Central ideals and Cartan invariants of symmetric algebras

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Received 17 September 2004
Available online 25 March 2005
Communicated by Michel Broué

Abstract

In this paper, we investigate certain ideals in the center of a symmetric algebra $A$ over an algebraically closed field of characteristic $p > 0$. These ideals include the Higman ideal and the Reynolds ideal. They are closely related to the $p$-power map on $A$. We generalize some results concerning these ideals from group algebras to symmetric algebras, and we obtain some new results as well. In case $p = 2$, these ideals detect odd diagonal entries in the Cartan matrix of $A$. In a sequel to this paper, we will apply our results to group algebras.

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doi:10.1016/j.jalgebra.2005.01.052
1. Introduction

Let $A$ be a symmetric algebra over an algebraically closed field $F$ of characteristic $p > 0$, with symmetrizing bilinear form $\langle . \mid . \rangle$. In this paper we investigate the following chain of ideals of the center $ZA$ of $A$:

$$ZA \supseteq T_1A^\perp \supseteq T_2A^\perp \supseteq \cdots \supseteq RA \supseteq HA \supseteq Z_0A \supseteq 0;$$

here $Z_0A := \sum_B ZB$ where $B$ ranges over the set of blocks of $A$ which are simple $F$-algebras. Thus $Z_0A$ is a direct product of copies of $F$, one for each simple block $B$ of $A$. Furthermore, $HA$ denotes the Higman ideal of $A$, defined as the image of the trace map

$$\tau : A \to A, \quad x \mapsto \sum_{i=1}^n b_i xa_i;$$

here $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are a pair of dual bases of $A$. Moreover, $RA$ is the Reynolds ideal of $A$, defined as the intersection of the socle $SA$ of $A$ and the center $ZA$ of $A$. The ideals $T_nA^\perp (n \in \mathbb{N})$ were introduced in [6, II]; they can be viewed as generalizations of the Reynolds ideal. In fact, $RA$ is their intersection. These ideals are defined in terms of the $p$-power map $A \to A, x \mapsto x^p$, and the bilinear form $\langle . \mid . \rangle$. The precise definition will be given below. Motivated by the special case of group algebras [8,9], we show that

$$Z_0A \subseteq (T_1A^\perp)^2 \subseteq HA,$$

so that $(T_1A^\perp)^2$ fits nicely into the chain of ideals above. When $p$ is odd then

$$(T_1A^\perp)^2 = Z_0A.$$

The case $p = 2$ behaves differently and turns out to have some interesting special features. We show that, in this case,

$$(T_1A^\perp)^3 = (T_1A^\perp)(T_2A^\perp) = Z_0A,$$

but that $(T_1A^\perp)^2 \neq Z_0A$ in general. We prove that, in case $p = 2$, the mysterious ideal $(T_1A^\perp)^2$ is a principal ideal of $ZA$. It is generated by the element $\xi_1(1)^2$ where $\xi_1 : ZA \to ZA$ is a certain natural semilinear map related to the $p$-power map. The map $\xi_1$ was first defined in [6, IV].

Moreover, in case $p = 2$, the dimension of $(T_1A^\perp)^2$ is the number of blocks $B$ of $A$ with the property that the Cartan matrix $C_B = (c_{ij})$ of $B$ contains an odd diagonal entry $c_{ii}$. A primitive idempotent $e$ in $A$ satisfies $e\xi_1(1)^2 \neq 0$ if and only if the dimension of $eAe$ is odd.

At the end of the paper, we investigate the behaviour of the ideals $T_nA^\perp$ under Morita and derived equivalences, and we dualize some of the results obtained in the previous sections. In a sequel [2] to this paper, we will apply our results to group algebras of finite
groups. We will see that a finite group \( G \) contains a real conjugacy class of 2-defect zero if and only if the Cartan matrix of \( G \) in characteristic 2 contains an odd diagonal entry. We will also prove a number of related facts.

2. The Reynolds ideal and its generalizations

In the following, let \( F \) be an algebraically closed field of characteristic \( p > 0 \), and let \( A \) be a symmetric \( F \)-algebra with symmetrizing bilinear form \((. \mid .)\). Thus \( A \) is a finite-dimensional associative unitary \( F \)-algebra, and \((. \mid .)\) is a non-degenerate symmetric bilinear form on \( A \) which is associative, in the sense that \((ab \mid c) = (a \mid bc)\) for \( a, b, c \in A \). We denote the center of \( A \) by \( Z_A \), the Jacobson radical of \( A \) by \( J_A \), the socle of \( A \) by \( S_A \), and the commutator subspace of \( A \) by \( K_A \). Thus \( K_A \) is the \( F \)-subspace of \( A \) spanned by all commutators \( ab - ba \) \((a, b \in A)\). For \( n \in \mathbb{N} \),

\[
T_n A := \{ x \in A : x^{p^n} \in K_A \}
\]

is a \( Z_A \)-submodule of \( A \), so that

\[
K_A = T_0 A \subseteq T_1 A \subseteq T_2 A \subseteq \cdots
\]

and

\[
\sum_{n=0}^{\infty} T_n A = J_A + K_A
\]

(cf. [7]). For any \( F \)-subspace \( X \) of \( A \), we set

\[
X^\perp := \{ y \in A : (x \mid y) = 0 \text{ for } x \in X \}.
\]

Then

\[
Z_A = K_A^\perp = T_0 A^\perp \supseteq T_1 A^\perp \supseteq T_2 A^\perp \supseteq \cdots
\]

is a chain of ideals of \( Z_A \) such that

\[
\bigcap_{n=0}^{\infty} T_n A^\perp = S_A \cap Z_A.
\]

We call \( R_A := S_A \cap Z_A \) the Reynolds ideal of \( Z_A \), in analogy to the terminology used for group algebras. For \( n \in \mathbb{N} \) and \( z \in Z_A \), there is a unique element \( \zeta_n(z) \in Z_A \) such that

\[
(\zeta_n(z) \mid x)^{p^n} = (z \mid x^{p^n}) \quad \text{for } x \in A.
\]

This defines a map \( \zeta_n = \zeta_n^A : Z_A \rightarrow Z_A \) with the following properties:
Lemma 2.1. Let \( m, n \in \mathbb{N} \), and let \( y, z \in ZA \). Then the following holds:

(i) \( \zeta_n(y + z) = \zeta_n(y) + \zeta_n(z) \) and \( \zeta_n(yz) = \zeta_n(yz^n) \).
(ii) \( \zeta_m \circ \zeta_n = \zeta_{m+n} \).
(iii) \( \text{Im}(\zeta_n) = T_nA^\perp \).
(iv) \( \zeta_n^A(z)e = \zeta_{eA}^n(ze) \) for every idempotent \( e \in A \).

