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Central ideals and Cartan invariants of symmetric algebras

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Abstract

In this paper, we investigate certain ideals in the center of a symmetric algebra A over an algebraically closed field of characteristic $p > 0$. These ideals include the Higman ideal and the Reynolds ideal. They are closely related to the p-power map on A. We generalize some results concerning these ideals from group algebras to symmetric algebras, and we obtain some new results as well. In case $p = 2$, these ideals detect odd diagonal entries in the Cartan matrix of A. In a sequel to this paper, we will apply our results to group algebras.

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1. Introduction

Let A be a symmetric algebra over an algebraically closed field F of characteristic $p > 0$, with symmetrizing bilinear form $(.\mid.)$. In this paper we investigate the following chain of ideals of the center $\mathbf{Z}A$ of A:

$$\mathbf{Z}A \supseteq \mathbf{T}_1A^\perp \supseteq \mathbf{T}_2A^\perp \supseteq \cdots \supseteq \mathbf{R}A \supseteq \mathbf{H}A \supseteq \mathbf{Z}_0A \supseteq 0;$$

here $\mathbf{Z}_0A := \sum_B \mathbf{Z}B$ where B ranges over the set of blocks of A which are simple F-algebras. Thus $\mathbf{Z}_0A$ is a direct product of copies of F, one for each simple block B of A. Furthermore, $\mathbf{H}A$ denotes the *Higman ideal* of A, defined as the image of the *trace map*

$$\tau : A \to A, \quad x \mapsto \sum_{i=1}^{n} b_i x a_i;$$

here $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are a pair of dual bases of A. Moreover, $\mathbf{R}A$ is the *Reynolds ideal* of A, defined as the intersection of the socle $\mathbf{S}A$ of A and the center $\mathbf{Z}A$ of A. The ideals $\mathbf{T}_nA^\perp$ ($n \in \mathbb{N}$) were introduced in [6, II]; they can be viewed as generalizations of the Reynolds ideal. In fact, $\mathbf{R}A$ is their intersection. These ideals are defined in terms of the p-power map $A \to A$, $x \mapsto x^p$, and the bilinear form $(.\mid.)$. The precise definition will be given below. Motivated by the special case of group algebras [8,9], we show that

$$\mathbf{Z}_0A \subseteq \left(\mathbf{T}_1A^\perp\right)^2 \subseteq \mathbf{H}A,$$

so that $(\mathbf{T}_1A^\perp)^2$ fits nicely into the chain of ideals above. When p is odd then

$$\left(\mathbf{T}_1A^\perp\right)^2 = \mathbf{Z}_0A.$$

The case $p = 2$ behaves differently and turns out to have some interesting special features. We show that, in this case,

$$\left(\mathbf{T}_1A^\perp\right)^3 = \left(\mathbf{T}_1A^\perp\right)\left(\mathbf{T}_2A^\perp\right) = \mathbf{Z}_0A,$$

but that $(\mathbf{T}_1A^\perp)^2 \neq \mathbf{Z}_0A$ in general. We prove that, in case $p = 2$, the mysterious ideal $(\mathbf{T}_1A^\perp)^2$ is a principal ideal of $\mathbf{Z}A$. It is generated by the element $\zeta_1(1)^2$ where $\zeta_1 : \mathbf{Z}A \to \mathbf{Z}A$ is a certain natural semilinear map related to the p-power map. The map ζ_1 was first defined in [6, IV].

Moreover, in case $p = 2$, the dimension of $(\mathbf{T}_1A^\perp)^2$ is the number of blocks B of A with the property that the Cartan matrix $C_B = (c_{ij})$ of B contains an odd diagonal entry c_{ii}. A primitive idempotent e in A satisfies $e\zeta_1(1)^2 \neq 0$ if and only if the dimension of eAe is odd.

At the end of the paper, we investigate the behaviour of the ideals $\mathbf{T}_nA^\perp$ under Morita and derived equivalences, and we dualize some of the results obtained in the previous sections. In a sequel [2] to this paper, we will apply our results to group algebras of finite groups. We will see that a finite group G contains a real conjugacy class of 2-defect zero if and only if the Cartan matrix of G in characteristic 2 contains an odd diagonal entry. We will also prove a number of related facts.

2. The Reynolds ideal and its generalizations

In the following, let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F-algebra with symmetrizing bilinear form $(.\mid.)$. Thus A is a finite-dimensional associative unitary F-algebra, and $(.\mid.)$ is a non-degenerate symmetric bilinear form on A which is associative, in the sense that $(ab \mid c) = (a \mid bc)$ for $a, b, c \in A$. We denote the center of A by $\mathbf{Z}A$, the Jacobson radical of A by $\mathbf{J}A$, the socle of A by $\mathbf{S}A$ and the commutator subspace of A by $\mathbf{K}A$. Thus $\mathbf{K}A$ is the F-subspace of A spanned by all commutators $ab - ba$ $(a, b \in A)$. For $n \in \mathbb{N}$,

$$\mathbf{T}_n A := \left\{x \in A\colon\ x^{p^n} \in \mathbf{K}A\right\}$$

is a $\mathbf{Z}A$-submodule of A, so that

$$\mathbf{K}A = \mathbf{T}_0 A \subseteq \mathbf{T}_1 A \subseteq \mathbf{T}_2 A \subseteq \cdots$$

and

$$\sum_{n=0}^{\infty} \mathbf{T}_n A = \mathbf{J}A + \mathbf{K}A$$

(cf. [7]). For any F-subspace X of A, we set

$$X^{\perp} := \left\{y \in A\colon\ (x \mid y) = 0 \text{ for } x \in X\right\}.$$

Then

$$\mathbf{Z}A = \mathbf{K}A^{\perp} = \mathbf{T}_0 A^{\perp} \supseteq \mathbf{T}_1 A^{\perp} \supseteq \mathbf{T}_2 A^{\perp} \supseteq \cdots$$

is a chain of ideals of $\mathbf{Z}A$ such that

$$\bigcap_{n=0}^{\infty} \mathbf{T}_n A^{\perp} = \mathbf{S}A \cap \mathbf{Z}A.$$

We call $\mathbf{R}A := \mathbf{S}A \cap \mathbf{Z}A$ the *Reynolds ideal* of $\mathbf{Z}A$, in analogy to the terminology used for group algebras. For $n \in \mathbb{N}$ and $z \in \mathbf{Z}A$, there is a unique element $\zeta_n(z) \in \mathbf{Z}A$ such that

$$\left(\zeta_n(z) \mid x\right)^{p^n} = \left(z \mid x^{p^n}\right) \quad \text{for } x \in A.$$

This defines a map $\zeta_n = \zeta_n^A : \mathbf{Z}A \to \mathbf{Z}A$ with the following properties:

Lemma 2.1. *Let $m, n \in \mathbb{N}$, and let $y, z \in \mathbf{Z}A$. Then the following holds*:

(i) $\zeta_n(y+z) = \zeta_n(y) + \zeta_n(z)$ *and* $\zeta_n(y)z = \zeta_n(yz^{p^n})$.
(ii) $\zeta_m \circ \zeta_n = \zeta_{m+n}$.
(iii) $\mathrm{Im}(\zeta_n) = \mathbf{T}_n A^{\perp}$.
(iv) $\zeta_n^A(z)e = \zeta_n^{eAe}(ze)$ *for every idempotent e in A.*

Proof. (i)–(iii) are proved in [7, (44)–(47)].

