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A linear equation for Minkowski sums of polytopes relatively in general position[☆]

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ABSTRACT

The objective of this paper is to study a special family of Minkowski sums, that is of polytopes relatively in general position. We show that the maximum number of faces in the sum can be attained by this family. We present a new linear equation that is satisfied by f-vectors of the sum and the summands. We study some of the implications of this equation.

1. Introduction

Minkowski sums of polytopes naturally arise in many domains, ranging from mechanical engineering [8] and robotics [7] to algebra [5,9]. In direct applications to physical models, the important factor is often the general shape of the sum, and some details can be approximated. In theory applications, though, it is often the combinatorial structure of the sum which is relevant.

The scope of this paper centers on that combinatorial structure. Few results are known as yet as to the structure of Minkowski sums. It is usually difficult to estimate even the number of k-dimensional faces (k-faces) of the sum, let alone the general structure. This paper focuses on a certain family of sums and its properties, which can be used to make general statements about Minkowski sums.

Every nonempty face of a Minkowski sum can be *decomposed* uniquely into a sum of faces of each summand [3]. We say this decomposition is *exact* when the dimension of the sum is equal to the sum of the dimensions of the summands. When all facets have an exact decomposition, we say the summands are *relatively in general position*.

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Our first observation is that the maximal number of faces in a sum can be attained when summands are relatively in general position.

For any polytope P, we denote its *f-vector* by $f(P)$ whose kth component $f_k(P)$ is the number of faces of dimension k in P for $k = -1, 0, \dots, \dim(P)$. For any set $S \subseteq \{-1, \dots, \dim(P)\}$, we define $f_S(P)$ as the number of chains in P in which the dimensions of elements of the chain are exactly the elements of S. The vector with components $f_S(P)$ for all $S \subseteq \{-1, \dots, \dim(P)\}$ is called the *extended f-vector* and denoted also as $f(P)$.

Theorem 1. *Let $P = P_1 + \cdots + P_r$ be a Minkowski sum. There is a Minkowski sum $P' = P'_1 + \cdots + P'_r$ of polytopes relatively in general position so that $f_k(P'_i) = f_k(P_i)$ for all i and k, and so that $f_k(P') \geq f_k(P)$ for all k.*

This family can therefore be used for computing the maximum complexity of Minkowski sums. We present now our main theorem.

For any sum of Minkowski $P = P_1 + \cdots + P_r$, and for any face F of P, we will define its f^δ-vector as

$$f_k^\delta(F) = f_k(F) - (f_k(F_1) + \cdots + f_k(F_r)),$$

where $F_1, \dots, F_r$ is the decomposition of F in faces of $P_1, \dots, P_r$ respectively.

Theorem 2. *Let $P_1, \dots, P_r$ be d-dimensional polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their Minkowski sum. Then*

$$\sum_{k=0}^{d-1} (-1)^k k f_k^\delta(P) = 0.$$

Note that the form is rather similar to Euler's Equation:

$$\sum_{k=0}^{d-1} (-1)^k f_k(P) = 1 - (-1)^d.$$

By using Euler's Equation, we can write the theorem slightly differently.

Corollary 3. *Let $P_1, \dots, P_r$ be d-dimensional polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their Minkowski sum. Then for all a,*

$$\sum_{k=0}^{d-1} (-1)^k (k + a) f_k^\delta(P) = a(1 - r)(1 - (-1)^d).$$

Additionally, the main theorem can be extended to sums of polytopes which are not full-dimensional:

Theorem 4. *Let $P_1, \dots, P_r$ be polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their d-dimensional Minkowski sum. Furthermore, let $S \subseteq \{1, \dots, r\}$ be the set of indices i for which* $\dim(P_i) < d$. *Then*

$$\sum_{k=0}^{d-1} (-1)^k k f_k^\delta(P) = (-1)^{d+1} \sum_{i \in S} \dim(P_i).$$

The main theorem has an interesting application when used in conjunction with the following theorem about perfectly centered polytopes (defined in the next section). For any face F of a polytope P containing the origin, we denote as F^D the associated dual face of the dual polytope P^*.