Proof. (i)–(iii) are proved in [7, (44)–(47)].

(iv) Recall that \( eAe \) is a symmetric \( F \)-algebra; a corresponding symmetric bilinear form is obtained by restricting \( (.,.) \) to \( eAe \). Note that \( ez = ez = eze \in ZAe \subseteq Z(eAe) \) and, similarly, \( \zeta_n^A(z)e \in Z(Ae) \). Moreover, for \( x \in eAe \), we have

\[
\left( \zeta_n^A(z)e \mid x \right)^p^n = \left( e_n^A(z) \mid ex \right)^p = \left( \zeta_n^A(z) \mid x \right)^p = \left( z \mid x^p \right)^n
\]

and the result follows. \( \square \)

We apply these properties in order to prove:

Lemma 2.2. Let \( m, n \in \mathbb{N} \). Then

\[
(T_mA^\perp)(T_nA^\perp) \subseteq \zeta_{m+n}(T_nA^\perp)^{p^n(p^{m-1})} \subseteq T_{m+n}A^\perp.
\]

Proof. Let \( y, z \in ZA \). Then Lemma 2.1 implies that

\[
\zeta_m(\zeta_n(z)) = \zeta_m(\zeta_n(z)^p) = \zeta_m(\zeta_j(\zeta_n(z)^p)z) = \zeta_m(\zeta_j(\zeta_n(z)^p)z)
\]

Thus the result follows from Lemma 2.1(iii). \( \square \)

Let \( B_1, \ldots, B_r \) denote the blocks of \( A \), so that \( A = B_1 \oplus \cdots \oplus B_r \). Each \( B_i \) is itself a symmetric \( F \)-algebra. If a block \( B_i \) is a simple \( F \)-algebra then \( B_i \cong \text{Mat}(d_i, F) \) for a positive integer \( d_i \), and thus \( ZB_i \cong F \). We set

\[
Z_0A := \sum_i ZB_i,
\]

where the sum ranges over all \( i \in \{1, \ldots, r\} \) such that \( B_i \) is a simple \( F \)-algebra. Then \( Z_0A \) is an ideal of \( ZA \) and an \( F \)-algebra which is isomorphic to a direct sum of copies of \( F \). Its dimension is the number of simple blocks of \( A \). We exploit Lemma 2.2 in order to prove:
Theorem 2.3.

(i) \((T_1 A^\perp)^2 \subseteq RA.\)

(ii) \((T_1 A^\perp)(T_2 A^\perp) = (T_1 A^\perp)^3 = Z_0 A.\)

(iii) If \(p\) is odd, then \((T_1 A^\perp)^2 = Z_0 A.\)

Proof. (i) Lemma 2.2 implies

\[
(T_1 A^\perp)^2 \subseteq \zeta_2((T_1 A^\perp)^{p(p-1)}) \subseteq \zeta_2((T_1 A^\perp)^2).
\]

Iteration yields

\[
(T_1 A^\perp)^2 \subseteq \zeta_2((T_1 A^\perp)^{2}) = \zeta_4((T_1 A^\perp)^2) \subseteq \zeta_6((T_1 A^\perp)^2) \subseteq \cdots.
\]

Thus

\[
(T_1 A^\perp)^2 \subseteq \bigcap_{n=0}^{\infty} \zeta_2^n(T_2n A^\perp) = SA \cap ZA = RA,
\]

by Lemma 2.1(iii).

(ii) It is easy to see that \(T_n A = T_n B_1 \oplus \cdots \oplus T_n B_r\) and \(T_n A^\perp = T_n B_1^\perp \oplus \cdots \oplus T_n B_r^\perp\)

for \(n \in \mathbb{N}\) where \(T_n B_i^\perp = \{x \in B_i : (x | T_n B_i) = 0\}\) for \(i = 1, \ldots, r.\) So we may assume that \(A\) itself is a block.

If \(A\) is simple, then \(JA = 0,\) so \(T_n A = KA\) and \(T_n A^\perp = ZA\) for all \(n \in \mathbb{N}.\) Hence

\[
ZA = (T_1 A^\perp)(T_2 A^\perp) = (T_1 A^\perp)^3
\]

in this case.

Now suppose that \(A\) is non-simple. Then \(JA \neq 0,\) so \(ZA \neq RA.\) It follows that \(JA + KA \neq KA,\) whence \(JA\) is not contained in \(KA.\) So \(T_1 A \neq KA.\) This means that \(T_1 A^\perp\) is a proper ideal of \(ZA.\) Since \(ZA\) is a local \(F\)-algebra this implies that \(T_1 A^\perp \subseteq JZA \subseteq JA.\) Thus we may conclude, using (i), that \((T_1 A^\perp)^3 \subseteq (RA)(JA) = 0.\) Hence Lemma 2.2 yields

\[
(T_1 A^\perp)(T_2 A^\perp) \subseteq \zeta_3((T_2 A^\perp)^{p^2(p-1)}) \subseteq \zeta_3((T_1 A^\perp)^3) = \zeta_3(0) = 0.
\]

(iii) Suppose that \(p\) is odd. As in the proof of (ii), we may assume that \(A\) is a block, and that \(A\) is non-simple. Then Lemma 2.2 and (ii) imply that

\[
(T_1 A^\perp)^2 \subseteq \zeta_2((T_1 A^\perp)^{p(p-1)}) \subseteq \zeta_2((T_1 A^\perp)^3) = \zeta_2(0) = 0,
\]

and the result is proved. \(\square\)

Theorem 2.3 extends [9, Theorem 9] from group algebras to symmetric algebras. We will later improve on part (i). But first we note the following consequence.
Corollary 2.4. Suppose that \( A \) is a block, and denote the central character of \( A \) by \( \omega : \mathbb{Z}A \to F \). Moreover, let \( m, n \in \mathbb{N} \) with \( m \neq 0 \neq n \), and let \( x, y \in \mathbb{Z}A \). Then

\[
\zeta_m(x)\zeta_n(y) = \omega(x)\omega(y)^{p^{-m}}\omega(y)^{p^{-n}}\zeta_m(1)\zeta_n(1).
\]

In particular, we have

\[
\left( T_mA^\perp \right)\left( T_nA^\perp \right) = F\zeta_m(1)\zeta_n(1),
\]

so that \( \dim(T_mA^\perp)(T_nA^\perp) \leq 1 \).