(iv) Recall that eAe is a symmetric F-algebra; a corresponding symmetric bilinear form is obtained by restricting $(. \mid .)$ to eAe. Note that $ez = eze \in e\mathbf{Z}Ae \subseteq \mathbf{Z}(eAe)$ and that, similarly, $\zeta_n^A(z)e \in \mathbf{Z}(eAe)$. Moreover, for $x \in eAe$, we have

$$\begin{aligned}\big(\zeta_n^A(z)e \mid x\big)^{p^n} &= \big(\zeta_n^A(z) \mid ex\big)^{p^n} = \big(\zeta_n^A(z) \mid x\big)^{p^n} = \big(z \mid x^{p^n}\big)\\ &= \big(z \mid ex^{p^n}\big) = \big(ze \mid x^{p^n}\big) = \big(\zeta_n^{eAe}(ze) \mid x\big)^{p^n},\end{aligned}$$

and the result follows. □

We apply these properties in order to prove:

Lemma 2.2. *Let $m, n \in \mathbb{N}$. Then*

$$\big(\mathbf{T}_m A^{\perp}\big)\big(\mathbf{T}_n A^{\perp}\big) \subseteq \zeta_{m+n}\Big(\big(\mathbf{T}_n A^{\perp}\big)^{p^n(p^m-1)}\Big) \subseteq \mathbf{T}_{m+n} A^{\perp}.$$

Proof. Let $y, z \in \mathbf{Z}A$. Then Lemma 2.1 implies that

$$\begin{aligned}\zeta_m(y)\zeta_n(z) &= \zeta_m\big(y\zeta_n(z)^{p^m}\big) = \zeta_m\big(\zeta_n\big(y^{p^n}z\big)\zeta_n(z)^{p^m-1}\big)\\ &= \zeta_m\big(\zeta_n\big(y^{p^n}z\zeta_n(z)^{p^n(p^m-1)}\big)\big) \in \zeta_{m+n}\Big(\big(\mathbf{T}_n A^{\perp}\big)^{p^n(p^m-1)}\Big).\end{aligned}$$

Thus the result follows from Lemma 2.1(iii). □

Let $B_1, \ldots, B_r$ denote the blocks of A, so that $A = B_1 \oplus \cdots \oplus B_r$. Each B_i is itself a symmetric F-algebra. If a block B_i is a simple F-algebra then $B_i \cong \mathrm{Mat}(d_i, F)$ for a positive integer d_i, and thus $\mathbf{Z}B_i \cong F$. We set

$$\mathbf{Z}_0 A := \sum_i \mathbf{Z}B_i,$$

where the sum ranges over all $i \in \{1, \ldots, r\}$ such that B_i is a simple F-algebra. Then $\mathbf{Z}_0 A$ is an ideal of $\mathbf{Z}A$ and an F-algebra which is isomorphic to a direct sum of copies of F. Its dimension is the number of simple blocks of A. We exploit Lemma 2.2 in order to prove:

Theorem 2.3.

(i) $(\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{R}A$.
(ii) $(\mathbf{T}_1 A^\perp)(\mathbf{T}_2 A^\perp) = (\mathbf{T}_1 A^\perp)^3 = \mathbf{Z}_0 A$.
(iii) *If p is odd, then* $(\mathbf{T}_1 A^\perp)^2 = \mathbf{Z}_0 A$.

Proof. (i) Lemma 2.2 implies

$$\left(\mathbf{T}_1 A^\perp\right)^2 \subseteq \zeta_2\left(\left(\mathbf{T}_1 A^\perp\right)^{p(p-1)}\right) \subseteq \zeta_2\left(\left(\mathbf{T}_1 A^\perp\right)^2\right).$$

Iteration yields

$$\left(\mathbf{T}_1 A^\perp\right)^2 \subseteq \zeta_2\left(\zeta_2\left(\left(\mathbf{T}_1 A^\perp\right)^2\right)\right) = \zeta_4\left(\left(\mathbf{T}_1 A^\perp\right)^2\right) \subseteq \zeta_6\left(\left(\mathbf{T}_1 A^\perp\right)^2\right) \subseteq \cdots.$$

Thus

$$\left(\mathbf{T}_1 A^\perp\right)^2 \subseteq \bigcap_{n=0}^{\infty} \Im(\zeta_{2n}) = \bigcap_{n=0}^{\infty} \mathbf{T}_{2n} A^\perp = \mathbf{S}A \cap \mathbf{Z}A = \mathbf{R}A,$$

by Lemma 2.1(iii).

(ii) It is easy to see that $\mathbf{T}_n A = \mathbf{T}_n B_1 \oplus \cdots \oplus \mathbf{T}_n B_r$ and $\mathbf{T}_n A^\perp = \mathbf{T}_n B_1^\perp \oplus \cdots \oplus \mathbf{T}_n B_r^\perp$ for $n \in \mathbb{N}$ where $\mathbf{T}_n B_i^\perp = \{x \in B_i\colon\ (x \mid \mathbf{T}_n B_i) = 0\}$ for $i = 1, \ldots, r$. So we may assume that A itself is a block.

If A is simple, then $\mathbf{J}A = 0$, so $\mathbf{T}_n A = \mathbf{K}A$ and $\mathbf{T}_n A^\perp = \mathbf{Z}A$ for all $n \in \mathbb{N}$. Hence

$$\mathbf{Z}A = \left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_2 A^\perp\right) = \left(\mathbf{T}_1 A^\perp\right)^3$$

in this case.

Now suppose that A is non-simple. Then $\mathbf{J}A \neq 0$. So $\mathbf{Z}A \neq \mathbf{R}A$. It follows that $\mathbf{J}A + \mathbf{K}A \neq \mathbf{K}A$, whence $\mathbf{J}A$ is not contained in $\mathbf{K}A$. So $\mathbf{T}_1 A \neq \mathbf{K}A$. This means that $\mathbf{T}_1 A^\perp$ is a proper ideal of $\mathbf{Z}A$. Since $\mathbf{Z}A$ is a local F-algebra this implies that $\mathbf{T}_1 A^\perp \subseteq \mathbf{JZ}A \subseteq \mathbf{J}A$. Thus we may conclude, using (i), that $(\mathbf{T}_1 A^\perp)^3 \subseteq (\mathbf{R}A)(\mathbf{J}A) = 0$. Hence Lemma 2.2 yields

$$\left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_2 A^\perp\right) \subseteq \zeta_3\left(\left(\mathbf{T}_2 A^\perp\right)^{p^2(p-1)}\right) \subseteq \zeta_3\left(\left(\mathbf{T}_1 A^\perp\right)^3\right) = \zeta_3(0) = 0.$$

(iii) Suppose that p is odd. As in the proof of (ii), we may assume that A is a block, and that A is non-simple. Then Lemma 2.2 and (ii) imply that

$$\left(\mathbf{T}_1 A^\perp\right)^2 \subseteq \zeta_2\left(\left(\mathbf{T}_1 A^\perp\right)^{p(p-1)}\right) \subseteq \zeta_2\left(\left(\mathbf{T}_1 A^\perp\right)^3\right) = \zeta_2(0) = 0,$$

and the result is proved. □

Theorem 2.3 extends [9, Theorem 9] from group algebras to symmetric algebras. We will later improve on part (i). But first we note the following consequence.

