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ALGEBRAS OVER FINE CATEGORIES

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1. Preface

Typical of what we define as fine categories are the categories of semiuniform spaces, bitopological spaces, preordered sets, and sets. Each admits taking finite products, has a base functor to **S**, and has a conjugation: a self functor of order 2. A fine functor between such fine categories preserves products, conjugation and respects the base functors.

In defining a nonnullary operation on such a fine category we also permit different variances (co, contra, variance) at different terms. Thus when considering a ordered group (additively written) an operation like $(a,b)\to(a-b)$ would be a binary operation with covariance in the first term and contravariance in the second (relative to the order). We use a variance function: here ν will take the indices $(1,2)$ of the binary operation to $(+1,-1)$.

When we have composite structures **GDF**, **HD′F′**, where F, F' are collections of algebraic operations, D, D' equations or identities for the algebras, and **G**, **H** types of fine category structures, we can have two types of functors between these. A type $T:\mathbf{GDF}\to\mathbf{HDF}$, which keeps the algebraic part unchanged, and a type $L:\mathbf{GDF}\to\mathbf{GD'F'}$ which keeps the fine part unchanged. What we prove in this paper is that, not only do these types of functors have left-adjoints, but more: the first type of functor is topological in the sense of Herrlich's definition [1], while the second type is algebraic (or monoidal or triplable).

2. Fine categories and fine functors

Definition. A category **G** is called a *fine category* if it has a *base functor* $B:\mathbf{G}\to\mathbf{S}$ and a *conjugation functor* $N:\mathbf{G}\to\mathbf{G}$ satisfying the following conditions: **G** is closed for formation of products for finite families of its objects (or morphisms), B is a faithful functor which preserves products and conjugation (assuming $1_{\mathbf{S}}$ as the conjugation in **S**, B preserves conjugation means that, for any object A of **G**, and any morphism f of **G**, $B(N(A)) = B(A)$, $B(N(f)) = B(f)$), and $N \cdot N = 1_G$ (thus $N(N(A)) = A$ and $N(N(f)) = f$, for any object A of **G** and any morphism f of **G**)).

When talking of different fine categories we use the same names B, N for the base functor and the conjugation functor for all of them.

Definition. A functor $K: \mathbf{G} \to \mathbf{H}$, where **G**, **H** are both fine categories, is called a *fine functor* if K preserves finite products, B and N; that is, $B = B \cdot K$ and $K \cdot N = N \cdot K$.

Clearly the identity functor is a fine functor for any fine category; and the base functor B of a fine category is a fine functor, when we set both $B, N = 1_S$ to make **S** a fine category. Also compositions of fine functors give fine functors.

Thus **S** is a fine category. Next we have the fine category **PO**, whose objects are preordered sets and morphisms are monotone maps; B being a forgetful functor, and N taking a $f: (X, \leq) \to (Y, \leq)$ to $f: (X, \geq) \to (Y, \geq)$.

For the next fine category, we define a bitopological space (X, T, T'). When T is a topology (= family of open sets for a topology) on the set X there is an associated preorder $\leq(T)$ on X defined by: $x \leq(T)\ y$ iff [$y \in G \in T$ implies $x \in G$]. When T, T' are two topologies on X such that $\leq(T)$ is the reverse of the preorder $\leq(T')$ [so that $x \leq(T)\ y$ iff $y \leq(T')\ x$], we call (X, T, T') a *bitopological space;* a map $f: X \to Y$ is called *doubly continuous* from (X, T, T') to (Y, S, S') when these are both bitopological spaces, if $f: (X, T) \to (Y, S)$ and $f: (X, T') \to (Y, S')$ are both continuous maps. We have then a category **BT** with bitopological spaces for objects and doubly continuous maps for morphisms. This is a fine category: for a typical morphism $f: (X, T, T') \to (Y, S, S')$ of **BT**, B takes the morphism to $f: X \to Y$ and N takes it to $f: (X, T', T) \to (Y, S', S)$.