Proof. Theorem 2.3(i) implies that \( \zeta_m(x)^{p^n} \in RA \subseteq SA \). Thus

\[
\zeta_m(x)^{p^n}y = \omega(y)\zeta_m(x)^{p^n}.
\]

Similarly, we have \( x\zeta_n(1)^{p^m} = \omega(x)\zeta_n(1)^{p^m} \). So we conclude that

\[
\zeta_m(x)\zeta_n(y) = \zeta_n\left( \zeta_m(x)^{p^n}y \right) = \zeta_n\left( \omega(y)\zeta_m(x)^{p^n} \right) = \omega(y)^{p^{-n}}\zeta_m(x)\zeta_n(1)
\]

\[
= \omega(y)^{p^{-n}}\omega(x)^{p^{-m}}\zeta_m(1)\zeta_n(1).
\]

The remaining assertions follow from Lemma 2.1(iii). \( \square \)

We can generalize part of Corollary 2.4 in the following way.

Proposition 2.5. Let \( m, n \in \mathbb{N} \) with \( m \neq 0 \neq n \). Then

\[
\left( T_mA^\perp \right)\left( T_nA^\perp \right) = \mathbb{Z}A \cdot \zeta_m(1)\zeta_n(1)
\]

is a principal ideal of \( \mathbb{Z}A \). If \( p \) is odd, or if \( m + n > 2 \), then the dimension of \( (T_mA^\perp)(T_nA^\perp) \) equals the number of simple blocks of \( A \) and in particular does not depend on \( m + n \).

Proof. It is easy to see that we may assume that \( A \) is a block. In this case the assertion follows from Corollary 2.4 and Theorem 2.3. \( \square \)

In the next two sections, we will handle the remaining case \( p = 2 \) and \( m = n = 1 \). Here we just illustrate this exceptional case by an example.

Let \( G \) be a finite group. Then the group algebra \( FG \) is a symmetric \( F \)-algebra; a symmetrizing bilinear form on \( FG \) satisfies

\[
(g \mid h) = \begin{cases} 
1, & \text{if } gh = 1, \\
0, & \text{otherwise},
\end{cases}
\]
for \( g, h \in G \). An element \( g \in G \) is called \textit{real} if \( g \) is conjugate to its inverse \( g^{-1} \), and \( g \) is said to be of \textit{\( p \)-defect zero} if \(|C_G(g)|\) is not divisible by \( p \). We denote the set of all real elements of \( 2 \)-defect zero in \( G \) by \( R_G \). For a subset \( X \) of \( G \), we set

\[
X^+ := \sum_{x \in X} x \in FG.
\]

It was proved in [8, Proposition 4.1] that \( R_G^+ = \zeta_1(1)^2 \in (T_1 FG^\perp)^2 \), in case \( p = 2 \).

**Example 2.6.** Let \( p = 2 \), and suppose that \( G \) is the symmetric group \( S_4 \) of degree 4. Then \( FG \) has no simple blocks; in fact, \( FG \) has just one block, the principal one. Thus \( Z_0 FG = 0 \). On the other hand, \( R_G \) is precisely the set of all 3-cycles in \( S_4 \). Thus \( 0 \neq R_G^+ \in (T_1 FG^\perp)^2 \). (In fact, \( (T_1 FG^\perp)^2 \) is one-dimensional, by Corollary 2.4.) This example shows that \( (T_1 A^\perp)^2 \neq Z_0 A \), in general.

### 3. Odd Cartan invariants

Let \( F \) be an algebraically closed field of characteristic \( p = 2 \), and let \( A \) be a symmetric \( F \)-algebra with symmetrizing bilinear form \((\cdot | \cdot)\). In this section, we will prove some remarkable properties of the ideal \((T_1 A^\perp)^2\) of \( ZA \). We start by recalling some known facts concerning symmetric bilinear forms over \( F \).

**Lemma 3.1.** Let \( V \) be a finite-dimensional vector space over \( F \), and let \((\cdot | \cdot)\) be a non-degenerate symmetric bilinear form on \( V \). Then either \((\cdot | \cdot)\) is symplectic (i.e., \( \langle v \mid v \rangle = 0 \) for every \( v \in V \)), or there exists an orthonormal basis \( v_1, \ldots, v_n \) of \( V \) (i.e., \( \langle v_i \mid v_j \rangle = \delta_{ij} \) for \( i, j = 1, \ldots, n \)).

**Proof.** This can be found in [4, Hauptsatz V.3.5], for example. \( \square \)

If \((\cdot | \cdot)\) is symplectic, then there exists a symplectic basis \( v_1, \ldots, v_m, v_{m+1}, \ldots, v_{2m} \) of \( V \), i.e.,

\[
\langle v_i \mid v_{m+i} \rangle = \langle v_{m+i} \mid v_i \rangle = 1 \quad \text{for} \quad i = 1, \ldots, m,
\]

\[
\langle v_i \mid v_j \rangle = 0 \quad \text{otherwise}
\]

(cf. [4, Hauptsatz V.4.10]). Thus there exist only two types of non-degenerate symmetric bilinear forms on a finite-dimensional vector space \( V \) over \( F \), a symplectic one and a non-symplectic one. In the symplectic case, the dimension of \( V \) has to be even.

We now apply Lemma 3.1 to the symmetrizing bilinear form \((\cdot | \cdot)\) on \( A \).

