1^k\chi_2} a(\chi_0,\chi_1,\chi_2)d\chi_0 d\chi_2\right)\chi_1^{2n-1}d\chi_1 = \lambda^{-\frac{2n}{k}}d(a) + \mathcal{O}(\lambda^{-\frac{2n+1}{k}}\log^2(\lambda)),$$

where the leading coefficient is given by

$$d(a) = \frac{1}{k}\Gamma\left(\frac{2n}{k}\right)\int_{\mathbb{R}^2}|\chi_0\chi_2|^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\mathrm{sign}\,(\chi_0\chi_2)\right)a(\chi_0,0,\chi_2)\,d\chi_0\,d\chi_2. \tag{24}$$

Proof. We use the Bernstein–Sato polynomial, see [20] for a detailed construction. We use variables (t,r,v) instead of (χ_0,χ_1,χ_2) and since $\chi_1 \geqslant 0$ for $tv \geqslant 0$ we can write

$$\frac{\partial^2}{\partial t\partial v}\frac{\partial^k}{\partial r^k}((tvr^k)^{1-z}r^{2n-1}) = b_k(z)(tvr^k)^{-z}r^{2n-1}, \tag{25}$$

$$b_k(z) = (1-z)^2\prod_{j=1}^{k}(j-kz+2n-1). \tag{26}$$

With the classical representation

$$\begin{aligned}&\int_0^\infty\int_{\{tv\geqslant 0\}} e^{i\lambda tvr^k}a(t,r,v)\,dt\,dv\,dr\\ &\quad = \frac{1}{2i\pi}\int_\gamma e^{i\frac{\pi z}{2}}\Gamma(z)\lambda^{-z}\left(\int_0^\infty\int_{\{tv\geqslant 0\}}(tvr^k)^{-z}a(t,r,v)\,dt\,dv\,dr\right)dz,\end{aligned}$$

where $\gamma =]c-i\infty, c+i\infty[$ and $\mathrm{Re}(c)<k^{-1}$, we can compute the asymptotic expansion with the residue method by pushing of the complex path of integration γ to the right. Note that, when the phase is negative, we have

$$\frac{1}{2i\pi}\int_\gamma e^{-i\frac{\pi z}{2}}\Gamma(z)\lambda^{-z}\left(\int_0^\infty\int_{\{tv\leqslant 0\}}(|tv|r^k)^{-z}a(t,r,v)\,dt\,dv\,dr\right)dz. \tag{27}$$

With Eqs. (25) and (26) the meromorphic extension is given by

$$\begin{aligned}&\int_0^\infty\int_{\{tv\geqslant 0\}}(tr^kv)^{-z}r^{2n-1}a(t,r,v)\,dt\,dr\,dv\\ &\quad = \frac{(-1)^k}{b_k(z)}\int_0^\infty\int_{\{tv\geqslant 0\}}(tvr^k)^{1-z}r^{2n-1}\frac{\partial^2}{\partial t\partial v}\frac{\partial^k}{\partial r^k}a(t,r,v)\,dt\,dv\,dr.\end{aligned}$$

Under our assumptions, the poles are $z = 1$ as a double pole and

$$z = \frac{j + 2n - 1}{k}, \quad j \in [1, \ldots, k].$$

Since $k > 2n$, the first pole is $\frac{2n}{k} \notin \mathbb{Z}$. The residue in this pole is

$$c_{k,n} \lambda^{-\frac{2n}{k}} \exp\left(i\pi \frac{n}{k}\right) \Gamma\left(\frac{2n}{k}\right) \int_{\{tv>0\}} (tv)^{1-\frac{2n}{k}} r^{k-1} \frac{\partial^2}{\partial t \partial v} \frac{\partial^k}{\partial r^k} a(t, r, v)\, dt\, dv\, dr,$$

$$c_{k,n} = (-1)^k \lim_{z \to \frac{2n}{k}} \frac{(z - \frac{2n}{k})}{b_k(z)} = (-1)^{k+1} \frac{k}{(k-2n)^2 \Gamma(k)}.$$