We call $(U, (J, \leq), c)$ a *semiuniformity* on the set X, when $(J, \leq)$ is a down-directed ordered set, c a map of J in itself, and U a monotone map of $(J, \leq)$ in $[P(X \times X), \subseteq]$ such that, for each j of J, 1_X is contained in $U(j)$ and $U(cj) \circ U(cj)$ is also contained in $U(j)$, where cj denotes the image of j under the map $c: J \to J$. When such a semiuniformity on X is given, we call $(X, U, (J, \leq), c)$ a *semiuniform space.* A map $f: X \to Y$ is a *uniform map* from the space $(X, U, (J, \leq), c)$ to $(Y, V, (K, \leq), c)$ if, for each k of K there is a j of J such that $(x, x') \in U(j)$ implies that $(f(x), f(x')) \in V(k)$. We then have a category **SU** with semiuniform spaces as objects and uniform maps as morphisms. A semiuniformity (U, J) on X determines a 'conjugate' semiuniformity (U_c, J) when we set $U_c(j) = [U(j)]^r = \{(x', x) : (x, x') \in U(j)\}$; and it also determines a symmetric semiuniformity (V, j) with $V(j) = U(j) \cap U_c(j)$, for each j. And a semiuniformity (U, J) defines on X a bitopology $(X, T(U), T(U_c))$, where the topology $T(U)$ is one with the base of neighbourhoods $\{U(j)(x) : j \text{ in } J\}$ at the point x, where $y \in U(j)(x)$ means the same as $(x, y) \in U(j)$. The topology $T(V)$ defined by the symmetric associate of U we call the *star topology* for (X, U) and denote it by $T^*(U)$; thus $T^*(U) = T^*(U_c) = T^*(V)$.

A map $s: D \to X$, when $(D, \leq)$ is any down directed set, is called a D-sequence in X. Such a sequence in X is called a *Cauchy D-sequence* in (X, U) if, for each j of J, a nonnull initial subset B of $(D, \leq)$ can be found such that $(s(d), s(e))$ is in $U(j)$ whenever d, e are from B. With the usual definition of convergence, [that a D-sequence in the space (X, T) converges to an x in X if, for each neighbourhood $U(x)$ of x, there is a nonnull initial subset B of $(D, \leq)$ such that $s(d)$ is in $U(x)$ for all d from B], it is easy to see that a D-sequence in X which converges to some point x of X in $(X, T^*(U))$ must be a Cauchy D-sequence of (X, U). We shall call the semiuniform space (X, U) a *complete space* if conversely, every Cauchy sequence of (X, U) converges to some point in the space $(X, T^*(U))$. Elsewhere I have proved the following results regarding these complete spaces (see [2, 3]):

(a) The product of a set-indexed family of complete semiuniform spaces is a complete semiuniform space; a closed subspace of a complete semiuniform space is also complete; one can also check that a coproduct of complete semiuniform spaces is complete, and that the conjugate (X, U_c) is complete when (X, U) is so.

(b) Given a semiuniform space $(X, U, (J, \leq), c)$, there is an associated complete, T_0, semiuniform space $(X^*, U^*, (J, \leq), c)$, called the *canonical completion* of (X, U), with a uniform map $h: (X, U) \to (X^*, U^*)$, such that, if $g: (X, U) \to (Y, V)$ is a uniform map of (X, U) in a complete, T_0, semiuniform space (Y, V), then there is a unique uniform map $g^*: (X^*, U^*) \to (Y, V)$ such that $g^* \cdot h = g$.

We have then a full subcategory **CSU** of **SU** whose objects are the complete T_0 semiuniform spaces. This category is also a fine category, with the same definition for the base functor and the conjugate as **SU**.

Besides the base functors from **PO**, **BT**, **SU** and **CSU** which we know are fine functors, we have the following functors, and combinations of these that are definable, as fine functors: N: **CSU** $\to$ **SU** [the inclusion functor], B: **SU** $\to$ **BT** [the functor taking a typical $f: (X, U) \to (Y, V)$ to $f: (X, T(U), T(U_c)) \to (Y, T(V), T(V_c))$] P: **BT** $\to$ **PO** [the functor taking a typical $f: (X, T, T') \to (Y, S, S')$ to $f: (X, \leq(T)) \to (Y, \leq(S))$]. The verifications of these are not difficult.

Theorem 1. *All the four fine functors N, B, P and S*: **PO** $\to$ **S**, (*and their composites that can be defined*) *have left-adjoints.*

Proof. We describe the nature of the left-adjoint; the verification that they are indeed left-adjoints is mostly routine.

(i) For N: **CSU** $\to$ **SU**, the left-adjoint C^*: **SU** $\to$ **CSU** is essentially what is given by the canonical completion, as described in (b) above. That is, $C^*(X, U) = (X^*, U^*)$, and $C^*[f: (X, U) \to (Y, V)] =$ the obvious extension $f^*: (X^*, U^*) \to (Y^*, V^*)$, for which $f^* \cdot h = h' \cdot f$, if $h: (X, U) \to (X^*, U^*)$ and $h': (Y, V) \to (Y^*, V^*)$ are as given in (b)) the canonical morphisms of the spaces in their completions.