**Lemma 3.2.**

\[
(\zeta_1(1) \mid \zeta_1(1)) = (\dim A) \cdot 1_F.
\]
Proof. By Lemma 3.1, there exists an $F$-basis

$$a_1, \ldots, a_m, a_{m+1}, \ldots, a_{2m}, a_{2m+1}, \ldots, a_n$$

of $A$ such that

$$(a_i | a_{m+i}) = (a_{m+i} | a_i) = 1 \quad \text{for } i = 1, \ldots, m,$$

$$(a_i | a_i) = 1 \quad \text{for } i = 2m + 1, \ldots, n,$$

$$(a_i | a_j) = 0 \quad \text{otherwise}$$

(and either $n = 2m$ or $m = 0$). Then the dual basis $b_1, \ldots, b_n$ of $a_1, \ldots, a_n$ is given by

$$a_{m+1}, \ldots, a_{2m}, a_1, \ldots, a_m, a_{2m+1}, \ldots, a_n.$$

Thus $(\zeta_1(1) | a_i)^2 = (1 | a_i^2) = (a_i | a_i) = (a_i | a_i)^2$ for $i = 1, \ldots, n$, so

$$\zeta_1(1) = \sum_{i=1}^{n} (\zeta_1(1) | a_i) b_i = \sum_{i=1}^{n} (a_i | a_i) b_i = \sum_{i=2m+1}^{n} a_i$$

and

$$\left(\zeta_1(1) | \zeta_1(1) \right) = \sum_{i,j=2m+1}^{n} (a_i | a_j) = \sum_{i=2m+1}^{n} (a_i | a_i) = (n - 2m) \cdot 1_F = n \cdot 1_F$$

and the result is proved. \qed

The next statement holds in arbitrary characteristic. It is essentially taken from [10, Corollary (1.G)].

Lemma 3.3. Let $e$ be a primitive idempotent in $A$, and let $r \in RA$. Then $er = 0$ if and only if $(e | r) = 0$.

Proof. If $er = 0$, then $0 = (er | 1) = (e | r)$. Conversely, if $(e | r) = 0$ then

$$(eAe \ | \ ere) = (eAe \ | r) = (Fe + J(eAe) \ | r) \subseteq F(e \ | r) + (JA \cdot r \ | 1) = 0.$$ 

Thus $0 = ere = er$ since the restriction of $(\cdot | \cdot)$ to $eAe$ is non-degenerate. \qed

Now we choose representatives $a_1 = e_1, \ldots, a_l = e_l$ for the conjugacy classes of primitive idempotents in $A$. (This means that $Ae_1, \ldots, Ae_l$ are representatives for the isomorphism classes of indecomposable projective left $A$-modules.) Moreover, we let $a_{l+1}, \ldots, a_n$ denote an $F$-basis of $JA + KA$. Then $a_1, \ldots, a_n$ form an $F$-basis of $A$. 


Let \( b_1, \ldots, b_n \) denote the dual basis of \( a_1, \ldots, a_n \). Then \( r_1 := b_1, \ldots, r_l := b_l \) are contained in \((JA + KA)^\perp = SA \cap ZA = RA\), so they form an \( F \)-basis of \( RA \). Moreover, Lemma 3.3 implies that \( e_i r_j = 0 \) for \( i \neq j \) and \( e_i r_i \neq 0 \) for \( i = 1, \ldots, l \).

**Lemma 3.4.** With \( e_1, \ldots, e_l \) as above, we have \( \zeta_1(1)^2 = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i \) and \( e_i \zeta_1(1)^2 = (\dim e_i A e_i) \cdot e_i r_i \) for \( i = 1, \ldots, l \).

**Proof.** Lemma 2.1(iii) and Theorem 2.3(i) imply that \( \zeta_1(1)^2 \in (T_1 A^\perp)^2 \subseteq RA \). By making use of Lemma 2.1(iv) and Lemma 3.2, we obtain

\[
\zeta_1(1)^2 = \sum_{i=1}^l (\zeta_1(1)^2 | e_i) r_i = \sum_{i=1}^l (\zeta_1(1) e_i | \zeta_1(1) e_i) r_i
\]

\[
= \sum_{i=1}^l \left( \zeta_1^e_i A e_i (e_i) | \zeta_1^e_i A e_i (e_i) \right) r_i = \sum_{i=1}^l (\dim e_i A e_i) \cdot r_i.
\]

Since \( e_i r_j = 0 \) for \( i \neq j \) the result follows. \( \square \)

The next theorem is the main result of this section.

**Theorem 3.5.** For \( A \) a symmetric algebra over an algebraically closed field \( F \) of characteristic 2 and for \( e \) a primitive idempotent in \( A \), the following assertions are equivalent:

1. \( \dim e A e \) is even.
2. \( e \zeta_1(1)^2 = 0 \).
3. \( (e | \zeta_1(1)^2) = 0 \).

**Proof.** We may assume that \( e = e_i \) for some \( i \in \{1, \ldots, l\} \). Then \( e_i \zeta_1(1)^2 = (\dim e_i A e_i) \cdot e_i r_i \) with \( e_i r_i \neq 0 \), by Lemma 3.4. This shows that (1) and (2) are equivalent. Since \( \zeta_1(1)^2 \in RA \), Lemma 3.3 implies that (2) and (3) are equivalent. \( \square \)

The Cartan matrix \( C := (c_{ij})_{i,j=1}^l \) of \( A \) is defined by

\[
c_{ij} := \dim e_i A e_j \quad \text{for } i, j = 1, \ldots, l.
\]

Thus \( C \) is a symmetric matrix with non-negative integer coefficients, the Cartan invariants of \( A \). Hence Theorem 3.5 has the following consequence.

**Corollary 3.6.** With the notation for the Cartan matrix of \( A \) as above, \( \zeta_1(1)^2 \neq 0 \) if and only if \( c_{ii} \) is odd for some \( i \). More precisely, for a block \( B \) of \( A \), we have \( \zeta_1(1)^2 1_B \neq 0 \) if and only if the Cartan matrix of \( B \) contains an odd diagonal entry.
In order to illustrate Corollary 3.6, recall that, by Example 2.6, the group algebra $FG$, for $G = S_4$, satisfies $\zeta_1(1)^2 = R_G^+ \neq 0$. Thus the Cartan matrix of $FG$ contains an odd diagonal entry, by Corollary 3.6. Indeed, the Cartan matrix of $FG$ is

$$C := \begin{pmatrix} 4 & 2 \\ 2 & 3 \end{pmatrix},$$

as is well known. More substantial examples will be presented in [2].

It may be of interest to note that the existence of odd diagonal Cartan invariants in characteristic 2 is invariant under derived equivalences (cf. [5]).

**Proposition 3.7.** Let $A'$ be a symmetric $F$-algebra which is derived equivalent to $A$. Then the Cartan matrix of $A'$ contains an odd diagonal entry if and only if the Cartan matrix of $A$ does.