Now, with the following relations:

$$\int_{t,v>0} (tv)^{1-\frac{2n}{k}} \frac{\partial^2 a}{\partial t \partial v}(t, v)\, dt\, dv = \left(1 - \frac{2n}{k}\right)^2 \int_{t,v>0} (tv)^{-\frac{2n}{k}} a(t, v)\, dt\, dv,$$

$$(-1)^k \int_{r>0} r^{k-1} \frac{\partial^k a}{\partial r^k}\, dr = (k-1)!a(0),$$

and also using that

$$\frac{k}{(k-2n)^2 \Gamma(k)} \left(1 - \frac{2n}{k}\right)^2 (k-1)! \Gamma\left(\frac{2n}{k}\right) = \frac{\Gamma(\frac{2n}{k})}{k},$$

we find that the first residue is given by

$$\frac{\exp(i\pi\frac{n}{k})}{k} \lambda^{-\frac{2n}{k}} \Gamma\left(\frac{2n}{k}\right) \int_{\{t,v \geqslant 0\}} (tv)^{-\frac{2n}{k}} a(t, 0, v)\, dt\, dv.$$

The summation over the other quadrants gives the result with Eq. (27). Finally, the remainder is of order $\mathcal{O}(\lambda^{-\frac{2n+1}{k}} \log^2(\lambda))$ if $2n + 1 = k$ and of order $\mathcal{O}(\lambda^{-\frac{2n+1}{k}})$ otherwise. $\square$

Remark 20. The preceding result holds again, in a weaker sense, if k is not a divisor of $2n$, as can be shown by iterating the process of meromorphic extension above and using the fact that our amplitude is $\mathcal{O}(\chi_1^{2n-1})$ near $\chi_1 = 0$. Now, if k divide $2n$ then terms $\lambda^{-j} \log^d(\lambda)$, with $d = 1, 2$, can appear for the leading term, since we obtain poles of order 3 for the first non-zero residue.

Example. If we consider Gaussian amplitude, we obtain easily

$$I(\lambda,k,n)=\int_{\mathbb{R}_+\times\mathbb{R}^2} e^{i\lambda tr^k v}e^{-(t^2+v^2+r^2)}r^{2n-1}\,dr\,dt\,dv=2\pi\int_0^\infty \frac{r^{2n-1}e^{-r^2}}{\sqrt{4+r^{2k}\lambda^2}}\,dr.$$

The choice of $k=5$ and $n=2$ leads to

$$I(\lambda,5,2)\sim 2\pi\frac{\Gamma(\frac{2}{5})\Gamma(\frac{11}{10})}{2^{\frac{1}{5}}\sqrt{\pi}}\lambda^{-\frac{4}{5}}+\mathcal{O}(\lambda^{-1}).$$

This extra Gamma-factor comes from the identity

$$\int_{\mathbb{R}^2}|tv|^{-\frac{4}{5}}e^{-t^2}e^{-v^2}\,dt\,dv=\Gamma\left(\frac{1}{10}\right)^2.$$

Lemma 21. *When $k=2n$ we have the particular result*

$$\int_0^\infty\left(\int_{\mathbb{R}^2} e^{i\lambda\chi_0\chi_1^{2n}\chi_2}a(\chi_0,\chi_1,\chi_2)d\chi_0 d\chi_2\right)\chi_1^{2n-1}d\chi_1=\lambda^{-1}\log(\lambda)d(a)+\mathcal{O}(\lambda^{-1}),$$

with the leading coefficient

$$d(a)=\frac{\pi}{n}a(0,0,0). \tag{28}$$

Proof. We use again the Bernstein–Sato polynomial method. We use variables (t,r,v) instead of (χ_0,χ_1,χ_2) and we define

$$J_+(\lambda)=\int_0^\infty\int_{\{tv>0\}} e^{i\lambda tvr^{2n}}a(t,r,v)r^{2n-1}\,dt\,dr\,dv,$$