Corollary 2.4. *Suppose that A is a block, and denote the central character of A by $\omega : \mathbf{Z}A \to F$. Moreover, let $m, n \in \mathbb{N}$ with $m \neq 0 \neq n$, and let $x, y \in \mathbf{Z}A$. Then*

$$\zeta_m(x)\zeta_n(y) = \omega(x)^{p^{-m}}\omega(y)^{p^{-n}}\zeta_m(1)\zeta_n(1).$$

In particular, we have

$$\left(\mathbf{T}_m A^{\perp}\right)\left(\mathbf{T}_n A^{\perp}\right) = F\zeta_m(1)\zeta_n(1),$$

so that $\dim(\mathbf{T}_m A^{\perp})(\mathbf{T}_n A^{\perp}) \leqslant 1$.

Proof. Theorem 2.3(i) implies that $\zeta_m(x)^{p^n} \in \mathbf{R}A \subseteq \mathbf{S}A$. Thus

$$\zeta_m(x)^{p^n} y = \omega(y)\zeta_m(x)^{p^n}.$$

Similarly, we have $x\zeta_n(1)^{p^m} = \omega(x)\zeta_n(1)^{p^m}$. So we conclude that

$$\begin{aligned}\zeta_m(x)\zeta_n(y) &= \zeta_n\left(\zeta_m(x)^{p^n} y\right) = \zeta_n\left(\omega(y)\zeta_m(x)^{p^n}\right) = \omega(y)^{p^{-n}}\zeta_m(x)\zeta_n(1)\\ &= \omega(y)^{p^{-n}}\zeta_m\left(x\zeta_n(1)^{p^m}\right) = \omega(y)^{p^{-n}}\zeta_m\left(\omega(x)\zeta_n(1)^{p^m}\right)\\ &= \omega(y)^{p^{-n}}\omega(x)^{p^{-m}}\zeta_m(1)\zeta_n(1).\end{aligned}$$

The remaining assertions follow from Lemma 2.1(iii). □

We can generalize part of Corollary 2.4 in the following way.

Proposition 2.5. *Let $m, n \in \mathbb{N}$ with $m \neq 0 \neq n$. Then*

$$\left(\mathbf{T}_m A^{\perp}\right)\left(\mathbf{T}_n A^{\perp}\right) = \mathbf{Z}A \cdot \zeta_m(1)\zeta_n(1)$$

is a principal ideal of $\mathbf{Z}A$. If p is odd, or if $m + n > 2$, then the dimension of $(\mathbf{T}_m A^{\perp})(\mathbf{T}_n A^{\perp})$ equals the number of simple blocks of A and in particular does not depend on $m + n$.

Proof. It is easy to see that we may assume that A is a block. In this case the assertion follows from Corollary 2.4 and Theorem 2.3. □

In the next two sections, we will handle the remaining case $p = 2$ and $m = n = 1$. Here we just illustrate this exceptional case by an example.

Let G be a finite group. Then the group algebra FG is a symmetric F-algebra; a symmetrizing bilinear form on FG satisfies

$$(g \mid h) = \begin{cases} 1, & \text{if } gh = 1,\\ 0, & \text{otherwise,}\end{cases}$$

for $g, h \in G$. An element $g \in G$ is called *real* if g is conjugate to its inverse g^{-1}, and g is said to be of *p-defect zero* if $|\mathbf{C}_G(g)|$ is not divisible by p. We denote the set of all real elements of 2-defect zero in G by R_G. For a subset X of G, we set

$$X^+ := \sum_{x \in X} x \in FG.$$

It was proved in [8, Proposition 4.1] that $R_G^+ = \zeta_1(1)^2 \in (\mathbf{T}_1 FG^\perp)^2$, in case $p = 2$.

Example 2.6. Let $p = 2$, and suppose that G is the symmetric group S_4 of degree 4. Then FG has no simple blocks; in fact, FG has just one block, the principal one. Thus $\mathbf{Z}_0 FG = 0$. On the other hand, R_G is precisely the set of all 3-cycles in S_4. Thus $0 \neq R_G^+ \in (\mathbf{T}_1 FG^\perp)^2$. (In fact, $(\mathbf{T}_1 FG^\perp)^2$ is one-dimensional, by Corollary 2.4.) This example shows that $(\mathbf{T}_1 A^\perp)^2 \neq \mathbf{Z}_0 A$, in general.

3. Odd Cartan invariants

Let F be an algebraically closed field of characteristic $p = 2$, and let A be a symmetric F-algebra with symmetrizing bilinear form $(. \mid .)$. In this section, we will prove some remarkable properties of the ideal $(\mathbf{T}_1 A^\perp)^2$ of $\mathbf{Z}A$. We start by recalling some known facts concerning symmetric bilinear forms over F.

Lemma 3.1. *Let V be a finite-dimensional vector space over F, and let $\langle . \mid . \rangle$ be a non-degenerate symmetric bilinear form on V. Then either $\langle . \mid . \rangle$ is symplectic (i.e., $\langle v \mid v \rangle = 0$ for every $v \in V$), or there exists an orthonormal basis $v_1, \ldots, v_n$ of V (i.e., $\langle v_i \mid v_j \rangle = \delta_{ij}$ for $i, j = 1, \ldots, n$).*

Proof. This can be found in [4, Hauptsatz V.3.5], for example. □

If $\langle . \mid . \rangle$ is symplectic, then there exists a symplectic basis $v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m}$ of V, i.e.,

$$\langle v_i \mid v_{m+i} \rangle = \langle v_{m+i} \mid v_i \rangle = 1 \quad \text{for } i = 1, \ldots, m,$$
$$\langle v_i \mid v_j \rangle = 0 \quad \text{otherwise}$$

(cf. [4, Hauptsatz V.4.10]). Thus there exist only two types of non-degenerate symmetric bilinear forms on a finite-dimensional vector space V over F, a symplectic one and a non-symplectic one. In the symplectic case, the dimension of V has to be even.

We now apply Lemma 3.1 to the symmetrizing bilinear form $(. \mid .)$ on A.