**Proof.** It is known that the Cartan matrices $C = (c_{ij})_{i,j=1}^l$ of $A$ and $C' = (c'_{ij})_{i,j=1}^l$ of $A'$ have the same format, and that they are related by an equation

$$C' = Q \cdot C \cdot Q^\top,$$

where $Q = (q_{ij})_{i,j=1}^l$ is an integral matrix with determinant $\pm 1$ (cf. [5]). Thus

$$c'_{ii} = \sum_{j,k=1}^l q_{ij}q_{ik}c_{jk} \equiv \sum_{j=1}^l q_{ij}^2c_{jj} \pmod 2$$

for $i = 1, \ldots, l$. If $c'_{ii}$ is odd then $c_{jj}$ has to be odd for some $j \in \{1, \ldots, l\}$ (and conversely). \[ \square \]

4. The Higman ideal

Let $F$ be an algebraically closed field, and let $A$ be a symmetric $F$-algebra with symmetrizing bilinear form $(.,.)$. Moreover, let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ denote a pair of dual bases of $A$. In the following, the $F$-linear map

$$\tau : A \to A, \quad x \mapsto \sum_{i=1}^n b_i xa_i,$$

will be of interest (cf. [3, §66]). We record the following properties of this trace map $\tau$:

**Lemma 4.1.**

(i) $\tau$ is independent of the choice of dual bases.

(ii) $\tau$ is self-adjoint with respect to $(.,.)$.

(iii) $\text{Im}(\tau) \subseteq \text{SA} \cap \text{ZA} = \text{RA}$ and $\text{JA} + \text{KA} \subseteq \text{Ker}(\tau)$. 


\textbf{Proof.} (i) Let $a_1', \ldots, a_n'$ and $b_1', \ldots, b_n'$ be another pair of dual bases of $A$. Then $b_j' = \sum_{j=1}^{n} (a_j | b_j') b_j$ and $a_i = \sum_{j=1}^{n} (a_i | b_j') a_j'$ for $i = 1, \ldots, n$. Thus

$$
\sum_{i=1}^{n} b_i' x a_i' = \sum_{i,j=1}^{n} (a_j | b_j') b_j x a_i' = \sum_{j=1}^{n} b_j x \sum_{i=1}^{n} (a_j | b_j') a_i' = \sum_{j=1}^{n} b_j x a_j
$$

for $x \in A$.

(ii) Let $x, y \in A$. Then, by (i), we get

$$
(\tau(x) | y) = \sum_{i=1}^{n} (b_i x a_i | y) = \sum_{i=1}^{n} (x | a_i y b_i) = (x | \tau(y)).
$$

(iii) Let $x, y \in A$. Then

$$
\tau(x) y = \sum_{i=1}^{n} b_i x a_i y = \sum_{i,j=1}^{n} b_i x (a_i y | b_j) a_j = \sum_{i,j=1}^{n} (a_i | y b_j) b_j x a_j
$$

$$
= \sum_{j=1}^{n} y b_j x a_j = y \tau(x).
$$

Hence $\text{Im}(\tau) \subseteq ZA$. In order to prove $\text{Im}(\tau) \subseteq SA$, we choose $a_1, \ldots, a_n$ appropriately. Indeed, we may assume that $a_1 + JA, \ldots, a_r + JA$ form an $F$-basis of $A/JA$, that $a_{r+1} + (JA^2), \ldots, a_s + (JA^2)$ form an $F$-basis of $(JA)/JA^2$, that $a_{s+1} + (JA)^3, \ldots, a_t + (JA)^3$ form an $F$-basis of $(JA^2)/(JA)^3$, etc. Then $b_1, \ldots, b_r$ are contained in $(JA)^\perp$, $b_1, \ldots, b_s$ are contained in $((JA)^2)^\perp$, $b_1, \ldots, b_t$ are contained in $((JA)^3)^\perp$, etc.

Now let $x \in A$ and $y \in JA$. Then $b_i x a_i y \in (JA)^\perp \cdot A \cdot (JA) = 0$ for $i = 1, \ldots, r$, $b_i x a_i y \in ((JA)^2)^\perp \cdot A \cdot (JA) \cdot (JA) = 0$ for $i = r+1, \ldots, s$, $b_i x a_i y \in ((JA)^3)^\perp \cdot A \cdot (JA)^2$. $(JA) = 0$ for $i = s+1, \ldots, t$, etc. We see that $\tau(x) y = 0$, so $\text{Im}(\tau) \subseteq SA$.

Since $\tau$ is self-adjoint (i.e., $\tau^* = \tau$), we conclude that

$$
\text{Ker}(\tau) = \text{Ker}(\tau^*) = \text{Im}(\tau)^\perp \supseteq (SA \cap ZA)^\perp = JA + KA.
$$

Thus $HA := \text{Im}(\tau)$ is an ideal of $ZA$ contained in $RA$, called the \textit{Higman ideal of ZA}. By Lemma 4.1, it is independent of the choice of dual bases. In the following, we write

$$
1_A = e_1 + \cdots + e_m
$$

with pairwise orthogonal primitive idempotents $e_1, \ldots, e_m$ of $A$.

\textbf{Lemma 4.2.} We have $(\tau(e_i) | e_j) = (\dim e_i A e_j) \cdot 1_F$ for $i, j = 1, \ldots, m$.

\textbf{Proof.} We consider the decomposition $A = \bigoplus_{i,j=1}^{m} e_i A e_j$. For $i, j = 1, \ldots, m$, let $X_{ij}$ be an $F$-basis of $e_i A e_j$. Then $X := \bigcup_{i,j=1}^{m} X_{ij}$ is an $F$-basis of $A$. We denote the dual basis
of $X$ by $X^*$. For $x \in X$, there is a unique $x^* \in X^*$ such that $(x \mid x^*) = 1$. Then the map $X \to X^*$, $x \mapsto x^*$, is a bijection. Moreover, for $i, j = 1, \ldots, m$, $X_{ij}^* := \{x^* : x \in X_{ij}\}$ is an $F$-basis of $e_j A e_i$. Thus

$$
\tau(e_i) e_j = e_j \tau(e_i) e_j = \sum_{x \in X} e_j x^* e_i x e_j = \sum_{x \in X_{ij}} e_j x^* e_i x e_j = \sum_{x \in X_{ij}} x^* x
$$

and

$$
(\tau(e_i) \mid e_j) = (\tau(e_i) e_j \mid 1) = \sum_{x \in X_{ij}} (x^* x \mid 1) = \sum_{x \in X_{ij}} (x^* \mid x) = |X_{ij}| \cdot 1_F
$$

so the result is proved. \quad \square

We may assume that $e_1, \ldots, e_m$ are numbered in such a way that $a_1 := e_1, \ldots, a_l := e_l$ represent the conjugacy classes of primitive idempotents in $A$. We choose an $F$-basis $a_{l+1}, \ldots, a_n$ of $JA + KA$, so that $a_1, \ldots, a_n$ form an $F$-basis of $A$. We denote the dual basis of $a_1, \ldots, a_n$ by $b_1, \ldots, b_n$. As above, $r_1 := b_1, \ldots, r_l := b_l$ form an $F$-basis of $RA = SA \capZA$.