similarly, we define $J_-(\lambda)$ on the set $\{tv<0\}$. Then we can write

$$J_+(\lambda)=\frac{1}{2i\pi}\int_\gamma e^{i\frac{\pi z}{2}}\Gamma(z)\lambda^{-z}\int_0^\infty\int_{\{tv>0\}}(tv)^{-z}r^{-2nz}a(t,r,v)r^{2n-1}\,dt\,dr\,dv,$$

where $\gamma=]c-i\infty,c+i\infty[$, $c<(2n)^{-1}$. Similar computations as in the proof of Lemma 19 show that the associated Bernstein–Sato polynomial is

$$b_{2n}(z)=(1-z)^2\prod_{j=1}^{2n}(j-2nz+2n-1),$$

under our assumptions the first pole is $z=1$ and is of order 3. We define two holomorphic functions, near $z=1$, via

$$G_n^\pm(z)=\frac{(z-1)^3}{b_{2n}(z)}e^{\pm i\frac{\pi z}{2}}\Gamma(z).$$

Hence, when using the residue method, the first terms of the asymptotic expansion of $J_+(\lambda)$ are given by the formula

$$\frac{1}{2}\lim_{z\to 1}\left(\frac{\partial^2}{\partial z^2}\left(G_n^+(z)\lambda^{-z}\int_0^\infty\int_{\{tv>0\}}(tv)^{1-z}r^{2n-2nz}r^{2n-1}\frac{\partial^2}{\partial t\partial v}\frac{\partial^{2n}}{\partial r^{2n}}a(t,r,v)\,dt\,dr\,dv\right)\right),$$

and we have a similar formula for $J_-(\lambda)$ if we use $G_n^-(z)$ and integration over $\{tv<0\}$. For all holomorphic application f and all $\lambda>0$, we have

$$\frac{\partial^2}{\partial z^2}(f(z)\lambda^{-z}) = \frac{\partial^2 f}{\partial z^2}(z)\lambda^{-z} - 2\log(\lambda)\frac{\partial f}{\partial z}(z)\lambda^{-z} + \log^2(\lambda)f(z)\lambda^{-z}. \tag{29}$$

The term involving $\log^2(\lambda)$ is computed as in Lemma 19 and we have

$$\lim_{z\to 1} G_n^+(z) = -\frac{i}{(2n)!}.$$

The distributional factor is here given by

$$\int_0^\infty\int_{\{tv>0\}} r^{2n-1}\frac{\partial^2}{\partial t\partial v}\frac{\partial^{2n}}{\partial r^{2n}}a(t,r,v)\,dt\,dr\,dv = 2(2n-1)!a(0,0,0).$$

Hence the contribution is given by

$$-\log^2(\lambda)\lambda^{-1}\frac{i}{n}a(0,0,0),$$

but when the phase is negative we have

$$\lim_{z\to 1} G_n^-(z) = +\frac{i}{(2n)!},$$

consequently, by summation, there is no term associated to $\log(\lambda)^2$.

Now, we compute the main term associated to the $\log(\lambda)$.

For the same reason as previously all terms obtained by derivation of $\Gamma(z)$ and $(z-1)^3/b_{2n}(z)$ will give a zero contribution by summation. This will not be true for terms obtained by derivation of the exponential. As concerns derivation of the meromorphic distributions, we obtain, respectively,

$$\begin{aligned}
&-\frac{i}{(2n)!}\int_0^\infty\int_{\{tv>0\}}\log(tvr^{2n})r^{2n-1}\frac{\partial^2}{\partial t\partial v}\frac{\partial^{2n}}{\partial r^{2n}}a(t,r,v)\,dt\,dr\,dv,\\
&+\frac{i}{(2n)!}\int_0^\infty\int_{\{tv<0\}}\log(|tv|r^{2n})r^{2n-1}\frac{\partial^2}{\partial t\partial v}\frac{\partial^{2n}}{\partial r^{2n}}a(t,r,v)\,dt\,dr\,dv.
\end{aligned}$$

By summation and using that $\log(|tv|r^{2n}) = \log(|tv|) + \log(r^{2n})$ we easily obtain that the associated factor is given by

$$-\frac{i}{2n}\left(\int_{\{tv>0\}} \log(tv)\frac{\partial^2}{\partial t\partial v}a(t,0,v)\,dt\,dv - \int_{\{tv<0\}} \log(|tv|)\frac{\partial^2}{\partial t\partial v}a(t,0,v)\,dt\,dv\right).$$

A new splitting of the logarithms and integrations by parts show that the associated contribution vanish.