Theorem 5 ([4]). *Let P be a perfectly centered polytope. A subset H of $P + P^*$ is a nontrivial face of $P + P^*$ if and only if $H = G + F^D$ for some ordered nontrivial faces $G \subseteq F$ of P.*

A perfectly centered polytope and its dual satisfy the general position condition posed by the main theorem, which makes it a statement about lattices of polytopes combinatorially similar to a perfectly centered polytope. We extend the statement to all polytopes and Eulerian posets:

Theorem 6. *Let f be the extended f-vector of an Eulerian poset of rank d. Then*

$$\sum_{k=0}^{d-1}(-1)^k k\left(\sum_{i=0}^{k} f_{i,i+d-1-k} - f_k - f_{d-1-k}\right) = 0.$$

2. Normal cones

For each nonempty face F of a convex polytope P, we define its *outer normal cone* $\mathcal{N}(F;P)$ as the set of vectors defining linear functions which are maximized over P on F. If P has full dimension, the dimension of $\mathcal{N}(F;P)$ is $\dim(P) - \dim(F)$.

We say a polytope P is *perfectly centered* if for any nonempty face F of P, the intersection $\mathit{relint}(F) \cap \mathcal{N}(F;P)$ is nonempty. We will use this definition in Section 5. Perfectly centered polytopes are further studied in [4].

The set of all normal cones of a polytope P is a polyhedral complex of relatively open cones whose body is $\mathbb{R}^d$, and which is known as the *normal fan* of P. The combinatorial structure (i.e. the face poset) of the normal fan is dual to that of the polytope, excluding the empty face.

It is not difficult to see that the normal fan of a Minkowski sum is the common refinement of normal fans of the summands, i.e. the set of nonempty intersections of normal cones of the faces of the summands (see e.g. Section 7.2 of [10]). For any face F of a Minkowski sum $P = P_1 + \cdots + P_r$, F decomposes into $F_1 + \cdots + F_r$ if and only if

$$\mathcal{N}(F;P) = \mathcal{N}(F_1;P_1) \cap \cdots \cap \mathcal{N}(F_r;P_r).$$

3. Maximality

We now show that the maximum number of faces of a Minkowski sum can always be attained when the summands are relatively in general position.

Theorem 1. *Let $P = P_1 + \cdots + P_r$ be a Minkowski sum. There is a Minkowski sum $P' = P'_1 + \cdots + P'_r$ of polytopes relatively in general position so that $f_k(P'_i) = f_k(P_i)$ for all i and k, and so that $f_k(P') \geq f_k(P)$ for all k.*

Proof. Let $P = P_1 + P_2$ be a Minkowski sum of polytopes not relatively in general position. We show that if we rotate F_1 by a small angle on an axis in general position, then the number of faces will not diminish.

Let $F = F_1 + F_2$ be a face whose decomposition is not exact, that is $\dim(F) < \dim(F_1) + \dim(F_2)$. In terms of normal cones, it means that $\mathcal{N}(F;P) = \mathcal{N}(F_1;P_1) \cap \mathcal{N}(F_2;P_2)$, and $\dim(\mathcal{N}(F;P)) > \dim(\mathcal{N}(F_1;P_1)) + \dim(\mathcal{N}(F_2;P_2)) - d$. If we perturb P_1 by a small enough rotation on an axe in general position, there is a superface $\mathcal{N}(G_2;P_2)$ of $\mathcal{N}(F_2;P_2)$ so that $\mathcal{N}(F_1;P_1) \cap \mathcal{N}(G_2;P_2) \neq \emptyset$, and so that $\dim(\mathcal{N}(F_1;P_1) \cap \mathcal{N}(G_2;P_2)) = \dim(\mathcal{N}(F;P))$. This means F_1 and G_2 sum to a face F' with $\dim(F') = \dim(F)$. So for every face with an inexact decomposition, there is a new face of the same dimension with an exact decomposition. If the angle is small enough, every face with an exact decomposition still exists. Therefore the number of faces will not diminish, and the new sum is relatively in general position.

By induction, we can slightly rotate the summands $P_1, \ldots, P_r$ so that they are relatively in general position, without diminishing the number of faces in their sum. □

4. Proof

We prove in this section the main theorem, first when all summands are full-dimensional, then extending it to the general case.