(ii) For B: **SU** $\to$ **BT**, the left-adjoint U^*: **BT** $\to$ **SU** is given as follows: for an object (X, T, T') of BT, there are semiuniformities $[U_k]$ on X such that T is finer than $T(U_k)$ and T' is finer than $T(U_{kc})$; and the lattice product of these gives the finest such semiuniformity, which we denote by $U(T,T')$; then if $f:(X,T,T')\to(Y,S,S')$ is in **BT**, one sees that $f:(X,U(T,T'))\to(Y,U(S,S'))$ is in **SU**; we define U^* such that it takes the first f in **BT** to this f in **SU**.

(iii) For $P:\mathbf{BT}\to\mathbf{PO}$, the left-adjoint $B^*:\mathbf{PO}\to\mathbf{BT}$ is defined as follows: given a preordered set $(X,\leq)$, the family of initial sets in $(X,\leq)$ is closed for arbitrary intersections and arbitrary unions, and contains the null set and X. It is thus a topology on X, denoted by $T^<$; similarly the family of final sets of $(X,\leq)$ determines another topology on X denoted by $T^>$; and a monotone map $f:(X,\leq)\to(Y,\leq)$ gives continuous maps $f:(X,T^<)\to(Y,T^<)$, $f:(X,T^>)\to(Y,T^>)$. Thus we set $B^*[f:(X,\leq)\to(Y,\leq)]$ to be $f:(X,T^<,T^>)\to(Y,T^<,T^>)$.

(iv) For $S:\mathbf{PO}\to\mathbf{S}$, the left adjoint $P^*:\mathbf{S}\to\mathbf{PO}$ is defined by setting, for a morphism $f:X\to Y$ in **S**, $P^*(f)=f:(X,=)\to(Y,=)$ in **PO**.

3. Algebras over fine categories and top-functors

In defining the algebras we shall consider a family of formal operations and split it into the subfamilies of the nullary ones and the nonnullary ones. Thus $F=$ the disjoint union $F_0\cup F_1$, where $F_0=\{0_j: j \text{ in } J\}$ consists of the symbols (or names) for a family of nullary operations, and $F_1=\{(\theta_k,n_k,v_k): k \text{ in } K\}$ consists of symbols for the nonnullary operations; each of these has a name θ_k, an arity n_k (= an integer ≥ 1) and a variance v_k, which is a map of the ordered set $(1,2,\ldots,n_k)$ in the set $[+1,-1]$. We say θ_k has covariance or contravariance at the place m $(1\leq m\leq n_k)$ according as $v_k(m)=+1$ or -1.

Given then an object A of a fine category **G**, we say that A is closed for a 0_j of F_0, if a unique element $0_j^A(\emptyset)$ [or just $0_j(A)$] of the set $B(A)$ is assigned; B denotes the base functor from **G** to **S**. And we say that A is closed for a (θ,n,v) of F_1 if a morphism $\theta^A:P[A_1,\ldots,A_n]\to A$ is assigned where $P[\ ]$ denotes a product in **G** of the A's, where $A_m=A$ or $N(A)$ according as $v(m)=+1$ or -1. We call A an *F-algebra over* **G** (or a *GF-algebra*) if A is closed for all the operations from F_0 and from F_1. Given two such F-algebras A, B over **G**, a map $f:X\to Y$ which is a morphism of **G** is called an *F-homomorphism* (or just a *GF-morphism*) if

(i) for each 0 of F_0, $f(0(A))=0(B)$;

(ii) for each (θ,n,v) of F_1, $f\cdot\theta^A=\theta^B\cdot f^n$, where f^n denotes the product morphism from $P[A_1,\ldots,A_n]$ to $P[B_1,\ldots,B_n]$ of the morphisms $f_i:A_i\to B_i$ in **G**, where f_i is f or $N(f)$ according as $v(i)=+1$ or -1.

We then get a category **GF** whose objects are the GF-algebras and whose morphisms are the GF-morphisms. When **G** is taken to be **S**, the category **SF** is usually just denoted by **F**; it is the familiar category of F-algebras. In this case, since the conjugation in **S** is trivial, the part v for a nonnullary operation plays no significant role.

When **G**, **H** are fine categories and $K:\mathbf{G}\to\mathbf{H}$ is a fine functor, it is clear that K takes products of objects to products of objects and products of morphisms to products of morphisms too. Hence it follows that when A is an F-algebra over **G**, we can consider $K(A) = B$ to be an F-algebra over **H** by setting: $0(B) = 0(A)$, for any 0 of F_0, and $\theta^B = K(\theta^A)$ for any (θ, n, v) of F_1. Thus we have an induced functor from **GF** to **HF** which we also denote by K. In particular, the base functor $B : \mathbf{G} \to \mathbf{S}$ determines thus a functor $B : \mathbf{GF} \to \mathbf{F}$. The base of an F-algebra over **G** is an F-algebra (over **S**).