It remains now to compute the term associated to the derivation of the exponential. An easy computation shows that the associated contribution is

$$\log(\lambda)\lambda^{-1}\left(i\frac{\pi}{2}\right)\frac{e^{i\frac{\pi}{2}}}{(2n)!}\int_0^{\infty}\int_{\{tv>0\}} r^{2n-1}\,\frac{\partial^2}{\partial t\partial v}\frac{\partial^{2n}}{\partial r^{2n}}a(t,r,v)\,dt\,dr\,dv,$$

by integrations by parts, we finally obtain that the contribution is given by

$$\log(\lambda)\lambda^{-1}\left(i\frac{\pi}{2}\right)\frac{e^{i\frac{\pi}{2}}}{2n}\int_{\{tv>0\}}\frac{\partial^2}{\partial t\partial v}\,a(t,0,v)\,dt\,dv = -\frac{\pi}{2n}\log(\lambda)\lambda^{-1}a(0,0,0).$$

With Eq. (29) we have

$$J_+(\lambda) = \frac{\pi}{2n}\log(\lambda)\lambda^{-1}a(0,0,0) + \mathcal{O}(\lambda^{-1}).$$

A totally similar calculation gives

$$J_-(\lambda) = \frac{\pi}{2n}\log(\lambda)\lambda^{-1}a(0,0,0) + \mathcal{O}(\lambda^{-1}),$$

and the result holds by summation of $J_-(\lambda)$ and $J_+(\lambda)$. $\square$

Example. We use again an amplitude that is a product of Gaussian-functions, we have

$$I(\lambda,k,n) = \int_{\mathbb{R}_+\times\mathbb{R}^2} e^{i\lambda tr^k v}e^{-(t^2+v^2+r^2)}r^{2n-1}\,dr\,dt\,dv = 2\pi\int_0^{\infty}\frac{r^{2n-1}e^{-r^2}}{\sqrt{4+r^{2k}\lambda^2}}\,dr.$$

Hence, for $k=2$ and $n=1$ we obtain

$$I(\lambda,2,1) = \frac{1}{4\lambda}\left(-2\pi^2\,Y_0\left(\frac{2}{\lambda}\right) + (2\pi-1)J_0\left(\frac{2}{\lambda}\right)\log\left(\frac{1}{\lambda^2}\right) + \pi\mathrm{H}_0\left(\frac{2}{\lambda}\right)\right),$$

where $J_v(x)$, $Y_v(x)$ are, respectively, the standard Bessel functions of first and second kind. Also, $\mathrm{H}_v(x)$ is the standard Struve function defined by

$$z^2y''(z) + zy'(z) + (z^2-v^2)y(z) = \frac{2}{\pi}\frac{z^{v+1}}{(2v-1)!!}.$$

From classical properties of these special functions, we obtain

$$I(\lambda, 2, 1) \sim \left(\frac{\pi \log(\lambda) - \gamma\pi}{\lambda} \right) + \mathcal{O}(\lambda^{-2}). \tag{30}$$

Here, γ is Euler's constant and is obtained by derivation of the $\Gamma(z)$ factor in the formula that gives meromorphic extensions of our distributions. $\square$

6. Proof of the main results

Directions where $\mathfrak{p}_k(\theta) \neq 0$. Following step by step the proof of Lemma 4 of [6] we obtain that the first non-zero coefficient is obtained for $l = 2n - 1$ (see Remark 16) and is given by

$$\frac{1}{k} \frac{1}{(2n-1)!} \left\langle \mathscr{F}(x_+^{\frac{2n-k}{k}}) \otimes \delta_0^{(2n-1)}, A_0^+(\chi_0, \chi_1) \right\rangle = \frac{1}{k} \int \mathscr{F}(x_+^{\frac{2n-k}{k}})(\chi_0) \tilde{A}_0^+(\chi_0, 0) d\chi_0.$$