We start by an important lemma about the f^δ-vector of faces of a Minkowski sum which have an exact decomposition:

Lemma 7. *Let $F = F_1 + \cdots + F_r$ be a nonempty face of the Minkowski sum $P = P_1 + \cdots + P_r$ with an exact decomposition. Then:*

$$\sum_{k=0}^{d-1} (-1)^k k f_k^\delta(F) = 0.$$

Proof. Since F has an exact decomposition, all of its subfaces also have one. Furthermore, for any set $(G_1, \ldots, G_r)$ so that $G_i \subseteq F_i$ for all i, the sum $G = G_1 + \cdots + G_r$ is a subface of F.

Let d_i be the dimension of F_i and f^i its f-vector. It can be written as $(f_0^i, \ldots, f_{d_i}^i)$, with $f_{d_i}^i = 1$. The f-vector f^i verifies Euler's equation, which means

$$\sum_{k=0}^{d_i} (-1)^k f_k^i = 1.$$

Let us define the characteristic function $p_i(x)$ of the vectors f^i as follows:

$$p_i(x) = f_0^i x^0 + \cdots + f_{d_i}^i x^{d_i}.$$

Euler's equation can now be written as $p_i(-1) = 1$.

Let f be the f-vector of F. Since any r-tuple of subfaces G_i of F_i sums to a subface of F, we can write:

$$f_i = \sum_{e_1+\cdots+e_r=i} (f_{e_1}^1 \cdots f_{e_r}^r).$$

Therefore, if we denote as $p(x)$ the characteristic function of the f-vector of F, we have that $p(x) = \prod_{i=1}^r p_i(x)$.

If we denote as $p_\delta(x)$ the characteristic function of the f^δ-vector of F, we have that $p_\delta(x) = f_0^\delta x^0 + \cdots + f_{\dim(F)}^\delta x^{\dim(F)} = p(x) - (p_1(x) + \cdots + p_r(x)) = \prod_{i=1}^r p_i(x) - \sum_{i=1}^r p_i(x)$. It is easy to see that

$$\sum_{k=0}^{\dim(F)} (-1)^k k f_k^\delta(F) = -(p_\delta)'(-1).$$

Since $p'(x) = \sum_{i=1}^r \left(p_i'(x) \prod_{j \neq i} p_j(x) \right)$, and we have $p_j(-1) = 1$ for all j, $p_\delta'(-1) = \sum_{i=1}^r p_i'(-1) - \sum_{i=1}^r p_i'(-1) = 0$. $\square$

As we can see, the f^δ-vector of the faces of a Minkowski sum which have an exact decomposition are so to say transparent to the equation of the final theorem. We will now show that the f^δ-vector of a Minkowski sum of d-polytopes can be written as an alternated sum of the f^δ-vector of its proper faces.

Theorem 8. *Let $P_1, \ldots, P_r$ be d-dimensional polytopes and $P = P_1 + \cdots + P_r$ their Minkowski sum. Then for any $k < d$,*

$$f_k^\delta(P) = -\sum_{F \subset P} (-1)^{d-\dim(F)} f_k^\delta(F).$$

Or equivalently, we can write

$$\sum_{F \subseteq P} (-1)^{d-\dim(F)} f_k^\delta(F) = 0.$$

The f^δ-vector of P consists of the difference between its f-vector and that of its summands. The equation above actually holds for each of these f-vectors. We are going to prove them separately in the two following lemmas.

First we prove that Theorem 8 holds for the f-vector of the sum:

Lemma 9. *Let P be a d-dimensional polytope. Then, for any $k < d$,*

$$\sum_{F \subseteq P} (-1)^{d-\dim(F)} f_k(F) = 0.$$

Proof.

$$\begin{aligned}
\sum_{F \subseteq P} (-1)^{d-\dim(F)} f_k(F) &= \sum_{F \subseteq P} \left(\sum_{\substack{G \subseteq F \\ \dim(G)=k}} (-1)^{d-\dim(F)} \right) \\
&= \sum_{\substack{G \subseteq P \\ \dim(G)=k}} \left(\sum_{F \supseteq G} (-1)^{d-\dim(F)} \right) = \sum_{\substack{G \subseteq P \\ \dim(G)=k}} \left(\sum_{j=k}^{d} (-1)^{d-j} f_j([G, P]) \right),
\end{aligned}$$

where $[G, P]$ denotes the set of faces of P which contain G. By Euler's equation, the internal sum is equal to zero. □

We now prove that Theorem 8 holds for the f-vector of the summands. The proof follows the same logic, but is slightly complicated by the fact we are summing on the faces of the sum.