Lemma 3.2.

$$\bigl(\zeta_1(1) \mid \zeta_1(1)\bigr) = (\dim A) \cdot 1_F.$$

Proof. By Lemma 3.1, there exists an F-basis

$$a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m}, a_{2m+1}, \ldots, a_n$$

of A such that

$$(a_i \mid a_{m+i}) = (a_{m+i} \mid a_i) = 1 \quad \text{for } i = 1, \ldots, m,$$
$$(a_i \mid a_i) = 1 \quad \text{for } i = 2m+1, \ldots, n,$$
$$(a_i \mid a_j) = 0 \quad \text{otherwise}$$

(and either $n = 2m$ or $m = 0$). Then the dual basis $b_1, \ldots, b_n$ of $a_1, \ldots, a_n$ is given by

$$a_{m+1}, \ldots, a_{2m}, a_1, \ldots, a_m, a_{2m+1}, \ldots, a_n.$$

Thus $(\zeta_1(1) \mid a_i)^2 = (1 \mid a_i^2) = (a_i \mid a_i) = (a_i \mid a_i)^2$ for $i = 1, \ldots, n$, so

$$\zeta_1(1) = \sum_{i=1}^{n} \bigl(\zeta_1(1) \mid a_i\bigr) b_i = \sum_{i=1}^{n} (a_i \mid a_i) b_i = \sum_{i=2m+1}^{n} a_i$$

and

$$\bigl(\zeta_1(1) \mid \zeta_1(1)\bigr) = \sum_{i,j=2m+1}^{n} (a_i \mid a_j) = \sum_{i=2m+1}^{n} (a_i \mid a_i) = (n-2m) \cdot 1_F = n \cdot 1_F$$
$$= (\dim A) \cdot 1_F,$$

and the result is proved. □

The next statement holds in arbitrary characteristic. It is essentially taken from [10, Corollary (1.G)].

Lemma 3.3. *Let e be a primitive idempotent in A, and let $r \in \mathbf{R}A$. Then $er = 0$ if and only if $(e \mid r) = 0$.*

Proof. If $er = 0$, then $0 = (er \mid 1) = (e \mid r)$. Conversely, if $(e \mid r) = 0$ then

$$(eAe \mid ere) = (eAe \mid r) = \bigl(Fe + \mathbf{J}(eAe) \mid r\bigr) \subseteq F(e \mid r) + (\mathbf{J}A \cdot r \mid 1) = 0.$$

Thus $0 = ere = er$ since the restriction of $(. \mid .)$ to eAe is non-degenerate. □

Now we choose representatives $a_1 = e_1, \ldots, a_l = e_l$ for the conjugacy classes of primitive idempotents in A. (This means that $Ae_1, \ldots, Ae_l$ are representatives for the isomorphism classes of indecomposable projective left A-modules.) Moreover, we let $a_{l+1}, \ldots, a_n$ denote an F-basis of $\mathbf{J}A + \mathbf{K}A$. Then $a_1, \ldots, a_n$ form an F-basis of A.

Let $b_1, \dots, b_n$ denote the dual basis of $a_1, \dots, a_n$. Then $r_1 := b_1, \dots, r_l := b_l$ are contained in $(\mathbf{J}A + \mathbf{K}A)^\perp = \mathbf{S}A \cap \mathbf{Z}A = \mathbf{R}A$, so they form an F-basis of $\mathbf{R}A$. Moreover, Lemma 3.3 implies that $e_i r_j = 0$ for $i \neq j$ and $e_i r_i \neq 0$ for $i = 1, \dots, l$.

Lemma 3.4. *With $e_1, \dots, e_l$ as above, we have $\zeta_1(1)^2 = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i$ and $e_i \zeta_1(1)^2 = (\dim e_i A e_i) \cdot e_i r_i$ for $i = 1, \dots, l$.*

Proof. Lemma 2.1(iii) and Theorem 2.3(i) imply that $\zeta_1(1)^2 \in (\mathbf{T}_1 A^\perp)^2 \subseteq \mathbf{R}A$. By making use of Lemma 2.1(iv) and Lemma 3.2, we obtain

$$\zeta_1(1)^2 = \sum_{i=1}^l \big(\zeta_1(1)^2 \mid e_i\big) r_i = \sum_{i=1}^l \big(\zeta_1(1) e_i \mid \zeta_1(1) e_i\big) r_i$$
$$= \sum_{i=1}^l \big(\zeta_1^{e_i A e_i}(e_i) \mid \zeta_1^{e_i A e_i}(e_i)\big) r_i = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i.$$

Since $e_i r_j = 0$ for $i \neq j$ the result follows. □

The next theorem is the main result of this section.

Theorem 3.5. *For A a symmetric algebra over an algebraically closed field F of characteristic 2 and for e a primitive idempotent in A, the following assertions are equivalent*:

(1) $\dim eAe$ *is even.*
(2) $e\zeta_1(1)^2 = 0$.
(3) $(e|\zeta_1(1)^2) = 0$.

Proof. We may assume that $e = e_i$ for some $i \in \{1, \dots, l\}$. Then $e_i \zeta_1(1)^2 = (\dim e_i A e_i) \cdot e_i r_i$ with $e_i r_i \neq 0$, by Lemma 3.4. This shows that (1) and (2) are equivalent. Since $\zeta_1(1)^2 \in \mathbf{R}A$, Lemma 3.3 implies that (2) and (3) are equivalent. □

The Cartan matrix $C := (c_{ij})_{i,j=1}^l$ of A is defined by

$$c_{ij} := \dim e_i A e_j \quad \text{for } i, j = 1, \dots, l.$$

Thus C is a symmetric matrix with non-negative integer coefficients, the Cartan invariants of A. Hence Theorem 3.5 has the following consequence.

Corollary 3.6. *With the notation for the Cartan matrix of A as above, $\zeta_1(1)^2 \neq 0$ if and only if c_{ii} is odd for some i. More precisely, for a block B of A, we have $\zeta_1(1)^2 1_B \neq 0$ if and only if the Cartan matrix of B contains an odd diagonal entry.*

In order to illustrate Corollary 3.6, recall that, by Example 2.6, the group algebra FG, for $G = S_4$, satisfies $\zeta_1(1)^2 = R_G^+ \neq 0$. Thus the Cartan matrix of FG contains an odd diagonal entry, by Corollary 3.6. Indeed, the Cartan matrix of FG is

$$C := \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix},$$

as is well known. More substantial examples will be presented in [2].

It may be of interest to note that the existence of odd diagonal Cartan invariants in characteristic 2 is invariant under derived equivalences (cf. [5]).