Starting with a set X (of variables) we construct the free F-algebra $P(X, F)$ over the set X as usual: it consists of formal polynomials over the set obtained by using the operations from F_1 a finite number of times. Each has a rank: the polynomials of rank 0 are the elements of X or F_0; a polynomial of rank r, ≥ 1, is a symbol of the form $\theta(y_1, \ldots, y_n)$ for some (θ, n, v) from F_1, with each y_i a polynomial of rank $\leq (r-1)$. To define 'equational' algebras or identical equations in an algebra, we select a set of pairs $D = [(p_i, q_i)]$ from this $P(X, F)$. When (Y, F) is any F-algebra (over the set base Y), any mapping $g : X \to Y$ evidently has a unique extension $g^* : (P(X, F), F) \to (Y, F)$ which is an F-homomorphism (with $g^*[(x)] = g(x)$ for each x of X). We call the F-algebra (Y, F) over **S**, a *DF-algebra* if, for each set map $g : X \to Y$, and for each pair (p_i, q_i) from D we have $g^*(p_i) = g^*(q_i)$. And when we have a GF-algebra A, we call it a *GDF-algebra* if the F-algebra $B(A)$ is a DF-algebra, where B is the usual base functor from **G** to **S**. We then have a full subcategory **GDF** of **DF** whose objects are these GDF-algebras. We noted that a fine functor $K : \mathbf{G} \to \mathbf{H}$ takes a GF-algebra A to a HF-algebra $K(A)$. Since K respects the base functors, it is not hard to see that K would also take a GDF-algebra A to a HDF-algebra $K(A)$. We still denote this functor K from **GDF** to **HDF** (really a restriction of the original K to objects and morphisms of **GDF**).

Theorem 2. *The four functors N, B, P and S defined in Theorem 1 give rise to similarly named functors* $N : \mathbf{CSUF} \to \mathbf{SUF}$, $N : \mathbf{CSUDF} \to \mathbf{SUDF}$, $B : \mathbf{SUF} \to \mathbf{BTF}$, $B : \mathbf{SUDF} \to \mathbf{BTDF}$, $P : \mathbf{BTF} \to \mathbf{POF}$, $P : \mathbf{BTDF} \to \mathbf{PODF}$, $S : \mathbf{POF} \to \mathbf{F}$, $P : \mathbf{PODF} \to \mathbf{DF}$. *All these eight functors also have left-adjoints.*

Proof. (i) *The functors N*: When $(X, U(J, \leq), c; F)$ is a semiuniform F-algebra, and we have the canonical completion $(X^*, U^*, (J, \leq), c)$ of (X, U), this X^* is obtained by taking the quotient semiuniformity of a complete semiuniform space $(X^\#, U^\#, (J, \leq), c)$ by an equivalence connecting topologically indistinguishable points, where $X^\#$ has elements which are Cauchy $(J, \leq)$ sequences from (X, U). Each point of X^* can be obtained as a limit of a Cauchy sequence of the form $\{h(x_j) : j \text{ in } (J, \leq)\}$, with h as the canonical uniform mapping of (X, U) in (X^*, U^*). We extend the F-algebraic structure to X^* as follows: for a nullary θ from F_0, we set $\theta^{X^*}(\emptyset) = h(\theta^X(\emptyset))$; for a (θ, n, v) from F_1, and an ordered n-tuple of elements $(x_1^*, \ldots, x_n^*)$ from X^*, we choose a Cauchy sequence $\{x_{r,j} : j \text{ in } (J, \leq)\}$ for each x_r^*, $r = 1, \ldots, n$, such that x_r^* is the limit of $h(x_{r,j})$, for each r. Then if $x_j = \theta^X(x_{1,j}, \ldots, x_{n,j})$ for each j of J, it is not hard to see that $\{x_j : j \text{ in } J\}$ is a Cauchy sequence of (X, U) too; and this determines a unique limit x^* for $\{h(x_j)\}$ in X^*. If $\theta^{X^*}(x_1^*, \ldots, x_n^*)$ is this

x^*, we get a definition of (θ, n, v) valid in (X^*, U^*). This gives a semiuniform F-algebra $(X^*, U^*, (J, \leq), c; F)$. We set $C^*(X, U)$ equal to this (X^*, U^*); and $C^*(f) = f^*$ is again defined, as earlier, as the extension of f to the completions.