Since by construction

$$\tilde{A}_0^+(\chi_0, 0) = \int_{\mathbb{S}^{2n-1}} a(\chi_0, 0) \Omega_0^+(\theta) |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta, \tag{31}$$

we obtain that the local contribution, associated to $\mathrm{supp}(\Omega_0^+)$, is

$$\frac{1}{k} \left\langle (\mathscr{F}(x_+^{\frac{2n-k}{k}})(\chi_0), a(\chi_0, 0) \right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_0^+(\theta) |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta. \tag{32}$$

A similar computation gives the contribution of $\mathrm{supp}(\Omega_0^-)$, via

$$\frac{1}{k} \left\langle (\mathscr{F}(x_-^{\frac{2n-k}{k}})(\chi_0), a(\chi_0, 0) \right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_0^-(\theta) |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta. \tag{33}$$

Now, since $a(t, 0) = \hat{\varphi}(t) \exp(itp_1(z_0))$, cf. Eq. (15), the contributions of the directions where $\mathfrak{p}_k(\theta) \neq 0$ are given, respectively, by

$$I_+(\lambda) \sim \frac{1}{k} \lambda^{-\frac{2n}{k}} \left\langle |t|_+^{\frac{2n-k}{k}}, \varphi(t + p_1(z_0)) \right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_0^+(\theta) |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta, \tag{34}$$

for the set of directions where $\mathfrak{p}_k(\theta) > 0$ and by

$$I_-(\lambda) \sim \frac{1}{k} \lambda^{-\frac{2n}{k}} \left\langle |t|_-^{\frac{2n-k}{k}}, \varphi(t + p_1(z_0)) \right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_0^-(\theta) |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}} d\theta, \tag{35}$$

for the directions where $\mathfrak{p}_k(\theta)$ is negative.

Microlocal contribution of the set $\mathfrak{p}_k(\theta) = 0$.

Case of $k > 2n$. Here we examine the contribution of terms

$$\int e^{\frac{i}{h}\chi_0\chi_1^k\chi_2} A_i(\chi_0,\chi_1,\chi_2)\, d\chi_0\, d\chi_1\, d\chi_2,$$

but Lemma 19 shows that these are given by

$$\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)h^{\frac{2n}{k}}\int |\chi_0\chi_2|^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\operatorname{sign}(\chi_0\chi_2)\right)\tilde{A}_i(\chi_0,0,\chi_2)\, d\chi_0\, d\chi_2 + \mathcal{O}(h^{\frac{2n+1}{k}}\log^2(h)).$$

Writing the delta-Dirac distribution as an oscillatory integral leads to

$$\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)h^{\frac{2n}{k}}\frac{1}{2\pi}\int e^{iz\chi_1}|\chi_0\chi_2|^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\operatorname{sign}\,(\chi_0\chi_2)\right)\tilde{A}_i(\chi_0,\chi_1,\chi_2)\, d\chi_0\, d\chi_1\, d\chi_2\, dz,$$

with the amplitude

$$\tilde{A}_i(\chi_0,\chi_1,\chi_2) = \int \chi^*(\Omega_i(\theta)a(t,r\theta)|J\chi|)d\chi_3\dots d\chi_{2n}.$$

Now, we return to the initial coordinates and with $\chi_1 = r$ this gives

$$\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)h^{\frac{2n}{k}}\frac{1}{2\pi}\int e^{izr}|(\chi_0\chi_2)(t,r,\theta)|^{-\frac{2n}{k}}\Omega_i(\theta)\exp\left(i\frac{\pi n}{k}\operatorname{sign}(\chi_0\chi_2)\right)a(t,r\theta)\, dt\, dr\, d\theta\, dz,$$

if we use that $(\chi_0,\chi_1,\chi_2)(t,0,\theta) = (t,0,\mathfrak{p}_k(\theta))$, we obtain

$$\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)h^{\frac{2n}{k}}\int_{\mathbb{R}\times\mathbb{S}^{2n-1}}|t\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\operatorname{sign}(t\mathfrak{p}_k(\theta))\right)\Omega_i(\theta)a(t,0)\, dt\, d\theta.$$