Lemma 10. *Let $P_1, \ldots, P_r$ be d-dimensional polytopes and $P = P_1 + \cdots + P_r$ their Minkowski sum. For any face F of P, we denote as $t_i(F)$ the face of P_i in the decomposition of F. Then for any $1 \le i \le r$ and $k < d$,*

$$\sum_{F \subseteq P} (-1)^{d-\dim(F)} f_k(t_i(F)) = 0.$$

Proof.

$$\sum_{F \subseteq P} (-1)^{d-\dim(F)} f_k(t_i(F)) = \sum_{F \subseteq P} \left(\sum_{\substack{G \subseteq t_i(F) \\ \dim(G)=k}} (-1)^{d-\dim(F)} \right).$$

For any F and any i, F_i is in the decomposition of F if and only if $\mathcal{N}(F; P) \subseteq \mathcal{N}(F_i; P_i)$. Furthermore, $G \subseteq F_i$ if and only if $\mathcal{N}(F_i; P_i) \subseteq cl(\mathcal{N}(G; P_i))$. Therefore, the above expression is equal to

$$\sum_{F \subseteq P} \left(\sum_{\substack{\mathrm{cl}(\mathcal{N}(G;P_i)) \supseteq \mathcal{N}(F;P) \\ \dim(G)=k}} (-1)^{d-\dim(F)} \right) = \sum_{\substack{G \subseteq P_i \\ \dim(G)=k}} \left(\sum_{\mathcal{N}(F;P) \subseteq \mathrm{cl}(\mathcal{N}(G;P_i))} (-1)^{d-\dim(F)} \right).$$

It is important to note that the sum inside the parentheses is on the set of faces of P which have their normal cones in the closure of the normal cone of G, which is a face of P_i. Here, the polyhedral cone $\mathrm{cl}(\mathcal{N}(G; P_i))$ is subdivided into a polyhedral complex, which we call $\Omega(G)$. We can now write the expression as

$$\sum_{\substack{G \subseteq P_i \\ \dim(G)=k}} \left(\sum_{j=0}^{d-k} (-1)^j f_j(\Omega(G)) \right) = \sum_{\substack{G \subseteq P_i \\ \dim(G)=k}} \chi(\Omega(G)),$$

where $\chi(\Omega(G))$ is the Euler characteristic of $\Omega(G)$. The Euler characteristic of any support being independent of the actual subdivision, we can replace $\Omega(G)$ by its support, which is $\mathrm{cl}(\mathcal{N}(G; P_i))$:

$$= \sum_{\substack{G\subseteq P_i\\ \dim(G)=k}} \chi(\mathrm{cl}(\mathcal{N}(G;P_i))) = \sum_{\substack{G\subseteq P_i\\ \dim(G)=k}} \left(\sum_{j=0}^{d-k}(-1)^j f_j(\mathrm{cl}(\mathcal{N}(G;P_i)))\right)$$
$$= \sum_{\substack{G\subseteq P_i\\ \dim(G)=k}} \left(\sum_{j=k}^{d}(-1)^{d-j} f_j([G,P_i])\right).$$

Since $\dim(G) = k < d = \dim(P_i)$, the internal sum is equal to zero. □

Theorem 8 is now proved. We can now prove the main theorem:

Theorem 2. *Let $P_1, \ldots, P_r$ be d-dimensional polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their Minkowski sum. Then*

$$\sum_{k=0}^{d-1}(-1)^k k f_k^\delta(P) = 0.$$

Proof. By Theorem 8, we can replace $f_k^\delta(P)$ by a sum over proper subfaces:

$$\sum_{k=0}^{d-1}(-1)^k k f_k^\delta(P) = \sum_{k=0}^{d-1}(-1)^k k\left(-\sum_{F\subset P}(-1)^{d-\dim(F)} f_k^\delta(F)\right)$$
$$= -\sum_{F\subset P}(-1)^{d-\dim(F)}\left(\sum_{k=0}^{d-1}(-1)^k k f_k^\delta(F)\right).$$

By Lemma 7, the internal sum is equal to zero. □

We have now proved the main theorem, which makes the assumption that the summands are full-dimensional. We now extend the result to the general case. For this, we write an extension of Lemma 10:

Lemma 11. *Let $P_1, \ldots, P_r$ be polytopes and $P = P_1 + \cdots + P_r$ their d-dimensional Minkowski sum. For any face F of P, we denote as $t_i(F)$ be the face of P_i in the decomposition of F. Then for any $1 \le i \le r$ and $k < d$,*

$$\sum_{F\subseteq P}(-1)^{d-\dim(F)} f_k(t_i(F)) = (-1)^{d-\dim(P_i)}\delta_{k,\dim(P_i)}.$$

Proof. As in the proof of Lemma 10, we prove that

$$\sum_{F\subseteq P}(-1)^{d-\dim(F)} f_k(t_i(F)) = \sum_{\substack{G\subseteq P_i\\ \dim(G)=k}}\left(\sum_{j=k}^{d}(-1)^{d-j} f_j([G,P_i])\right).$$

If $k < \dim(P_i)$, the internal sum is zero as before. If $k = \dim(P_i)$, then the sums reduce to the single term where $G = P_i$, and the result is $(-1)^{d-\dim(P_i)}$. If $k > \dim(P_i)$, then the sum contains no terms. □

The equation of Theorem 8 reads now as

$$f_k^\delta(P) = -\sum_{F\subset P}(-1)^{d-\dim(F)} f_k^\delta(F) - (-1)^{d-k}\left|\{P_i : \dim(P_i) = k\}\right|.$$

We can now prove the extension:

Theorem 4. *Let $P_1, \dots, P_r$ be polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their d-dimensional Minkowski sum. Furthermore, let $S \subseteq \{1, \dots, r\}$ be the set of indices i for which $\dim(P_i) < d$. Then*

$$\sum_{k=0}^{d-1} (-1)^k k f_k^\delta(P) = (-1)^{d+1} \sum_{i \in S} \dim(P_i).$$

Proof. Following the same lines as in the proof of Theorem 2, we get:

$$\begin{aligned}
\sum_{k=0}^{d-1} (-1)^k k f_k^\delta(P) &= \sum_{k=0}^{d-1} (-1)^k k \left(-\sum_{F \subset P} (-1)^{d-\dim(F)} f_k^\delta(F) - (-1)^{d-k} \left|\{P_i : \dim(P_i) = k\}\right| \right) \\
&= -\sum_{F \subset P} (-1)^{d-\dim(F)} \underbrace{\left(\sum_{k=0}^{d-1} (-1)^k k f_k^\delta(F) \right)}_{0} - \sum_{k=0}^{d-1} (-1)^k k (-1)^{d-k} \left|\{P_i : \dim(P_i) = k\}\right| \\
&= (-1)^{d+1} \sum_{i \in S} \dim(P_i). \quad \square
\end{aligned}$$

5. Application to perfectly centered polytopes

Theorem 12. *Let P be a perfectly centered polytope and f its extended f-vector. Then the f-vector of $P+P^*$ can be written as:*

$$f_k(P + P^*) = \sum_{i=0}^{k} f_{i,i+d-1-k}, \quad \forall k = 0, \dots, d-1.$$

Proof. Let P be a perfectly centered polytope. From Theorem 5, we know that for every k, the number of k-faces of $P+P^*$ is equal to the number of pairs of faces (G, F) of P, $G \subseteq F$ so that $\dim(G)+\dim(F^D) = k$, which means $\dim(G) + d - 1 - k = \dim(F)$. This is the number of chains of two nontrivial faces of dimensions i and $i + d - 1 - k$. $\square$

We can apply the main theorem to perfectly centered polytopes, which proves that if f is the f-vector of a perfectly centered polytope, then

$$\sum_{k=0}^{d-1} (-1)^k k \left(\sum_{i=0}^{k} f_{i,i+d-1-k} - f_k - f_{d-1-k} \right) = 0.$$

It turns out that we can extend this result to polytopes and Eulerian posets in general.