It can also be verified that when we start with a semiuniform DF-algebra this construction leads to a semiuniform DF-algebra. These describe the left-adjoint C^* of N in the two cases of N.

(ii) *The functors B*: Given a bitopological F-algebra $(X, T, T'; F)$ there are semiuniformities (U, J) on X such that:

(a) T is finer than $T(U)$,

(b) T' is finer than $T(U_c)$,

(c) $(X, U; F)$ is a semiuniform F-algebra,

when the operations from F are as given originally in X; for surely the coarsest semiuniformity on X satisfies all three conditions. Then we see that the lattice product of all such U on X is also a semiuniformity $(U^{\#}, J^{\#})$ with the same properties. We set $U^*(X, T, T'; F) = (X, U^{\#}; F)$ and $U^*(f : (X, T, T'; F) \to (Y, S, S^{\pm}; F) = f : (X, U^{\#}; F) \to (Y, V^{\#}; F)$, where $V^{\#}$ is defined for Y similarly in terms of S, S'. Again the same U^* gives the left-adjoint for the other B from **SUDF** to **BTDF**.

(iii) *The functors P*: Using the same definition of B^* as in the proof of Theorem 1, we see that, when $(X, \leq)$ is a preordered F-algebra or a preordered DF-algebra, then $(X, T^<, T^>)$ can also be considered as a bitopological F- or DF-algebra, with the same effect on the operations for $\emptyset$ (when the operation is nullary) or for n-tuples of elements of X, as earlier. So we again have a left-adjoint named B^* for these P.

(iv) *The functors S*: In these cases also we repeat the definition of the left-adjoint P^* of S, as in Theorem 1.

We formulate a definition of top-functors and top-categories that imply that a top-functor is a special case of a topological functor in Herrlich's sense, and then show that all the above functors are top-functors.

Definition. A *top-category* $(\mathbf{A}, E, M)$ is a category $\mathbf{A}$ which is (small) complete, and admits an (E, M)-factorization that is unique; E, M are classes of epi-, monomorphisms of $\mathbf{A}$ closed for compositions with isomorphisms. A functor $K : \mathbf{A} \to \mathbf{B}$, is a *top-functor* between the top-categories $(\mathbf{A}, E, M)$ and $(\mathbf{B}, E^*, M^*)$ if: F preserves products, and $[A \in 0(\mathbf{A}),\ \mathbf{m}^* \in M^* \cap \hom_{\mathbf{B}}(-, K(A))] \Rightarrow$ [there is an $m \in M \cap \hom_{\mathbf{A}}(-, A)$ such that (i) $K(m) = m^*$, and (ii) when $K(f) = m^* \cdot h^*$ for an f from $\hom_{\mathbf{A}}(-, A)$ then $f = m \cdot h$ for an h with $K(h) = h^*$].

The basic result in this connection (that is easily proved, given the definitions) is the following:

Lemma 1. *When $K : (\mathbf{A}, E, M) \to (\mathbf{B}, E^*, M^*)$ is a top-functor between top-categories, then K is a topological functor from $\mathbf{A}$ to $\mathbf{B}$, which is given the (E^*, M^{**}) category structure (in Herrlich's sense), with M^{**} consisting of sources of the form $(A, p_i \cdot m)$*

where m is from M^* and in $\hom_{\mathbf{A}}(A, P(A_i))$, p_i *being the canonical maps of a product* $P(A_i)$ *in the factors* A_i.

We now associate suitable classes E, M with the various categories of the form **GDF**, **GF**, or **G** where **G** is one of the types **CSU**, **SU**, **BT**, or **PO**, so that each gives a top-category.

When **G** is **CSU**, we use the classes $E(U)$, $M(U)$ defined as follows: $E(U)$ consists of the morphisms from the category **G** which are onto subspaces dense in the codomain space (under the topology defined by the symmetrised form $U \cap U^r$ from the semiuniformity); while $M(U)$ consists of isomorphisms in the given **G** on a closed subspace of the codomain space.

For **G** = **SU** the above $E(U)$, $M(U)$ can be used; a second pair $E^*(U)$, $M^*(U)$ that can be used also is defined by: $E^*(U)$ consists of morphisms which are surjective as set maps and $M^*(U)$ of morphisms which are isomorphisms on subspaces of the codomains.

When **G** is **BT**, we use $E(B)$ = the class of morphisms which are surjective as set maps, and $M(B)$ = the class of isomorphisms onto subspaces (for both the topologies).