Hence, when $\mathfrak{p}_k(\theta)$ is positive, we have

$$\begin{aligned}&\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)\int_{\mathbb{R}}|t|^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\operatorname{sign}(t)\right)\hat{\varphi}(t)e^{itp_1(z_0)}\, dt\\ &\quad=\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)\left\langle |t|_+^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\right)+|t|_-^{-\frac{2n}{k}}\exp\left(-i\frac{\pi n}{k}\right),\hat{\varphi}(t)e^{itp_1(z_0)}\right\rangle.\end{aligned}$$

Similarly, when $\mathfrak{p}_k(\theta)$ is negative we have

$$\begin{aligned}&\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)\int_{\mathbb{R}}|t|^{-\frac{2n}{k}}\exp\left(-i\frac{\pi n}{k}\operatorname{sign}(t)\right)\hat{\varphi}(t)e^{itp_1(z_0)}\, dt\\ &\quad=\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)\left\langle |t|_+^{-\frac{2n}{k}}\exp\left(-i\frac{\pi n}{k}\right)+|t|_-^{-\frac{2n}{k}}\exp\left(i\frac{\pi n}{k}\right),\hat{\varphi}(t)e^{itp_1(z_0)}\right\rangle.\end{aligned}$$

Hence, by summation the contribution is

$$\frac{1}{k}\Gamma\left(\frac{2n}{k}\right)h^{\frac{2n}{k}}\left(\cos\left(\frac{\pi n}{k}\right)\left\langle |t|^{-\frac{2n}{k}}, \hat{\varphi}(t)e^{itp_1(z_0)}\right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_i(\theta)|\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}d\theta\right.$$

$$\left. + i\sin\left(\frac{\pi n}{k}\right)\left\langle |t|^{-\frac{2n}{k}}\mathrm{sign}(t), \hat{\varphi}(t)e^{itp_1(z_0)}\right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_i(\theta)|\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}\mathrm{sign}(\mathfrak{p}_k(\theta))\, d\theta\right).$$

If we use the classical relations

$$\mathscr{F}(|x|^{\lambda})(\xi) = -2\sin\left(\frac{\lambda\pi}{2}\right)\Gamma(\lambda+1)|\xi|^{-\lambda-1},$$

$$\mathscr{F}(|x|^{\lambda}\mathrm{sign}(x))(\xi) = 2i\cos\left(\frac{\lambda\pi}{2}\right)\Gamma(\lambda+1)|\xi|^{-\lambda-1}\,\mathrm{sign}(\xi),$$

we obtain, after some manipulations, that the contribution is

$$\frac{1}{k}h^{\frac{2n}{k}}\left(\left\langle |t|^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_i(\theta)|\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}d\theta\right.$$

$$\left. + \left\langle |t|^{\frac{2n}{k}-1}\mathrm{sign}(t), \varphi(t+p_1(z_0))\right\rangle \int_{\mathbb{S}^{2n-1}} \Omega_i(\theta)|\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}\mathrm{sign}(\mathfrak{p}_k(\theta))\, d\theta\right).$$

We can split the integral with respect to $d\theta$ into two parts to finally obtain

$$\frac{1}{k}h^{\frac{2n}{k}}\left(\left\langle |t|_+^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \int_{\mathbb{S}^{2n-1}\cap\{\mathfrak{p}_k\geqslant 0\}} \Omega_i(\theta)|\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}d\theta\right.$$

$$\left. + \left\langle |t|_-^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \int_{\mathbb{S}^{2n-1}\cap\{\mathfrak{p}_k\leqslant 0\}} \Omega_i(\theta)|\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}d\theta\right).$$