First, let us introduce the Bayer–Billera relations for extended f-vectors:

Lemma 13 ([1,6]). *Let P be an Eulerian poset of rank d, $S \subset \{0, \dots, d-1\}$, $\{i, k\} \subseteq S \cup \{-1, d\}$, $i < k - 1$, and S contains no j so that $i < j < k$. Then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup j}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

These relations have been presented as an extension of the Dehn–Sommerville relations. If we examine the special case where $S = \{i, k\} \subseteq \{-1, \dots, d\}$, we can write the following equation:

Lemma 14.

$$\sum_{j=i}^{k}(-1)^{j}f_{i,j,k}(P)=0.$$

Now for the theorem:

Theorem 6. *Let f be the extended f-vector of an Eulerian poset of rank d. Then*

$$\sum_{k=0}^{d-1}(-1)^{k}k\left(\sum_{i=0}^{k}f_{i,i+d-1-k}-f_{k}-f_{d-1-k}\right)=0.$$

Proof. Let P be an Eulerian poset of rank d. Below, we evaluate and rewrite each of the three terms in the parentheses multiplied by $(-1)^{(d-1-k)}k$. (The change of exponent simplifies computations.):

$$-\sum_{k=0}^{d-1}(-1)^{(d-1-k)}kf_{k}(P)=\sum_{i=0}^{d-1}(-1)^{d+i}if_{i,d}(P). \tag{1}$$

Then, by using $k'=d-1-k$:

$$-\sum_{k=0}^{d-1}(-1)^{(d-1-k)}kf_{d-1-k}(P)=\sum_{k'=0}^{d-1}(-1)^{k'-1}(d-1-k')f_{-1,k'}(P). \tag{2}$$

Finally, by using $k'=i+d-1-k$:

$$\begin{aligned}\sum_{k=0}^{d-1}\sum_{i=0}^{k}(-1)^{(d-1-k)}kf_{i,i+d-1-k}(P)&=\sum_{i=0}^{d-1}\sum_{k'=i}^{d-1}(-1)^{k'-i}(i+d-1-k')f_{i,k'}(P)\\&=\sum_{i=0}^{d-1}\sum_{k'=i}^{d-1}\left((-1)^{k'+i}if_{i,k'}(P)+(-1)^{k'+i}(d-1-k')f_{i,k'}(P)\right)\\&=\sum_{i=0}^{d-1}\sum_{k'=i}^{d-1}(-1)^{k'+i}if_{i,k'}(P)+\sum_{k'=0}^{d-1}\sum_{i=0}^{k'}(-1)^{k'+i}(d-1-k')f_{i,k'}(P).\end{aligned} \tag{3}$$

Combining Eqs. (1)–(3), we get:

$$\begin{aligned}&\sum_{k=0}^{d-1}(-1)^{(d-1-k)}k\left(\sum_{i=0}^{k}f_{i,i+d-1-k}-f_{k}-f_{d-1-k}\right)\\&=\sum_{i=0}^{d-1}\sum_{k=i}^{d}(-1)^{k+i}if_{i,k}(P)+\sum_{k'=0}^{d-1}\sum_{i=-1}^{k}(-1)^{k'+i}(d-1-k')f_{i,k}(P)\\&=\sum_{i=0}^{d-1}i\sum_{k=i}^{d}(-1)^{k+i}f_{i,k,d}(P)+\sum_{k=0}^{d-1}(d-1-k)\sum_{i=-1}^{k}(-1)^{k+i}f_{-1,i,k}(P)=0.\end{aligned}$$

The internal sums are zero by the Bayer–Billera relations (Lemma 14). □

So we see that this linear relation is a consequence of the Bayer–Billera relations. This is not surprising, in view of the theorem of their authors stating that all linear equalities holding for the extended f-vector of Eulerian posets are derived from these equalities ([2]).

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