Proposition 3.7. *Let A' be a symmetric F-algebra which is derived equivalent to A. Then the Cartan matrix of A' contains an odd diagonal entry if and only if the Cartan matrix of A does.*

Proof. It is known that the Cartan matrices $C = (c_{ij})_{i,j=1}^{l}$ of A and $C' = (c'_{ij})_{i,j=1}^{l}$ of A' have the same format, and that they are related by an equation

$$C' = Q \cdot C \cdot Q^{\top},$$

where $Q = (q_{ij})_{i,j=1}^{l}$ is an integral matrix with determinant ± 1 (cf. [5]). Thus

$$c'_{ii} = \sum_{j,k=1}^{l} q_{ij} q_{ik} c_{jk} \equiv \sum_{j=1}^{l} q_{ij}^2 c_{jj} \pmod 2$$

for $i = 1, \ldots, l$. If c'_{ii} is odd then c_{jj} has to be odd for some $j \in \{1, \ldots, l\}$ (and conversely). □

4. The Higman ideal

Let F be an algebraically closed field, and let A be a symmetric F-algebra with symmetrizing bilinear form $(.\,|\,.)$. Moreover, let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ denote a pair of dual bases of A. In the following, the F-linear map

$$\tau : A \to A, \quad x \mapsto \sum_{i=1}^{n} b_i x a_i,$$

will be of interest (cf. [3, §66]). We record the following properties of this *trace map* τ:

Lemma 4.1.

(i) *τ is independent of the choice of dual bases.*
(ii) *τ is self-adjoint with respect to* $(.\,|\,.)$.
(iii) $\mathrm{Im}(\tau) \subseteq \mathbf{S}A \cap \mathbf{Z}A = \mathbf{R}A$ *and* $\mathbf{J}A + \mathbf{K}A \subseteq \mathrm{Ker}(\tau)$.

Proof. (i) Let $a'_1,\dots,a'_n$ and $b'_1,\dots,b'_n$ be another pair of dual bases of A. Then $b'_i=\sum_{j=1}^n(a_j\mid b'_i)b_j$ and $a_i=\sum_{j=1}^n(a_i\mid b'_j)a'_j$ for $i=1,\dots,n$. Thus

$$\sum_{i=1}^n b'_i x a'_i=\sum_{i,j=1}^n \bigl(a_j\mid b'_i\bigr)b_j x a'_i=\sum_{j=1}^n b_j x\sum_{i=1}^n\bigl(a_j\mid b'_i\bigr)a'_i=\sum_{j=1}^n b_j x a_j$$

for $x\in A$.

(ii) Let $x,y\in A$. Then, by (i), we get

$$\bigl(\tau(x)\mid y\bigr)=\sum_{i=1}^n(b_i x a_i\mid y)=\sum_{i=1}^n(x\mid a_i y b_i)=\bigl(x\mid\tau(y)\bigr).$$

(iii) Let $x,y\in A$. Then

$$\tau(x)y=\sum_{i=1}^n b_i x a_i y=\sum_{i,j=1}^n b_i x(a_i y\mid b_j)a_j=\sum_{i,j=1}^n(a_i\mid y b_j)b_i x a_j$$
$$=\sum_{j=1}^n y b_j x a_j=y\tau(x).$$

Hence $\mathrm{Im}(\tau)\subseteq\mathbf{Z}A$. In order to prove $\mathrm{Im}(\tau)\subseteq\mathbf{S}A$, we choose $a_1,\dots,a_n$ appropriately. Indeed, we may assume that $a_1+\mathbf{J}A,\dots,a_r+\mathbf{J}A$ form an F-basis of $A/\mathbf{J}A$, that $a_{r+1}+(\mathbf{J}A)^2,\dots,a_s+(\mathbf{J}A)^2$ form an F-basis of $(\mathbf{J}A)/(\mathbf{J}A)^2$, that $a_{s+1}+(\mathbf{J}A)^3,\dots,a_t+(\mathbf{J}A)^3$ form an F-basis of $(\mathbf{J}A)^2/(\mathbf{J}A)^3$, etc. Then $b_1,\dots,b_r$ are contained in $(\mathbf{J}A)^\perp$, $b_1,\dots,b_s$ are contained in $((\mathbf{J}A)^2)^\perp$, $b_1,\dots,b_t$ are contained in $((\mathbf{J}A)^3)^\perp$, etc.

Now let $x\in A$ and $y\in\mathbf{J}A$. Then $b_i x a_i y\in(\mathbf{J}A)^\perp\cdot A\cdot A\cdot(\mathbf{J}A)=0$ for $i=1,\dots,r$, $b_i x a_i y\in((\mathbf{J}A)^2)^\perp\cdot A\cdot(\mathbf{J}A)\cdot(\mathbf{J}A)=0$ for $i=r+1,\dots,s$, $b_i x a_i y\in((\mathbf{J}A)^3)^\perp\cdot A\cdot(\mathbf{J}A)^2\cdot(\mathbf{J}A)=0$ for $i=s+1,\dots,t$, etc. We see that $\tau(x)y=0$, so $\mathrm{Im}(\tau)\subseteq\mathbf{S}A$.

Since τ is self-adjoint (i.e., $\tau^*=\tau$), we conclude that

$$\mathrm{Ker}(\tau)=\mathrm{Ker}(\tau^*)=\mathrm{Im}(\tau)^\perp\supseteq(\mathbf{S}A\cap\mathbf{Z}A)^\perp=\mathbf{J}A+\mathbf{K}A.\qquad\square$$

Thus $\mathbf{H}A:=\mathrm{Im}(\tau)$ is an ideal of $\mathbf{Z}A$ contained in $\mathbf{R}A$, called the *Higman ideal* of $\mathbf{Z}A$. By Lemma 4.1, it is independent of the choice of dual bases. In the following, we write

$$1_A=e_1+\cdots+e_m$$

with pairwise orthogonal primitive idempotents $e_1,\dots,e_m$ of A.

Lemma 4.2. *We have* $(\tau(e_i)\mid e_j)=(\dim e_i A e_j)\cdot 1_F$ *for* $i,j=1,\dots,m$.

Proof. We consider the decomposition $A=\bigoplus_{i,j=1}^m e_i A e_j$. For $i,j=1,\dots,m$, let X_{ij} be an F-basis of $e_i A e_j$. Then $X:=\bigcup_{i,j=1}^m X_{ij}$ is an F-basis of A. We denote the dual basis of X by X^*. For $x \in X$, there is a unique $x^* \in X^*$ such that $(x \mid x^*) = 1$. Then the map $X \to X^*$, $x \mapsto x^*$, is a bijection. Moreover, for $i, j = 1, \dots, m$, $X_{ij}^* := \{x^*\colon\ x \in X_{ij}\}$ is an F-basis of $e_j A e_i$. Thus

$$\tau(e_i)e_j = e_j\tau(e_i)e_j = \sum_{x\in X} e_j x^* e_i x e_j = \sum_{x\in X_{ij}} e_j x^* e_i x e_j = \sum_{x\in X_{ij}} x^* x$$

and

$$\begin{aligned}\big(\tau(e_i) \mid e_j\big) &= \big(\tau(e_i)e_j \mid 1\big) = \sum_{x\in X_{ij}} (x^*x \mid 1) = \sum_{x\in X_{ij}} (x^* \mid x) = |X_{ij}| \cdot 1_F \\ &= (\dim e_i A e_j) \cdot 1_F,\end{aligned}$$

so the result is proved. □

We may assume that $e_1, \dots, e_m$ are numbered in such a way that $a_1 := e_1, \dots, a_l := e_l$ represent the conjugacy classes of primitive idempotents in A. We choose an F-basis $a_{l+1}, \dots, a_n$ of $\mathbf{J}A + \mathbf{K}A$, so that $a_1, \dots, a_n$ form an F-basis of A. We denote the dual basis of $a_1, \dots, a_n$ by $b_1, \dots, b_n$. As above, $r_1 := b_1, \dots, r_l := b_l$ form an F-basis of $\mathbf{R}A = \mathbf{S}A \cap \mathbf{Z}A$.