With Eqs. (34) and (35), by summation on the partition of unity the main contribution to the trace formula is given by

$$\gamma_{z_0}(E_c,h) \simeq \frac{1}{k}\frac{h^{\frac{2n}{k}-n}}{(2\pi)^n}\left(\left\langle |t|_+^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \int_{\mathbb{S}^{2n-1}\cap\{\mathfrak{p}_k\geqslant 0\}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}d\theta\right.$$

$$\left. + \left\langle |t|_-^{\frac{2n}{k}-1}, \varphi(t+p_1(z_0))\right\rangle \int_{\mathbb{S}^{2n-1}\cap\{\mathfrak{p}_k\leqslant 0\}} |\mathfrak{p}_k(\theta)|^{-\frac{2n}{k}}d\theta\right). \tag{36}$$

And this proves the first statement of Theorem 4 for $k > 2n$. $\square$

Case of $k = 2n$. Here the contribution is given by

$$\int_{\mathbb{R}\times\mathbb{R}_+\times\mathbb{S}^{2n-1}} e^{\frac{i}{h}\Psi(t,r,\theta)}\Omega_i(\theta)a(t,r\theta)r^{2n-1}\,dt\,dr\,d\theta = \int e^{\frac{i}{h}\chi_0\chi_1^k\chi_2}A_i(\chi_0,\chi_1,\chi_2)d\chi_0\,d\chi_1\,d\chi_2,$$

and we recall that the amplitude is given by

$$A_i(\chi_0,\chi_1,\chi_2) = \int \chi^*(\Omega_i(\theta)a(t,r\theta)r^{2n-1}|J\chi|)\,d\chi_3\ldots d\chi_{2n}.$$

With $A_i = \chi_1^{2n-1}\tilde{A}_i$, from Lemma 21 we know that we have

$$\int e^{\frac{i}{h}\chi_0\chi_1^k\chi_2}\tilde{A}_i(\chi_0,\chi_1,\chi_2)\chi_1^{2n-1}\,d\chi_0\,d\chi_1\,d\chi_2 = \frac{\pi h}{n}\log(h)\tilde{A}_i(0,0,0) + \mathcal{O}(h).$$

Where, by construction, we have defined

$$\tilde{A}_i(\chi_0,\chi_1,\chi_2) = \chi_1^{-(2n-1)}\int \chi^*(\Omega_i(\theta)a(t,r\theta)r^{2n-1}|J\chi|)\,d\chi_3\ldots d\chi_{2n}.$$

With $z = (z_1, z_2, z_3)$, we write the amplitude as

$$\tilde{A}_i(0,0,0) = \frac{1}{(2\pi)^3}\int e^{-i\langle z,(\chi_0,\chi_1,\chi_2)\rangle}\tilde{A}_i(\chi_0,\chi_1,\chi_2)\,dz\,d\chi_0\,d\chi_1\,d\chi_2,$$

and by inversion of our diffeomorphism we obtain:

$$\tilde{A}_i(0,0,0) = \frac{1}{(2\pi)^3}\int e^{-i\langle z,(\chi_0,\chi_1,\chi_2)\rangle}\Omega_i(\theta)a(t,r\theta)\,dr\,dt\,d\theta\,dz.$$

Using that $(\chi_0,\chi_1) = (t,r)$, by integration w.r.t. (t,r,z_1,z_2) we get

$$\tilde{A}_i(0,0,0) = \frac{1}{2\pi}\int e^{-iz_3\chi_2(0,0,\theta)}\Omega_i(\theta)a(0,0)\,d\theta\,dz_3.$$

By construction $\chi_2(t,0,\theta) = \mathfrak{p}_k(\theta)$ and, since $\mathfrak{p}_k(\theta)$ is an admissible coordinate on $\operatorname{supp}(\Omega_i)$, we can use the change of coordinates $u = \mathfrak{p}_k(\theta)$, this leads to

$$\tilde{A}_i(0,0,0) = \frac{1}{2\pi}a(0,0)\int e^{-iz_3u}\Omega_i(\theta)\,dL_{\mathfrak{p}_k}(\theta)\,du\,dz_3,$$

with $d\mathfrak{p}_k\wedge dL_{\mathfrak{p}_k}(\theta) = d\theta$. If we introduce