For **G** = **PO**, $E(P)$ is the class of morphisms which are surjective as set maps, and $M(P)$ the class of isomorphisms onto relatively ordered subalgebras, of the codomains.

Definition. A functor $K: A \to B$ is said to:

(i) *raise codomains* if [A in $0(\mathbf{A})$, g in $\hom_{\mathbf{B}}(K(A), B')$] ⇒ [there is an f in $\hom_{\mathbf{A}}(A, A')$ such that $K(f) = g$],

(ii) *raise factorizations* if [f in $M(\mathbf{A})$, $K(f) = g_1 \cdot g_2$ in $M(\mathbf{B})$] ⇒ [$f = f_1 \cdot f_2$ in $M(\mathbf{A})$ where $K(f_1) = g_1$, $K(f_2) = g_2$],

(iii) be *latticiel* if

(a) for each B in $0(\mathbf{B})$, $K^{-1}(B) = \{A \text{ in } 0(\mathbf{A}): K(A) = B\}$ is a set and a complete lattice under the preorder $\leq(K)$ on it defined by setting $A \leq(K)\, A'$ iff there is a $g: A \to A'$ in $M(\mathbf{A})$ with $K(g) = 1_B$, and

(b) [$K(f_i)$ = a given g of $M(\mathbf{B})$, for a set of morphisms $f_i: A \to A_i$, all with the same domain A] ⇒ [there is a morphism $\Pi(f_i): A \to \Pi(A_i)$ in $M(\mathbf{A})$ with $K(\Pi(f_i)) = g$ where $\Pi(A_i)$ denotes a lattice product of the A_i all from K^{-1} (codomain g)].

Theorem 3. (a) *The categories* **CSUDF**, **CSUF**, **CSU** *become top-categories when* $E(U)$, $M(U)$ *are used for E and M. The categories* **SUDF**, **SUF**, **SU** *also are top-categories with the same* $E(U)$, $M(U)$; *they are also top-categories when* $E^*(U)$, $M^*(U)$ *are used for E, M. The categories* **PODF**, **POF**, **PO** *are top-categories under* $E(P)$, $M(P)$. *The categories* **BTDF**, **BTF**, **BT** *are top-categories under* $E(B)$, $M(B)$. *The categories* **DF**, **F**, **S** *are top-categories under E, M (the usual classes of surjective, or injective morphisms).*

(b) *The twelve functors of the form N, B, P and S mentioned in Theorems 1, 2 are faithful, raise codomains, raise factorizations, and are latticiel. They are top-functors for suitable choice of the E, M for the domain, and codomain categories.*

Proof. All the proofs are quite straightforward and follow from the definitions involved.

To prove that these are top-functors, the part that they preserve products follows from the fact that they all have left-adjoints. The way the M's are defined, it would be seen that in each case when m^* is in $\hom_{\mathbf{B}}(\text{-},K(A))\cap M^*$, there is an m in $\hom_{\mathbf{A}}(\text{-},A)\cap M$ for which $K(m)=m^*$; and then the second part would follow from the fact that K raises factorisations.

4. Algebraic functors and free algebras

If F and D are as before, while F', D' are subsets of F and D respectively such that the pairs of D' are from $P(X,F')$, then we could consider DF-algebras as a subclass of the $D'F'$-algebras and similarly the F-algebras as a subclass of the F'-algebras. Thus we have inclusion functors $M:\mathbf{DF}\to\mathbf{D'F'}$, $L:\mathbf{F}\to\mathbf{F'}$, $G:\mathbf{DF}\to\mathbf{F}$, $G':\mathbf{D'F'}\to\mathbf{F'}$, $T:\mathbf{F}\to\mathbf{S}$ and $T':\mathbf{F'}\to\mathbf{S}$. We use the same symbols for the corresponding functors when we have a G-structure along with the algebraic one, where G is one of **CSU**, **SU**, **BT**, or **PO**; thus there is an $M:\mathbf{SUDF}\to\mathbf{SUD'F'}$, and so on. (The same G is tacked on to the domain and codomain). We claim that all these thirty functors are algebraic ones. To prove this, we prove later a result regarding raising 'shortest paths'. We relate these paths to existence of adjoints as follows:

Lemma 2. *A functor $K:A\to B$ has a left-adjoint $J:B\to A$ iff, for each object B of $\mathbf{B}$ there is a shortest K-path $(h(B),J(B))$: meaning thereby, that $J(B)$ is an object of $\mathbf{A}$, $h(B):B\to K(J(B))$ is a morphism in $\mathbf{B}$, and for any morphism $g:B\to K(A')$ for an A' of $0(\mathbf{A})$, there is a unique $g^*:J(B)\to A'$ in $M(\mathbf{A})$ for which $K(g^*)\cdot h(B)=g$.*

This result is well known, and the proof is therefore omitted.