Lemma 4.3. *We have $\tau(e_i) = \sum_{j=1}^{l} (\dim e_i A e_j) \cdot r_j$ for $i = 1, \dots, l$.*

Proof. Let $i \in \{1, \dots, l\}$. Then $\tau(e_i) \in \mathbf{H}A \subseteq \mathbf{R}A$, so

$$\tau(e_i) = \sum_{j=1}^{l} \big(\tau(e_i) \mid e_j\big) r_j = \sum_{j=1}^{l} (\dim e_i A e_j) \cdot r_j$$

by Lemma 4.2. □

In the following, suppose that $\operatorname{char} F = p > 0$. We know from Theorem 2.3 that $(\mathbf{T}_1 A^{\perp})^2 \subseteq \mathbf{R}A$. We are going to show that, more precisely, $(\mathbf{T}_1 A^{\perp})^2 \subseteq \mathbf{H}A$. In the proof, we will make use of the following fact.

Lemma 4.4. *Let $C = (c_{ij})$ be a symmetric $(n \times n)$-matrix with coefficients in the field $\mathbb{F}_2$ with two elements. Then its main diagonal $c := (c_{11}, c_{22}, \dots, c_{nn})$, considered as a vector in $\mathbb{F}_2^n$, is a linear combination of the rows of C.*

Proof. Arguing by induction on n, we may assume that $n > 1$. If $c = 0$, then there is nothing to prove. So we may assume that $c_{ii} = 1$ for some $i \in \{1, \dots, l\}$. Permuting the rows and columns of C, if necessary, we may assume that $c_{11} = 1$. We now perform elementary row operations on C. For $k = 2, \dots, n$, we subtract the first row, multiplied by c_{k1}, from the kth row. The resulting matrix C' has the entries

$$0, c_{k2} - c_{k1}c_{12}, \dots, c_{kn} - c_{k1}c_{1n}$$

in its kth row and the entries

$$c_{1k}, c_{2k} - c_{21}c_{1k}, \ldots, c_{nk} - c_{n1}c_{1k}$$

in its kth column. We now remove the first row and the first column from C' and end up with a symmetric $((n-1)\times(n-1))$-matrix D with diagonal entries

$$c_{kk} - c_{k1}c_{1k} = c_{kk} - c_{1k}^2 = c_{kk} - c_{1k} \quad (k = 2, \ldots, n).$$

On the other hand, if we subtract the first row of C from c, then we obtain the vector

$$c' := (0, c_{22} - c_{12}, \ldots, c_{nn} - c_{1n}).$$

Thus the vector $d := (c_{22} - c_{12}, \ldots, c_{nn} - c_{1n})$ coincides with the main diagonal of D. By induction, d is a linear combination of the rows of D, so c is a linear combination of the rows of C. $\square$

As Gary McGuire kindly pointed out to us, a different proof of Lemma 4.4 can be found in [1, Proposition 4.6.2]. We apply Lemma 4.4 in the proof of the following result which is a refinement of Theorem 2.3(i). The special case of group algebras was first proved in [8, Lemma 5.1].

Theorem 4.5. *We always have* $(\mathbf{T}_1 A^{\perp})^2 \subseteq \mathbf{H}A$.

Proof. If p is odd then, by Theorem 2.3(iii), we have

$$\big(\mathbf{T}_1 A^{\perp}\big)^2 \subseteq \mathbf{Z}_0 A = \sum_B \mathbf{Z}B = \sum_B \mathbf{H}B \subseteq \mathbf{H}A,$$

where B ranges over the simple blocks of A; in fact, if $B = \mathrm{Mat}(d, F)$ for a positive integer d then $\mathbf{H}B = \mathbf{Z}B$.

Thus we may assume that $p = 2$. Then Lemma 2.2 gives us elements $\alpha_1, \ldots, \alpha_l$ in the prime field of F such that

$$\sum_{j=1}^{l} (\dim e_i A e_j) \cdot \alpha_j = (\dim e_i A e_i) \cdot 1_F \quad \text{for } i = 1, \ldots, l.$$

Thus Lemmas 3.4 and 4.3 imply that

$$\zeta_1(1)^2 = \sum_{i=1}^{l} (\dim e_i A e_i) \cdot r_i = \sum_{i,j=1}^{l} (\dim e_i A e_j) \cdot \alpha_j r_i = \sum_{j=1}^{l} \alpha_j \tau(e_j) \in \mathbf{H}A.$$

Hence Proposition 2.5 implies that $(\mathbf{T}_1 A^{\perp})^2 = \mathbf{Z}A \cdot \zeta_1(1)^2 \subseteq \mathbf{H}A$. $\square$

5. Morita invariance

Let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F-algebra. In this section we investigate the behaviour of the ideals $\mathbf{T}_n A^{\perp}$ of $\mathbf{Z}A$ under Morita equivalences. These results will be used in [2].

Proposition 5.1. *Let e be an idempotent in A such that $AeA = A$. Then the map*

$$f : \mathbf{Z}A \to \mathbf{Z}(eAe), \quad z \mapsto ez = ze,$$

is an isomorphism of F-algebras mapping $\mathbf{T}_n A^{\perp}$ onto $\mathbf{T}_n(eAe)^{\perp}$, for $n \in \mathbb{N}$.

Proof. Certainly f is a homomorphism of F-algebras. Let $z \in \mathbf{Z}A$ such that $0 = f(z) = ez$. Then $0 = AezA = AeAz = Az$, so that $z = 0$. Thus f is injective. Since $AeA = A$ the F-algebras A and eAe are Morita equivalent; in particular, their centers are isomorphic. Hence f is an isomorphism of F-algebras. Lemma 2.1(iv) implies that $f \circ \zeta_n^A = \zeta_n^{eAe} \circ f$, so

$$f\big(\mathbf{T}_n A^{\perp}\big) = f\big(\zeta_n^A(\mathbf{Z}A)\big) = \zeta_n^{eAe}\big(f(\mathbf{Z}A)\big) = \zeta_n^{eAe}\big(\mathbf{Z}(eAe)\big) = \mathbf{T}_n(eAe)^{\perp}$$

by Lemma 2.1(iii). □

We mention two consequences of Proposition 5.1.

Corollary 5.2. *Let d be a positive integer, and let A_d denote the symmetric F-algebra* $\mathrm{Mat}(d, A)$. *Then the map*

$$h : \mathbf{Z}A \to \mathbf{Z}A_d, \quad z \mapsto z1_d,$$

is an isomorphism of F-algebras mapping $\mathbf{T}_n A^{\perp}$ onto $(\mathbf{T}_n A_d)^{\perp}$, for $n \in \mathbb{N}$.