**Lemma 4.3.** We have $\tau(e_i) = \sum_{j=1}^l (\dim e_i A e_j) \cdot r_j$ for $i = 1, \ldots, l$.

**Proof.** Let $i \in \{1, \ldots, l\}$. Then $\tau(e_i) \in HA \subset RA$, so

$$
\tau(e_i) = \sum_{j=1}^l (\tau(e_i) \mid e_j) r_j = \sum_{j=1}^l (\dim e_i A e_j) \cdot r_j
$$

by Lemma 4.2. \quad \square

In the following, suppose that $\text{char } F = p > 0$. We know from Theorem 2.3 that $(T_1 A^\perp)^2 \subset RA$. We are going to show that, more precisely, $(T_1 A^\perp)^2 \subset HA$. In the proof, we will make use of the following fact.

**Lemma 4.4.** Let $C = (c_{ij})$ be a symmetric $(n \times n)$-matrix with coefficients in the field $\mathbb{F}_2$ with two elements. Then its main diagonal $c := (c_{11}, c_{22}, \ldots, c_{nn})$, considered as a vector in $\mathbb{F}_2^n$, is a linear combination of the rows of $C$.

**Proof.** Arguing by induction on $n$, we may assume that $n > 1$. If $c = 0$, then there is nothing to prove. So we may assume that $c_{ii} = 1$ for some $i \in \{1, \ldots, l\}$. Permuting the rows and columns of $C$, if necessary, we may assume that $c_{11} = 1$. We now perform elementary row operations on $C$. For $k = 2, \ldots, n$, we subtract the first row, multiplied by $c_{k1}$, from the $k$th row. The resulting matrix $C'$ has the entries

$$
0, c_{k2} - c_{k1}c_{12}, \ldots, c_{kn} - c_{k1}c_{1n}
$$
in its $k$th row and the entries
\[ c_{1k}, c_{2k} - c_{21}c_{1k}, \ldots, c_{nk} - c_{n1}c_{1k} \]
in its $k$th column. We now remove the first row and the first column from $C'$ and end up with a symmetric $((n - 1) \times (n - 1))$-matrix $D$ with diagonal entries
\[ c_{kk} - c_{k1}c_{1k} = c_{kk} - c_{1k}^2 = c_{kk} - c_{1k} \quad (k = 2, \ldots, n). \]
On the other hand, if we subtract the first row of $C$ from $c$, then we obtain the vector
\[ c' := (0, c_{22} - c_{12}, \ldots, c_{nn} - c_{1n}). \]
Thus the vector $d := (c_{22} - c_{12}, \ldots, c_{nn} - c_{1n})$ coincides with the main diagonal of $D$. By induction, $d$ is a linear combination of the rows of $D$, so $c$ is a linear combination of the rows of $C$. □

As Gary McGuire kindly pointed out to us, a different proof of Lemma 4.4 can be found in [1, Proposition 4.6.2]. We apply Lemma 4.4 in the proof of the following result which is a refinement of Theorem 2.3(i). The special case of group algebras was first proved in [8, Lemma 5.1].

**Theorem 4.5.** We always have $(T_1 A^\perp)^2 \subseteq HA$.

**Proof.** If $p$ is odd then, by Theorem 2.3(iii), we have
\[ (T_1 A^\perp)^2 \subseteq Z_0 A = \sum_B ZB = \sum_B HB \subseteq HA, \]
where $B$ ranges over the simple blocks of $A$; in fact, if $B = \text{Mat}(d, F)$ for a positive integer $d$ then $HB = ZB$.

Thus we may assume that $p = 2$. Then Lemma 2.2 gives us elements $\alpha_1, \ldots, \alpha_l$ in the prime field of $F$ such that
\[ \sum_{j=1}^l (\dim e_i Ae_j) \cdot \alpha_j = (\dim e_i Ae_i) \cdot 1_F \quad \text{for } i = 1, \ldots, l. \]

Thus Lemmas 3.4 and 4.3 imply that
\[ \xi_1(1)^2 = \sum_{i=1}^l (\dim e_i Ae_i) \cdot r_i = \sum_{i,j=1}^l (\dim e_i Ae_j) \cdot \alpha_j r_i = \sum_{j=1}^l \alpha_j \tau(e_j) \in HA. \]

Hence Proposition 2.5 implies that $(T_1 A^\perp)^2 = ZA \cdot \xi_1(1)^2 \subseteq HA$. □
5. Morita invariance

Let $F$ be an algebraically closed field of characteristic $p > 0$, and let $A$ be a symmetric $F$-algebra. In this section we investigate the behaviour of the ideals $T_nA^\perp$ of $ZA$ under Morita equivalences. These results will be used in [2].

**Proposition 5.1.** Let $e$ be an idempotent in $A$ such that $AeA = A$. Then the map

$$f : ZA \to Z(eAe), \quad z \mapsto ez = ze,$$

is an isomorphism of $F$-algebras mapping $T_nA^\perp$ onto $T_n(eAe)^\perp$, for $n \in \mathbb{N}$.

**Proof.** Certainly $f$ is a homomorphism of $F$-algebras. Let $z \in ZA$ such that $0 = f(z) = ez$. Then $0 = AeAZ = Aez = Aze$, so that $z = 0$. Thus $f$ is injective. Since $AeA = A$ the $F$-algebras $A$ and $eAe$ are Morita equivalent; in particular, their centers are isomorphic. Hence $f$ is an isomorphism of $F$-algebras. Lemma 2.1(iv) implies that $f \circ \zeta_n^A = \zeta_n^{eAe} \circ f$, so

$$f(T_nA^\perp) = f(\zeta_n^A(ZA)) = \zeta_n^{eAe}(f(ZA)) = \zeta_n^{eAe}(Z(eAe)) = T_n(eAe)^\perp$$

by Lemma 2.1(iii). $\square$

We mention two consequences of Proposition 5.1.

