AN INVARIANT OF QUADRATIC FORMS MOD 8

BY

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In a short note W. LEDERMANN [1] proved the

Theorem 1. *If f is an integral quadratic form in n variables with odd determinant D, then*

$$f(w, w) \equiv D+\tau-\operatorname{sgn} D \pmod 4, \tag{1}$$

for every integral vector w, which satisfies $f(x,x) \equiv f(x, w) \pmod 2$ for all integral vectors x. Here τ is the signature of f and $\operatorname{sgn} D = D|D|^{-1}$.

As a special case he found that if f is unimodular we have

$$f(w, w) \equiv \tau \pmod 4, \tag{2}$$

this was a corollary of topological investigations of F. HIRZEBRUCH and H. HOPF [2]. Now it may be readily verified that $f(w, w)$ is an invariant mod 8, since from $f(x, w) \equiv f(x, t) \pmod 2$ we find $t = w+2z$, with an integral vector z and $f(t, t) = f(w, w)+4f(w, z)+4f(z, z) \equiv f(w, w) \pmod 8$.

From the theory of the transformation of theta functions [3] one finds if f is definite positive and unimodular that

$$f(w, w) \equiv n \pmod 8. \tag{3}$$

And now one can ask for a suitable generalization of (1) modulo 8.

We define a gaussian sum by

$$G = \sum_{x \in M'/M} e^{\pi i f(x+\frac{1}{2}w,\, x+\frac{1}{2}w)}, \tag{4}$$

here M is the lattice of all integral vectors and M' is the dual one with respect to f, that is to say M' consists of all vectors z such that $f(z, a)$ is integral for all $a \in M$. Using classical theorems on Fourier expansions one finds

$$G = |D| \sum_{y \in M} \int_T e^{\pi i f(t+\frac{1}{2}w,\, t+\frac{1}{2}w)}\, e^{-2\pi i f(t, y)}\, dt,$$

where T is a fundamental domain of M.

$$G = |D| \sum_{y \in M} \int_T e^{\pi i f(t-y+\frac{1}{2}w,\, t-y+\frac{1}{2}w)}\, dt,$$

$$G = |D| \int e^{\pi i f(z,z)}\, dz.$$

here the integration must be extended for all coordinates of z from $-\infty$ to ∞. Transforming f in a diagonal form one can easily calculate this integral. We find then

$$G = |D|^{\frac{1}{2}} e^{\pi i\tau/4}. \tag{5}$$

On the other hand we find

$$G = e^{\frac{1}{4}\pi i f(w,w)} \sum_{x \in M'/M} e^{\pi i f(x,x)} e^{\pi i f(x,w)}$$

and since D is odd we may replace x by $2y$ and thus

$$G = e^{\frac{1}{4}\pi i f(w,w)} \sum_{y \in M'/M} e^{4\pi i f(y,y)}.$$

Now we write

$$G_0 = \sum_{y \in M'/M} e^{4\pi i f(y,y)} \tag{6}$$

and we have proved:

Theorem 2.

$$e^{\frac{1}{4}\pi i(f(w,w)-\tau)} = |D|^{\frac{1}{2}} G_0^{-1}.$$

We consider some special cases. First let $D = \pm 1$, then $G_0 = 1$ and hence

$$f(w, w) \equiv \tau \pmod 8. \tag{7}$$

Further, one can easily calculate G_0^2 (e.g. by transforming f in a diagonal form modulo D)

$$G_0^2 = \left(\frac{-1}{|D|}\right) |D|$$

and thus we obtain

$$f(w, w) \equiv |D| + \tau - 1 \pmod 4. \tag{8}$$

Since $|D| + \tau - 1 \equiv D + n - 1 \equiv D + \tau - \operatorname{sgn} D \pmod 4$ this is equivalent with theorem 1.

Theorem 2 gives a relation between certain local invariants of f; $f(w, w)$ mod 8 is an invariant for the place 2, which can be described as a character of the local Witt group which has on the form αx^2, with odd α the value $e^{\frac{1}{4}\pi i\alpha}$ and which equals 1 on the even forms with odd determinant.

If $p|D$ we can write f at the place p as $\sum_{i=1}^{n} f_i p^{\alpha_i} x_i^2$ and this prime gives as contribution to the expression $G_0^{-1}|D|^{\frac{1}{2}}$ a factor (we use the multiplicity properties of gaussian sums)

$$\varphi_p(f) = \prod_{i=1}^{n} \left(\frac{f_i}{p^{\alpha_i}}\right) e^{\frac{1}{4}\pi i(p^{\alpha_i}-1)}.$$

We can express φ_p in a simple way in the local invariants of f. With every p-adic form f, there are two forms f_0 and f_1 over the field with p elements, such that the Witt class of f is determined by the form $f_0^* + pf_1^*$ where f_0^* and f_1^* are p-adic forms, which give by restriction modulo p the forms f_0 and f_1 (vid [4]).

The Witt group of the field with p elements consists of four elements. If $p \equiv 3 \pmod 4$ the Witt group is a cyclic group, if $p \equiv 1 \pmod 4$ this group is the direct product of two cyclic groups of order two. Let $\varepsilon_p(f_1)$ be the character of this group which equals $e^{\frac{1}{4}\pi i(p-1)}$ for $f_1(x, x) = x^2$ and in the second case also $-e^{\frac{1}{4}\pi i(p-1)}$ for εx^2, where ε is a non-square modulo p. Now it can be readily verified that $\varphi_p(f) = \varepsilon_p(f_1)$. Thus we have proved

Theorem 3.

$$e^{\frac{1}{4}\pi i(f(w,w)-r)} = \prod_{p \mid D} \varphi_p(f),$$

where $\varphi_p(f) = \varepsilon_p(f_1)$ *is a character of the Witt group over the p-adic field.*

This is an example of a relation between the local invariants of quadratic forms. Another example is given by EICHLER [5]. It may be an interesting question to ask for other relations of this type.

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