Proof. We denote the matrix units of A_d by e_{ij} ($i, j = 1, \ldots, d$). Then the map

$$f : A \to e_{11} A_d e_{11}, \quad a \mapsto a e_{11},$$

is an isomorphism of F-algebras. This implies that $f(\mathbf{Z}A) = \mathbf{Z}(e_{11} A_d e_{11})$ and $f(\mathbf{T}_n A^{\perp}) = \mathbf{T}_n(e_{11} A_d e_{11})^{\perp}$ for $n \in \mathbb{N}$. On the other hand, Proposition 5.1 implies that the map

$$g : \mathbf{Z}A_d \to \mathbf{Z}(e_{11} A_d e_{11}), \quad z \mapsto z e_{11} = e_{11} z,$$

is an isomorphism of F-algebras such that $g((\mathbf{T}_n A_d)^{\perp}) = \mathbf{T}_n(e_{11} A_d e_{11})^{\perp}$ for $n \in \mathbb{N}$. Now observe that h is an isomorphism of F-algebras such that $g \circ h$ is the restriction of f to $\mathbf{Z}A$. Thus $h(\mathbf{T}_n A^{\perp}) = (\mathbf{T}_n A_d)^{\perp}$ for $n \in \mathbb{N}$. □

Corollary 5.3. *Let B be a symmetric F-algebra which is Morita equivalent to A. Then there is an isomorphism of F-algebras $\mathbf{Z}A \to \mathbf{Z}B$ mapping $\mathbf{T}_n A^{\perp}$ onto $\mathbf{T}_n B^{\perp}$, for $n \in \mathbb{N}$.*

Proof. Let e be an idempotent in A such that eAe is a basic algebra of A, and let f be an idempotent in B such that fBf is a basic algebra of B. Then $AeA = A$ and $BfB = B$. Moreover, eAe and fBf are isomorphic since A and B are Morita equivalent. Thus Proposition 5.1 yields a chain of isomorphisms

$$\mathbf{Z}A \to \mathbf{Z}(eAe) \to \mathbf{Z}(fBf) \to \mathbf{Z}B$$

mapping $\mathbf{T}_n A^{\perp}$ onto $\mathbf{T}_n B^{\perp}$, for $n \in \mathbb{N}$. □

It would be interesting to know whether Corollary 5.3 extends to symmetric F-algebras which are derived equivalent (cf. [5]).

Question 5.4. Suppose that A and B are derived equivalent symmetric F-algebras. Is there an isomorphism of F-algebras $\mathbf{Z}A \to \mathbf{Z}B$ mapping $\mathbf{T}_n A^{\perp}$ onto $\mathbf{T}_n B^{\perp}$, for $n \in \mathbb{N}$?

6. Some dual results

Let F be an algebraically closed field of characteristic $p > 0$, and let A be a symmetric F-algebra. For $n \in \mathbb{N}$,

$$\mathbf{T}_n\mathbf{Z}A := \left\{z \in \mathbf{Z}A\colon\ z^{p^n} = 0\right\}$$

is an ideal of $\mathbf{Z}A$. In this way we obtain an ascending chain of ideals

$$0 = \mathbf{T}_0\mathbf{Z}A \subseteq \mathbf{T}_1\mathbf{Z}A \subseteq \mathbf{T}_2\mathbf{Z}A \subseteq \cdots \subseteq \mathbf{JZ}A \subseteq \mathbf{Z}A$$

of $\mathbf{Z}A$ such that

$$\sum_{n=0}^{\infty} \mathbf{T}_n\mathbf{Z}A = \mathbf{JZ}A.$$

This ascending chain of ideals turns out to be related to the descending chain of ideals

$$\mathbf{Z}A = \mathbf{T}_0 A^{\perp} \supseteq \mathbf{T}_1 A^{\perp} \supseteq \mathbf{T}_2 A^{\perp} \supseteq \cdots \supseteq \mathbf{R}A \supseteq 0$$

of $\mathbf{Z}A$ considered before.

Proposition 6.1. *Let $n \in \mathbb{N}$. Then* $(\mathbf{T}_n A^{\perp})(\mathbf{T}_n\mathbf{Z}A) = 0$.

Proof. Let $y \in \mathbf{Z}A$ and $z \in \mathbf{T}_n\mathbf{Z}A$, so that $z^{p^n} = 0$. Then Lemma 2.1(i) implies that

$$\zeta_n(y)z = \zeta_n\left(yz^{p^n}\right) = \zeta_n(y0) = \zeta_n(0) = 0.$$

Hence $(\mathbf{T}_n A^{\perp})(\mathbf{T}_n\mathbf{Z}A) = (\operatorname{Im}\zeta_n)(\mathbf{T}_n\mathbf{Z}A) = 0$, by Lemma 2.1(iii). □

The result above is essentially [9, Proposition 4]. We conclude that

$$\mathbf{T}_n\mathbf{Z}A \subseteq \{z \in \mathbf{Z}A\colon\ z(\mathbf{T}_nA^{\perp}) = 0\} \subseteq \{z \in \mathbf{Z}A\colon\ z\zeta_n(1) = 0\}.$$

In [2], we will see that these inclusions are proper in general, even for group algebras of finite groups. If n is sufficiently large then $\mathbf{T}_n\mathbf{Z}A = \mathbf{JZ}A$ and $\mathbf{T}_nA^{\perp} = \mathbf{R}A$, and certainly

$$\mathbf{JZ}A = \{z \in \mathbf{Z}A\colon\ z \cdot \mathbf{R}A = 0\}.$$

Also, if n is large and $A = FG$ for a finite group G then $\zeta_n(1) = G_p^+$ where G_p denotes the set of p-elements in G (cf. [7, (48)]), and it is known that

$$\mathbf{JZ}FG = \{z \in \mathbf{Z}FG\colon\ zG_p^+ = 0\}$$

(cf. [7, (59)]). However, it is easy to construct an example of a symmetric F-algebra A such that

$$\mathbf{JZ}A \neq \{z \in \mathbf{Z}A\colon\ z\zeta_n(1) = 0\}$$

for all sufficiently large n.

For $n \in \mathbb{N}$, the ideal $\mathbf{T}_n\mathbf{Z}A$ of $\mathbf{Z}A$ is related to a semilinear map $\kappa_n : A/\mathbf{K}A \to A/\mathbf{K}A$ first constructed in [6, IV]; κ_n is defined in such a way that

$$(z^{p^n} \mid x) = (z \mid \kappa_n(x))^{p^n} \quad \text{for } z \in \mathbf{Z}A \quad \text{and} \quad x \in A/\mathbf{K}A;$$

here we set $(z \mid a + \mathbf{K}A) := (z \mid a)$ for $z \in \mathbf{Z}A$ and $a \in A$. Also, we set $(a + \mathbf{K}A)^{p^n} := a^{p^n} + \mathbf{K}A$ for $a \in A$. We recall the following properties of κ_n (cf. [7, (50)–(53)]).