**Corollary 5.2.** Let $d$ be a positive integer, and let $A_d$ denote the symmetric $F$-algebra $\text{Mat}(d, A)$. Then the map

$$h : ZA \to ZA_d, \quad z \mapsto z1_d,$$

is an isomorphism of $F$-algebras mapping $T_nA^\perp$ onto $(T_nA_d)^\perp$, for $n \in \mathbb{N}$.

**Proof.** We denote the matrix units of $A_d$ by $e_{ij}$ ($i, j = 1, \ldots, d$). Then the map

$$f : A \to e_{11}Ade_{11}, \quad a \mapsto ae_{11},$$

is an isomorphism of $F$-algebras. This implies that $f(ZA) = Z(e_{11}Ade_{11})$ and $f(T_nA^\perp) = T_n(e_{11}Ade_{11})^\perp$ for $n \in \mathbb{N}$. On the other hand, Proposition 5.1 implies that the map

$$g : ZA_d \to Z(e_{11}Ade_{11}), \quad z \mapsto ze_{11} = e_{11}z,$$

is an isomorphism of $F$-algebras such that $g((T_nA_d)^\perp) = T_n(e_{11}Ade_{11})^\perp$ for $n \in \mathbb{N}$. Now observe that $h$ is an isomorphism of $F$-algebras such that $g \circ h$ is the restriction of $f$ to $ZA$. Thus $h(T_nA^\perp) = (T_nA_d)^\perp$ for $n \in \mathbb{N}$. $\square$

**Corollary 5.3.** Let $B$ be a symmetric $F$-algebra which is Morita equivalent to $A$. Then there is an isomorphism of $F$-algebras $ZA \to ZB$ mapping $T_nA^\perp$ onto $T_nB^\perp$, for $n \in \mathbb{N}$. 
Proof. Let $e$ be an idempotent in $A$ such that $eAe$ is a basic algebra of $A$, and let $f$ be an idempotent in $B$ such that $fBf$ is a basic algebra of $B$. Then $AeA = A$ and $BfB = B$. Moreover, $eAe$ and $fBf$ are isomorphic since $A$ and $B$ are Morita equivalent. Thus Proposition 5.1 yields a chain of isomorphisms

$$
Z_A \to Z(eAe) \to Z(fBf) \to Z_B
$$

mapping $T_nA$ onto $T_nB$, for $n \in \mathbb{N}$. □

It would be interesting to know whether Corollary 5.3 extends to symmetric $F$-algebras which are derived equivalent (cf. [5]).

Question 5.4. Suppose that $A$ and $B$ are derived equivalent symmetric $F$-algebras. Is there an isomorphism of $F$-algebras $Z_A \to Z_B$ mapping $T_nA$ onto $T_nB$, for $n \in \mathbb{N}$?

6. Some dual results

Let $F$ be an algebraically closed field of characteristic $p > 0$, and let $A$ be a symmetric $F$-algebra. For $n \in \mathbb{N},$

$$
T_nZA := \{ z \in ZA: z^{p^n} = 0 \}
$$
is an ideal of $ZA$. In this way we obtain an ascending chain of ideals

$$
0 = T_0ZA \subseteq T_1ZA \subseteq T_2ZA \subseteq \cdots \subseteq JZA \subseteq ZA
$$
of $ZA$ such that

$$
\sum_{n=0}^{\infty} T_nZA = JZA.
$$

This ascending chain of ideals turns out to be related to the descending chain of ideals

$$
ZA = T_0A \supseteq T_1A \supseteq T_2A \supseteq \cdots \supseteq RA \supseteq 0
$$
of $ZA$ considered before.

Proposition 6.1. Let $n \in \mathbb{N}$. Then $(T_nA)TA(ZA) = 0$.

Proof. Let $y \in ZA$ and $z \in T_nZA$, so that $z^{p^n} = 0$. Then Lemma 2.1(i) implies that

$$
\zeta_n(yz) = \zeta_n(yz^{p^n}) = \zeta_n(y0) = \zeta_n(0) = 0.
$$
Hence $(T_nA)TA(ZA) = (\text{Im} \, \zeta_n)(T_nZA) = 0$, by Lemma 2.1(iii). □
The result above is essentially [9, Proposition 4]. We conclude that

\[ T_n ZA \subseteq \{ z \in ZA : z(T_nA^\perp) = 0 \} \subseteq \{ z \in ZA : z \zeta_n(1) = 0 \}. \]

In [2], we will see that these inclusions are proper in general, even for group algebras of finite groups. If \( n \) is sufficiently large then \( T_n ZA = JZA \) and \( T_n A^\perp = RA \), and certainly

\[ JZA = \{ z \in ZA : z \cdot RA = 0 \}. \]

Also, if \( n \) is large and \( A = FG \) for a finite group \( G \) then \( \zeta_n(1) = G_p^+ \) where \( G_p \) denotes the set of \( p \)-elements in \( G \) (cf. [7, (48)]), and it is known that

\[ JZFG = \{ z \in ZFG : zG_p^+ = 0 \} \]

(cf. [7, (59)]). However, it is easy to construct an example of a symmetric \( F \)-algebra \( A \) such that

\[ JZA \neq \{ z \in ZA : z \zeta_n(1) = 0 \} \]

for all sufficiently large \( n \).

For \( n \in \mathbb{N} \), the ideal \( T_n ZA \) of \( ZA \) is related to a semilinear map \( \kappa_n : A/KA \to A/KA \) first constructed in [6, IV]; \( \kappa_n \) is defined in such a way that

\[ (z^{p^n} | x) = (z | \kappa_n(x))^{p^n} \text{ for } z \in ZA \text{ and } x \in A/KA; \]

here we set \((z + KA) := (z | a)\) for \( z \in ZA \) and \( a \in A \). Also, we set \((a + KA)^{p^n} := a^{p^n} + KA\) for \( a \in A \). We recall the following properties of \( \kappa_n \) (cf. [7, (50)–(53)]).