$$C_{\mathfrak{p}_k} = \{\theta\in\mathbb{S}^{2n-1}/\mathfrak{p}_k(\theta) = 0\},$$

the sum over the partition of unity gives

$$\sum_i \int_{C_{\mathfrak{p}_k}} \Omega_i(\theta)\, dL_{\mathfrak{p}_k}(\theta) = \mathrm{LVol}(C_{\mathfrak{p}_k}),$$

where LVol is the Liouville volume attached to $\mathfrak{p}_k(\theta)$ on $\mathbb{S}^{2n-1}$. Finally, since we have $a(t,0) = \hat{\varphi}(t)\exp(itp_1(z_0))$, the second statement of Theorem 4 holds.

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References

[1] R. Balian, C. Bloch, Solution of the Schrödinger equation in term of classical path, Ann. of Phys. 85 (1974) 514–545.
[2] R. Brummelhuis, A. Uribe, A semi-classical trace formula for Schrödinger operators, Comm. Math. Phys. 136 (3) (1991) 567–584.
[3] R. Brummelhuis, T. Paul, A. Uribe, Spectral estimate near a critical level, Duke Math. J. 78 (3) (1995) 477–530.
[4] B. Camus, Formule des traces semi-classique au niveau d'une énergie critique, Thèse de l'université de Reims, 2001.
[5] B. Camus, A semi-classical trace formula at a non-degenerate critical level, J. Funct. Anal. 208 (2) (2004) 446–481.
[6] B. Camus, A semi-classical trace formula at a totally degenerate critical level. Contributions of extremums, accepted for publication in Communications in Mathematical Physics.
[7] A.M. Charbonnel, G. Popov, A semi-classical trace formula for several commuting operators, Comm. Partial Differential Equations 24 (1–2) (1999) 283–323.
[8] M. Combescure, J. Ralston, D. Robert, A proof of the Gutzwiller semi-classical trace formula using coherent states decomposition, Comm. Math. Phys. 202 (2) (1999) 463–480.
[9] J.J. Duistermaat, Oscillatory integrals Lagrange immersions and unfolding of singularities, Comm. Pure Appl. Math. 27 (1974) 207–281.
[10] J.J. Duistermaat, L. Hörmander, Fourier integral operators, Acta Math. 128 (3–4) (1972) 183–269.
[11] M. Gutzwiller, Periodic orbits and classical quantization conditions, J. Math. Phys. 12 (1971) 343–358.
[12] B. Helffer, D. Robert, Comportement semi-classique du spectre des hamiltoniens quantiques élliptiques, Ann. Fourier 31 (3) (1981) 169–223.
[13] L. Hörmander, Seminar on Singularities of Solutions of Linear Partial Differential Equations, Annals of Mathematical Studies, Vol. 91, Princeton University Press, Princeton, NJ, 1979, pp. 3–49.
[14] L. Hörmander, The Analysis of Linear Partial Operators 1,2,3,4, Springer, Berlin, 1985.
[15] D. Khuat-Duy, A semi-classical trace formula for Schrödinger operators in the case of a critical level, Thèse de l'université Paris 9, 1996.
[16] D. Khuat-Duy, Semi-classical trace formula at a critical level, J. Funct. Anal. 146 (2) (1997) 299–351.
[17] T. Paul, A. Uribe, Sur la formule semi-classique des traces, C. R. Séances Acad. Sci. Sér. I 313 (5) (1991) 217–222.

[18] V. Petkov, G. Popov, Semi-classical trace formula and clustering of the eigenvalues for Schrödinger operators, Ann. Inst. H. Poincaré Phys. Théori. 68 (1) (1998) 17–83.
[19] D. Robert, Autour de l'approximation semi-classique, Progr. Math. 68 (1987).
[20] R. Wong, Asymptotic Approximation of Integrals, Academic Press Inc., New York, 1989.
[21] J.A. Yorke, Periods of periodic solutions and the Lipschitz constant, Proc. Amer. Math. Soc. 22 (1969) 509–512.