Lemma 6.2. *Let $m, n \in \mathbb{N}$, let $x, y \in A/\mathbf{K}A$, and let $z \in \mathbf{Z}A$. Then the following holds*:

(i) $\kappa_n(x + y) = \kappa_n(x) + \kappa_n(y)$, $z\kappa_n(x) = \kappa_n(z^{p^n}x)$ *and* $\kappa_n(zx^{p^n}) = \zeta_n(z)x$.
(ii) $\kappa_m \circ \kappa_n = \kappa_{m+n}$.
(iii) $\mathrm{Im}(\kappa_n) = \mathbf{T}_n\mathbf{Z}A^{\perp}/\mathbf{K}A$.

Our next result is a dual version of Theorem 2.3. For simplicity, we concentrate on the case where A is a non-simple block. (If A is a simple block then $\mathbf{T}_1\mathbf{Z}A = 0$, so $\mathbf{T}_1\mathbf{Z}A^{\perp} = A$. Moreover, we have $\mathbf{T}_2A^{\perp} = \mathbf{T}_1A^{\perp} = \mathbf{Z}A$ in this case.)

Proposition 6.3. *Suppose that A is a non-simple block. Then the following holds*:

(i) $(\mathbf{T}_1A^{\perp})(\mathbf{T}_1\mathbf{Z}A^{\perp}) \subseteq \mathbf{K}A$ *for* $p \neq 2$.
(ii) $(\mathbf{T}_2A^{\perp})(\mathbf{T}_1\mathbf{Z}A^{\perp}) \subseteq \mathbf{K}A$ *and* $(\mathbf{T}_1A^{\perp})(\mathbf{T}_2\mathbf{Z}A^{\perp}) \subseteq \mathbf{K}A$ *for* $p = 2$.
(iii) $(\mathbf{T}_1A^{\perp})(\mathbf{T}_1\mathbf{Z}A^{\perp}) \subseteq \mathbf{JZ}A^{\perp}$ *for* $p = 2$. *Moreover, in this case we have* $(\mathbf{T}_1A^{\perp}) \times (\mathbf{T}_1\mathbf{Z}A^{\perp}) \subseteq \mathbf{K}A$ *if and only if* $\zeta_1(1)^2 = 0$.

Proof. (i) Let $y \in \mathbf{Z}A$ and $x \in A/\mathbf{K}A$. Then $\zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^p x) = 0$ since $\zeta_1(y)^p \in (\mathbf{T}_1 A^\perp)^p = 0$ by Theorem 2.3(iii). Thus

$$\left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_1 \mathbf{Z}A^\perp/\mathbf{K}A\right) = (\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1) = 0,$$

and (i) is proved.

(ii) Let x, y be as in (i). Then $\zeta_2(y)\kappa_1(x) = \kappa_1(\zeta_2(y)^2 x) = 0$ since $\zeta_2(y)^2 \in (\mathbf{T}_2 A^\perp)^2 = 0$, by Theorem 2.3(ii). Thus

$$\left(\mathbf{T}_2 A^\perp\right)\left(\mathbf{T}_1 \mathbf{Z}A^\perp/\mathbf{K}A\right) = (\operatorname{Im}\zeta_2)(\operatorname{Im}\kappa_1) = 0.$$

Similarly, we have $\zeta_1(y)\kappa_2(x) = \kappa_2(\zeta_1(y)^4 x) = 0$ since $\zeta_1(y)^3 \in (\mathbf{T}_1 A^\perp)^3 = 0$ by Theorem 2.3(ii). Thus

$$\left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_2 \mathbf{Z}A^\perp/\mathbf{K}A\right) = (\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_2) = 0,$$

and (ii) follows.

(iii) Again, let x, y be as in (i). Then

$$\zeta_1(y)\kappa_1(x) = \kappa_1\left(\zeta_1(y)^2 x\right) = \kappa_1\left(\zeta_1(y)\kappa_1(yx^2)\right) \in \kappa_1\left((\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1)\right).$$

Iteration yields

$$\begin{aligned}(\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1) &\subseteq \kappa_1\left((\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1)\right) \subseteq \kappa_1\left(\kappa_1\left((\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1)\right)\right)\\ &= \kappa_2\left((\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1)\right) \subseteq \cdots.\end{aligned}$$

Thus

$$\left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_1 \mathbf{Z}A^\perp/\mathbf{K}A\right) = (\operatorname{Im}\zeta_1)(\operatorname{Im}\kappa_1) \subseteq \bigcap_{n=0}^{\infty} \operatorname{Im}(\kappa_n) = \bigcap_{n=0}^{\infty} \mathbf{T}_n \mathbf{Z}A^\perp/\mathbf{K}A = \mathbf{J}\mathbf{Z}A^\perp/\mathbf{K}A,$$

and the first assertion of (iii) is proved. Now note that $(\mathbf{T}_1 A^\perp)(\mathbf{T}_1 \mathbf{Z}A^\perp) \subseteq \mathbf{K}A$ if and only if

$$0 = \left(\left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_1 \mathbf{Z}A^\perp\right) \mid \mathbf{Z}A\right) = \left(\mathbf{T}_1 A^\perp \mid \mathbf{T}_1 \mathbf{Z}A^\perp\right)$$

if and only if $\mathbf{T}_1 A^\perp \subseteq \mathbf{T}_1 \mathbf{Z}A$ if and only if $z^2 = 0$ for all $z \in \mathbf{T}_1 A^\perp$. But $(\mathbf{T}_1 A^\perp)^2 = F\zeta_1(1)^2$ by Corollary 2.4, so $z^2 = 0$ for all $z \in \mathbf{T}_1 A^\perp$ if and only if $\zeta_1(1)^2 = 0$. □

Note that, in the situation of Proposition 6.3(iii), we have $\zeta_1(1)^2 = 0$ if and only if all diagonal Cartan invariants of A are even, by Lemma 3.4. Also, we have

$$\dim\left(\mathbf{T}_1 A^\perp\right)\left(\mathbf{T}_1 \mathbf{Z}A^\perp\right) + \mathbf{K}A/\mathbf{K}A \leqslant 1.$$

There is the following dual of Proposition 6.1.

Proposition 6.4. *Let* $n \in \mathbb{N}$. *Then* $(\mathbf{T}_n\mathbf{Z}A)(\mathbf{T}_n\mathbf{Z}A^{\perp}) \subseteq \mathbf{K}A$.

Proof. Let $z \in \mathbf{T}_n\mathbf{Z}A$ and $x \in A/\mathbf{K}A$. Then

$$z\kappa_n(x) = \kappa_n\left(z^{p^n}x\right) = \kappa_n(0x) = 0.$$

Thus $(\mathbf{T}_n\mathbf{Z}A)(\mathbf{T}_n\mathbf{Z}A^{\perp}/\mathbf{K}A) = (\mathbf{T}_n\mathbf{Z}A)(\operatorname{Im}\kappa_n) = 0$, and the result follows. □

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