Lemma 3. *All the functors $M:\mathbf{DF}\to\mathbf{D'F'}$, $L:\mathbf{F}\to\mathbf{F'}$, $G:\mathbf{DF}\to\mathbf{F}$, $G':\mathbf{D'F'}\to\mathbf{F'}$, $T:\mathbf{F}\to\mathbf{S}$ and $T':\mathbf{F'}\to\mathbf{S}$ have left-adjoints.*

Proof. Since G', T' are essentially the same as G, T we can omit them. For $T:\mathbf{F}\to\mathbf{S}$, a left-adjoint $T^*:\mathbf{S}\to\mathbf{F}$ is obtained by noting that there is a shortest T-path to any set X: $[j,(P(X,F),F)]$, where j is the inclusion map of X in the free algebra $P(X,F)$. To get a left-adjoint $G^*:\mathbf{F}\to\mathbf{DF}$ for G, for an object (X,F) of $\mathbf{F}$, we first construct the free algebra $(P(X,F),F)$ over X and then seek a congruence on it which (i) would ensure that the quotient algebra was a DF-algebra and (ii) would be larger than the congruence on $(P(X,F),F)$ defined by the natural homomorphism of the free algebra on (X,F). Using the smallest such congruence, the quotient algebra (Y,F), and the obvious homomorphism of (X,F) on (Y,F) provide a shortest G-path at (X,F). The procedures for finding left-adjoints for L or M are similar to the last, with some obvious modifications.

From the above result, to deduce the existence of left-adjoints for similar functors between categories of the form **GDF**, **GF**, **GD′F′**, etc., we use the following lemma.

Lemma 4. *Given a functor $K: A \to B$ which raises codomains, raises factorizations, and is faithful and latticiel, and given subcategories C, D of A, B with the inclusion functors $I_1: C \to A$, $I_2: D \to B$, such that, for an f of $M(\mathbf{A})$, f belongs to $M(C)$ iff $K(f)$ belongs to $M(D)$, the following is true: [for an object A of* **A**, *if $K(A)$ has a shortest I_2-path (g, D), then A has a shortest I_1-path (g^*, C) such that $K(g^*) = g$]. We then say that K raises the shortest I_2-path (g, D) to the shortest I_1-path (g^*, C).*

Proof. Since K raises factorizations and since $K(A')$ is in **D** gives A' is in **C**, the morphism $g: K(A) \to D$ (with D in **D**) leads to a morphism $g': A \to C$ with C in **C** and with $K(g') = g$. The morphisms $\{g'_i : A \to C_i \mid C_i \text{ in } \mathbf{C} \text{ and } K(g'_i) = g\}$ form a set, and since K is latticiel we also have a $g^* = \Pi(g'_i): A \to \Pi(C_i)$ with $K(g^*) = g$. This implies $K(\Pi(C_i)) = D$ in **D**, so that $\Pi(C_i)$ is in **C**. We now show that this $(g^*, \Pi(C_i))$ is a shortest I_1-path at A. For, if $f: A \to C$ is any morphism with C in **C**, then $K(f)$ resolves in the form $K(f) = h \cdot g$, since, by assumption, (g, D) is a shortest I_2-path at $K(A)$. As K raises factorizations, this leads to a factorization $f = h' \cdot g'$ with $K(g') = g$. Then it is clear that this g' must be one of the g'_i. But then since $\Pi(C_i) \leq (K)C_i$, there is a $j: \Pi(C_i) \to C_i$ such that $K(j) = 1_D$. Hence we have $K(j \cdot g^*) = K(j) \cdot K(g^*) = g = K(g')$. As K is faithful, it follows that $j \cdot g^* = g'$, and so $f = h' \cdot j \cdot g^*$. That is, f factors out through g^*; the complementary factor $(h' \cdot j)$ must be unique since K is faithful. These prove that $(g^*, \Pi(C_i))$ is indeed a shortest I_1-path at A.