**Lemma 6.2.** Let \( m, n \in \mathbb{N} \), let \( x, y \in A/KA \), and let \( z \in ZA \). Then the following holds:

(i) \( \kappa_n(x + y) = \kappa_n(x) + \kappa_n(y) \), \( z\kappa_n(x) = \kappa_n(z^{p^n} x) \) and \( \kappa_n(zx^{p^n}) = \zeta_n(z)x \).

(ii) \( \kappa_m \circ \kappa_n = \kappa_{m+n} \).

(iii) \( \text{Im}(\kappa_n) = T_n ZA^\perp / KA \).

Our next result is a dual version of Theorem 2.3. For simplicity, we concentrate on the case where \( A \) is a non-simple block. (If \( A \) is a simple block then \( T_1 ZA = 0 \), so \( T_1 ZA^\perp = A \). Moreover, we have \( T_2 A^\perp = T_1 A^\perp = ZA \) in this case.)

**Proposition 6.3.** Suppose that \( A \) is a non-simple block. Then the following holds:

(i) \( (T_1 A^\perp)(T_1 ZA^\perp) \subseteq KA \) for \( p \neq 2 \).

(ii) \( (T_2 A^\perp)(T_1 ZA^\perp) \subseteq KA \) and \( (T_1 A^\perp)(T_2 ZA^\perp) \subseteq KA \) for \( p = 2 \).

(iii) \( (T_1 A^\perp)(T_1 ZA^\perp) \subseteq JZA^\perp \) for \( p = 2 \). Moreover, in this case we have \( (T_1 A^\perp) \times (T_1 ZA^\perp) \subseteq KA \) if and only if \( \xi_1(1)^2 = 0 \).
Proof. (i) Let $y \in ZA$ and $x \in A/KA$. Then $\zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^px) = 0$ since $\zeta_1(y)^p \in (T_1 A^\perp)^p = 0$ by Theorem 2.3(iii). Thus

$$\left( T_1 A^\perp \right) (T_1 ZA^\perp/KA) = (\text{Im} \, \zeta_1) (\text{Im} \, \kappa_1) = 0,$$

and (i) is proved.

(ii) Let $x, y$ be as in (i). Then $\zeta_2(y)\kappa_1(x) = \kappa_1(\zeta_2(y)^2x) = 0$ since $\zeta_2(y)^2 \in (T_2 A^\perp)^2 = 0$, by Theorem 2.3(ii). Thus

$$\left( T_2 A^\perp \right) (T_1 ZA^\perp/KA) = (\text{Im} \, \zeta_2) (\text{Im} \, \kappa_1) = 0.$$

Similarly, we have $\zeta_1(y)\kappa_2(x) = \kappa_2(\zeta_1(y)^4x) = 0$ since $\zeta_1(y)^3 \in (T_1 A^\perp)^3 = 0$ by Theorem 2.3(ii). Thus

$$\left( T_1 A^\perp \right) (T_2 ZA^\perp/KA) = (\text{Im} \, \zeta_1) (\text{Im} \, \kappa_2) = 0,$$

and (ii) follows.

(iii) Again, let $x, y$ be as in (i). Then

$$\zeta_1(y)\kappa_1(x) = \kappa_1(\zeta_1(y)^2x) = \kappa_1(\zeta_1(y)\kappa_1(y^2x)) \in \kappa_1((\text{Im} \, \zeta_1)(\text{Im} \, \kappa_1)).$$

Iteration yields

$$(\text{Im} \, \zeta_1) (\text{Im} \, \kappa_1) \subseteq \kappa_1((\text{Im} \, \zeta_1)(\text{Im} \, \kappa_1)) \subseteq \kappa_1(\kappa_1((\text{Im} \, \zeta_1)(\text{Im} \, \kappa_1))) $$

$$= \kappa_2((\text{Im} \, \zeta_1)(\text{Im} \, \kappa_1)) \subseteq \cdots.$$ 

Thus

$$\left( T_1 A^\perp \right) (T_1 ZA^\perp/KA) = (\text{Im} \, \zeta_1) (\text{Im} \, \kappa_1) \subseteq \bigcap_{n=0}^{\infty} \text{Im}(\kappa_n) = \bigcap_{n=0}^{\infty} T_n ZA^\perp/KA = JZA^\perp/KA,$$

and the first assertion of (iii) is proved. Now note that $(T_1 A^\perp)(T_1 ZA^\perp) \subseteq KA$ if and only if

$$0 = \left(\left( T_1 A^\perp \right) \left( T_1 ZA^\perp \right) \mid ZA \right) = \left( T_1 A^\perp \mid T_1 ZA^\perp \right)$$

if and only if $T_1 A^\perp \subseteq T_1 ZA$ if and only if $z^2 = 0$ for all $z \in T_1 A^\perp$. But $(T_1 A^\perp)^2 = F\zeta_1(1)^2$ by Corollary 2.4, so $z^2 = 0$ for all $z \in T_1 A^\perp$ if and only if $\zeta_1(1)^2 = 0$. \[ \square \]

Note that, in the situation of Proposition 6.3(iii), we have $\zeta_1(1)^2 = 0$ if and only if all diagonal Cartan invariants of $A$ are even, by Lemma 3.4. Also, we have

$$\dim(\left( T_1 A^\perp \right) \left( T_1 ZA^\perp \right) + KA/KA \leq 1.$$ 

There is the following dual of Proposition 6.1.
Proposition 6.4. Let \( n \in \mathbb{N} \). Then \((T_nZA)(T_nZA)\perp \subseteq KA\).

Proof. Let \( z \in T_nZA \) and \( x \in A/KA \). Then
\[
z\kappa_n(x) = \kappa_n(z^{P^n}x) = \kappa_n(0) = 0.
\]
Thus \((T_nZA)(T_nZA)\perp/KA) = (T_nZA)(\text{Im} \kappa_n) = 0\), and the result follows. □

Acknowledgments

The ideas in this paper have their origin in visits of B. Külshammer to the National University of Ireland, Maynooth, and to the Technical University of Budapest, in September 2003. B. Külshammer is very grateful for the invitation to Maynooth and for the hospitality received there. His visit to Maynooth was partially funded by a New Researcher Award from the National University of Ireland, Maynooth. B. Külshammer’s visit to Budapest was kindly supported by the German–Hungarian exchange project No. D-4/99 (TéT-BMBF) and by the Hungarian National Science Foundation Research Grant T034878 and T042481.

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