From Lemmas 2 and 4 we get the following consequence:

Corollary. *Under the conditions assumed in Lemma 4 for A, B, C, D, K, I_1 and I_2, the functor I_1 has a left-adjoint provided the functor I_2 has one.*

Using this corollary with Lemma 2 we can deduce many functors have left-adjoints. For example a functor M: **GDF** → **GD′F′**, when **G** is any one of **CSU**, **SU**, **BT** or **PO**, has a left-adjoint; we have only to check that the functor $K: GD'F' \to D'F'$ (the forgetful functor) has the requisite properties from Lemma 4, to apply the corollary. This verification is in all cases quite simple. Thus we have:

Lemma 5. *Besides the six functors listed in Lemma 3, the similarly named ones between categories of the form* **GDF**, **GD′F′**, **GF**, **GF′**, *and* **G**, *where G is any one of* **CSU**, **SU**, **BT** *or* **PO** *all have left-adjoints.*

More is true. All these functors are algebraic functors. To prove this, we note, from S. MacLane's book the following version of Beck's theorem [4, Theorem 1, p. 147]:

Beck's Theorem. *A functor $G:A\to X$ which has a left-adjoint is an algebraic functor provided that G creates coequalisers for those parallel pairs f, g in A for which Gf, Gg has a split coequaliser in X.*

Theorem 4. *All the thirty functors named in Lemma 5 are algebraic functors.*

Proof. We shall see how a proof goes for a typical functor $M:\mathbf{GDF}\to\mathbf{GD'F'}$; the other functors are really simpler and the proofs are quite analoguous. All the cases of $\mathbf{G}$ we consider are fine categories. Given any fine category $\mathbf{A}$ and a operation $\theta^*=(\theta,n,v)$ of arity≥ 1, we note that we have a functor $P(\theta^*):A\to A$ taking a typical morphism $f:A\to B$ of $\mathbf{A}$ to $f^n=P(f_1,\dots,f_n):P(A_1,\dots,A_n)\to P(B_1,\dots,B_n)$, where $f_i:A_i\to B_i$ is either $f_i:A_i\to B_i$ or $f'_i:A'_i\to B'_i$ according as $v(i)=+1$ or -1.

If now f, g are a pair of parallel pair of morphisms in $\mathbf{GDF}$ from (X,G,F) to (Y,G,F) for which $M(g)$, $M(f)$ have a split coequaliser $(e,(Z,G,F'))$ in $\mathbf{GD'F'}$, this split coequaliser being an absolute coequaliser, gives rise to a coequaliser $(P(\theta^*)(e)$, $P(\theta^*)(Z,G))$ for $P(\theta^*)(M(f))$ and $P(\theta^*)(M(g))$. If θ^* is one of the operations from F, since (X,G,F) and (Y,G,F) are closed for θ^*, we have morphisms θ^{*Y}: $P(\theta^*)(Y,G))\to(Y,G)$ and $\theta^{*X}:P(\theta^*)(X,G)\to(X,G)$ such that $\theta^{*Y}\cdot P(\theta^*)(M(f))=M(f)\cdot\theta^{*X}$ and $\theta^{*Y}\cdot P(\theta^*)(M(g))=M(g)\cdot\theta^{*X}$, for the $M(f)$, $M(g)$ which are homomorphisms for θ^* (from F), even as f, g are, since M is an inclusion functor.

These equations give the equality $e\cdot\theta^{*Y}\cdot P(\theta^*)(M(f))=e\cdot\theta^{*Y}\cdot P(\theta^*)(M(g))$, since $e\cdot M(f)=e\cdot M(g)$. Since $P(\theta^*)(e)$ is a coequaliser for the pair $P(\theta^*)(M(f))$, $P(\theta^*)(M(g))$, it follows that there must be a unique morphism $h:P(\theta^*)(Z,G)\to(Z,G)$ such that $e\cdot\theta^{*Y}=h\cdot P(\theta^*)(e)$. We call this θ^{*Z}; so now e is a homomorphism of (Y,G) in (Z,G) relative to θ^*. This way each nonnullary θ^* of F can be defined on (Z,G) to make e a homomorphism. For a nullary θ from F, we set $\theta(Z)=e(\theta(Y))$, so that we finally have a GF-algebra (Z,G,F) with e a F-homomorphism, and an epimorphism from (Y,G,F) to (Z,G,F). From the nature of the G's we are using, the fact that e is an epimorphism (being a coequaliser) ensures that the GF-algebra (Z,F) is a DF-algebra just as (Y,F) is. Thus the split coequaliser e for f, g has been created, as required in Beck's theorem. The uniqueness of the e and (Z,G,F) would follow from the fact that M is a faithful functor.

Given one of these categories $\mathbf{G}$ and an object A from it, we can now call an object A^* of $\mathbf{GF}$ (resp. of $\mathbf{GDF}$) a *free F-algebra* (a *free DF-algebra*) over A, if A^* is the image of A under a left-adjoint of the functor $T:\mathbf{GF}\to\mathbf{G}$ (the functor $T\cdot G:\mathbf{GDF}\to\mathbf{G}$).

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